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MONOIDS AND POINTED S-PROTOMODULAR CATEGORIES

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(*communicated by George Janelidze*)

Abstract

We investigate the notion of pointed S-protomodular category, with respect to a suitable class S of points, and we prove that these categories satisfy, relatively to the class S, many partial aspects of the properties of Mal'tsev and protomodular categories, like the split short five lemma for S-split exact sequences, or the fact that a reflexive S-relation is transitive. The main examples of S-protomodular categories are the category of monoids and, more generally, any category of monoids with operations, where the class S is the class of Schreier points.

1. Introduction

The notions of protomodular [**2**] and semi-abelian [**13**] category allowed to intrinsically describe many classical properties and results in group theory (see, for example, [**1**]), and to point out the similarities with other algebraic structures, like rings, associative algebras, Lie algebras and many others.

From a categorical point of view, much less is known for other algebraic structures, like monoids. However, the notion of monoid is fundamental in category theory. Until now, the most important categorical property of monoids that has been pointed out is unitality [**3**]; this property allows to describe the algebraic notion of commutativity of subobjects and, more generally, of morphisms.

In the recent paper [**15**], the three last authors introduced the algebraic context of *monoids with operations*, inspired by the analogous notion, introduced by Porter [**19**], of groups with operations (see also [**17**], where this kind of structures was first considered). This new context includes, among other examples, monoids, commutative monoids, semirings, join-semilattices with a bottom element and distributive lattices with a bottom element (or a top one). The study of the semidirect products in this setting allowed to identify a class of points, called *Schreier points* (the name is inspired by the notion of Schreier internal category introduced by Patchkoria [**18**] in the category of monoids). Schreier points correspond to actions via the semidirect product construction, as it is proved in [**15**].

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In the monograph [**9**], and in the paper [**8**], the present authors observed moreover that, in the case of monoids, Schreier points (also called Schreier split epimorphisms) satisfy some important properties that are classically known to be satisfied by all split epimorphisms of groups (but not by all split epimorphisms of monoids), like the split short five lemma. Defining a *Schreier reflexive relation* as a reflexive relation such that the point given by the first projection (with section given by the reflexivity morphism) is a Schreier one, it was proved that any Schreier reflexive relation is transitive. Moreover, in [**9**] other interesting properties of Schreier split epimorphisms of monoids were studied, and they were extended to the case of semirings. In particular, it was shown that special Schreier extensions with a fixed abelian kernel form an abelian group, as it happens for all extensions with abelian kernel in the category of groups. We recall that a special Schreier extension is a surjective homomorphism whose kernel pair is a Schreier equivalence relation [**9**, Chapter 7].

All these results gave evidence of the need of a conceptual notion which captures this algebraic context; it was introduced, in the pointed case, in [**9**], under the name of *S-protomodularity*, where S is a suitable class of points. In the present paper we investigate the properties of this intrinsic setting and we show that it conceptually allows to recover many partial aspects of the properties of Mal'tsev [**10**] and protomodular categories, in particular with respect to the internal structures and to centrality for reflexive relations. As mentioned in [**9**], the main examples of such a situation are the category Mon of monoids and $SRng$ of semirings with the class S of Schreier points. Here we enlarge the class of examples, showing that all categories of monoids with operations are S-protomodular. Note that our approach to relative non-abelian homological algebra is different from the one initiated by T. Janelidze in [**14**] and developed by her in several later papers: in our work, the word "relative" refers to a chosen class of points, i.e. of split epimorphisms with specified splitting, while in T. Janelidze's papers it refers to a chosen class of (not necessarily split) regular epimorphisms.

A useful notion that we introduce in the present paper is the one of S-special morphism. We show that it allows to associate with any S-protomodular category $\mathbb{C}$ a protomodular subcategory $S^\sharp\mathbb{C}$, called the *protomodular core* of $\mathbb{C}$ relatively to S. If $\mathbb{C}$ is the category of monoids, equipped with the class of Schreier points, its protomodular core is the category Gp of groups. This gives then a characterization of groups among monoids. In the same way, we prove that the protomodular core of the category of semirings is the category of rings, and we generalize this result to any category of monoids with operations, with respect to the class of Schreier points. We also prove that the notion of S-special morphism allows a characterization of reflexive graphs (relatively to the class S) that are internal groupoids. This characterization is completely analogous to the one known for Mal'tsev categories (see [**11**]).

The paper is organized as follows. In Section 2 we recall from [**3**] the notion of unital category, and from [**9**] a generalization of it, namely the notion of $\mathbb{C}'$-unital category, which will be used later to describe some Mal'tsev-type properties of S-protomodular categories. In Section 3 we define S-protomodular categories and we study their first properties. In Section 4 we recall the notion of monoids with operations and of the class S of Schreier points, and we show that they are examples of S-protomodular categories. In Section 5 we prove that an S-reflexive graph has at most one structure of internal category, and that any S-reflexive relation is transitive, relating S-protomodular categories to Mal'tsev ones. In Section 6 we define S-special morphisms and we use them to characterize internal groupoids among internal S-categories and equivalence relations among S-reflexive relations. Moreover, we define the protomodular core of an S-protomodular category, and describe it in the examples of monoids with operations. In Section 7 we describe other Mal'tsev aspects of S-protomodular categories, mainly related to the centrality of reflexive relations. Moreover, we show that an S-reflexive graph such that the domain morphism is S-special is an internal groupoid if and only if the kernel pairs of the domain and the codomain morphisms centralize each other.

2. Unital and $\mathbb{C}'$-unital categories

We start by recalling from [**3**] the following definition.

Definition 2.1. Let $\mathbb{C}$ be a pointed category with finite products. Given two objects A and B in $\mathbb{C}$, consider the following diagram

$$A \underset{\langle 1_A, 0\rangle}{\overset{p_A}{\rightleftarrows}} A \times B \underset{\langle 0, 1_B\rangle}{\overset{p_B}{\rightleftarrows}} B.$$

The category $\mathbb{C}$ is said to be *unital* if, for every pair of objects $A, B \in \mathbb{C}$, the morphisms $\langle 1_A, 0\rangle$ and $\langle 0, 1_B\rangle$ are jointly extremally epimorphic.

This means that, if $\langle 1_A, 0\rangle$ and $\langle 0, 1_B\rangle$ factor jointly through a monomorphism $m\colon M \to A \times B$, then m is an isomorphism. In other terms, the join of the two subobjects A and B in $A \times B$ is $A \times B$ itself.

If $\mathbb{C}$ is finitely complete, then a pair of morphisms with the same codomain is jointly extremally epimorphic if and only if it is jointly strongly epimorphic (see, for example, the Appendix of [**1**] for more details), hence we will use this second name throughout the paper. Moreover, every jointly strongly epimorphic pair is jointly epimorphic. Hence finitely complete unital categories are a setting where it is possible to express a categorical notion of commutativity.

Definition 2.2 ([**12**])**.** Let $\mathbb{C}$ be a finitely complete unital category. Two morphisms with the same codomain $f\colon X \to Z$ and $g\colon Y \to Z$ are said to *commute* (or to *cooperate*, as in [**4**]) if there exists a morphism $\varphi\colon X \times Y \to Z$ such that both triangles in the following diagram commute:

$$\begin{array}{ccccc} X & \xrightarrow{\langle 1_X, 0\rangle} & X \times Y & \xleftarrow{\langle 0, 1_Y\rangle} & Y \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \varphi} & \swarrow{\scriptstyle g} & \\ & & Z. & & \end{array}$$

The morphism φ is necessarily unique, because $\langle 1_X, 0\rangle$ and $\langle 0, 1_Y\rangle$ are jointly epimorphic, and it is called the *cooperator* of f and g.

