Homology, Homotopy and Applications, vol. 18(1), 2016, pp.49–70

EQUIVARIANT K-THEORY OF CENTRAL EXTENSIONS AND TWISTED EQUIVARIANT K-THEORY: $SL_3\mathbb{Z}$ AND $St_3\mathbb{Z}$.

NOÉ BÁRCENAS AND MARIO VELÁSQUEZ

(*communicated by Graham Ellis*)

Abstract

We compare twisted equivariant K-theory of $SL_3\mathbb{Z}$ with untwisted equivariant K-theory of a central extension $St_3\mathbb{Z}$. We compute all twisted equivariant K-theory groups of $SL_3\mathbb{Z}$, and compare them with previous work on the equivariant K-theory of $BSt_3\mathbb{Z}$ by Tezuka and Yagita.

Using a universal coefficient theorem by the authors, the computations explained here give the domain of Baum–Connes assembly maps landing on the topological K-theory of twisted group C^*-algebras related to $SL_3\mathbb{Z}$, for which a version of KK-theoretic duality studied by Echterhoff, Emerson, and Kim is verified.

1. Introduction

In this note, we compare versions of twisted equivariant K-theory with respect to a discrete group G, and *untwisted* equivariant K-theory of a central extension of G coding a discrete torsion twist data.

Given a discrete group G, a proper G-CW complex X and a cohomology class α in the third Borel cohomology group $H^3(X \times_G EG, \mathbb{Z})$, twisted equivariant K-theory, denoted by ${}^{\alpha}K^*_G(X)$, was defined in [**BEJU14**].

Specializing to the classifying space $\underline{E}G$ of proper actions of G and performing the Borel construction $\underline{E}G \times_G EG$ gives a model for BG, and thus all twistings agree with elements in the cohomology groups $H^3(BG, \mathbb{Z})$.

In the case of a discrete group G (compare [**Moo64, Moo68**]), a class $\alpha \in H^3(BG, \mathbb{Z}) \cong H^2(BG, S^1)$ determines a central extension

$$1 \to S^1 \to \widetilde{G}_\alpha \overset{p_\alpha}{\to} G \to 1.$$

The space $\underline{E}G$ with the $\widetilde{G}_\alpha$-action given by precomposition with p_α is a model for the classifying space of proper actions of $\widetilde{G}_\alpha$, denoted by $\underline{E}\widetilde{G}_\alpha$. We compare the

The second author was financially supported under the project *Aplicaciones de la K-teoría en teoría del índice y las conjeturas de isomorfismo* with ID 00006552 of Faculty of Sciences of *Pontificia Universidad Javeriana*.

Received November 28, 2013, revised August 15, 2014, March 18, 2015; published on February 24, 2016.

2010 Mathematics Subject Classification: 19L64, 19K33, 19L47.

Key words and phrases: Twisted equivariant K-theory, Bredon cohomology, Baum-Connes conjecture with coefficients, twisted group C^*-algebra, KK-theoretic duality.

Article available at `http://dx.doi.org/10.4310/HHA.2016.v18.n1.a4`

abelian groups $K^*_{\widetilde{G}_\alpha}(\underline{E}\widetilde{G}_\alpha)$ and ${}^\alpha K^*_G(\underline{E}G)$.

We pay specific attention to the groups $SL_3\mathbb{Z}$ and $St_3\mathbb{Z}$, related by a central extension of the form

$$1 \to \mathbb{Z}/2 \to St_3\mathbb{Z} \to SL_3\mathbb{Z} \to 1.$$

The integral cohomology of both groups $St_3\mathbb{Z}$, and $SL_3\mathbb{Z}$ has been extensively studied in [**Sou78**], where also a model for the classifying space for proper actions $\underline{E}SL_3\mathbb{Z}$ was constructed. In degree 3, the cohomology groups are finitely generated, 2-torsion, and generated by classes u_1, u_2 in the case of $SL_3\mathbb{Z}$ and a single class w_1 in the case of $St_3\mathbb{Z}$.

We describe the restriction of the classes u_1 and u_2 to the cohomology of finite subgroups of $SL_3\mathbb{Z}$ in Section 5, where the relation to the generating class w_1 is also stated. We follow these classes to their restrictions on finite subgroups of $St_3\mathbb{Z}$, which are covers 2 to 1 of finite subgroups of $SL_3\mathbb{Z}$.

It turns out that the torsion class $u_1 + u_2$ represents the central extension

$$1 \to \mathbb{Z}/2\mathbb{Z} \to St_3\mathbb{Z} \xrightarrow{p} SL_3\mathbb{Z} \to 1,$$

and its restriction to finite groups $H \leqslant G$ gives a model for Schur covering groups of H:

$$1 \to \mathbb{Z}/2\mathbb{Z} \to p^{-1}(H) \to H.$$

(However, more finite subgroups appear in $St_3\mathbb{Z}$ that are not a Schur covering group for any subfinite group of $SL_3\mathbb{Z}$.)

Thus, a cocycle representing $u_1 + u_2$ and the central extension satisfy the hypotheses of the following theorem, which is to be proved in Section 4.

Theorem (4.4)**.** *Let G be a discrete group, and let $\alpha \in Z^2(G; S^1)$ be a cocycle taking values in $\mathbb{Z}/n\mathbb{Z} \subseteq S^1$. Consider the central extension associated to α:*

$$1 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow G_\alpha \xrightarrow{\ \rho\ } G \longrightarrow 1.$$

Denote by $\underline{E}G$ a model for the classifying space of proper actions and notice that the action of G_α via ρ on $\underline{E}G$ exhibits the latter space as a model for $\underline{E}G_\alpha$.

Then the map ρ gives an isomorphism of abelian groups between the Bredon cohomology groups of $\underline{E}G$ with coefficients in the α-twisted representation group and the Bredon cohomology groups of $\underline{E}G_\alpha$ with coefficients in the 1-central group representation Bredon module (defined in 4.2). In symbols,

$$H^*(\underline{E}G; \mathcal{R}^G_\alpha) \xrightarrow{\rho^*} H^*(\underline{E}G_\alpha; \mathcal{R}^{G_\alpha}_1)$$

is an isomorphism.

We use the (Bredon) cohomological description to feed a spectral sequence constructed, in [**BV14**], to compute twisted equivariant K-theory. The input of the spectral sequence is the Bredon Cohomology groups with coefficients in twisted representations, as briefly introduced in Section 2. The spectral sequence is seen to collapse at the E_2-term and the twisted equivariant K-theory groups are determined.

- (Theorem 6.1) The twisted equivariant K-theory groups with respect to u_1 are as follows:
$$ {}^{u_1}K^0_{SL_3\mathbb{Z}}(\underline{\mathrm{E}}SL_3\mathbb{Z}) \cong \mathbb{Z}^{\oplus 13} \quad {}^{u_1}K^1_{SL_3\mathbb{Z}}(\underline{\mathrm{E}}SL_3\mathbb{Z}) = 0. $$
- (Theorem 6.3) The twisted equivariant K-theory groups with respect to u_2 are as follows:
$$ {}^{u_2}K^0_{SL_3\mathbb{Z}}(\underline{\mathrm{E}}SL_3\mathbb{Z}) \cong \mathbb{Z}^{\oplus 7}, \quad {}^{u_2}K^1_{SL_3\mathbb{Z}}(\underline{\mathrm{E}}SL_3\mathbb{Z}) = 0. $$
- (Theorem 6.7) The twisted equivariant K-theory groups with respect to $u_1 + u_2$ are as follows:
$$ {}^{u_1+u_2}K^0_{SL_3\mathbb{Z}}(\underline{\mathrm{E}}SL_3\mathbb{Z}) \cong \mathbb{Z}^{\oplus 5}, \quad {}^{u_1+u_2}K^1_{SL_3\mathbb{Z}}(\underline{\mathrm{E}}SL_3\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}. $$

Using the Universal Coefficient Theorem for Bredon cohomology with coefficients in twisted representations [**BV14**, Theorem 1.13] the previous groups are verified to be isomorphic to some equivariant K-Homology groups with coefficients defined in terms of Kasparov KK-Theory groups in Section 7, Theorem 7.2. This extends and generalizes work by Sánchez-García in [**SG08**] in the untwisted setting.

A version of the Baum–Connes Conjecture with coefficients, [**CE01**] relates these groups to the topological K-theory of twisted group C^*-algebras. We see that the input of the Baum–Connes assembly map with coefficients given by the twistings u_1, u_2 and $u_1 + u_2$ satisfy a version of KK-theoretic Duality studied in [**EEK08**] and verified in [**BV14**] for the twist u_1.

The result is interpreted in terms of twisted equivariant K-theory of the classifying space $BSL_3\mathbb{Z}$ using results by Tezuka and Yagita [**TY92**], the Atiyah–Segal Completion Theorem [**LO01**, Theorem 4.4].

This work is organized as follows: In Section 2, we introduce Bredon (co)homology, focusing on coefficients in twisted representations. In Section 3, we review spectral sequences relating Bredon cohomology groups to versions of twisted equivariant K-theory. Section 4 deals with the proof of Theorem 4.4, relating twisted equivariant K-theory and untwisted K-theory that is equivariant with respect to a central extension coding the twist. Section 5 describes cohomological information determining the twists, as well as some misunderstandings in the literature concerning the universal central extension of $SL_3\mathbb{Z}$ and $St_3\mathbb{Z}$; see 5.3. Section 6 deals with the computations in Bredon cohomology. Finally, Section 7 gives interpretations of the results as computations of twisted equivariant K-homology related to versions with coefficients of the Baum–Connes Conjecture, as well as computations of the complex K-theory of the classifying space $BSt_3\mathbb{Z}$ by Tezuka and Yagita.

