

LIN-WANG TYPE FORMULA FOR THE HAEFLIGER INVARIANT

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Abstract

In this paper we study the Haefliger invariant for long embeddings $\mathbb{R}^{4k-1} \hookrightarrow \mathbb{R}^{6k}$ in terms of the self-intersections of their projections to \mathbb{R}^{6k-1} , under the condition that the projection is a generic long immersion $\mathbb{R}^{4k-1} \looparrowright \mathbb{R}^{6k-1}$. We define the notion of “crossing changes” of the embeddings at the self-intersections and describe the change of the isotopy classes under crossing changes using the linking numbers of the double point sets in \mathbb{R}^{4k-1} . This formula is a higher-dimensional analogue to that of X.-S. Lin and Z. Wang for the order 2 invariant for classical knots. As a consequence, we show that the Haefliger invariant is of order 2 in a similar sense to Birman and Lin. We also give an alternative proof for the result of M. Murai and K. Ohba concerning “unknotting numbers” of embeddings $\mathbb{R}^3 \hookrightarrow \mathbb{R}^6$. Our formula enables us to define an invariant for generic long immersions $\mathbb{R}^{4k-1} \looparrowright \mathbb{R}^{6k-1}$ which are liftable to embeddings $\mathbb{R}^{4k-1} \hookrightarrow \mathbb{R}^{6k}$. This invariant corresponds to V. Arnold’s plane curve invariant in Lin–Wang theory, but in general our invariant does not coincide with the order 1 invariant of T. Ekholm.

1. Introduction

A long j -embedding in \mathbb{R}^n is an embedding $\mathbb{R}^j \hookrightarrow \mathbb{R}^n$ that is the standard inclusion outside a compact set. We denote by $\mathcal{K}_{n,j}$ the space of long j -embeddings in \mathbb{R}^n . Similarly, we denote the space of long immersions $\mathbb{R}^j \looparrowright \mathbb{R}^n$ by $\mathcal{I}_{n,j}$.

In [25] the author constructed, for some pairs (n, j) , a cochain map $I: \mathcal{D}^* \rightarrow \Omega_{DR}^*(\mathcal{K}_{n,j})$ from a complex \mathcal{D}^* of graphs to the de Rham complex of $\mathcal{K}_{n,j}$ via configuration space integrals associated with graphs. For other interesting pairs—in particular, for $(n, j) = (6k, 4k - 1)$ —the map I has not yet been proved to be a cochain map, and it is not clear whether graph cocycles in \mathcal{D}^* yield closed forms of $\mathcal{K}_{6k,4k-1}$. But in [25] we found a cocycle $H \in \mathcal{D}^*$ and a differential form $c \in \Omega_{DR}^0(\mathcal{K}_{6k,4k-1})$ such that $\mathcal{H} := I(H) + c \in \Omega_{DR}^0(\mathcal{K}_{6k,4k-1})$ is closed and is equal (up to sign) to the Haefliger invariant that gives a group isomorphism $\pi_0(\mathcal{K}_{6k,4k-1}) \cong \mathbb{Z}$. This integral expression \mathcal{H} looks very similar to that for the finite type invariant v_2 of order 2 (the Casson invariant) for classical knots [4, 15].

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In this paper, based on the integral expression \mathcal{H} , we show that the Haefliger invariant indeed behaves similarly to v_2 . To do this, we study $\mathcal{H}(f)$ under the condition that its projection $p \circ f \in \mathcal{I}_{6k-1,4k-1}$ is a generic immersion, where $p: \mathbb{R}^{6k} \rightarrow \mathbb{R}^{6k-1}$ denotes the projection forgetting the last $6k$ -th coordinate. In contrast to the case of knots in \mathbb{R}^3 , this is not a generic condition. But it is possible to move the generator of $\pi_0(\mathcal{K}_{6k,4k-1})$ by an isotopy so that the above condition is satisfied (see §3), and hence it is possible for all the embeddings. A generic immersion $g \in \mathcal{I}_{6k-1,4k-1}$ has only (possibly empty) transverse two-fold self-intersection $A = A_1 \sqcup \cdots \sqcup A_m \subset \mathbb{R}^{6k-1}$, where each A_i is a connected, closed oriented $(2k-1)$ -dimensional manifold. If $g = p \circ f$ for some $f \in \mathcal{K}_{6k,4k-1}$, then $g: g^{-1}(A_i) \rightarrow A_i$ is a trivial double covering and we write $g^{-1}(A_i) = L_i^0(f) \sqcup L_i^1(f) \subset \mathbb{R}^{4k-1}$. We define the notion of the *crossing changes* at the “crossings” A_i , similarly as in the classical knot theory, and denote by $f_S \in \mathcal{K}_{6k,4k-1}$ the embedding obtained from f by crossing changes at A_i , $i \in S \subset \{1, \dots, m\}$. In Theorem 2.5 we show that the difference $\mathcal{H}(f) - \mathcal{H}(f_S)$ can be described as a signed sum of the linking numbers $lk(L_i^\epsilon(f), L_j^{\epsilon'}(f))$ and $lk(L_i^\epsilon(f_S), L_j^{\epsilon'}(f_S))$, $\epsilon, \epsilon' \in \{0, 1\}$, $i, j \in \{1, \dots, m\}$. This formula is a higher-dimensional analogue to those for v_2 [16, (4.3)], [19, (3.2)], [20, (2.6)], [23].