The uniqueness of the cooperator makes commutativity a property and not an additional structure in the category $\mathbb{C}$.

A generalization of the notion of unital category, that we shall need later on, is given by the following definition, that we recall from [**9**].

Definition 2.3. Let $\mathbb{C}'$ be a full subcategory of a pointed finitely complete category $\mathbb{C}$. The category $\mathbb{C}$ is said to be $\mathbb{C}'$-*unital* when, for any object $A \in \mathbb{C}'$ and any object $B \in \mathbb{C}$, the morphisms $\langle 1_A, 0\rangle$ and $\langle 0, 1_B\rangle$ in the following diagram are jointly strongly epimorphic:

$$A \underset{\langle 1_A,0\rangle}{\overset{p_A}{\rightleftarrows}} A \times B \underset{\langle 0,1_B\rangle}{\overset{p_B}{\rightleftarrows}} B.$$

In a finitely complete $\mathbb{C}'$-unital category we can still speak of cooperating pairs (f, g) of morphisms, provided that the domain X of f belongs to $\mathbb{C}'$. More generally, $X \times Y$ being isomorphic to $Y \times X$, we can speak of cooperating pair of morphisms as soon as the domain of one of the two maps is in $\mathbb{C}'$.

Proposition 2.4. *Suppose that* $\mathbb{C}$ *is* $\mathbb{C}'$*-unital and that* $\mathbb{C}'$ *is closed under finite products (in particular, it contains the zero object* 0*). Then* $\mathbb{C}'$ *is unital.*

Proof. Straightforward. □

3. S-protomodular categories

From now on, we will denote by $\mathbb{C}$ a pointed finitely complete category. By a *point* in $\mathbb{C}$ we mean a pair (f, s) of morphisms in $\mathbb{C}$ such that $fs = 1$; in other terms, f is a split epimorphism with fixed section s. Let us consider the functor $\P_{\mathbb{C}} \colon Pt\mathbb{C} \to \mathbb{C}$ which associates with every point (f, s) the codomain of f. This functor is a fibration, since split epimorphisms are stable under pullbacks.

Let S be a class of points in $\mathbb{C}$ which is stable under pullbacks in the following sense: given a downward pullback

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow\uparrow{\scriptstyle s'} & & {\scriptstyle f}\downarrow\uparrow{\scriptstyle s} \\ Y' & \xrightarrow[g]{} & Y, \end{array}$$

where the two vertical morphisms are split epimorphisms and the upward square commutes (or, in other terms, the pair (g, g') is a morphism of points), if (f, s) belongs to S, then (f', s') belongs to S too. We will denote by $SPt\mathbb{C}$ the full subcategory of $Pt\mathbb{C}$ whose objects are those which are in S. Since S is stable under pullbacks, this class determines a subfibration $\P_{\mathbb{C}}^S$ of the fibration of points:

$$\begin{array}{ccc} SPt\mathbb{C} & \xrightarrow{j} & Pt\mathbb{C} \\ & {\scriptstyle \P_{\mathbb{C}}^S}\searrow \quad \swarrow{\scriptstyle \P_{\mathbb{C}}} & \\ & \mathbb{C}. & \end{array}$$

Given a point (f, s) in $\mathbb{C}$, we say that it is a *strong point* if the pair (k, s), where k is a kernel of f, is jointly strongly epimorphic. Strong points were introduced independently in [**5**], under the name of strongly split epimorphisms, and in [**16**], under the name of regular points.

Definition 3.1 ([**9**, Definition 8.1.1])**.** Let $\mathbb{C}$ be a pointed finitely complete category, and S a class of points stable under pullbacks. $\mathbb{C}$ is said to be *S-protomodular* when:

(1) any object in $SPt\mathbb{C}$ is a strong point;

(2) $SPt\mathbb{C}$ is closed under finite limits in $Pt\mathbb{C}$ (in particular, it contains the terminal object $0 \leftrightarrows 0$ of $Pt\mathbb{C}$).

So, S is a class of strong points. Condition (2) implies that any fiber $SPt_Y\mathbb{C}$ is closed under finite limits in the fiber $Pt_Y\mathbb{C}$ and that any change-of-base functor with respect to $¶^S_{\mathbb{C}}$ is left exact. The fact that $SPt\mathbb{C}$ contains the terminal object implies that the class S contains the isomorphisms (because S is stable under pullbacks, and any isomorphism can be seen as a pullback of the terminal object $0 \leftrightarrows 0$ of $Pt\mathbb{C}$). Hence, any fiber $SPt_Y\mathbb{C}$ is pointed, with zero object given by the point $Y \underset{1_Y}{\overset{1_Y}{\leftrightarrows}} Y$.
We observe that the name *S-protomodular* is justified by the fact that a pointed finitely complete category $\mathbb{C}$ is protomodular if and only if every point in $\mathbb{C}$ is a strong point. Hence our notion is a relative version of the one of pointed protomodular category, with respect to the class S (the general non-pointed case needs further tools and notions).

Protomodular categories are also characterized by the fact that all change-of-base functors of the fibration of points $¶_{\mathbb{C}}$ are conservative. The relative version of this property is considered in the following

Proposition 3.2 ([**9**, Proposition 8.1.2])**.** *Let $\mathbb{C}$ be a pointed finitely complete category and S a class of points stable under pullbacks. Then:*

(i) when S satisfies Condition (1) *of Definition 3.1, any fiber $Pt_Y\mathbb{C}$ is $SPt_Y\mathbb{C}$-unital;*

(ii) when $\mathbb{C}$ is S-protomodular, any fiber $SPt_Y\mathbb{C}$ is unital;

(iii) when $\mathbb{C}$ is S-protomodular, any change-of-base functor with respect to the fibration $¶^S_{\mathbb{C}}$ is conservative.

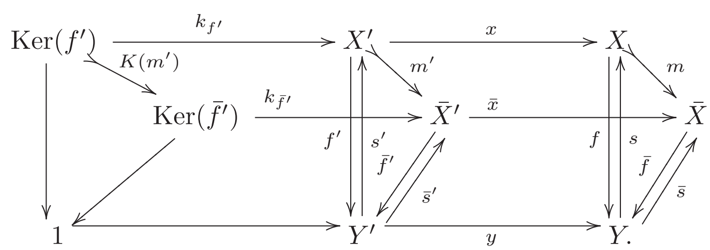
Proof. (i) Consider the following left hand side downward pullback of split epimorphisms, where the point (f, s) is in the fiber $SPt_Y\mathbb{C}$ and k_f is a kernel of f:

$$\begin{array}{ccc} X' & \underset{g'}{\overset{t'}{\leftrightarrows}} & X \\ {\scriptstyle f'}\downarrow\uparrow{\scriptstyle s'} & & {\scriptstyle f}\downarrow\uparrow{\scriptstyle s} \\ Y' & \underset{g}{\overset{t}{\leftrightarrows}} & Y, \end{array} \qquad\qquad \begin{array}{ccccc} \mathrm{Ker}(f) & \overset{k_f}{\rightarrowtail} & X & \overset{t'}{\longrightarrow} & X' \\ & & {\scriptstyle f}\downarrow\uparrow{\scriptstyle s} & & {\scriptstyle f'}\downarrow\uparrow{\scriptstyle s'} \\ & & Y & \underset{t}{\longrightarrow} & Y'. \end{array}$$

The left hand side pullback above represents the product of (f, s) and (g, t) in $Pt_Y\mathbb{C}$, hence we have to prove that the pair (t', s') is jointly strongly epimorphic. We have that (f', s') belongs to $SPt_{Y'}\mathbb{C}$, since $¶^S_{\mathbb{C}}$ is a subfibration of $¶_{\mathbb{C}}$. So the point (f', s') is strong. On the other hand, the right hand side square is still a pullback, so the map $t'k_f$ is a kernel of f'. Accordingly the pair $(t'k, s')$ is jointly strongly epimorphic. So this is equally the case for the pair (t', s'). Accordingly the fibre $Pt_Y\mathbb{C}$ is $SPt_Y\mathbb{C}$-unital.