Acknowledgments

The first author thanks the support of a CONACYT Postdoctoral fellowship. The second author thanks the partial support of a UNAM Postdoctoral Fellowship, as well as partial support by the project *Aplicaciones de la K-teoría en Teoría del Índice y las Conjeturas de Isomorfismo* with ID 6552 from Faculty of Sciences of Pontificia Universidad Javeriana, Bogotá, Colombia.

The first author thanks Prof. Pierre de la Harpe for enlightening correspondence related to the difference between $St_3\mathbb{Z}$ and the universal central extension of $SL_3\mathbb{Z}$.

Both authors thank an anonymous referee for making crucial suggestions about both the presentation and the mathematical content of this note, particularly the suggestion of the material in Section 4, which helped the authors to identify a mistake in a previous version of this work.

2. Bredon (co)homology

We recall briefly some definitions relevant to Bredon homology and cohomology; see [**MV03**] for more details. Let G be a discrete group. A G-CW-complex is a CW-complex with a G-action permuting the cells and such that if a cell is sent to itself, this is done by the identity map. We call the G-action proper if all cell stabilizers are finite subgroups of G.

Definition 2.1. A model for $\underline{E}G$ is a proper G-CW-complex X such that for any proper G-CW-complex Y there is a unique G-map $Y \to X$, up to G-homotopy equivalence.

One can prove that a proper G-CW-complex X is a model of $\underline{E}G$ if and only if the subcomplex of fixed points X^H is contractible for each finite subgroup $H \subseteq G$. It can be shown that classifying spaces for proper actions always exist.

Let $\mathsf{Or}_{\mathcal{FIN}}(G)$ be the orbit category of finite subgroups of G; a category with one object G/H for each finite subgroup $H \subseteq G$ and where morphisms are given by G-equivariant maps. There exists a morphism $\phi : G/H \to G/K$ if and only if H is conjugate in G to a subgroup of K.

Definition 2.2 (Bredon chain complex)**.** Let X be a proper G-CW-complex. The contravariant functor $\underline{C}_*(X) : \mathsf{Or}_{\mathcal{FIN}}(G) \to \mathbb{Z} - \mathsf{CHCOM}$ assigns to every object G/H the cellular $\mathbb{Z}$-chain complex of the H-fixed point complex $\underline{C}_*(X^H) \cong C_*(\mathrm{Map}_G(G/H, X))$ with respect to the cellular boundary maps $\underline{\partial}_*$.

We will use homological algebra to define Bredon cohomology.

A contravariant coefficient system is a contravariant functor $M : \mathsf{Or}_{\mathcal{FIN}}(G) \to \mathbb{Z}$–$\mathsf{MODULES}$. Given a contravariant coefficient system M, the Bredon cochain module $C^n_G(X; M)$ is defined as the abelian group of natural transformations of functors defined on the orbit category $\underline{C}_n(X) \to M$. In symbols,

$$C^n_G(X; M) = \mathrm{Mor}_{\mathrm{Funct}(\mathsf{Or}_{\mathcal{FIN}}(G), \mathbb{Z}-\mathsf{MODULES})}(\underline{C}_n(X), M).$$

Given a set $\{e_\lambda\}$ of orbit representatives of the n-cells of the G-CW complex X, and isotropy groups H_λ in G of the cells e_λ, the abelian groups $C^n_G(X, M)$ satisfy

$$C^n_G(X, M) = \bigoplus_\lambda Hom_{\mathbb{Z}}(\mathbb{Z}[e_\lambda], M(G/H_\lambda))$$

with one summand for each orbit representative e_λ. They afford a differential $\delta^n : C^n_G(X, M) \to C^{n+1}_G(X, M)$ determined by $\underline{\partial}_*$ and pullback maps $M(\phi) : M(G/H_\mu) \to M(G/H_\lambda)$ for morphisms $\phi : G/H_\lambda \to G/H_\mu$.

Definition 2.3 (Bredon cohomology)**.** Let M be a contravariant coefficient system. The Bredon cohomology groups with coefficients in M, denoted by $H^*_G(X, M)$, are the cohomology groups of the cochain complex $\left(C^*_G(X, M), \delta^*\right)$.

A covariant coefficient system is a covariant functor $N : \mathsf{Or}_{\mathcal{FIN}}(G) \to \mathbb{Z}\text{–}\mathsf{MODULES}$. Let N be a covariant coefficient system and X be a proper G-CW-complex. Dually to the cohomological situation, one can define the Bredon homology groups with coefficients in N. We denote these by $H_*^G(X, N)$. Details can be found in [**MV03**, pp. 14-15].

Bredon (co)homology with coefficients in twisted representations

Definition 2.4. Let K be a finite subgroup in the discrete group G. Let V be a complex vector space, and let S^1 be the unit circle in the complex numbers. Given a cocycle $\alpha : K \times K \to S^1$ representing a class in $H^2(BK, S^1) \cong H^3(BK, \mathbb{Z})$, an α-twisted representation is a function to the general linear group of V, $P : K \to \mathrm{GL}(V)$ satisfying:

$$P(e) = 1,$$
$$P(x)P(y) = \alpha(x, y)P(xy).$$

The Grothendieck group of isomorphism classes of α-twisted representations is called the *α-twisted representation group*, and it is denoted by $R_\alpha(K)$.

Two α, α'-twisted representations are isomorphic if the cocycles α, α' are cohomologous in $H^2(BK, S^1)$.

Definition 2.5. Let H be a finite group, and let $\alpha \in Z^2(H, S^1)$ be a cocycle. Recall that the α-twisted Complex group algebra $\mathbb{C}^\alpha H$ is generated as a complex vector space by the elements $\{h \mid h \in H\}$. The multiplication is given by the following formula on representatives,

$$h_1 h_2 = \alpha(h_1, h_2) h_1 h_2,$$

and extended $\mathbb{C}$-linearly to define a complex algebra structure on $\mathbb{C}^\alpha H$.

It is a consequence of [**Kar94**, Theorem 3.2, p. 112], that the K_0 group of the α-twisted complex group algebra $\mathbb{C}^\alpha H$ agrees with the α-twisted representation group $R_\alpha(H)$.

We define a contravariant and a covariant coefficient system for the family $\mathcal{F}_G = \mathcal{FIN}$ of finite subgroups agreeing on objects by using the K_0-group of the twisted group algebra, using restriction to define the contravariant functoriality, and using induction to define the covariant functoriality.

Definition 2.6. Let G be a discrete group, and let $\alpha \in Z^2(G, S^1)$ be a cocycle. Let $i : H \to G$ be an inclusion of a finite subgroup H.

Define $\mathcal{R}_\alpha$ on objects G/H by

$$\mathcal{R}_\alpha(G/H) := K_0(\mathbb{C}^{i^*\alpha}(H)) \cong R_{i^*(\alpha)}(H).$$

Let $\phi : G/H \to G/K$ be a G-equivariant map; we denote by $\mathcal{R}_{\alpha ?}(\phi) : R_{\alpha|}(H) \to R_{\alpha|}(K)$ the induction of α-twisted representations for the covariant functor. For the contravariant functor, we denote by ${\mathcal{R}_\alpha}^?(\phi) : R_{\alpha|}(K) \to R_{\alpha|}(H)$ the restriction of α-twisted representations.

Definition 2.7. Let G be a discrete group, let X be a proper G-CW complex, and let $\alpha \in Z^2(G, S^1)$ be a cocycle. The α-twisted Bredon (co)homology groups of X are the Bredon (co)homology groups with respect to the functors described in Definition 2.6.

Remark 2.8. Notice the role of the family of finite groups in Definition 2.7. More generally, one can define Bredon (co)homology groups for a family $\mathcal{F}$ of subgroups that contains the isotropy groups of a G-CW complex X, and a functor $\mathcal{F} \to \mathbb{Z}$–MODULES. Since we are dealing with proper actions on G-CW complexes, we can concentrate on Bredon cohomology for the family of finite subgroups.

3. Spectral sequences for twisted equivariant K-theory

Twisted equivariant K-theory for proper and discrete actions has been defined in a variety of ways. For a torsion cocycle $\alpha \in Z^2(G, S^1)$, it is possible to define it in terms of finite-dimensional, so-called α-twisted vector bundles, as for example in [**Dwy08**]. This is not possible for twistings of infinite order, and the general approach of [**BEJU14**] or C^*-algebraic methods are needed.

Definition 3.1. Let $\alpha \in Z^2(G, S^1)$ be a normalized torsion cocycle of order n for the discrete group G, with associated central extension

$$0 \to \mathbb{Z}/n \to G_\alpha \to G.$$

An *α-twisted vector bundle* is a finite-dimensional G_α-equivariant complex vector bundle such that $\mathbb{Z}/n$ acts by multiplication by a primitive nth root of unity. The *α-twisted, G-equivariant K-theory* groups ${}^{\alpha}K^0_G(X)$ are defined as the Grothendieck groups of the isomorphism classes of α-twisted vector bundles over X.

Given a proper G-CW complex X, define ${}^{\alpha}K^{-n}_G(X)$ as the kernel of the induced map

$${}^{\alpha}K^0_G(X \times S^n) \overset{\text{incl}^*}{\to} {}^{\alpha}K^0_G(X).$$

The α-twisted equivariant K-theory catches information relevant to the class of twistings coming from the torsion part of the group cohomology of the group, in the sense that the K-groups are zero for cocycles representing non-torsion classes. In contrast, the approach discussed in [**BEJU14**] overcomes this difficulty.