We define the notion of *finite type invariants for $\mathcal{K}_{6k,4k-1}$* in a similar manner to the Birman–Lin characterization [3], and as a corollary of Theorem 2.5 we see in Theorem 2.9 that the Haefliger invariant is *of order 2*. In this sense, the Haefliger invariant can be seen as a higher-dimensional analogue to v_2 . It seems that, in some aspects, the geometric meaning of the Haefliger invariant is understood better (see, for example, [14, 5, 17, 28, 29]) than that of finite type invariants for classical knots. Thus more detailed studies on the Haefliger invariant (and other cohomology classes in higher dimensions that can be described by some integrals) might shed light on the geometric meaning of finite type invariants. It might be possible that the notion of “finite type invariants” can also be characterized from the perspective of *manifold calculus* [12] (see also [6, 7, 18, 31]). See Remark 2.10 for some discussion. As another consequence, we reprove the result of Murai and Ohba [22] concerning the “unknotting numbers” of long embeddings $\mathbb{R}^3 \hookrightarrow \mathbb{R}^6$.

Similarly to v_2 , the invariant \mathcal{H} is essentially the sum of two integrals $I(X), I(Y)$ over some configuration spaces, which correspond, respectively, to the graphs X and Y (see Figure 4.1). Our formula in Theorem 2.5 is obtained by clarifying the geometric meaning of $I(X)$ to some extent; we see in Proposition 5.3 that $I(X)$ is essentially a signed sum of the linking numbers of $L_i^{\epsilon'}$ s and can be thought of as a high-dimensional analogue to the Gauss diagram term in the formulas in [16, 19, 20, 23]. Using our formula, we can define an invariant E of generic immersions $\mathbb{R}^{4k-1} \looparrowright \mathbb{R}^{6k-1}$ that can be lifted to embeddings $\mathbb{R}^{4k-1} \hookrightarrow \mathbb{R}^{6k}$ (see Theorem 2.11). In fact, E is the essential part of $I(Y)$ (see (6.1)). Our invariant E is a high-dimensional analogue to Lin–Wang invariant α for generic plane curves [16, Definition 5.4]. The invariant α turns out to be a linear combination of the *Arnold invariants* for generic plane curves [1]. One might therefore expect that E would be a linear combination of *Ekholm’s order 1 invariants* [10, 11], which look analogous to the Arnold invariants. Contrary to the expectation, we see in Theorem 2.11 that in general E is not of order 1. In fact, we see that, when $k = 1$, the jump of E at self-tangency singularities can be computed using the linking numbers of links that arise as the double point set of immersion $\mathbb{R}^3 \looparrowright \mathbb{R}^5$, and the linking numbers can be arbitrarily large by the result of Ogasa [21].

This paper is organized as follows. In §2 we fix the notation and state the results. The main results are Theorems 2.5, 2.9 (proved in §5), and 2.11 (proved in §6). In §3 we show an explicit computation using Theorem 2.5. We review the graph complex \mathcal{D} and our construction of \mathcal{H} in §4.

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2. Notation and results

The *self-intersection* A of an immersion $g: M \looparrowright N$ is $A := \{q \in N \mid |g^{-1}(q)| \geq 2\}$, where $|S|$ is the cardinality of a set S . If $g \in \mathcal{I}_{6k-1,4k-1}$ is generic, then $|g^{-1}(q)| = 2$ for any $q \in A$. Moreover, A is a $(2k - 1)$ -dimensional closed submanifold and $g: g^{-1}(A) \rightarrow A$ is a double covering. We call $g^{-1}(A) \subset \mathbb{R}^{4k-1}$ the *double point set*. Suppose that g is *liftable* to $f \in \mathcal{K}_{6k,4k-1}$ —namely, $g = p \circ f$, where in general $p: \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ is given by $p(x_1, \dots, x_n) = (x_1, \dots, x_{n-1})$ —then $g: g^{-1}(A) \rightarrow A$ is a trivial double covering. Let $A_i \subset \mathbb{R}^{6k-1}$ ($i = 1, 2, \dots$) be path components of A , and we call each A_i a *crossing* of (the “knot diagram” $p \circ f$ of) f . We set $g^{-1}(A_i) = L_i^0 \sqcup L_i^1 = L_i^0(f) \sqcup L_i^1(f)$. Each $L_i^\epsilon \subset \mathbb{R}^{4k-1}$ is a $(2k - 1)$ -dimensional connected closed submanifold. By convention, $f(L_i^1) \subset \mathbb{R}^{6k}$ sits “above” $f(L_i^0)$ —namely, if $\mathbf{x}^\epsilon = (x_1, \dots, x_{6k-1}, x_{6k}^\epsilon) \in f(L_i^\epsilon)$, $\epsilon = 0, 1$ (so $p(\mathbf{x}^0) = p(\mathbf{x}^1)$), then $x_{6k}^0 < x_{6k}^1$.