(ii) This comes immediately from (1), because, as we already observed, if $\mathbb{C}$ is S-protomodular, then $SPt_Y\mathbb{C}$ is closed under finite products in $Pt_Y\mathbb{C}$, so that $SPt_Y\mathbb{C}$ is unital by Proposition 2.4.

(iii) Since any change-of-base functor with respect to $\P^S_{\mathbb{C}}$ is left exact, it is enough to prove that, given a morphism $y\colon Y' \to Y$ in $\mathbb{C}$, the change-of-base functor $y^*\colon SPt_Y\mathbb{C} \to SPt_{Y'}\mathbb{C}$ is conservative on monomorphisms (see Lemma 3.3 below). Let us consider the following diagram, where all the quadrangles are pullbacks, all the points are in $SPt\mathbb{C}$, and m is a monomorphism:



Suppose moreover that $y^*(m) = m'$, and consequently $K(m')$ (which is the restriction of m' to the kernels), are isomorphisms. We need to prove that m is an isomorphism too. This comes from the facts that $K(m) \simeq K(m')$ are isomorphisms, the pair $(k_{\bar{f}}, \bar{s})$ is jointly epimorphic and both morphisms factor through m, indeed $k_{\bar{f}} = mk_f K(m)^{-1}$, and $\bar{s} = ms$. □

Lemma 3.3. *Suppose that $U\colon \mathbb{C} \to \mathbb{D}$ is a left exact functor such that, for any monomorphism m in $\mathbb{C}$, if Um is an isomorphism in $\mathbb{D}$ then m is an isomorphism. Then U is conservative.*

Proof. Given any morphism f in $\mathbb{C}$, consider the kernel pair of f:

$$R[f] \underset{p_1}{\overset{p_0}{\underset{\overset{s_0}{\longleftarrow}}{\rightrightarrows}}} X \xrightarrow{\ f\ } Y.$$

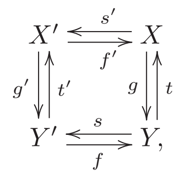
Since U is left exact, we have that $UR[f]$ is the kernel pair of Uf:

$$UR[f] = R[Uf] \underset{Up_1}{\overset{Up_0}{\underset{\overset{Us_0}{\longleftarrow}}{\rightrightarrows}}} UX \xrightarrow{\ Uf\ } UY.$$

Suppose that Uf is an isomorphism. Then Us_0 is an isomorphism. Since s_0 is a monomorphism, our hypothesis implies that s_0 is an isomorphism. But then f is a monomorphism, hence an isomorphism by our hypothesis. □

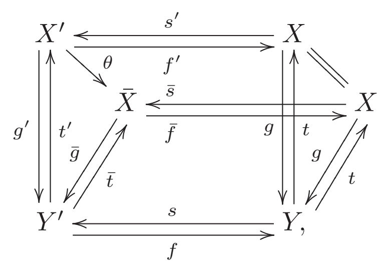
The following result is the relative version of another property which is known to hold in protomodular categories, and more generally in Mal'tsev ones (see [**1**]).

Proposition 3.4. *Let $\mathbb{C}$ be a pointed finitely complete category, and S a class of strong points which is stable under pullbacks. Given any commutative square of split* *epimorphisms*



where the point (g,t) is in S, the induced factorization to the pullback of (g,t) along f is an extremal epimorphism.

Proof. Consider the following diagram



where the square $f\bar{g} = g\bar{f}$ is a pullback. Since S is stable under pullbacks, $(\bar{g}, \bar{t})$ belongs to S. Moreover, since the category $Pt_Y\mathbb{C}$ is $SPt_Y\mathbb{C}$-unital (by Proposition 3.2) and the pullback considered above is actually the product of the two points $Y' \underset{f}{\overset{s}{\leftrightarrows}} Y$ and $X \underset{g}{\overset{t}{\leftrightarrows}} Y$ in the category $Pt_Y\mathbb{C}$, the pair $(\bar{t}, \bar{s})$ is jointly strongly epimorphic. Now let θ be the factorization in question. Suppose $j\colon \bar{U} \rightarrowtail \bar{X}$ is a monomorphism such that θ factors through it by a map $\theta'\colon j\theta' = \theta$. Then we have that both $\bar{t}$ and $\bar{s}$ factor through j: $\bar{t} = j\theta' t'$, and $\bar{s} = j\theta' s'$. Since the pair $(\bar{t}, \bar{s})$ is jointly strongly epimorphic, j is an isomorphism. $\square$

4. Schreier points in monoids with operations

The aim of this section is to introduce an important class of examples of the situation described in the previous one. We start by recalling from [**15**] the following definition, which was inspired by the analogous one of *groups with operations* introduced by Porter in [**19**] (we observe that the axioms defining groups with operations were first considered in [**17**], although no name was given there for such structures).

Definition 4.1 ([**15**, Definition 4.1])**.** Let Ω be a set of finitary operations such that the following conditions hold: if Ω_i is the set of i-ary operations in Ω, then:

(1) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;

(2) there is a binary operation $+ \in \Omega_2$ (not necessarily commutative) and a constant $0 \in \Omega_0$ satisfying the usual axioms for monoids;

(3) $\Omega_0 = \{0\}$;

(4) let $\Omega_2' = \Omega_2 \backslash \{+\}$; if $* \in \Omega_2'$, then $*^\circ$ defined by $x *^\circ y = y * x$ is also in Ω_2';

(5) any $* \in \Omega_2'$ is left distributive w.r.t. $+$, i.e.:

$$a * (b + c) = a * b + a * c;$$

(6) for any $* \in \Omega_2'$ we have $b * 0 = 0$;

(7) any $\omega \in \Omega_1$ satisfies the following conditions:
   - $\omega(x + y) = \omega(x) + \omega(y)$;
   - for any $* \in \Omega_2'$, $\omega(a * b) = \omega(a) * b$.

Let moreover E be a set of axioms including the ones above. We will denote by $\mathbb{C}$ the category of (Ω, E)-algebras. We call the objects of $\mathbb{C}$ *monoids with operations*.

As already observed in [**15**], the definition above does not include the case of groups, or more generally, the one of groups with operations. Indeed, the unary operation given by the group inverses, denoted by $-$, does not satisfy Condition (7). In order to recover all these structures, it suffices to add another condition (already considered in Porter's definition of groups with operations, and, even before, in [**17**]): if the base monoid structure (given by the operations $+$ and 0) is a group, then the operation $-$ should be distinguished from the other unary operations. In other terms, Condition (7) should be satisfied only by operations in $\Omega_1' = \Omega_1 \backslash \{-\}$. Adding this condition, the definition above includes all groups with operations.

Example 4.2. Apart from the known structures covered by Porter's definition, such as groups, rings, associative algebras, Lie algebras and many others, our definition includes the cases of monoids, commutative monoids, semirings (i.e. rings where the additive structure is not necessarily a group, but just a commutative monoid), join-semilattices with a bottom element, distributive lattices with a bottom element (or a top one).

Let us observe that, if $\mathbb{C}$ is a category of monoids with operations, then it is pointed, complete and unital.