As noted in [**BEUV13**], there is a spectral sequence connecting the α-twisted Bredon cohomology and the α-twisted equivariant K-theory of finite proper G-CW complexes. When the twisting is given by a torsion element of $H^3(BG, \mathbb{Z})$, this spectral sequence is a special case of the Atiyah–Hirzebruch spectral sequence for *untwisted* G-cohomology theories constructed by Davis and Lück [**DL98**]. In particular, it collapses rationally.

Theorem 3.2 ([**BEUV13**])**.** *Let X be a finite proper G-CW complex for a discrete group G, and let $\alpha \in Z^2(G, S^1)$ be a normalized torsion cocycle. Then there is a spectral sequence with*

$$E_2^{p,q} = \begin{cases} H^p_G(X, \mathcal{R}^?_\alpha) & \text{if } q \text{ is even} \\ 0 & \text{if } q \text{ is odd} \end{cases}$$

so that $E_\infty^{p,q} \Rightarrow {}^{\alpha}K_G^{p+q}(X)$.

4. S^1-central extensions and torsion cocycles

Definition 4.1. Let $1 \to \mathbb{Z}/n\mathbb{Z} \to \tilde{H} \to H \to 1$ be a central extension. Let k be a natural number with $0 \leqslant k \leqslant n$. Let V be a complex vector space. A k-central representation of $\tilde{H}$ is a homomorphism $\tilde{H} \to \mathrm{GL}(V)$, where the generator $t \in \mathbb{Z}/n\mathbb{Z}$ acts by multiplication by $e^{2\pi ik/n}$.

Definition 4.2. The k-central representation group of $\tilde{H}$, denoted by $R_k(\tilde{H})$, is the Grothendieck group of isomorphism classes of k-central representations of $\tilde{H}$.

The k-central representation group is a contravariant coefficient system. Given a central extension of discrete groups, $1 \to \mathbb{Z}/n\mathbb{Z} \to \tilde{G} \to H \to 1$, we denote by $\mathcal{R}_k^?$ the functor

$$\mathcal{R}_k^? : \mathsf{Or}_{\mathcal{FIN}}(\tilde{G}) \to \mathbb{Z}-\mathsf{MODULES}$$
$$\tilde{G}/\tilde{H} \mapsto R_k(\tilde{H}).$$

Lemma 4.3. *Let G be a discrete group, and let $\alpha \in Z^2(G; S^1)$ be a torsion cocycle of order n. Then:*

(i) There exists a cocycle γ with values on $\mathbb{Z}/n \subset S^1$, which is cohomologous to α.

(ii) There exists a central extension of the form

$$1 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow G_\alpha \xrightarrow{\ \rho\ } G \longrightarrow 1$$

with the property that for each finite group $H \leqslant G$, the 1-central representation group $\mathcal{R}_1(\rho^{-1}(H))$ is isomorphic to the $\alpha\,|_H$- twisted representation group $\mathcal{R}_\alpha(H)$ as an abelian group.

(iii) Moreover, this extends to a natural transformation of contravariant functors defined over the orbit category of G,

$$T : \mathcal{R}_\alpha^? \cong \mathcal{R}_1^? \circ \rho,$$

which consists of group isomorphisms on each orbit.

Proof. (i) Let $\alpha \in Z^2(G; S^1)$ be a torsion cocycle of order n. Then α^n is cohomologous to the trivial cocycle, i.e., there is a cochain $t \in C^1(G, S^1)$ with $\alpha^n = \delta t$, where δ is the coboundary map.
Define a cochain $u \in C^1(G, S^1)$ by $u(g) = (t(g))^{-\frac{1}{n}}$. The cocycle $\gamma = \alpha \cdot \delta u$ is again torsion of order n. The cocycle γ takes values in $\mathbb{Z}/n\mathbb{Z}$ and it is cohomologous to α.

(ii) We use the $\mathbb{Z}/n\mathbb{Z}$-valued cocycle γ to define a group structure on the set $G \times \mathbb{Z}/n\mathbb{Z}$, and obtain a central extension of G by $\mathbb{Z}/n\mathbb{Z}$, denoted by G_α (the notation being justified by the fact that α is cohomologous to γ).
Let σ be the generator of $\mathbb{Z}/n\mathbb{Z}$ and $0 \leqslant j, k \leqslant n-1$. The multiplication on the group G_α is given on elements (g, σ^j) by

$$(g, \sigma^j) \star (h, \sigma^k) = (gh, \gamma(g,h)\sigma^{j+k}),$$

thus defining a central extension

$$1 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow G_\alpha \xrightarrow{\ \rho\ } G \longrightarrow 1.$$

Let H be a finite subgroup of G. Let us denote $H_\alpha = \rho^{-1}(H)$, with ρ being the

projection from the above central extension.
Let $\beta : H \to \mathrm{GL}(V)$ be a γ-twisted representation. We define the 1-central representation $T(\beta) : H_\alpha \to \mathrm{GL}(V)$ as $T(\beta)(h, \sigma^j) = \sigma^j(\beta(h))$. Note that $T(\beta)$ is a 1-central representation of H_α because

$$\begin{aligned} T(\beta)((h_1,\sigma^j)\star(h_2,\sigma^k)) &= T(\beta)(h_1h_2,\gamma(h_1,h_2)\sigma^{j+k}) \\ &= \gamma(h_1,h_2)\sigma^{j+k}\gamma(h_1,h_2)^{-1}\beta(h_1,H_2) \\ &= \sigma^{j+k}\beta(h_1)\beta(h_2) \\ &= (\sigma^j\beta(h_1))(\sigma^k\beta(h_2)) \\ &= \left(T(\beta)(h_1,\sigma^j)\right)\left(T(\beta)(h_2,\sigma^k)\right). \end{aligned}$$

This defines a group homomorphism

$$T : {}^\alpha\mathcal{R}(G/H) \to \mathcal{R}_1(G_\alpha/\rho^{-1}(H)).$$

An inverse to the homomorphism T is given by assigning to the 1-central representation $\epsilon : H_\alpha \to \mathrm{GL}(V)$ the γ-twisted representation $\kappa(\epsilon) : H \to \mathrm{GL}(V)$ given by $\kappa(\epsilon)(h) = \epsilon(h, 1)$ (here, we denote by 1 the complex number $1 \in S^1$). Notice that $\kappa(\epsilon)$ is a γ-twisted representation, because if $h_1, h_2 \in H$ then

$$\begin{aligned} \kappa(\epsilon)(h_1)\kappa(\epsilon)(h_2) &= \epsilon(h_1,1)\epsilon(h_2,1) \\ &= \epsilon\left((h_1,1)\star(h_2,1)\right) \\ &= \epsilon(h_1h_2,\gamma(h_1,h_2)) \\ &= \epsilon\left((h_1h_2,1)\star(e,\gamma(h_1,h_2))\right) \\ &= \epsilon(h_1h_2,1)\epsilon(e,\gamma(h_1h_2)) \\ &= \gamma(h_1,h_2)\epsilon(h_1h_2,1) \\ &= \gamma(h_1,h_2)\kappa(\epsilon)(h_1h_2). \end{aligned}$$

The map κ is an inverse of T, because if ϵ is a 1-central representation of H_β, then

$$\begin{aligned} T(\kappa(\epsilon))(h,\sigma^j) &= \sigma^j(\kappa(\epsilon)(h)) \\ &= \sigma^j(\epsilon(h,1)) \\ &= \epsilon(h,\sigma^j), \end{aligned}$$

where the last equality comes from the fact that ϵ is a 1-central representation. On the other hand, if β is a γ-twisted representation of H, we have $\kappa(T(\beta))(h) = T(\beta)(h,1) = \beta(h)$.

(iii) Let H and K be finite subgroups of G. The map $\rho : G_\alpha \to G$ defines a functor

$$\begin{aligned} \mathsf{Or}_{\mathcal{FIN}}(G) &\xrightarrow{\rho^*} \mathsf{Or}_{\mathcal{FIN}}(G_\alpha), \\ G/H &\mapsto G_\alpha/\rho^{-1}(H) \end{aligned}$$

between the orbit categories with respect to the family of finite subgroups.
We will analyze the behaviour of the functor T with respect to restriction.
Let $\phi : G/H \to G/K$ be a G-equivariant map. Recall that such a map is determined up to G-conjugacy by an inclusion $H \to K$ of finite subgroups of G.

Given an α-twisted representation $\beta : H \to \mathrm{GL}(V)$, the following diagram is commutative:

$$\begin{array}{ccccc} K & \longrightarrow & H & \xrightarrow{\beta} & \mathrm{GL}(V) \\ \uparrow{\scriptstyle\rho} & & \uparrow{\scriptstyle\rho} & \nearrow{\scriptstyle T(\beta)} & \\ \rho^{-1}(K) & \longrightarrow & \rho^{-1}(H) & & \end{array}$$

where the unlabelled arrow denote inclusions.
Hence, the functor T is compatible with restrictions and thus defines a natural transformation of contravariant functors over the orbit category. □

Theorem 4.4. *Let G be a discrete group, and let $\alpha \in Z^2(G; S^1)$ be a cocycle taking values in $\mathbb{Z}/n\mathbb{Z}$. Consider the central extension associated to α:*

$$1 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow G_\alpha \xrightarrow{\rho} G \longrightarrow 1.$$

Denote by $\underline{E}G$ a model for the classifying space of proper actions and notice that the action of G_α via ρ on $\underline{E}G$ exhibits the later space as a model for $\underline{E}G_\alpha$.