Remark 2.1. Any $f \in \mathcal{K}_{6k,4k-1}$ can be moved by an isotopy so that $p \circ f$ is a generic immersion; indeed such an isotopy exists for the embedding \mathcal{S} which generates $\pi_0(\mathcal{K}_{6k,4k-1})$ (see §3). But the condition for f that $p \circ f$ is a generic immersion is not generic, and in general such an isotopy is not “small.”

Remark 2.2. Not all $g \in \mathcal{I}_{6k-1,4k-1}$ are regularly homotopic to any liftable immersion, in contrast to the case of plane curves. Indeed, as shown in [30, §3], $g \in \mathcal{I}_{5,3}$ is regularly homotopic to a liftable immersion if and only if its Smale invariant $\pi_0(\mathcal{I}_{5,3}) \xrightarrow{\cong} \mathbb{Z}$ is even.

Lemma 2.3 ([10, Lemma 5.1.3], [11, Proposition 3.3]). *For any $f \in \mathcal{K}_{6k,4k-1}$ as above, the submanifolds $A_i \subset \mathbb{R}^{6k}$ and $L_i^\epsilon \subset \mathbb{R}^{4k-1}$ admit natural orientations.*

Proof. Given a basis $\vec{u} = (\mathbf{u}_1, \dots, \mathbf{u}_{2k-1})$ of $T_x A_i$ ($x \in A_i$), we can choose tangent frames $\vec{v} = (\mathbf{v}_{2k}, \dots, \mathbf{v}_{4k-1})$ and $\vec{w} = (\mathbf{w}_{2k}, \dots, \mathbf{w}_{4k-1})$ of the two sheets of $p \circ f$ meeting at $x \in A_i$ so that (\vec{u}, \vec{v}) and (\vec{u}, \vec{w}) are the positive bases of these two sheets. We say \vec{u} represents the positive orientation of A_i if $(\vec{u}, \vec{v}, \vec{w})$ is a positive basis of \mathbb{R}^{6k-1} . Since the codimension of $p \circ f$ is even, this definition is independent of the order of the two sheets. We orient L_i^ϵ , $\epsilon = 0, 1$, so that $p \circ f: L_i^\epsilon \rightarrow A_i$ preserves the orientation. □

To simplify the computations, we often move $f \in \mathcal{K}_{6k,4k-1}$ to a special position.

Definition 2.4 (see Figure 2.1). We say an embedding $f \in \mathcal{K}_{6k,4k-1}$ is *almost planar* if

- (i) the composite $p \circ f: \mathbb{R}^{4k-1} \looparrowright \mathbb{R}^{6k-1}$ is a generic immersion,
- (ii) $f(\mathbb{R}^{4k-1}) \subset \mathbb{R}^{6k-1} \times [0, \delta]$ for a small $\delta > 0$, and
- (iii) $f(\mathbb{R}^{4k-1} \setminus \bigcup_i N(L_i^1)) \subset \mathbb{R}^{6k-1} \times \{0\}$, where $N(L_i^\epsilon) \subset \mathbb{R}^{4k-1}$ are closed tubular neighborhoods of L_i^ϵ in \mathbb{R}^{4k-1} such that $N(L_i^\epsilon) \cap N(L_j^{\epsilon'}) = \emptyset$ if $(i, \epsilon) \neq (j, \epsilon')$.

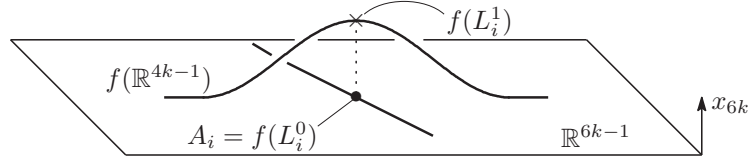


Figure 2.1: An almost planar embedding

If $f \in \mathcal{K}_{6k,4k-1}$ is such that $p \circ f \in \mathcal{I}_{6k-1,4k-1}$ is generic, then we can transform f to be almost planar without changing $p \circ f$, by an isotopy in the x_{6k} -direction. Notice that if f is almost planar, then the crossings A_i are equal to $f(L_i^0)$.