We now introduce the points that will form the desired class S. For the rest of the section, $\mathbb{C}$ will denote a category of monoids with operations.

Definition 4.3 ([**15**, Definition 2.6])**.** A point $A \underset{f}{\overset{s}{\rightleftarrows}} B$ in $\mathbb{C}$ is said to be a *Schreier point* (or a *Schreier split epimorphism*, as in [**9**]) when, for any $a \in A$, there exists a unique α in the kernel $\mathrm{Ker}(f)$ of f such that $a = \alpha + sf(a)$.

As shown in [**15**], in the category Mon, Schreier points are equivalent to monoid actions, where an action of a monoid B on a monoid X is a monoid homomorphism $B \to End(X)$, with $End(X)$ denoting the monoid of endomorphisms of X.

Example 4.4. We denote by $\mathbb{Z}^*$ the monoid of non-zero integers with the usual multiplication, and by $\mathbb{N}^*$ its submonoid whose elements are the numbers greater than 0. Then the point

$$\mathbb{Z}^* \underset{abs}{\overset{i}{\rightleftarrows}} \mathbb{N}^*,$$

where i is the inclusion and abs associates with any integer its absolute value, is a Schreier point. In fact $\mathrm{Ker}(abs) = \{\pm 1\}$, and it is immediate to see that any non-zero integer z can be written in a unique way as $z = \pm 1 \cdot |z|$.

The following result was already proved in [**9**] for the particular case of monoids (Proposition 2.1.4 there). We don't repeat the proof, since it is the same also in this more general framework.

Proposition 4.5. *A point* $A \underset{f}{\overset{s}{\leftrightarrows}} B$ *is a Schreier point if and only if there exists a map (which is not a morphism, in general)* $q\colon A \dashrightarrow Ker(f)$ *such that:*

$$q(a) + sf(a) = a,$$
$$q(\alpha + s(b)) = \alpha,$$

for every $a \in A$*,* $\alpha \in Ker(f)$ *and* $b \in B$*.*

We shall call the following diagram

$$\mathrm{Ker}(f) \underset{k}{\overset{q}{\leftrightarrows}} A \underset{f}{\overset{s}{\leftrightarrows}} B$$

the canonical *Schreier split sequence* associated with the Schreier point (f, s) and q the associated *Schreier retraction*. The following properties of the retraction q (already proved in the case of monoids, see Proposition 2.1.5 in [**9**]) will be useful later. For the sake of simplicity, we consider k just as an inclusion.

Proposition 4.6. *Given a Schreier point* $A \underset{f}{\overset{s}{\leftrightarrows}} B$*, we have:*

(i) $qk = 1_{Ker(f)}$*;*
(ii) $qs = 0$*;*
(iii) $q(0) = 0$*;*
(iv) if $b \in B$ *and* $\alpha \in Ker(f)$*, then* $q(s(b) + \alpha) + s(b) = s(b) + \alpha$*;*
(v) for every $a, a' \in A$*,* $q(a + a') = q(a) + q(sf(a) + q(a'))$*.*

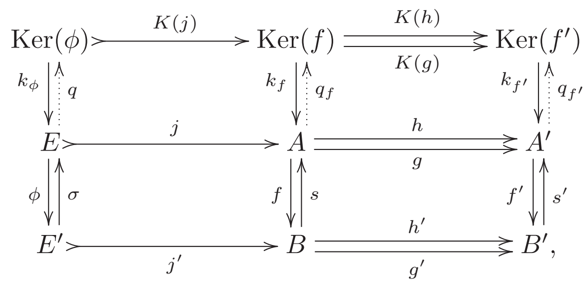
In Chapter 2 of [**9**] it was shown that, in the case of monoids, Schreier points are strong points, stable under pullbacks and closed under finite limits in the category of all points. The proofs for any category of monoids with operations are the same, so we will not repeat them. This means that any category $\mathbb{C}$ of monoids with operations is S-protomodular, when S is the class of Schreier points. We only give in full details the following result, which is an improvement of Proposition 2.4.5 in [**9**]:

Proposition 4.7. *Schreier points are closed under equalizers, i.e. the equalizer of two parallel morphisms between Schreier points is a Schreier one.*

Proof. Given two parallel morphisms of Schreier points

$$\begin{array}{ccc} A & \underset{g}{\overset{h}{\rightrightarrows}} & A' \\ f \downarrow \uparrow s & & f' \downarrow \uparrow s' \\ B & \underset{g'}{\overset{h'}{\rightrightarrows}} & B' \end{array}$$

consider the following diagram



where j is an equalizer of h and g and j' is an equalizer of h' and g' in $\mathbb{C}$. Then the lower part of the diagram is an equalizer diagram in $Pt\mathbb{C}$. Since the kernel functor preserves equalizers, $K(j)$ is an equalizer of $K(h)$ and $K(g)$ in $\mathbb{C}$, and hence in the category *Set* of sets. By Proposition 2.3.1 in [**9**], the Schreier retractions q_f and $q_{f'}$ make the upward right hand side square commute; hence we get a factorization q which satisfies the conditions of a Schreier retraction for the point (ϕ, σ) and makes it a Schreier point. □

5. S-reflexive relations and S-categories

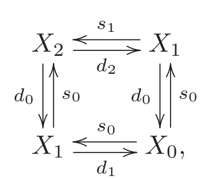
We recall that a reflexive graph in a category $\mathbb{C}$ is a diagram of the form

$$X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0, \quad s_0 \colon X_0 \to X_1 \tag{1}$$

such that $d_0 s_0 = 1_{X_0} = d_1 s_0$. A reflexive relation is a reflexive graph such that the pair (d_0, d_1) is jointly monomorphic.

Definition 5.1. A reflexive graph (resp. reflexive relation) (1) in an S-protomodular category $\mathbb{C}$ is said to be an *S-reflexive graph* (resp. *S-reflexive relation*) if the point (d_0, s_0) is in S.

As a consequence of Condition (2) of the definition of an S-protomodular category $\mathbb{C}$, S-reflexive graphs are closed under finite limits inside the category of reflexive graphs in $\mathbb{C}$. We are now ready to study the relationship between reflexive graphs and categories in an S-protomodular category. Let us recall that an internal category $\underline{X}_1$ in $\mathbb{C}$ is a reflexive graph (1) such that the object X_2 of composable pairs of morphisms, defined by the following pullback of split epimorphisms



(2)

is endowed with a composition map $d_1 \colon X_2 \to X_1$ satisfying the remaining simplicial identities:

(C1) $d_0d_1 = d_0d_0$, $d_1d_1 = d_1d_2$ (incidence axioms);

(C2) $d_1s_0 = 1_{X_1}$, $d_1s_1 = 1_{X_1}$ (composition with identities).

This composition must satisfy the associativity axiom. For that, consider the following pullback of split epimorphisms (where X_3 is the object of composable triples):

$$\begin{array}{ccc} X_3 & \underset{d_3}{\overset{s_2}{\leftrightarrows}} & X_2 \\ {\scriptstyle d_0}\downarrow\uparrow{\scriptstyle s_0} & & {\scriptstyle d_0}\downarrow\uparrow{\scriptstyle s_0} \\ X_2 & \underset{d_2}{\overset{s_1}{\leftrightarrows}} & X_1. \end{array} \tag{3}$$

The composition map d_1 induces a couple of maps $(d_1, d_2)\colon X_3 \rightrightarrows X_2$ such that $d_0d_1 = d_0d_0$, $d_2d_1 = d_1d_3$ and $d_0d_2 = d_1d_0$, $d_2d_2 = d_2d_3$. The associativity is given by the remaining simplicial axiom:

(C3) $d_1d_1 = d_1d_2$.