Then the map ρ gives an isomorphism of abelian groups between the Bredon cohomology groups of $\underline{E}G$ with coefficients in the α-twisted representation ring and the Bredon cohomology groups of $\underline{E}G_\alpha$ with coefficients in the 1-central group representation group. In symbols,

$$H^*(\underline{E}G; \mathcal{R}^G_\alpha) \xrightarrow{\rho^*} H^*(\underline{E}G_\alpha; \mathcal{R}_1^{G_\alpha}).$$

Proof. Fix a G-cellular structure of $\underline{\mathrm{E}}G$. Associate to each orbit of m-cells in $\underline{\mathrm{E}}G$ of the form $G/H \times D^m$ an orbit of m-cells in $\underline{\mathrm{E}}G_\alpha$ of the form $G_\alpha/\rho^{-1}(H) \times D^m$.

Consider the cellular cochain complex of $\underline{\mathrm{E}}G$. Following the notation of Definition 2.2, we have that in degree m it has the form

$$C^m_G(\underline{E}G, M) = \bigoplus_\lambda Hom_{\mathbb{Z}}(\mathbb{Z}[e_\lambda], {}^\alpha\mathcal{R}(G/H_\lambda)).$$

From Lemma 4.3, this term is isomorphic (via ρ^*) to

$$C^m_G(\underline{E}G, M) = \bigoplus_\lambda Hom_{\mathbb{Z}}(\mathbb{Z}[e_\lambda], \mathcal{R}_1(G_\alpha/\rho^{-1}(H_\lambda))),$$

and the isomorphism commutes with the cellular boundary, thus determining a chain isomorphism

$$C^*(\underline{\mathrm{E}}G; {}^\alpha\mathcal{R}^?) \xrightarrow{\rho^*} C^*(\underline{\mathrm{E}}G_\alpha; \mathcal{R}_1^?),$$

which induces an isomorphism in Bredon cohomology. □

Corollary 4.5. *Let G be a discrete group, and let $\alpha \in Z^2(G; S^1)$ be a cocycle taking* *values in* $\mathbb{Z}/n\mathbb{Z}$*. Consider the extension associated to* α*,*

$$1 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow G_\alpha \xrightarrow{\rho} G \longrightarrow 1.$$

Then there exists an isomorphism of abelian groups

$${}^{\alpha}K^*_G(\underline{E}G) \cong K^*_{G_\alpha}(\underline{E}G_\alpha)$$

between the α*-twisted,* G*-equivariant* K*-theory and the untwisted* G_α*-equivariant* K*-theory of the classifying spaces for proper actions* $\underline{E}G = \underline{E}G_\alpha$*.*

Proof. From Theorem 4.4, the Bredon cohomology groups are all isomorphic. The spectral sequence 3.2 lets us conclude the desired isomorphism. □

5. Twistings in $SL_3\mathbb{Z}$ and $St_3\mathbb{Z}$

The cohomology of $SL_3\mathbb{Z}$

We recall the analysis of the cohomology of $SL_3\mathbb{Z}$ in [**BV14**]. Soulé proved in [**Sou78**] that the integral cohomology of $SL_3\mathbb{Z}$ consists only of 2- and 3-torsion. The 3-primary part is isomorphic to the graded algebra

$$\mathbb{Z}[x_1, x_2]/\langle 3x_1, 3x_2\rangle,$$

with both generators in degree 4.

The two-primary component is isomorphic to the graded algebra

$$\mathbb{Z}[u_1, \dots, u_7],$$

with respective degrees 3, 3, 4, 4, 5, 6, 6, subject to the relations

$$2u_1 = 2u_3 = 4u_3 = 4u_4 = 2u_5 = 2u_6 = 2u_7 = 0,$$

$$u_7u_1 = u_7u_4 = u_7u_5 = u_7u_6 = u_2u_5 = u_2u_6 = 0,$$

$$u_7^2 + u_7u_2^2 = u_3u_4 + u_1u_5 = u_3u_6 + u_3u_1^2 = u_3u_6 + u_5^2 = 0,$$

$$u_1u_6 + u_4u_5 = u_3^0u_4^2 + u_6^2 = u_5u_6 + u_5u_1^2 = 0.$$

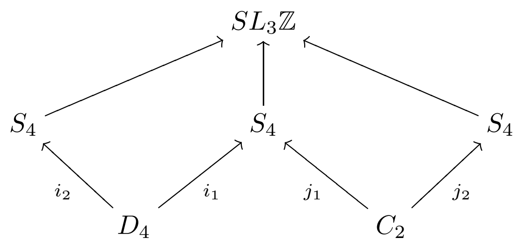
The twistings in equivariant K-theory are given by classes in $H^3(SL_3\mathbb{Z}, \mathbb{Z})$, all of which are 2-torsion. For this reason, we shall restrict to the two-primary component (we indicate this with the subscript (2)) in integral cohomology. In order to have a local description of these classes, we describe the cohomology of some finite subgroups inside $SL_3\mathbb{Z}$.

The finite groups of $SL_3\mathbb{Z}$ include S_4, the symmetric group in four letters, D_4, the dihedral group of order 8, the dihedral group of order 12, D_6, as well as the group of order 2 denoted by C_2.

Theorem 4 of [**Sou78**] gives the following result: For all $n \in \mathbb{N}$ there exists an exact sequence of abelian groups

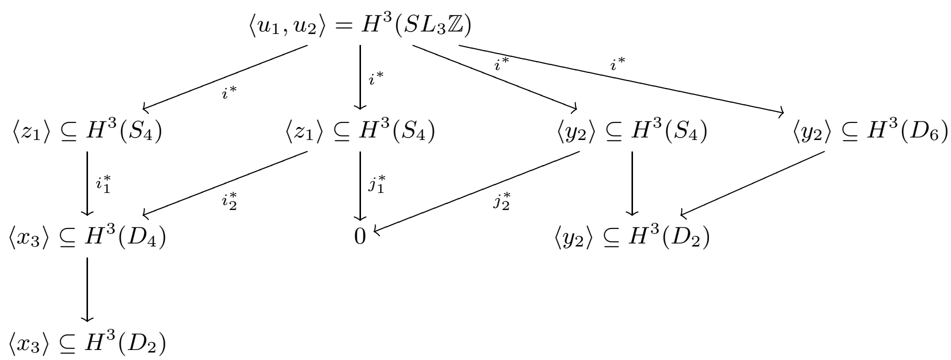
$$0 \to H^n(SL_3\mathbb{Z})_{(2)} \xrightarrow{\phi} H^n(S_4)_{(2)} \oplus H^n(S_4)_{(2)} \oplus H^n(S_4)_{(2)} \xrightarrow{\delta} H^n(D_4) \oplus H^n(C_2) \to 0$$

where ϕ is given by restrictions (see [**Sou78**, Corollary of Lemma 8]) and δ by the system of embeddings



If R is as in [**Sou78**, Proposition 4], the image of the morphism $\phi : H^*(SL_3\mathbb{Z})_{(2)} \to H^*(S_4)_{(2)} \oplus (i_1^*)^{-1}(R)$ is the set of elements (y, z) such that $j_2^*(y) = j_1^*(z)$. From Soulé's paper, we know that $H^*(S_4)_{(2)} = \mathbb{Z}[y_1, y_2, y_3]$, with $2y_1 = 2y_z = 4y_3 = y_1^4 + y_2^2 y_1 + y_3 y_1^2 = 0$, and, $(i_1^*)^{-1}(R) = \mathbb{Z}[z_1, z_2, z_3]$, with $2z_1 = 4z_2 = 2z_3 = z_3^2 + z_3 z_1^2 = 0$. Furthermore, $j_2^*(y_1) = t$, $j_2^*(y_2) = 0$, $j_2^*(y_3) = t^2$, $j_1^*(z_1) = 0$, $j_1^*(z_2) = t^2$, and $j_1^*(z_3) = 0$. Then the elements $u_1 = y_2$, $u_2 = z_1$, $u_3 = y_1^2 + z_2$, $u_4 = y_1^2 + y_3$, $u_5 = y_1 y_2$, $u_6 = y_1 y_3 + y_1^3$, and $u_7 = z_3$ generate $\phi(H^*(SL_3\mathbb{Z})_{(2)})$.

In $H^3(\;)$, the above discussion can be summarized in the following diagram:



(5.1)

In the following section, we will give explicit generators and analyze the depicted embeddings in $SL_3\mathbb{Z}$.

The cohomology of $St_3\mathbb{Z}$

The following result was published as Theorem 8 in [**Sou78**]; see also [**TY92**, Section 4, p. 92], for a more precise account.