Suppose that f is almost planar and that the self-intersection of $p \circ f$ has m components. For $S \subset \{1, \dots, m\}$, let $f_S \in \mathcal{K}_{6k,4k-1}$ be defined by

$$f_S(x) := \begin{cases} \iota(f(x)) & x \in N(L_i^1), i \in S, \\ f(x) & \text{otherwise,} \end{cases}$$

where $\iota: \mathbb{R}^{6k} \rightarrow \mathbb{R}^{6k}$ is given by $\iota(x_1, \dots, x_{6k}) = (x_1, \dots, x_{6k-1}, -x_{6k})$. We say f_S is an embedding obtained from f by *crossing changes* at the crossings $\{A_i\}_{i \in S}$. Notice that $p \circ f = p \circ f_S$ and

$$L_i^\epsilon(f_S) = \begin{cases} L_i^{\epsilon+1}(f) & i \in S, \\ L_i^\epsilon(f) & i \notin S; \end{cases}$$

here ϵ is understood to be in $\mathbb{Z}/2 = \{0, 1\}$ and $1 + 1 = 0$.

Let $\mathcal{H}: \mathcal{K}_{6k,4k-1} \rightarrow \mathbb{Z}$ be the Haefliger invariant (see §4 for our construction). Our main theorem describes the difference $\mathcal{H}(f) - \mathcal{H}(f_S)$ using the linking numbers of L_i^ϵ 's.

Theorem 2.5. *Let $f \in \mathcal{K}_{6k,4k-1}$ be such that $p \circ f$ is a generic immersion and has the nonempty self-intersection $A = A_1 \sqcup \dots \sqcup A_m$. Then for any $S \subset \{1, \dots, m\}$,*

$$\begin{aligned} & \mathcal{H}(f) - \mathcal{H}(f_S) \\ &= \frac{1}{4} \left(\sum_{(i,\epsilon) < (j,\epsilon')} (-1)^{\epsilon+\epsilon'} lk(L_i^\epsilon(f), L_j^{\epsilon'}(f)) - \sum_{(i,\epsilon) < (j,\epsilon')} (-1)^{\epsilon+\epsilon'} lk(L_i^\epsilon(f_S), L_j^{\epsilon'}(f_S)) \right) \end{aligned} \quad (2.1)$$

$$= \frac{1}{2} \sum_{\substack{(i,\epsilon) < (j,\epsilon'), \\ (\text{exactly one of } i,j) \in S}} (-1)^{\epsilon+\epsilon'} lk(L_i^\epsilon(f), L_j^{\epsilon'}(f)), \quad (2.2)$$

where lk stands for the linking number, and we write $(i, \epsilon) < (j, \epsilon')$ if $i < j$ or if $i = j$, $\epsilon = 0$, $\epsilon' = 1$.

(2.2) follows from (2.1); if $i, j \notin S$, then $lk(L_i^\epsilon(f), L_j^{\epsilon'}(f)) = lk(L_i^\epsilon(f_S), L_j^{\epsilon'}(f_S))$ is contained in both sums in (2.1) with the same sign $(-1)^{\epsilon+\epsilon'}$ and cancels out. If $i, j \in S$, then $lk(L_i^\epsilon(f), L_j^{\epsilon'}(f)) = lk(L_i^{\epsilon+1}(f_S), L_j^{\epsilon'+1}(f_S))$ is contained in both sums with the same sign $(-1)^{\epsilon+\epsilon'}$ and cancels out. If $i \in S$ and $j \notin S$, then $lk(L_i^\epsilon(f), L_j^{\epsilon'}(f))$ is contained in the first sum with sign $(-1)^{\epsilon+\epsilon'}$ while $lk(L_i^\epsilon(f), L_j^{\epsilon'}(f)) = lk(L_i^{\epsilon+1}(f_S), L_j^{\epsilon'}(f_S))$ is contained in the second sum with the opposite sign $(-1)^{\epsilon+1+\epsilon'}$.

Remark 2.6. The formula in Theorem 2.5 is similar to Lin and Wang’s formula for the Casson invariant v_2 [16]; given a diagram of $f \in \mathcal{K}_{3,1}$, let $f_S \in \mathcal{K}_{3,1}$ be obtained by changing the crossings c_i , $i \in S$. Then a slight generalization of [16, (4.3)] can be written as

$$v_2(f) - v_2(f_S) = \frac{1}{4} \left\langle \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \curvearrowright \quad \curvearrowright \\ \bullet \quad \bullet \quad \bullet \end{array}, G(f) \right\rangle - \frac{1}{4} \left\langle \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \curvearrowright \quad \curvearrowright \\ \bullet \quad \bullet \quad \bullet \end{array}, G(f_S) \right\rangle, \tag{2.3}$$

where $G(f)$ is the Gauss diagram of (the diagram of) f and $\left\langle \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \curvearrowright \quad \curvearrowright \\ \bullet \quad \bullet \quad \bullet \end{array}, G(f) \right\rangle$ is the sum of $\epsilon_1 \epsilon_2$ for all the subdiagrams of $G(f)$ of the shape $\begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \curvearrowright \quad \curvearrowright \\ \bullet \quad \bullet \quad \bullet \end{array}$ ($\epsilon_i = \pm 1$ are the signs of the corresponding crossings of the diagrams). This kind of pairing appears elsewhere, for example, in [23]. Choosing an S that yields a “descending diagram” f_S (and hence f_S is trivial), and computing the right-hand side of (2.3), we reprove the Polyak–Viro formula [23, Theorem 1.A]. Regarding the pairing as the sum of the linking numbers of “0-dimensional Hopf links $S^0 \sqcup S^0 \hookrightarrow \mathbb{R}^1$ ” determined by the crossings of the diagrams, we can say that Theorem 2.5 is a higher-dimensional analogue to (2.3) and to the Polyak–Viro formula.