It is well known that on a reflexive graph in a Mal'tsev category there is at most one structure of internal category (see Theorem 2.2 in [**11**]). A similar result holds for S-protomodular categories:

Proposition 5.2. *Let $\mathbb{C}$ be an S-protomodular category. On an S-reflexive graph there is at most one structure of internal category. It is sufficient to have the composition map $d_1\colon X_2 \to X_1$ satisfying Axiom (C2); Axioms (C1) and (C3) come for free.*

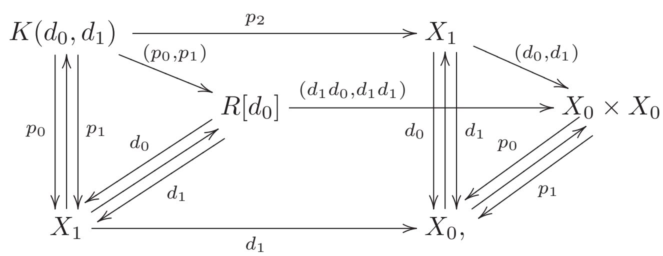
Proof. Give an S-reflexive graph (1), consider diagram (2). Since the rightward horizontal square is a pullback, and the right hand side point is in S, the left hand side one is in S, too. Moreover, since the category $Pt_{X_0}\mathbb{C}$ is $SPt_{X_0}\mathbb{C}$-unital (by Proposition 3.2) and the pullback above is actually the product of the two points $X_1 \underset{d_0}{\overset{s_0}{\leftrightarrows}} X_0$ and $X_1 \underset{d_1}{\overset{s_0}{\leftrightarrows}} X_0$ in the category $Pt_{X_0}\mathbb{C}$, the pair (s_0, s_1) is jointly (strongly) epimorphic. Hence there is at most one map d_1 satisfying Axiom (C2). Axiom (C1) can be also verified by composition with the pair (s_0, s_1). Axiom (C3) comes by composition with the pair (s_0, s_2) of diagram (3), which is jointly (strongly) epimorphic as well. □

If an S-reflexive graph is endowed with a structure of internal category, then it will be called an *S-category*.

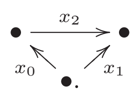
Let a reflexive graph (1) be given. Let us recall that its simplicial kernel is the universal 2-simplicial object associated with it, namely the universal solution relatively to the incidence (C1) and associative (C3) axioms:

$$K(d_0,d_1) \; \begin{array}{c} \xrightarrow{p_0} \\ \xleftarrow{s_0} \\ \xrightarrow{p_1} \\ \xleftarrow{s_1} \\ \xrightarrow{p_2} \end{array} \; X_1 \; \begin{array}{c} \xrightarrow{d_0} \\ \xleftarrow{s_0} \\ \xrightarrow{d_1} \end{array} \; X_0.$$

In a finitely complete category $\mathbb{C}$, it is obtained by the following pullback of reflexive graphs

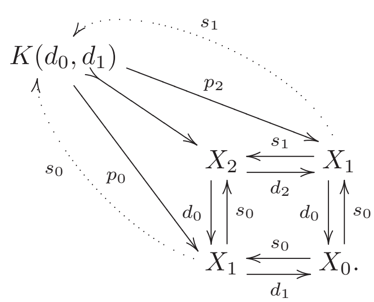


where $p_0, p_1 \colon X_0 \times X_0 \to X_0$ are the product projections, and $R[d_0]$ denotes the kernel equivalence relation of d_0. In set-theoretical terms, $K(d_0, d_1)$ is the set of triples (x_0, x_1, x_2) of elements of X_1 whose incidence conditions are given by the following drawing:



Proposition 5.3. *Let $\mathbb{C}$ be an S-protomodular category. Any S-reflexive relation is transitive.*

Proof. Let us consider an S-reflexive relation (1). The square $d_1 p_0 = d_0 p_2$ in the diagram above determines a factorization $(p_0, p_2) \colon K(d_0, d_1) \to X_2$ to the pullback of split epimorphisms (2):



Since $(d_0, d_1) \colon X_1 \rightarrowtail X_0 \times X_0$ is a relation, and hence d_0 and d_1 are jointly monomorphic, the factorization (p_0, p_2) is a monomorphism. In order to prove this fact, it suffices to observe that it is true in set-theoretical terms, and that it is invariant under the Yoneda embedding. The right hand side point (d_0, s_0) is in S, because $X_1 \underset{d_1}{\overset{d_0}{\underset{\longleftarrow s_0 -}{\rightrightarrows}}} X_0$ is an S-reflexive relation, and according to Proposition 3.4, the factorization $(p_0, p_2) \colon K(d_0, d_1) \rightarrowtail X_2$ is an extremal epimorphism, and hence an isomorphism; accordingly the morphism $X_2 \xrightarrow{(p_0,p_2)^{-1}} K(d_0, d_1) \xrightarrow{p_1} X_1$ produces the desired transitivity map. $\square$

In any Mal'tsev category, and in particular in any protomodular category, reflexive relations are always equivalence relations. In an S-protomodular category this is false, even for S-reflexive relations: an S-reflexive relation doesn't need to be an equivalence relation, because symmetry can fail. The following is a concrete counterexample in the category Mon, equipped with the class of Schreier points described in Section 4.

Example 5.4 ([**8**, Example 5.3]). The order relation in Mon given by the usual order between natural numbers:

$$\mathcal{O}_{\mathbb{N}} \underset{d_1}{\overset{d_0}{\rightrightarrows}} \mathbb{N}, \quad s_0 \colon \mathbb{N} \to \mathcal{O}_{\mathbb{N}}$$

where

$$\mathcal{O}_{\mathbb{N}} = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x \leqslant y\}$$

is a Schreier order relation, hence it is reflexive and transitive, but not symmetric.

6. S-special morphisms and internal S-groupoids

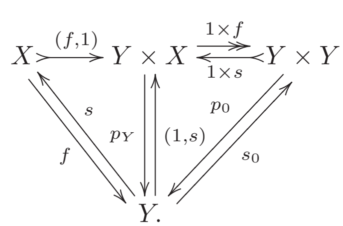
It is well known that every internal category in a Mal'tsev category is a groupoid. This is not true in S-protomodular categories. The aim of this section is to describe what are the additional conditions for an S-category to be a groupoid. For that we need the following notion.

Definition 6.1. Let $\mathbb{C}$ be an S-protomodular category. A morphism $f \colon X \to Y$ in $\mathbb{C}$ will be called *S-special* when the kernel equivalence relation $R[f]$ is an S-equivalence relation (which means that its underlying reflexive graph is an S-reflexive graph). An object X will be called *S-special* when the terminal morphism $\tau_X \colon X \to 1$ is S-special.

In an S-protomodular category, the S-special morphisms are stable under pullbacks (because the class S is stable under pullbacks). Moreover, the full subcategory $S^\sharp\mathbb{C} \subseteq \mathbb{C}$ of S-special objects is closed under finite limits in $\mathbb{C}$ (this comes from Condition (2) of Definition 3.1).