Theorem 5.2.
- *There exists a 3-torsion cohomology class $\xi \in H^4(St_3\mathbb{Z}, \mathbb{Z})$ such that, for any $St_3\mathbb{Z}$-module A, the cup product by ξ induces an isomorphism*

$$\cdot \cup \xi : H^k(St_3\mathbb{Z}, A) \to H^{k+4}(St_3\mathbb{Z}, A)$$

as soon as $k > 3$ and $k > 0$ when A is constant.
- *The ring $H^*(St_3\mathbb{Z}, \mathbb{Z})_2$ is generated by elements w_1, w_2, w_3 with respective degrees 3, 4, 4, submitted to the defining relations $2w_1 = 4w_2 = 16w_3 = {w_1}^2 = w_1 w_2 = w_2 w_3 = 0$. Hence $H^1(St_3\mathbb{Z}, \mathbb{Z}) = H^2(St_3\mathbb{Z}, \mathbb{Z}) = 0$, $H^3(St_3\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/2$, $H^4(St_3\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/16 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3$.*

The cohomology of $St_3\mathbb{Z}$ is seen to be completely determined by the classes w_1, w_2, w_3, as well as the periodicity class ξ. The classes w_i restrict non-trivially to some specific generators of the cohomology of finite subgroups. We will analyze briefly how they relate to the generating classes u_1, u_2 of $H^3(SL_3\mathbb{Z}, \mathbb{Z})$. This is a summary of the discussion in Lemma 9 and the proof of Theorem 8 in [**Sou78**].

The group $St_3(\mathbb{R})$ fits as a central extension $1 \to \mathbb{Z}/2 \to St_3(\mathbb{R}) \to SL_3(\mathbb{R}) \to 1$, which restricts to a central extension of lattices $1 \to \mathbb{Z}/2 \to St_3\mathbb{Z} \to SL_3\mathbb{Z} \to 1$.

The maximal compact subgroups of $St_3(\mathbb{R})$, (respectively $SL_3(\mathbb{R})$) are Spin_3, (respectively $SO(3)$). Hence, all finite subgroups of $St_3\mathbb{Z}$ are contained in Spin_3, which is homeomorphic to the 3-dimensional sphere, thus the cohomology of all finite subgroups in $St_3\mathbb{Z}$ is 4-periodic. This is the origin of the periodicity class ξ.

The class w_1 restricts non-trivially under a system of inclusions of finite groups

$$\begin{array}{ccc} S_4^* & & S_4^* \\ & \nwarrow \quad \nearrow & \\ & D_4^* & \end{array}$$

which covers the inclusions $i_1, i_2 : D_4 \rightrightarrows S_4$ in $SL_3\mathbb{Z}$.

Thus, u_1 maps to w_1, and u_2 maps to the trivial class under the map induced by the universal cover $St_3\mathbb{Z} \to SL_3\mathbb{Z}$ in cohomology.

Remark 5.3. [The universal central extension of $SL_3\mathbb{Z}$ and $St_3\mathbb{Z}$.]

In the early literature on the Steinberg group (particularly Steinberg' s Yale notes [**Ste68**]), there is an unfortunate identification of $St_3\mathbb{Z}$ with the universal central extension of $SL_3\mathbb{Z}$. This mistake has been repeated in the literature [**Sou78**, Section 2.4] and [**BdlH13**, Example IV].

Denote by $\widetilde{SL_n(\mathbb{Z})}$ the universal central extension of $SL_n(\mathbb{Z})$. It fits in an exact sequence

$$1 \to H_2(SL_n(\mathbb{Z}), \mathbb{Z}) \to \widetilde{SL_n(\mathbb{Z})} \to SL_n(\mathbb{Z}).$$

While there is an identification of $St_n(\mathbb{Z})$ with $\widetilde{SL_n(\mathbb{Z})}$ for $n \geqslant 5$, Van der Kallen [**vdK75**] computes the Schur Multiplier $H_2(G, \mathbb{Z})$ for G $SL_3\mathbb{Z}$ and $SL_4(\mathbb{Z})$, being in both cases isomorphic to Klein's four group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Thus, the universal central extension defining $St_3\mathbb{Z}$,

$$1 \to \mathbb{Z}/2\mathbb{Z} \to St_3\mathbb{Z} \to SL_3\mathbb{Z},$$

and the one defining $\widetilde{SL_3\mathbb{Z}}$,

$$1 \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to \widetilde{SL_3\mathbb{Z}} \to SL_3\mathbb{Z},$$

are not the same. We thank Prof. Pierre De la Harpe for pointing this fact to us in personal correspondence, leading to the correction of a mistake in a previous version of this note.

6. Twisted K-theory of $SL_3\mathbb{Z}$

We use the following notation: $\{1\}$ denotes the trivial group, C_n the cyclic group of n elements, D_n the dihedral group with $2n$ elements, and S_n the Symmetric group of permutations on n objects.

There are four twistings for $SL_3\mathbb{Z}$ up to cohomology—namely, $0, u_1, u_2, u_1 + u_2$; continuing the work started in [**BV14**], we will calculate the twisted K-theory for the twistings u_2 and $u_1 + u_2$.

From diagram 5.1, one can see that the class u_2 restricts non-trivially to two copies of S_4 corresponding to the stabilizer of the vertices v_3 and v_5. We recall the $SL_3\mathbb{Z}$-CW-complex structure of $\underline{E}SL_3\mathbb{Z}$ as is given in [**Sou78**]. The labels O, Q, M, N, P of the vertices refer to the Figure 2 of [**SG08**], where also Soulés matrices $g_1, \dots, g_{14}$ are recalled.

vertices				2-cells			
v_1	O	$g_2,\ g_3$	S_4	t_1	OQM	g_2	C_2
v_2	Q	$g_4,\ g_5$	D_6	t_2	$QM'N$	g_1	$\{1\}$
v_3	M	$g_6,\ g_7$	S_4	t_3	$MN'P$	g_{12}, g_{14}	$C_2 \times C_2$
v_4	N	$g_6,\ g_8$	D_4	t_4	$OQN'P$	g_5	C_2
v_5	P	$g_5\ ,\ g_9$	S_4	t_5	$OMM'P$	g_6	C_2
edges				3-cells			
e_1	OQ	$g_2,\ g_5$	$C_2 \times C_2$	T_1	g_1	$\{1\}$	
e_2	OM	g_6, g_{10}	D_3				
e_3	OP	g_6, g_5	D_3				
e_4	QM	g_2	C_2				
e_5	QN'	g_5	C_2				
e_6	MN	g_6, g_{11}	$C_2 \times C_2$				
e_7	$M'P$	g_6, g_{12}	D_4				
e_8	$N'P$	g_5, g_{13}	D_4				

The first column is an enumeration of equivalence classes of cells; the second lists a representative of each class; the third column gives generating elements for the stabilizer of the given representative; and the last one is the isomorphism type of the stabilizer. The generating elements referred to above are the same as in [**BV14**].

The twisting u_1

The following theorem was proved in [**BV14**]:

Proposition 6.1.

$$
\begin{aligned}
{}^{u_1}K^0_{SL_3\mathbb{Z}}(\underline{E}SL_3\mathbb{Z}) &\cong \mathbb{Z}^{\oplus 13},\\
{}^{u_1}K^1_{SL_3\mathbb{Z}}(\underline{E}SL_3\mathbb{Z}) &= 0.
\end{aligned}
$$

The twisting u_2

In order to determine the twisted K-theory, we calculate Bredon cohomology.

Determination of Φ_1

In order to determine the morphism Φ_1, we need to recall the projective character tables of the groups where u_2 restricts *non-trivially*.

Here, we denote by z the generator of the central copy of $\mathbb{Z}_2$. The linear character table of a Schur covering group S_4^* is obtained in [**Kar94**, p. 254], by considering the group with presentation

$$S_4^* = \langle h_1, h_2, h_3, z \mid h_i^2 = (h_j h_{j+1})^3 = (h_k h_l)^2 = z, z^2 = [z, h_i] = 1\rangle$$
$$1 \leqslant i \leqslant 3, j = 1, k \leqslant l-2$$

and the central extension

$$1 \to \langle z \rangle \to S_4^* \xrightarrow{f} S_4 \to 1$$

given by $f(h_i) = g_i$, as well as the choice of representatives of regular conjugacy classes

S_4^*	e	z	h_1	h_1h_3	h_1h_2	h_1h_2z	$h_1h_2h_3$	$h_1h_2h_3z$
ϵ_1	1	1	1	1	1	1	1	1
ϵ_2	1	1	-1	1	1	1	-1	-1
ϵ_3	2	2	0	2	-1	-1	0	0
ϵ_4	3	3	1	-1	0	0	-1	-1
ϵ_5	3	3	-1	-1	0	0	1	1
ϵ_6	2	-2	0	0	1	-1	$\sqrt{2}$	$-\sqrt{2}$
ϵ_7	2	-2	0	0	1	-1	$-\sqrt{2}$	$\sqrt{2}$
ϵ_8	4	-4	0	0	-1	1	0	0

where the first five lines are characters associated to S_4 and where ϵ_6 is the *Spin* representation.

We take the following presentation of Dihedral groups, $D_n = \langle g_i, g_j \rangle = \langle g_i, g_j \mid g_i^2 = g_j^2 = (g_i g_j)^n = 1\rangle$.

The dihedral group of order 6 has trivial 3-dimensional integer cohomology. Thus, its twisted representations do agree with the linear ones. The dihedral subgroups with n even in $SL_3\mathbb{Z}$ are $C_2 \times C_2 = D_2$ and D_4.