Theorem 2.5 together with the result of Ogasa [21] gives an alternative proof for the following result of Murai and Ohba [22], which states that the “unknotting number” of any nontrivial embedding $f \in \mathcal{K}_{6,3}$ is 1 (see §6.4 for the proof).

Corollary 2.7 ([22]). *Any nontrivial $f \in \mathcal{K}_{6,3}$ can be unknotted by a crossing change at a single crossing. Namely, f is isotopic to some f' , with $f'_{\{1\}}$ isotopic to the trivial inclusion.*

The proof is outlined as follows: Given any two-component link $L \subset \mathbb{R}^3$, by a result of Ogasa [21] we can find $f_0: \mathbb{R}^3 \hookrightarrow \mathbb{R}^6$ such that it is isotopic to the trivial inclusion and its projection $p \circ f: \mathbb{R}^3 \looparrowright \mathbb{R}^5$ has L as its double point set. By Theorem 2.5 the embedding f_1 obtained by the crossing change along L satisfies $\mathcal{H}(f_1) = -lk(L)$. This means that any $f \in \mathcal{K}_{6,3}$ with arbitrary $\mathcal{H}(f)$ can be obtained by a single crossing change from f_0 . See §6.4 for details. In [22] an explicit way to unknot the generator of $\pi_0(\mathcal{K}_{6,3})$ (and its connected sums) by a single crossing change is given.

Below we introduce the notion of *finite type invariants* for $\mathcal{K}_{6k,4k-1}$. As a consequence of Theorem 2.5, we prove in §5 that \mathcal{H} is an invariant of order 2.

Definition 2.8. Let $u: \mathcal{K}_{6k,4k-1} \rightarrow \mathbb{R}$ be a function, and let $A = \{A_i\}_{1 \leq i \leq s+1}$ be a (sub)set of crossings of $p \circ f$, where $f \in \mathcal{K}_{6k,4k-1}$ is almost planar. Define

$$V_{s+1}(u)(f) := \sum_{S \subset \{1, \dots, s+1\}} (-1)^{|S|} u(f_S).$$

An isotopy invariant u is said to be of order s if $V_{s+1}(u) = 0$.

Theorem 2.9. *The Haefliger invariant is of order 2.*

Remark 2.10. Our Definition 2.8 of “finite type invariants” is modeled after [3] and is similar to those in [13, 33]. Finite type invariants can also in some cases be characterized as the functions that factor through stages of the *Taylor tower* [12]. Volić uses Bott–Taubes integrals to prove in [32, Theorem 4.5] that \mathbb{R} -valued invariants of order k factor through the $2k$ -th stage of the homology Taylor tower for $\mathcal{K}_{3,1}$. In [6] it is proved that $v_2: \mathcal{K}_{3,1} \rightarrow \mathbb{Z}$ factors through the third stage of the homotopy Taylor tower for $\mathcal{K}_{3,1}$. Budney, Conant, Koytcheff, and Sinha [7] show that the $(k+1)$ -st stage of the homotopy tower defines order k invariants, and based on spectral sequence calculations conjecture any additive invariant of order k factors through this tower.

In general, the equivalence between these two characterizations of finiteness, Birman–Lin and Goodwillie–Weiss, is not known. Theorem 2.9 together with a result of Munson [18] proves the equivalence for $\mathcal{K}_{6k,4k-1}$.

In §6 we define an invariant for generic *liftable* immersion $\mathbb{R}^{4k-1} \looparrowright \mathbb{R}^{6k-1}$ and discuss some of its properties.

Theorem 2.11. *Let $g \in \mathcal{I}_{6k-1,4k-1}$ be a liftable generic immersion—namely, $g = p \circ f$ for some $f \in \mathcal{K}_{6k,4k-1}$. Choose a lift $f \in \mathcal{K}_{6k,4k-1}$ and define*

$$E(g) := \mathcal{H}(f) - \frac{1}{4} \sum_{(i,\epsilon) < (j,\epsilon')} (-1)^{\epsilon+\epsilon'} lk(L_i^\epsilon, L_j^{\epsilon'}). \quad (2.4)$$

Then E is independent of the choice of f and is invariant of generic immersions. E varies at the strata of non-generic immersions as described in Lemmas 6.4, 6.5, and 6.7. In the case $k = 1$, the invariant E is not of order 1 in the sense of Ekholm [10, 11].

E is an invariant of generic immersions because the linking numbers are constant unless the isotopy class of the self-intersection of $p \circ f$ changes. That E is well defined follows from our formula in Theorem 2.5. See §6.1.

