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ERRATUM TO "INVERSE LIMITS OF FINITE TOPOLOGICAL SPACES"

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(*communicated by Nicholas J. Kuhn*)

Abstract

In the paper [**1**], there is an error in the proof of Lemma 3.1, which was noticed by M. Thibault. We give a corrected (and simplified) proof, suggested by N. Kuhn. The statements of all results remain unchanged.

Given a finite simplicial complex K, one can construct an inverse system $\{X_n\}$ of finite topological spaces that are all weakly homotopy equivalent to $|K|$. The main result of [**1**] is that $|K|$ is homotopy equivalent to the inverse limit $\tilde{X}$ of these spaces. The proof of this theorem proceeds by first showing that $|K|$ is homeomorphic to a quotient space of $\tilde{X}$ (Lemma 3.1), and then exhibiting a deformation retraction from $\tilde{X}$ onto this quotient (Lemma 3.2).

We thank Matthew Thibault for pointing out an error in the proof of Lemma 3.1. A correction in a more general setting was given by him in [**2**]. Below, we present a simpler argument, for which we thank Nicholas Kuhn.

We assume the notation as in [**1**]. In particular, for each $n \geqslant 0$, X_n is the finite space whose points are faces of simplices of the nth barycentric subdivision of K. Given a point $x \in X_n$, we let B_x denote the minimal open set containing x, which exists by finiteness. There is a sequence of natural maps

$$q_n : X_n \to X_{n-1}$$

given by inclusion of simplices. These fit together into an inverse system with inverse limit $\tilde{X}$. We denote by

$$\pi_n : \tilde{X} \to X_n.$$

the projection maps.

The homeomorphism of Lemma 3.1 is constructed as a quotient of the map

$$G : \tilde{X} \to |K|$$
$$G(\overline{x}) = \bigcap_{n=0}^{\infty} |x_n|,$$

where $\overline{x} := (x_0, x_1, x_2, \ldots)$. Note that $\bigcap_{n=0}^{\infty} |x_n|$ consists of a single point since the

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diameters of the geometric realizations $|x_n|$ go to zero.

The error in [**1**] occurred in the proof that G is continuous, where it was implicitly assumed that a specific infinite product of open sets is open in the infinite product topology.

A correct proof is as follows. Let $U \subset |K|$ be an open set and let $\overline{x} \in G^{-1}(U)$. Denote $a := G(\overline{x})$. By definition,

$$a \in |x_n| \subset |B_{x_n}|$$

for every n. Since the diameters of the sets $|B_{x_n}|$ approach zero and all intersect U, we must have $|B_{x_N}| \subset U$ for some N.

We claim that $\pi_N^{-1}(B_{x_N}) \subset G^{-1}(U)$. Indeed, if $\overline{y} = (y_0, y_1, \ldots) \in \pi_N^{-1}(B_{X_N})$, then $y_N \in B_{x_N}$, and hence

$$G(\overline{y}) = \bigcap_{n=0}^{\infty} |y_n| \subset |y_N| \subset |B_{x_N}| \subset U.$$

That is, $\overline{y} \in G^{-1}(U)$. This shows that $G^{-1}(U)$ is open, and hence completes the proof.

Note that this is the only part of the proof in which finiteness of K is needed. With the correction, the proof in [**1**] extends verbatim to the setting of locally finite simplicial complexes and Alexandroff spaces, as discussed in [**2**].

References

[**1**] E. Clader. Inverse limits of finite topological spaces. *Homotopy, Homology, and Applications*, 11(2):223–227, 2009.

[**2**] M. Thibault. *Homotopy Theory of Combinatorial Categories*. PhD thesis, The University of Chicago, June 2013.

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