The following is the linear character table for D_n:

D_n	$\langle (g_i, g_j)^k \rangle$	$\langle g_j (g_i g_j)^k \rangle$
ξ_1	1	1
ξ_2	1	-1
$\hat{\xi_3}$	-1^k	-1^k
$\hat{\xi_4}$	-1^k	-1^{k+1}
ϕ_p	$2\cos(2\pi pk/n)$	0

where $0 \leqslant k \leqslant n-1$, p varies from 1 to $(n/2)-1$ (n even) or $(n-1)/2$ (n odd), and the hat denotes a representation that only appears in the case n even. The group $D_2^* = \langle h_1, h_3, z\rangle$ is isomorphic to the quaternion group of order eight, and a linear character table is given by

D_2^*	1	z	$\{h_1, h_1^{-1}\}$	$\{h_3, h_3^{-1}\}$	$\{h_1h_3, (h_1h_3)^{-1}\}$
η_1	1	1	1	1	1
η_2	1	1	1	-1	-1
η_3	1	1	-1	1	-1
η_4	1	1	-1	-1	1
η_5	2	-2	0	0	0

A Schur cover of D_4 can be taken as $D_8 = \langle a, x \mid a^4 = x^2 = e, xax^{-1} = a^{-1}\rangle$, whose character table is

D_8	e	$a^4(=z)$	a^2	a	$a^3(=az)$	x	ax
λ_1	1	1	1	1	1	1	1
λ_2	1	1	1	1	1	-1	-1
λ_3	1	1	1	-1	-1	1	-1
λ_4	1	1	1	-1	-1	-1	1
λ_5	2	2	-2	0	0	0	0
λ_6	2	-2	0	$\sqrt{2}$	$-\sqrt{2}$	0	0
λ_7	2	-2	0	$-\sqrt{2}$	$\sqrt{2}$	0	0

The relevant inclusions among stabilizers are the following. We give a conjugacy representative appearing in the corresponding character table when necessary.

$$\begin{array}{ll} stab(e_2) & \xrightarrow{i} stab(v_3) \\ \langle g_6, g_{10}\rangle & \to \langle g_6, g_7\rangle \\ g_6 & \mapsto g_6 \\ g_{10} & \mapsto g_7^{-1} g_6 g_7 \end{array} \qquad \begin{array}{ll} stab(e_3) & \xrightarrow{i} stab(v_5) \\ \langle g_6, g_5\rangle & \to \langle g_5, g_9\rangle \\ g_6 & \mapsto g_9^{-1} g_5 g_9 \\ g_5 & \mapsto g_5 \end{array}$$

$$\begin{array}{ll} stab(e_4) & \xrightarrow{i} stab(v_3) \\ \langle g_2\rangle & \to \langle g_6, g_7\rangle \\ g_2 & \mapsto g_6 g_7^2 g_6 g_7^{-1} \end{array} \qquad \begin{array}{ll} stab(e_5) & \xrightarrow{i} stab(v_4) \\ \langle g_5\rangle & \to q_2^{-1} \cdot \langle g_6, g_7\rangle \cdot q_2^{-1} \\ g_5 & \mapsto q_2^{-1} \cdot g_8 \cdot q_2^{-1} \end{array}$$

$$\begin{array}{ll} stab(e_6) & \xrightarrow{i} stab(v_3) \\ \langle g_6, g_{11}\rangle & \to \langle g_6, g_7\rangle \\ g_6 & \mapsto g_6 \\ g_{11} & \mapsto g_7 g_6 g_7^{-1} g_6 g_7 \sim g_6 \end{array} \qquad \begin{array}{ll} stab(e_6) & \xrightarrow{i} stab(v_4) \\ \langle g_6, g_{11}\rangle & \to \langle g_6, g_8\rangle \\ g_6 & \mapsto g_6 = x \\ g_{11} & \mapsto (g_6 g_8)^2 = a^2 \end{array}$$

$$\begin{array}{ll} stab(e_7) & \xrightarrow{i} stab(v_3) \\ \langle g_6, g_{12}\rangle & \to q_1^{-1} \cdot \langle g_6, g_7\rangle \cdot q_1 \\ g_6(=x) & \mapsto q_1^{-1} \cdot (g_6 g_7^2 g_6) \cdot q_1 \sim g_7^2 \\ g_{12}(=ax) & \mapsto q_1^{-1} \cdot g_6 \cdot q_1 \sim g_6 \end{array} \qquad \begin{array}{ll} stab(e_7) & \xrightarrow{i} stab(v_5) \\ \langle g_6, g_{12}\rangle & \to \langle g_5, g_9\rangle \\ g_6 & \mapsto g_9^{-1} g_5 g_9 \\ g_{12} & \mapsto g_9^2 \end{array}$$

$$\begin{array}{ll} stab(e_8) & \xrightarrow{i} stab(v_4) \\ \langle g_5, g_{13}\rangle & \to q_2^{-1} \cdot \langle g_6, g_8\rangle \cdot q_2 \\ g_5 & \mapsto q_2^{-1} \cdot g_8 \cdot q_2 \\ g_{13} & \mapsto q_2^{-1} \cdot g_6 \cdot q_2 \end{array} \qquad \begin{array}{ll} stab(e_8) & \xrightarrow{i} stab(v_5) \\ \langle g_5, g_{13}\rangle & \to \langle g_5, g_9\rangle \\ g_5 & \mapsto g_5 \\ g_{13} & \mapsto g_5 g_9^2 g_5 \sim g_9^2 \end{array}$$

Using the above inclusions and elementary calculations with characters, particularly the rectification procedure [**BV14**, Theorem 1.7], we obtain a matrix of size 34×33 representing the morphism Φ_1. The matrices representing the restrictions among stabilizers are the following. The signs corresponding to the coboundary map as in [**SG08**].

	e_1				e_2			e_3		
v_1	-1	0	0	0	-1	0	0	-1	0	0
	0	-1	0	0	0	-1	0	0	-1	0
	-1	-1	0	0	0	0	-1	0	0	-1
	-1	0	-1	-1	-1	0	-1	-1	0	-1
	0	-1	-1	-1	0	-1	-1	0	-1	-1

	e_1				e_4		e_5	
v_2	1	0	0	0	−1	0	−1	0
	0	1	0	0	0	−1	0	−1
	0	0	1	0	0	−1	−1	0
	0	0	0	1	−1	0	0	−1
	0	0	1	1	−1	−1	−1	−1
	1	1	0	0	−1	−1	−1	−1

	e_2			e_4		e_6	e_7	
v_3	0	0	1	1	1	−1	−1	0
	0	0	1	1	1	−1	0	−1
	1	1	1	2	2	−2	−1	−1

	e_5		e_6	e_8	
v_4	1	1	1	−1	0
	1	1	1	0	−1

	e_3			e_7		e_8	
v_5	0	0	1	1	0	1	0
	0	0	1	0	1	0	1
	1	1	1	1	1	1	1

The elementary divisors of the matrix representing the morphism ϕ is 1 repeated twelve times. The rank of this matrix is 12.

Determination of Φ_2

The relevant inclusions among stabilizers are the following. We give a conjugacy representative appearing in the corresponding character table when necessary.

$$\begin{array}{ll} stab(t_3) & \xrightarrow{i} stab(e_6) \\ \langle g_{12}, g_{14}\rangle & \to q_1^{-1}\cdot\langle g_6, g_7\rangle\cdot q_1 \\ g_{12} & \mapsto q_1^{-1}\cdot g_6\cdot q_1 \\ g_{14} & \mapsto q_1^{-1}\cdot g_{11}\cdot q_1 \end{array}$$

$$\begin{array}{ll} stab(t_3) & \xrightarrow{i} stab(e_7) \\ \langle g_{12}, g_{14}\rangle & \to \langle g_6, g_{12}\rangle \\ g_{12} & \mapsto g_{12} = ax \\ g_{14} & \mapsto g_{12}(g_6 g_{12})^2 = xa \end{array}$$

$$\begin{array}{ll} stab(t_3) & \xrightarrow{i} stab(e_8) \\ \langle g_{12}, g_{14}\rangle & \to \langle g_5, g_{13}\rangle \\ g_{12} & \mapsto g_{13}(g_5 g_{13})^2 = xa \\ g_{14} & \mapsto (g_5 g_{13})^2 = a^2 \end{array}$$

$$\begin{array}{ll} stab(t_4) & \xrightarrow{i} stab(e_8) \\ \langle g_5\rangle & \to \langle g_5, g_{13}\rangle \\ g_5 & \mapsto g_5 = x \end{array}$$

$$\begin{array}{ll} stab(t_5) & \xrightarrow{i} stab(e_6) \\ \langle g_6\rangle & \to (q_1 q_2)^{-1}\cdot\langle g_6, g_{12}\rangle\cdot(q_1 q_2) \\ g_6 & \mapsto (q_1 q_2)^{-1}\cdot(g_6 g_1 1)\cdot(q_1 q_2) \end{array}$$

$$\begin{array}{ll} stab(t_5) & \xrightarrow{i} stab(e_7) \\ \langle g_6\rangle & \to \langle g_6, g_{12}\rangle \\ g_6 \mapsto & g_6. \end{array}$$

Using the above inclusions, an elementary calculation yields a matrix of size 33×12 representing the morphism Φ_2. The matrices representing the restrictions among stabilizers are the following:

	t_1		t_4	
e_1	1	0	1	0
	0	1	0	1
	0	1	1	0
	1	0	0	1

	t_1		t_5	
e_2	−1	0	1	0
	0	−1	0	1
	−1	−1	1	1

	t_4		t_5	
e_3	−1	0	−1	0
	0	−1	0	−1
	−1	−1	−1	−1

	t_1		t_2
e_4	1	0	1
	0	1	1

	t_2	t_4	
e_5	−1	1	0
	−1	0	1

	t_2	t_3	t_4		t_5	
e_6	2	1	0	0	0	0

	t_3	t_5	
e_7	−1	1	1
	−1	1	1

	t_3	t_4	
e_8	1	1	1
	1	1	1

The elementary divisors of the matrix representing the morphism ϕ is 1,repeated seven times. The rank of this matrix is 7.