Remark 2.12. The invariant E is a high-dimensional analogue to the Lin–Wang invariant α [16, Definition 5.4] for generic plane curves; let $g \in \mathcal{I}_{2,1}$ be a generic plane curve, and let $f \in \mathcal{K}_{3,1}$ be its lift—namely, $g = p \circ f$. Then (2.3) implies that

$$\alpha(g) := v_2(f) - \frac{1}{4} \left\langle \begin{array}{c} \curvearrowright \\ \bullet \bullet \bullet \\ \curvearrowright \end{array}, G(f) \right\rangle$$

is independent of the choice f . The invariant α is in fact equal to a linear combination of the Arnold invariants J^\pm and St of a plane curve [1]. On the other hand, Ekholm [10, §6.1], [11, §4] defined invariants for generic immersions $M^{nm-1} \looparrowright N^{n(m+1)-1}$ that behave similarly to the Arnold invariants (for us, $m = 2$, $n = 2k$). One may thus expect that E might be a linear combination of Ekholm invariants, but Theorem 2.11 says that in general it is not the case.

3. Example

Using Theorem 2.5, we show $\mathcal{H}(\mathcal{S}) = \pm 1$ for Haefliger's generator \mathcal{S} of $\pi_0(\mathcal{K}_{6k,4k-1}) \cong \mathbb{Z}$ [14]. Fix $\alpha, \beta > 0$ so that $2\beta < \alpha$. Consider the *Borromean ring* $X \sqcup Y \sqcup Z \subset \mathbb{R}^{6k}$, where

$$\begin{aligned} X &:= \partial\{(\mathbf{0}, \mathbf{y}, \mathbf{z}) \in (\mathbb{R}^{2k})^{\times 3} \mid |\mathbf{y}| \leq \alpha, |\mathbf{z}| \leq \beta\} \approx S^{4k-1}, \\ Y &:= \partial\{(\mathbf{x}, \mathbf{0}, \mathbf{z}) \in (\mathbb{R}^{2k})^{\times 3} \mid |\mathbf{z}| \leq \alpha, |\mathbf{x}| \leq \beta\} \approx S^{4k-1}, \\ Z &:= \partial\{(\mathbf{x}, \mathbf{y}, \mathbf{0}) \in (\mathbb{R}^{2k})^{\times 3} \mid |\mathbf{x}| \leq \alpha, |\mathbf{y}| \leq \beta\} \approx S^{4k-1} \end{aligned}$$

(see Figure 3.1), and smooth their corners to get smooth $(4k - 1)$ -spheres (denoted by X, Y, Z again). \mathcal{S} is defined as the connected-sum $\mathcal{S} := X \# Y \# Z \# f_0$, where $f_0: \mathbb{R}^{4k-1} \subset \mathbb{R}^{6k}$ is (isotopic to) the standard inclusion.

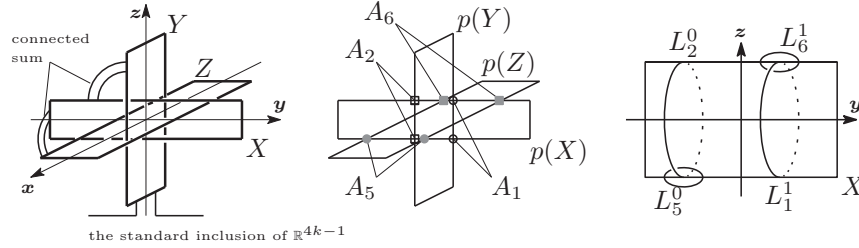


Figure 3.1: Haefliger's generator \mathcal{S} and the self-intersection of $p \circ \mathcal{S}$

Let $\mathbf{n} := (1, \dots, 1) \in \mathbb{R}^{6k}$, and consider the projection $p: \mathbb{R}^{6k} \rightarrow (\mathbb{R}^{\mathbf{n}})^\perp$, instead of $\mathbb{R}^{6k} \rightarrow \mathbb{R}^{6k-1} \times \{0\}$. Then $p \circ \mathcal{S}$ is generic, as seen in Figure 3.1. To detect $p(X) \cap p(Y)$, find $(\mathbf{0}, \mathbf{y}, \mathbf{z}) \in X$ and $t \in \mathbb{R}$ satisfying $(\mathbf{0}, \mathbf{y}, \mathbf{z}) + t\mathbf{n} \in Y$. In fact, $p(X) \cap p(Y) = A_1 \sqcup A_2$ has two components, and the double point set $L_i^\epsilon \subset \mathbb{R}^{4k-1}$ satisfying $A_i = p(\mathcal{S}(L_i^0)) = p(\mathcal{S}(L_i^1))$ ($i = 1, 2$) is given as follows: put $\mathbf{n}' := (1, \dots, 1) \in \mathbb{R}^{2k}$ and $\beta' := \beta/\sqrt{2k}$; then

$$\begin{aligned} L_1^0 &= \{(-\beta'\mathbf{n}', \mathbf{0}, \mathbf{z}) \in Y \mid |\mathbf{z} + \beta'\mathbf{n}'| = \beta\}, & L_1^1 &= \{(\mathbf{0}, \beta'\mathbf{n}', \mathbf{z}) \in X \mid |\mathbf{z}| = \beta\}, \\ L_2^0 &= \{(\mathbf{0}, -\beta'\mathbf{n}', \mathbf{z}) \in X \mid |\mathbf{z}| = \beta\}, & L_2^1 &= \{(\beta'\mathbf{n}', \mathbf{0}, \mathbf{z}) \in Y \mid |\mathbf{z} - \beta'\mathbf{n}'| = \beta\} \end{aligned}$$