Proposition 6.2. *Let $\mathbb{C}$ be an S-protomodular category. Any split epimorphism between S-special objects is in S and, consequently, is an S-special morphism. The subcategory $S^\sharp\mathbb{C}$ of S-special objects is protomodular.*

Proof. Let us observe that any point (f, s) produces a kernel diagram in the fibre $Pt_Y\mathbb{C}$:



When Y is in $S^\sharp\mathbb{C}$, the right hand side point is in S. The following pullback

$$\begin{array}{ccc} Y \times X & \xrightarrow{s\times 1} & X \times X \\ {\scriptstyle p_Y}\downarrow\uparrow{\scriptstyle (1,s)} & & {\scriptstyle p_0}\downarrow\uparrow{\scriptstyle s_0} \\ Y & \xrightarrow[s]{} & X \end{array}$$

shows that, when X is in $S^\sharp\mathbb{C}$, the middle point is in S. Since the fibre $SPt_Y\mathbb{C}$ is closed under finite limits, the kernel (f,s) is in S. Hence, in the subcategory $S^\sharp\mathbb{C}$ the subfibration $¶^S_{S^\sharp\mathbb{C}}$ coincides with the fibration $¶_{S^\sharp\mathbb{C}}$ of all points. According to Proposition 3.2, the change-of-base functor with respect to the fibration of all points in $S^\sharp\mathbb{C}$ is conservative, and consequently $S^\sharp\mathbb{C}$ is protomodular. Furthermore, since $PtS^\sharp\mathbb{C}$ is closed under finite limits, the kernel equivalence relation of f lies in $PtS^\sharp\mathbb{C}$, and the point $(p_0,s_0)\colon R[f] \leftrightarrows X$ is in S. Accordingly $R[f]$ is an S-equivalence relation, and f is an S-special morphism. □

Definition 6.3. Given an S-protomodular category $\mathbb{C}$, we will call the subcategory $S^\sharp\mathbb{C}$ the *protomodular core* of $\mathbb{C}$ relatively to S.

We are now going to describe the protomodular core when $\mathbb{C}$ is a category of monoids with operations and S is the class of Schreier points.

Proposition 6.4. *Let $\mathbb{C}$ be a category of monoids with operations and S the class of Schreier points. Given an object $X \in \mathbb{C}$, it is S-special if and only if $(X,+)$ is a group.*

Proof. Suppose that X is S-special. Consider the following diagram

$$X \underset{\langle 0,1\rangle}{\overset{q}{\leftrightarrows}} X \times X \underset{p_1}{\overset{p_0}{\rightrightarrows}} X, \quad (s_0\colon X \to X\times X)$$

where q is the Schreier retraction associated with the Schreier point (p_0,s_0). Let $x \in X$. According to the Schreier condition (as in Proposition 4.5), the pair $(x,0) \in X \times X$ can be written as

$$(x,0) = q(x,0) + s_0p_0(x,0) = q(x,0) + (x,x).$$

Since $q(x,0)$ is an element of the kernel of p_0, it is an element of the form $(0,y)$, for some $y \in X$. Hence we have

$$(x,0) = (0,y) + (x,x) = (x, y + x),$$

and from this equality we get $y + x = 0$. So y is a left inverse for x. Doing the same thing for all $x \in X$ we prove that $(X,+)$ is a group.

Conversely, suppose that $(X,+)$ is a group. The needed Schreier retraction is simply given by

$$q(x_1,x_2) = (0, x_2 - x_1).$$ □

As a consequence, we have the following:

Corollary 6.5. *If $\mathbb{C}$ is the category Mon of monoids and S is the class of Schreier points, the protomodular core of $\mathbb{C}$ is the category Gp of groups. If $\mathbb{C}$ is the category*

SRng of semirings, the protomodular core is the category Rng of (not necessarily unitary) rings. More generally, given any category $\mathbb{C}$ of monoids with operations, the protomodular core with respect to the class S of Schreier points is the corresponding category of groups with operations, obtained from $\mathbb{C}$ by adding the condition that $+$ is a group operation.

Proposition 6.6. *Let $\mathbb{C}$ be an S-protomodular category. Any point (f,s) such that f is an S-special morphism belongs to S, and the kernel of f is an S-special object.*

Proof. Consider the following diagram

$$\begin{array}{ccccc}
\mathrm{Ker}(f) & \xrightarrow{k_f} & X & \xrightarrow{s_1} & R[f] \\
\downarrow & & {\scriptstyle f}\downarrow\uparrow{\scriptstyle s} & & {\scriptstyle p_0}\downarrow\uparrow{\scriptstyle s_0} \\
1 & \xrightarrow[\alpha_Y]{} & Y & \xrightarrow[s]{} & X,
\end{array}$$

where s_1 is the morphism $(sf,1)\colon X \to R[f]$. The right hand side square is a pullback. If the morphism f is S-special then, by definition, the point (p_0,s_0) is in S. By stability under pullbacks, the point (f,s) is in S, too. The left hand side square is a pullback as well, so the terminal morphism $\mathrm{Ker}(f) \to 1$ is S-special as so is f, and then $\mathrm{Ker}(f)$ is an S-special object. $\square$

An internal category $\underline{X}_1$ as in (1) in a finitely complete category $\mathbb{C}$ is a groupoid when the following square determined by the composition map d_1 is a pullback:

$$\begin{array}{ccc}
X_2 & \xrightarrow{d_1} & X_1 \\
{\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle d_0} \\
X_1 & \xrightarrow[d_0]{} & X_0,
\end{array}$$

or, in other words, when the following vertical comparison morphism j is an isomorphism:

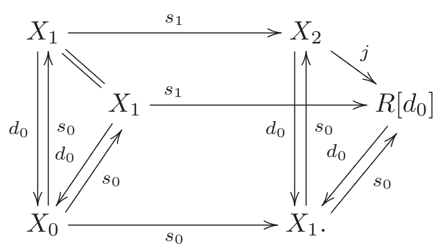
$$\begin{array}{ccccc}
X_2 & & & & \\
{\scriptstyle j}\vdots & \searrow^{d_0}\ \nwarrow^{s_0}\ \searrow^{d_1} & & & \\
R[d_0] & \underset{d_1}{\overset{d_0}{\underset{\longleftarrow s_0 \longrightarrow}{\rightrightarrows}}} & X_1 & \xrightarrow{d_0} & X_0.
\end{array} \tag{4}$$

We recall that this condition can be equivalently formulated by saying that X_2 is isomorphic to the kernel pair of $d_1\colon X_1 \to X_0$. In this case we have a discrete fibration between groupoids:

$$\begin{array}{ccc}
R[d_0] & \xrightarrow{d_2} & X_1 \\
{\scriptstyle d_0}\downarrow\uparrow\downarrow{\scriptstyle d_1} & & {\scriptstyle d_0}\downarrow\uparrow\downarrow{\scriptstyle d_1} \\
X_1 & \xrightarrow[d_1]{} & X_0.
\end{array} \tag{5}$$

Proposition 6.7. *Let $\mathbb{C}$ be an S-protomodular category. An S-category $\underline{X}_1$ is a groupoid if and only if the morphism $d_0\colon X_1 \to X_0$ is S-special.*

Proof. Suppose that the S-category $\underline{X}_1$ is a groupoid. The point $(d_0, s_0)\colon X_1 \leftrightarrows X_0$ is in S. By the pullbacks of the discrete fibration (5), the point $(d_0, s_0)\colon R[d_0] \leftrightarrows X_1$ is in S and consequently the morphism $d_0\colon X_1 \to X_0$ is S-special. Conversely, suppose that the map $d_0\colon X_1 \to X_0$ of the S-category $\underline{X}_1$ is S-special. In the following diagram, consider the comparison morphism j as in (4):



The two right hand side points are in S, because $\underline{X}_1$ is an S-category and the morphism $d_0\colon X_1 \to X_0$ is S-special. All quadrangles are pullbacks. Hence we have that the image of the map j by the change-of-base functor along s_0 is the isomorphism 1_{X_1}. According to Proposition 3.2, the map j is an isomorphism, and $\underline{X}_1$ is a groupoid. □

An S-category which is a groupoid will be called an *S-groupoid*. The previous proposition, together with Proposition 5.3, gives the following:

Corollary 6.8. *Let $\mathbb{C}$ be an S-protomodular category. An S-reflexive relation (1) is an S-equivalence relation if and only if the morphism $d_0\colon X_1 \to X_0$ is S-special.*

When $\mathbb{C}$ is a category of monoids with operations, and S is the class of Schreier points, the converse of Proposition 6.6 also holds:

Proposition 6.9. *Let $\mathbb{C}$ be a category of monoids with operations, and let S be the class of Schreier points. Given a Schreier point*

$$Ker(f) \underset{k_f}{\overset{q_f}{\rightleftarrows}} X \underset{f}{\overset{s}{\leftrightarrows}} Y,$$

f is an S-special morphism if and only if $Ker(f)$ is an S-special object (or, in other terms, if $(Ker(f), +)$ is a group).