Determination of Φ_3

The morphism Φ_3 is given by blocks which are represented by the following matrices:

	T_1
t_1	−1
	−1

	T_1
t_2	1

	T_1
t_3	−1
	−2

	T_1
t_4	1
	1

	T_1
t_5	−1
	−1

We have the Bredon cochain complex

$$0 \to \mathbb{Z}^{\oplus 19} \xrightarrow{\Phi_1^{u_2}} \mathbb{Z}^{\oplus 19} \xrightarrow{\Phi_2^{u_2}} \mathbb{Z}^{\oplus 8} \xrightarrow{\Phi_3^{u_2}} \mathbb{Z} \to 0.$$

Using the information concerning ranks and elementary divisors of $\Phi_i^{u_2}$, we obtain

$$H^p_{SL_3\mathbb{Z}}(\underline{\mathrm{E}}SL_3\mathbb{Z}, \mathcal{R}_{u_2}) = 0, \text{ if } p > 0, \quad H^0_{SL_3\mathbb{Z}}(\underline{\mathrm{E}}SL_3\mathbb{Z}, \mathcal{R}_{u_2}) \cong \mathbb{Z}^{\oplus 7}. \tag{6.2}$$

Since the Bredon cohomology concentrates at low degree, the spectral sequence described in Section 3.2 collapses at level 2 and we conclude the following

Proposition 6.3.

$$^{u_2}K^0_{SL_3\mathbb{Z}}(\underline{E}SL_3\mathbb{Z}) \cong \mathbb{Z}^{\oplus 7},$$
$$^{u_2}K^1_{SL_3\mathbb{Z}}(\underline{E}SL_3\mathbb{Z}) = 0.$$

The twisting $u_1 + u_2$

Now we continue with the calculation of $^{u_1+u_2}K_{SL_3\mathbb{Z}}(\underline{\mathrm{E}}SL_3\mathbb{Z})$. Notice that the classes u_1 and u_2 are disjoint, i.e., they do not restrict simultaneously to a non-zero element in the cohomology of any subgroup of $SL_3\mathbb{Z}$. This observation and diagram 5.1 lead to the following.

Remark 6.4. The matrix $\Phi_1^{u_1+u_2}$ corresponding to the twisting u_1+u_2 can be obtained as

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
v_1	u_1	u_1	u_1	0	0	0	0	0
v_2	u_1	0	0	u_1	u_1	0	0	0
v_3	0	u_2	0	u_2	0	u_2	u_2	0
v_4	0	0	0	0	u_2	u_2	0	u_2
v_5	0	0	u_2	0	0	0	u_2	u_2

where a u_i in position (j,k) means that we take the corresponding submatrix of $\Phi_1^{u_i}$ associated to the inclusion $stab(e_j) \to stab(v_k)$.

This matrix has size 14×16 and it has elementary divisors $(1,1,1,1,1,1,1,1,2)$ and its rank is 9.

Remark 6.5. The matrix $\Phi_2^{u_1+u_2}$ corresponding to the twisting u_1+u_2 can be obtained as

	t_1	t_2	t_3	t_4	t_5
e_1	u_1	0	0	u_1	0
e_2	u_0	0	0	0	u_0
e_3	0	0	0	u_0	u_0.
e_4	u_0	u_0	0	0	0
e_5	0	u_0	0	u_0	u_0
e_6	0	u_2	u_2	0	u_2
e_7	0	0	u_2	0	u_2
e_8	0	0	u_2	u_2	0

where a u_i in position (j,k) means that we take the corresponding submatrix of $\Phi_1^{u_i}$ associated to the inclusion $stab(t_j) \to stab(e_k)$ (u_0 denotes the trivial cocycle).

This matrix has size 16×8 and it has 1 as elementary divisor seven times and its rank is 7.

Finally the matrix $\Phi_3^{u_1+u_2}$ corresponding to the twisting u_1+u_2 is the same as the matrix $\Phi_3^{u_2}$.

We have the following cochain complex;

$$0 \to \mathbb{Z}^{\oplus 14} \xrightarrow{\Phi_1^{u_1+u_2}} \mathbb{Z}^{\oplus 16} \xrightarrow{\Phi_2^{u_1+u_2}} \mathbb{Z}^{\oplus 8} \xrightarrow{\Phi_3^{u_1+u_2}} \mathbb{Z} \to 0.$$

Using the data of $\Phi_i^{u_1+u_2}$ concerning ranks and elementary divisors, we obtain

$$\begin{aligned} &H^p_{SL_3\mathbb{Z}}(\underline{\mathrm{E}}SL_3\mathbb{Z}, \mathcal{R}_{u_1+u_2}) = 0, \text{ if } p > 1,\\ &H^0_{SL_3\mathbb{Z}}(\underline{\mathrm{E}}SL_3\mathbb{Z}, \mathcal{R}_{u_1+u_2}) \cong \mathbb{Z}^{\oplus 5},\\ &H^1_{SL_3\mathbb{Z}}(\underline{\mathrm{E}}SL_3\mathbb{Z}, \mathcal{R}_{u_1+u_2}) \cong \mathbb{Z}/2\mathbb{Z}. \end{aligned} \tag{6.6}$$

Since the Bredon cohomology concentrates at low degree, the spectral sequence described in Section 3.2 collapses at level 2 and we conclude the following.

Proposition 6.7.

$$\begin{aligned} {}^{u_1+u_2}K^0_{SL_3\mathbb{Z}}(\underline{E}SL_3\mathbb{Z}) &\cong \mathbb{Z}^{\oplus 5},\\ {}^{u_1+u_2}K^1_{SL_3\mathbb{Z}}(\underline{E}SL_3\mathbb{Z}) &= \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

7. Applications

Twisted equivariant K-Homology and the Baum–Connes Conjecture

The Baum–Connes Conjecture [**BCH94, MV03**] predicts for a discrete group G the existence of an isomorphism

$$\mu_i : K_i^G(\underline{E}G) \to K_i(C_r^*(G))$$

given by the (analytical) assembly map, where $C_r^*(G)$ is the reduced C^*-algebra of the group G.

More generally, given any G-C^*-algebra, the Baum–Connes conjecture with coefficients predicts an isomorphism given by an assembly map

$$\mu_i : K_i^G(\underline{E}G, A) \to K_i(A \rtimes G)),$$

where $K_i^G(\underline{E}G, A)$ is defined in terms of equivariant and bivariant KK-groups,

$$K_*^G(\underline{E}G, A) = \underset{G-\text{compact} X \subset \underline{E}G}{\operatorname{colim}} KK_*(C_0(X), A)$$

and $A \rtimes G$ denotes the crossed product C^*-algebra, $X \subset \underline{E}G$ is a cocompact subcomplex. See [**CE01, Ech08**] for more details.

Definition 7.1. Let G be a discrete group. Given a cocycle $\omega \in Z^2(G, S^1)$, an ω-twisted representation on a Hilbert space $\mathcal{H}$ is a map $V : G \to \mathcal{U}(\mathcal{H})$ satisfying $V(s)V(t) = \omega(s,t)V(st)$.

Consider the quotient map $\mathcal{U}(\mathcal{H}) \to PU(\mathcal{H}) = \mathcal{U}(\mathcal{H})/S^1$. Recall that the group $PU(\mathcal{H})$ is the outer automorphism group $Out(\mathcal{K})$ of the C^*-algebra of compact operators on $\mathcal{H}$, denoted by $\mathcal{K}$. The cocycle ω defines in this way an action of G on $\mathcal{K}$. This algebra is denoted by $\mathcal{K}_\omega$.

Let G be a discrete group with a finite model for $\underline{E}G$. Let $\omega \in Z^2(G, S^1)$ be a cocycle and assume that the bredon cohomology groups $H^*(\underline{E}G, \mathcal{R}^{-\omega})$ are concentrated in degrees 0 and 1.

Then the Universal Coefficient Theorem for Bredon cohomology [**BV14**, Theorem 1.13] identifies the Bredon homology groups $H_*^{SL_3\mathbb{Z}}(\underline{E}SL_3\mathbb{Z}, \mathcal{R}^\alpha)$ with the Bredon cohomology groups $H^*_{SL_3\mathbb{Z}}(\underline{E}SL_3\mathbb{Z}, \mathcal{R}^\alpha)$. By inspecting the Bredon cohomology groups computed in 6.6 and 6.2, the hypotheses of Corollary 7.3 in [**BV14**] are satisfied for the twistings u_2 and $u_1 + u_2$. This gives a duality isomorphism

$${}^{\omega}K_G^*(\underline{E}G) \to K_*^G(\underline{E}G, \mathcal{K}_{-\omega}).$$

Similar forms of Poincaré duality for proper and twisted actions have been studied by Echterhoff, Emerson, and Kim in [**EEK08**, Theorem 3.1] under assumptions concerning the Baum–Connes conjecture, particularly the validity of the Dirac–Dual-Dirac Method for the group G.