(in the computation we use $2\beta < \alpha$). By symmetry we see that $p(Y) \cap p(Z) = A_3 \sqcup A_4$ has two components, and the double point sets satisfying $A_i = p(\mathcal{S}(L_i^0)) = p(\mathcal{S}(L_i^1))$ ($i = 3, 4$) are given as

$$\begin{aligned} L_3^0 &= \{(\mathbf{x}, -\beta'\mathbf{n}', \mathbf{0}) \in Z \mid |\mathbf{x} + \beta'\mathbf{n}'| = \beta\}, & L_3^1 &= \{(\mathbf{x}, \mathbf{0}, \beta'\mathbf{n}') \in Y \mid |\mathbf{x}| = \beta\}, \\ L_4^0 &= \{(\mathbf{x}, \mathbf{0}, -\beta'\mathbf{n}') \in Y \mid |\mathbf{x}| = \beta\}, & L_4^1 &= \{(\mathbf{x}, \beta'\mathbf{n}', \mathbf{0}) \in Z \mid |\mathbf{x} - \beta'\mathbf{n}'| = \beta\}. \end{aligned}$$

Similarly, $p(Z) \cap p(X) = A_5 \sqcup A_6$ satisfies $A_i = p(\mathcal{S}(L_i^0)) = p(\mathcal{S}(L_i^1))$ ($i = 5, 6$), where

$$\begin{aligned} L_5^0 &= \{(\mathbf{0}, \mathbf{y}, -\beta'\mathbf{n}') \in X \mid |\mathbf{y} + \beta'\mathbf{n}'| = \beta\}, & L_5^1 &= \{(\beta'\mathbf{n}', \mathbf{y}, \mathbf{0}) \in Z \mid |\mathbf{y}| = \beta\}, \\ L_6^0 &= \{(-\beta'\mathbf{n}', \mathbf{y}, \mathbf{0}) \in Z \mid |\mathbf{y}| = \beta\}, & L_6^1 &= \{(\mathbf{0}, \mathbf{y}, \beta'\mathbf{n}') \in X \mid |\mathbf{y} - \beta'\mathbf{n}'| = \beta\}. \end{aligned}$$

If we take the connected-sum in the construction of \mathcal{S} suitably, then there are no self-intersections of $p \circ \mathcal{S}$ other than $A_1 \sqcup \dots \sqcup A_6$. The corner smoothing does not cause any trouble; for example, any $(\mathbf{0}, \mathbf{y}, -\beta'\mathbf{n}') \in L_5^0$ satisfies $|\mathbf{y}| \leq 2\beta < \alpha$ and L_5^0

does not touch the corner. All L_i^ϵ are S^{2k-1} and they form six disjoint Hopf links

$$L_1^1 \sqcup L_6^1, L_2^0 \sqcup L_5^0 \subset X, \quad L_1^0 \sqcup L_4^0, L_2^1 \sqcup L_3^1 \subset Y, \quad L_3^0 \sqcup L_6^0, L_4^1 \sqcup L_5^1 \subset Z,$$

all of whose linking numbers are by symmetry equal to each other. Since we can unknot \mathcal{S} by the crossing change at A_1 , by (2.2)

$$\mathcal{H}(\mathcal{S}) = \frac{1}{2}((-1)^{1+1}lk(L_1^1, L_6^1) + (-1)^{0+0}lk(L_1^0, L_4^0)) = \pm \frac{1}{2}(1 + 1) = \pm 1.$$

4. A review of an integral expression of the Haefliger invariant

4.1. Graph complex and configuration space integral

Here we briefly recall the cochain complex $\mathcal{D}^* = \mathcal{D}_{n,j}^*$ of *oriented graphs* and the linear map $I: \mathcal{D}^* \rightarrow \Omega_{DR}^*(\mathcal{K}_{n,j})$ that the author defined in [25] generalizing those in [8, 9, 33]. The map I is a cochain map under some conditions on n, j and graphs. See [25, 26] for details.