Proof. Thanks to Proposition 6.6, we only have to show that the point $R[f] \underset{p_0}{\overset{s_0}{\leftrightarrows}} X$ is a Schreier point. Let us define $q_{p_0}(x, x') = (0, q_f(x') - q_f(x))$. We can check that

$$\begin{aligned} q_{p_0}(x, x') + s_0 p_0(x, x') &= (0, q_f(x') - q_f(x)) + (x, x) = (x, q_f(x') - q_f(x) + x) \\ &= (x, q_f(x') + sf(x)) = (x, q_f(x') + sf(x')) = (x, x'), \end{aligned}$$

and, thanks to Proposition 4.6,

$$\begin{aligned} q_{p_0}((0, k) + s_0(x)) &= q_{p_0}((0, k) + (x, x)) = q_{p_0}(x, k + x) = (0, q_f(k + x) - q_f(x)) \\ &= (0, q_f(k) + q_f(sf(k) + q_f(x)) - q_f(x)) = (0, q_f(k) + q_f(x) - q_f(x)) = (0, k). \end{aligned}$$

The thesis follows then from Proposition 4.5. □

Example 6.10. The previous proposition implies that the morphism $abs\colon \mathbb{Z}^* \to \mathbb{N}^*$ of Example 4.4 is an S-special morphism.

Corollary 6.11. *Let $\mathbb{C}$ be a category of monoids with operations, and let S be the class of Schreier points. An S-category (resp. S-reflexive relation) (1) is an S-groupoid (resp. S-equivalence relation) if and only if $Ker(d_0)$ is an S-special object, which is equivalent to say that $(Ker(d_0), +)$ is a group.*

Proof. Thanks to the previous proposition, this is an immediate consequence of Proposition 6.7 and Corollary 6.8. □

7. Other Mal'tsev aspects of S-protomodular categories

7.1. Mal'tsev categories

We recall that a category $\mathbb{C}$ is a Mal'tsev category [**10, 11**] when any internal reflexive relation is an equivalence relation; this is equivalent to the property that any fiber $Pt_Y\mathbb{C}$ of the fibration $¶_{\mathbb{C}}$ is unital (see [**3**]). The category Gp of groups is a Mal'tsev one. The natural order $\mathcal{O}_{\mathbb{N}}$ of natural numbers (Example 5.4) shows that the category Mon of monoids is not a Mal'tsev one.

On the other hand, in the context of S-protomodular categories, any fiber $Pt_Y\mathbb{C}$ is $SPt_Y\mathbb{C}$-unital and, consequently, any fiber $SPt_Y\mathbb{C}$ is unital (Proposition 3.2). In this section, we shall be interested in exploring some properties of Mal'tsev categories that are partially valid in this new structural context.

7.2. Elementary observations

We already observed that, in an S-protomodular category, any S-reflexive relation (1) is only necessarily transitive (Proposition 5.3). The natural order on $\mathbb{N}$ gives an example of an S-reflexive relation (in the category of monoids) which is not an equivalence relation. An S-reflexive relation (1) is an equivalence relation if and only if d_0 is S-special (Corollary 6.8 above).

In a Mal'tsev category, on a reflexive graph (1) there is at most one structure of internal category, which is necessarily an internal groupoid [**11**]. In Section 5 we showed that, on an S-reflexive graph, there is again at most one structure of internal category, but there are S-categories which are not groupoids. An internal S-category is a groupoid if and only if, again, d_0 is S-special (Proposition 6.7).

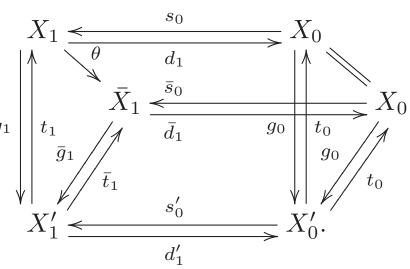
In a Mal'tsev category we have also the following useful result (see [**3**]): given any split epimorphism of reflexive graphs

$$\begin{array}{ccc}
X_1 & \overset{d_0}{\underset{d_1}{\rightrightarrows}} \; \overset{s_0}{\leftarrow} & X_0 \\
g_1 \downarrow \uparrow t_1 & & g_0 \downarrow \uparrow t_0 \\
X'_1 & \overset{d'_0}{\underset{d'_1}{\rightrightarrows}} \; \overset{s'_0}{\leftarrow} & X'_0
\end{array}$$

the commutative square $g_0 d_1 = d'_1 g_1$ is a pullback if and only if so is the square $g_0 d_0 = d'_0 g_1$. Here we have:

Proposition 7.1. *Let $\mathbb{C}$ be an S-protomodular category. Given a split epimorphism of reflexive graphs in $\mathbb{C}$ as in the diagram above, where the point (g_0, t_0) is in S, the commutative square $g_0 d_1 = d_1' g_1$ is a pullback if and only if so is the square $g_0 d_0 = d_0' g_1$.*

Proof. If the square $g_0 d_0 = d_0' g_1$ is a pullback and the point (g_0, t_0) is in S, so are the point (g_1, t_1) and the pullback $(\bar{g}_1, \bar{t}_1)$ of (g_0, t_0) along d_1' in the following diagram:



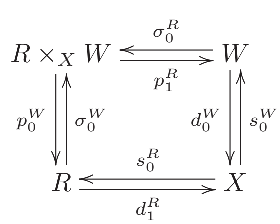
Let θ be the induced factorization. All leftward quadrangles are pullbacks. This means that the image of θ by the change-of-base functor along s_0' is the isomorphism 1_{X_0}. According to Proposition 3.2, the map θ is itself an isomorphism. The other implication can be proved similarly. □

7.3. Centrality with respect to S-reflexive relations

The context of Mal'tsev categories proved to be suitable for the study of centrality between equivalence relations (see [**6**] and [**7**]), thanks to the fact that a finitely complete category $\mathbb{C}$ is a Mal'tsev one if and only if all fibers $Pt_X\mathbb{C}$ of the fibration of points are unital [**3**]. It is well known that, in the category Gp, two equivalence relations R and W on a group G centralize each other if and only if the normal subgroups $\overline{1}_R$ and $\overline{1}_W$ given by the equivalence classes of the unit element commute inside the group G.

In an S-protomodular category $\mathbb{C}$, since any fiber $Pt_Y\mathbb{C}$ is $SPt_Y\mathbb{C}$-unital, we can keep the same definition of reflexive relations centralizing each other as in [**6**] and [**7**], provided that one of the relations is an S-reflexive relation:

Definition 7.2. Given a reflexive relation R and an S-reflexive relation W on the same object X in an S-protomodular category $\mathbb{C}$, we say that R and W *centralize each other* when there is a (necessarily unique) connector $p\colon R \times_X W \to X$, where $R \times_X W$ is defined by the following pullback



such that $p\sigma_0^R = d_1^W$ and $p\sigma_0^W = d_0^R$. In set-theoretical terms, this means that we have both $p(xRxWy) = y$ and $p(xRyWy) = x$. The morphisms σ_0^R and σ_0^W, defined by the universal property of the pullback, are explicitly given by $\sigma_0^R(yWz) = yRyWz$ and $\sigma_0^W(xRy) = xRyWy$. We denote this situation by $[R, W] = 0$.