Theorem 7.2. *The equivariant K-homology groups with coefficients in the G-C^**

algebra $\mathcal{K}_\omega$ are given as follows:

- $$K_0^{SL_3\mathbb{Z}}(\underline{E}SL_3\mathbb{Z},\mathcal{K}_{u_1}) \cong \mathbb{Z}^{\oplus 13},$$ $$K_1^{SL_3\mathbb{Z}}(\underline{E}SL_3\mathbb{Z},\mathcal{K}_{u_1}) = 0.$$
- $$K_0^{SL_3\mathbb{Z}}(\underline{E}SL_3\mathbb{Z},\mathcal{K}_{u_2}) \cong \mathbb{Z}^{\oplus 7},$$ $$K_1^{SL_3\mathbb{Z}}(\underline{E}SL_3\mathbb{Z},\mathcal{K}_{u_2}) = 0.$$
- $$K_0^{SL_3\mathbb{Z}}(\underline{E}SL_3\mathbb{Z},\mathcal{K}_{u_1+u_2}) \cong \mathbb{Z}^{\oplus 5},$$ $$K_1^{SL_3\mathbb{Z}}(\underline{E}SL_3\mathbb{Z},\mathcal{K}_{u_1+u_2}) \cong \mathbb{Z}/2\mathbb{Z}.$$

Relation to the work of Tezuka and Yagita

In the case of finite-order twists given by cocycles $\alpha \in Z^2(G, S^1)$, the finite-dimensional, α-twisted vector bundle model of twisted equivariant K-theory is related to untwisted equivariant K-theory groups in a way we will describe below.

Recall that given a normalized torsion cocycle α, there exists a central extension

$$1 \to \mathbb{Z}/n \to G_\alpha \to G \to 1.$$

Let X be a G-connected G-CW complex. The α-Twisted K-theory groups are seen to agree with the abelian group of G_α-equivariant complex vector bundles for which the generator of $\mathbb{Z}/n$ acts by complex multiplication by $e^{2\pi i n}$. There is a splitting

$$K^0_{G_\alpha}(X) \cong \bigoplus_{V\in \mathrm{Irr}(\mathbb{Z}/n)} K^0_{G_\alpha}(X,V), \tag{7.3}$$

where $K^0_{G_\alpha}(X,V)$ is the subgroup of G_α-equivariant complex vector bundles for which the action of the central $\mathbb{Z}/n$ on each fiber restricts to the irreducible representation V, and the definition is extended to other degrees using the remarks following Definition 3.1.

Given a discrete group G and a normalized torsion cocycle α, Theorem 3.4 in [**Dwy08**] proves that the groups ${}^\alpha K^*_G(X)$ extend to a $\mathbb{Z}/2$-graded equivariant cohomology theory on the category of finite, proper G-CW pairs. This theory restricts to equivariant K-theory [**LO01**] in the case of a trivial cocycle. The groups ${}^\alpha K^*_G(X)$ have a natural graded $K^*_G(X)$-module structure.

The multiplicative structure on the graded ring $K^*_G(\underline{E}G)$ is well known. Recall the definition of the augmentation ideal

$$I_G = \ker(K^0_G(\underline{E}G) \xrightarrow{i^*} K^0_G(\underline{E}G_0) \to K^0_{\{e\}}(\underline{E}G_0),$$

where $\underline{E}G_0 \to \underline{E}G$ denotes the inclusion of the $0th$-skeleton and the last map is the restriction map associated to the trivial group $\{e\} \subset G$.

The following result is a generalization of the Atiyah–Segal completion theorem and it is proved in [**LO01**, Theorem 4.4(b)].

Theorem 7.4. *Let $\underline{E}G$ be the classifying space for proper actions.*

- *If $\underline{E}G$ has the homotopy type of a finite G-CW complex, then there is an isomorphism*

$$K^*(BG) \cong K^*_G(\underline{E}G)_{\hat{I}_G},$$

where the right-hand side denotes the completion with respect to the ideal I_G.

Specializing to the case of $St_3\mathbb{Z}$, the topological K-theory ring $K^*(BSt_3\mathbb{Z})$ is known after computations by Tezuka and Yagita using Brown–Peterson spectra and its Conner–Floyd isomorphism [**TY92**, Corollary 4.7], which we recall here:

Theorem 7.5. *Localized at the prime 2, the topological K-theory of $BSt_3\mathbb{Z}$ is given as follows:*

- $K^0(BSt_3\mathbb{Z}) = \mathbb{Z}_{\hat{2}}^6 \oplus \mathbb{Z}_{(2)}$,
- $K^1(BSt_3\mathbb{Z}) = \mathbb{Z}_{\hat{2}}$,

where $\mathbb{Z}_{(2)}$ is the localization at 2 and $\mathbb{Z}_{\hat{2}}$ denotes the 2-adic completion of the integers.

Putting toghether Theorems 6.7, 7.4, and 7.5, one obtains the following:

Corollary 7.6. *The completion of the equivariant K-theory groups $K_G^*(\underline{E}St_3\mathbb{Z})$ computed in 6.7 with respect to the augmentation ideal $I_{St_3\mathbb{Z}}$ is given as follows:*

- $K_G^0(\underline{E}St_3\mathbb{Z})_{\hat{I}_{St_3\mathbb{Z}}} = \mathbb{Z}_{\hat{2}}^6 \oplus \mathbb{Z}_{(2)}$,
- $K_G^1(\underline{E}St_3\mathbb{Z})_{\hat{I}_{St_3\mathbb{Z}}} = \mathbb{Z}_{\hat{2}}$.

References

[BCH94] Paul Baum, Alain Connes, and Nigel Higson. Classifying space for proper actions and K-theory of group C^*-algebras. In *C^*-algebras: 1943–1993 (San Antonio, TX, 1993)*, volume 167 of *Contemp. Math.*, pages 240–291. Amer. Math. Soc., Providence, RI, 1994.

[BdlH13] Bachir Bekka and Pierre de la Harpe. Groups with faithful irreducible projective unitary representations. *Forum Math.* **25**(3):505–531, 2013.

[BEJU14] Noé Bárcenas, Jesús Espinoza, Michael Joachim, and Bernardo Uribe. Universal twist in equivariant K-theory for proper and discrete actions. *Proc. Lond. Math. Soc. (3)* **108**(5):1313–1350, 2014.

[BEUV13] N. Barcenas, Espinoza, B. Uribe, and M. Velasquez. Segal's spectral sequence in twisted equivariant K theory for proper actions. Preprint, arXiv:1307.1003 [math.AT], 2013.

[BV14] Noé Bárcenas and Mario Velásquez. Twisted equivariant K-theory and K-homology of $\mathrm{Sl}_3\mathbb{Z}$. *Algebr. Geom. Topol.* **14**(2):823–852, 2014.

[CE01] Jérôme Chabert and Siegfried Echterhoff. Twisted equivariant KK-theory and the Baum–Connes conjecture for group extensions. *K-Theory* **23**(2):157–200, 2001.

[DL98] James F. Davis and Wolfgang Lück. Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory. *K-Theory* **15**(3):201–252, 1998.

[Dwy08] Christopher Dwyer. Twisted equivariant K-theory for proper actions of discrete groups. *K-Theory* **38**(2):95–111, 2008.

[Ech08] Siegfried Echterhoff. The K-theory of twisted group algebras. In *C^*-algebras and elliptic theory II*, Trends Math., pages 67–86. Birkhäuser, Basel, 2008.

[EEK08] Siegfried Echterhoff, Heath Emerson, and Hyun Jeong Kim. *KK*-theoretic duality for proper twisted actions. *Math. Ann.* **340**(4):839–873, 2008.

[Kar94] Gregory Karpilovsky. *Group Representations. Vol. 3.* Volume 180 of *North-Holland Mathematics Studies.* North-Holland Publishing Co., Amsterdam, 1994.

[LO01] Wolfgang Lück and Bob Oliver. The completion theorem in K-theory for proper actions of a discrete group. *Topology* **40**(3):585–616, 2001.

[Moo64] Calvin C. Moore. Extensions and low-dimensional cohomology theory of locally compact groups, I, II. *Trans. Amer. Math. Soc.* **113**:40–63, 1964.

[Moo68] Calvin C. Moore. Group extensions of p-adic and adelic linear groups. *Inst. Hautes Études Sci. Publ. Math.* (35):157–222, 1968.

[MV03] Guido Mislin and Alain Valette. *Proper group actions and the Baum–Connes conjecture.* Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2003.

[SG08] Rubén Sánchez-García. Bredon homology and equivariant K-homology of $\mathrm{SL}(3,\mathbb{Z})$. *J. Pure Appl. Algebra* **212**(5):1046–1059, 2008.

[Sou78] Christophe Soulé. The cohomology of $\mathrm{SL}_3(\mathbf{Z})$. *Topology* **17**(1):1–22, 1978.

[Ste68] Robert Steinberg. *Lectures on Chevalley Groups.* Yale University, New Haven, Conn., 1968. Notes prepared by John Faulkner and Robert Wilson.

[TY92] M. Tezuka and N. Yagita. Complex K-theory of $B\mathrm{SL}_3(\mathbf{Z})$. *K-Theory* **6**(1):87–95, 1992.

[vdK75] Wilberd van der Kallen. The Schur multipliers of $\mathrm{SL}(3,\mathbf{Z})$ and $\mathrm{SL}(4,\mathbf{Z})$. *Math. Ann.* **212**:47–49, 1974/75.

Noé Bárcenas barcenas@matmor.unam.mx

Centro de Ciencias Matemáticas, Universidad Nacional Autónoma de México, Ap. Postal 61-3 Xangari, Morelia, Michoacán, México 58089

Mario Velásquez mario.velasquez@javeriana.edu.co

Departamento de Matemáticas, Pontificia Universidad Javeriana, Cra. 7 No. 43-82 - Edificio Carlos Ortíz 5to piso, Bogotá D.C, Colombia