Since W is an S-reflexive relation, the point (d_0^W, s_0^W) is in S, and consequently the pair (σ_0^R, σ_0^W) is jointly strongly epimorphic. This implies that the connector p is unique.

Example 7.3. Given the order $\mathcal{O}_{\mathbb{N}}$ on $\mathbb{N}$ in Mon, with the class S of Schreier points, we have that $[\mathcal{O}_{\mathbb{N}}, \mathcal{O}_{\mathbb{N}}] = 0$; in this case, the connector is the morphism p defined by $p(x \leqslant y \leqslant z) = z - y + x$.

When we have $[R, W] = 0$, we recover a well-known result in Mal'tsev categories [**6, 7**]:

Proposition 7.4. *Let $\mathbb{C}$ be an S-protomodular category. Suppose the reflexive relation R and the S-reflexive relation W on X centralize each other in $\mathbb{C}$. We have necessarily $xWp(xRyWz)$ and $p(xRyWz)Rz$.*

Proof. Let us consider the following pullback:

$$\begin{array}{ccc} U & \overset{j}{\rightarrowtail} & R \times_X W \\ \downarrow & & \downarrow{\scriptstyle (d_0^R p_0^W,\, p)} \\ W & \underset{(d_0^W,\, d_1^W)}{\rightarrowtail} & X \times X. \end{array}$$

In set-theoretical terms, it defines U as the subobject of those $xRyWz \in R \times_X W$ such that we have $xWp(xRyWz)$. For any yWz, the element $yRyWz \in R \times_X W$ belongs to U, since we have $y = p(yRyWz)$ (as we observed in Definition 7.2). This means that σ_0^R factors through j. In the same way, for any xRy, the element $xRyWy \in R \times_X W$ belongs to U, since we have $x = p(xRyWy)$. This means that σ_0^W factors through j. Since the pair (σ_0^R, σ_0^W) is jointly strongly epimorphic, the morphism j is an isomorphism, and for every $xRyWz \in R \times_X W$ we have $xWp(xRyWz)$.

We have a similar result concerning the subobject $V \rightarrowtail R \times_X W$ defined by the following pullback:

$$\begin{array}{ccc} V & \overset{j}{\rightarrowtail} & R \times_X W \\ \downarrow & & \downarrow{\scriptstyle (p,\, d_1^W p_1^R)} \\ R & \underset{(d_0^R,\, d_1^R)}{\rightarrowtail} & X \times X. \end{array}$$

This give us $p(xRyWz)Rz$ for any $xRyWz \in R \times_X W$. □

In set-theoretical terms, the previous proposition says that, with any triple $xRyWz$, we can associate a square of related elements:

$$\begin{array}{ccc} x & \overset{W}{\dashrightarrow} & p(x, y, z) \\ {\scriptstyle R}\downarrow & & \vdots{\scriptstyle R} \\ y & \underset{W}{\longrightarrow} & z. \end{array}$$

This says that any connected pair of reflexive relations (R, W) on the object X, where W is an S-reflexive relation, produces the following diagram of double reflexive relations on R and W:

$$\begin{array}{ccc} R\times_X W & \underset{(d_0^R p_0^R,p)}{\overset{p_1^W}{\rightleftarrows}} & W \\ {\scriptstyle p_0^R}\downarrow\uparrow\downarrow{\scriptstyle (p,d_1^W p_1^W)} & & {\scriptstyle d_0^W}\downarrow\uparrow\downarrow{\scriptstyle d_1^W} \\ R & \underset{d_0^R}{\overset{d_1^R}{\rightleftarrows}} & X. \end{array}$$

It is called the *centralizing double relation* associated with the connector p [**11**]. When R and W are equivalence relations, all the reflexive relations in this diagram are equivalence relations, and, moreover, any commutative square is a pullback (thanks to Proposition 7.1).

As in the case of Mal'tsev categories (see Lemma 2.1 in [**7**]), in the context of S-protomodular categories the existence of a double centralizing relation between a reflexive relation R and an S-reflexive relation W characterizes the fact that $[R,W]=0$. Indeed, given a double centralizing relation

$$\begin{array}{ccc} C & \underset{p_0^W}{\overset{p_1^W}{\rightleftarrows}} & W \\ {\scriptstyle p_0^R}\downarrow\uparrow\downarrow{\scriptstyle p_1^R} & & {\scriptstyle d_0^W}\downarrow\uparrow\downarrow{\scriptstyle d_1^W} \\ R & \underset{d_0^R}{\overset{d_1^R}{\rightleftarrows}} & X, \end{array}$$

i.e., a reflexive relation C both on R and W such that the square $d_1^R p_0^R = d_0^W p_1^W$ is a pullback, the morphism $d_1^R p_0^R\colon C\to X$ is the (necessarily unique) connector.

We can now prove the following result, which is the relative version of the characterization of internal groupoids in Mal'tsev categories (see [**11**]):

Proposition 7.5. *Let $\mathbb{C}$ be an S-protomodular category. Consider a reflexive graph (1) in $\mathbb{C}$ such that d_0 is S-special. The following conditions are equivalent:*

(i) the graph is underlying an S-category;

(ii) the graph is underlying an S-groupoid;

(iii) the kernel equivalence relations $R[d_0]$ and $R[d_1]$ centralize each other.

Proof. Since d_0 is S-special, the graph is an S-reflexive graph (thanks to Proposition 6.6). Moreover $R[d_0]$ is an S-equivalence relation, and we can talk about centralization of it with any reflexive relation on X_1.

The equivalence between conditions (i) and (ii) was already proved (see Proposition 6.7).

To prove that (ii) implies (iii), consider the following diagram:

$$\begin{array}{ccccc} R[d_2] & \underset{p_0}{\overset{p_1}{\rightleftarrows}} & R[d_0] & \xrightarrow{d_2} & X_1 \\ {\scriptstyle R(d_0)}\downarrow\uparrow\downarrow{\scriptstyle R(d_1)} & & {\scriptstyle d_0}\downarrow\uparrow\downarrow{\scriptstyle d_1} & & {\scriptstyle d_0}\downarrow\uparrow{\scriptstyle s_0}\downarrow{\scriptstyle d_1} \\ R[d_1] & \underset{p_0}{\overset{p_1}{\rightleftarrows}} & X_1 & \xrightarrow[d_1]{} & X_0. \end{array}$$

As we observed in Section 6, diagram (5), the right hand side square is a pullback, and hence the left hand side part of the diagram gives a double centralizing relation, which says that $[R[d_0], R[d_1]] = 0$.

To prove that (iii) implies (i), suppose that we have $[R[d_0], R[d_1]] = 0$ and p is the associated connector. By Proposition 5.2, in order to equip our S-reflexive graph with a structure of internal category, we only need to give the composition map $d_1 \colon X_2 \to X_1$ satisfying the equalities $d_1 s_0 = d_1 s_1 = 1_{X_1}$. The map d_1 can be defined as in the case of Mal'tsev categories (Theorem 3.6 in [**11**]). In set-theoretical terms, d_1 is given by

$$d_1(\alpha, \beta) = p(\beta R[d_0] 1_{d_1(\alpha)} R[d_1] \alpha).$$

It is easy to verify that it satisfies the desired equalities. □

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