By a *graph* we mean a graph with two types of vertices (called *i-* and *e-vertices*) and two types of edges (called η - and θ -edges). The sets of *i-* and *e-vertices*, η - and θ -edges of a graph Γ are denoted by, respectively, $V_i(\Gamma), V_e(\Gamma), E_\eta(\Gamma)$, and $E_\theta(\Gamma)$. We give the weights $j, n, j - 1$, and $n - 1$ to the elements of $V_i(\Gamma), V_e(\Gamma), E_\eta(\Gamma)$, and $E_\theta(\Gamma)$. An *orientation* of a graph is a choice of ordering of the weighted set $V_i(\Gamma) \sqcup V_e(\Gamma) \sqcup E_\eta(\Gamma) \sqcup E_\theta(\Gamma)$ together with the orientations of edges, modulo even permutations. Reversing η - and θ -edges changes the sign of orientation of graphs by $(-1)^j$ and $(-1)^n$, respectively. Figure 4.1 shows examples of graphs; *i-* and *e-vertices* are depicted by \bullet and \circ , respectively, and η - and θ -edges are depicted by solid and dotted arrows, respectively (Figure 4.1 shows graphs for even n and the orientation of the θ -edges are omitted). The numbers assigned to vertices and edges indicate the ordering. Denote by $\mathcal{D}^{p,q}$ the vector space spanned by oriented graphs of *order* p and

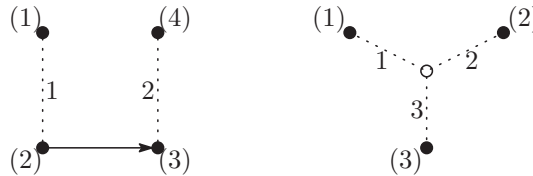


Figure 4.1: Graphs X and Y

of *degree* q , where

$$\text{ord}(\Gamma) := |E_\theta(\Gamma)| - |E_e(\Gamma)|, \quad \text{deg}(\Gamma) := 2|E_\theta(\Gamma)| - 3|E_e(\Gamma)| - |E_i(\Gamma)|.$$

For an oriented graph Γ consider the *configuration space*

$$C_\Gamma^\circ := \{(f; \mathbf{x}; \mathbf{y}) \in \mathcal{K}_{n,j} \times \text{Conf}_{|V_i(\Gamma)|}^\circ(\mathbb{R}^j) \times \text{Conf}_{|V_e(\Gamma)|}^\circ(\mathbb{R}^n) \mid f(x_i) \neq y_j \ (\forall i, j)\},$$

where $\text{Conf}_m^\circ(M) := M^{\times m} \setminus \bigcup_{i < j} \{x_i \neq x_j\}$ stands for the usual configuration space.

To each η -edge \overrightarrow{ij} and θ -edge \overrightarrow{st} , we assign the *generalized Gauss maps*

$$\begin{aligned} \varphi_{ij}^\eta: C_\Gamma^\circ &\rightarrow S^{j-1}, & (f; \mathbf{x}; \mathbf{y}) &\mapsto \frac{\mathbf{x}_j - \mathbf{x}_i}{|\mathbf{x}_j - \mathbf{x}_i|}, \\ \varphi_{st}^\theta: C_\Gamma^\circ &\rightarrow S^{n-1}, & (f; \mathbf{x}; \mathbf{y}) &\mapsto \frac{\mathbf{z}_t - \mathbf{z}_s}{|\mathbf{z}_t - \mathbf{z}_s|}, \end{aligned}$$

where

$$z_s := \begin{cases} f(x_s) & \text{if } s \text{ is an i-vertex,} \\ y_s & \text{if } s \text{ is an e-vertex.} \end{cases}$$

Choose a representative of the orientation of Γ so that θ -edges follow after η -edges. Let $\varphi_\Gamma: C_\Gamma^\circ \rightarrow (S^{j-1})^{\times |E_\eta(\Gamma)|} \times (S^{n-1})^{\times |E_\theta(\Gamma)|}$ be the product of all these Gauss maps assigned to the edges of Γ . The product is taken in the order of the orientation of Γ . Let $\text{vol}_{S^{N-1}} \in \Omega_{DR}^{N-1}(S^{N-1})$ ($N = n, j$) be a unit volume form of S^{N-1} that is (anti-)invariant under the action of $O(N)$ fixing the poles $\{\pm e_N\} \subset S^{N-1}$, where $e_N := (0, \dots, 0, 1) \in \mathbb{R}^N$. Define $\omega_\Gamma \in \Omega_{DR}^*(C_\Gamma^\circ)$ by

$$\omega_\Gamma := \varphi_\Gamma^*(\text{vol}_{S^{j-1}}^{\times |E_\eta(\Gamma)|} \times \text{vol}_{S^{n-1}}^{\times |E_\theta(\Gamma)|}),$$

where $\text{vol}_{S^{a-1}} \times \text{vol}_{S^{b-1}}$ is the product of volume forms pulled back on $S^{a-1} \times S^{b-1}$ by the projections. Integrating ω_Γ along the fibers of the natural projections

$$\pi_\Gamma: C_\Gamma^\circ \rightarrow \mathcal{K}_{n,j}$$

(whose fibers are subspaces of usual configuration spaces), we obtain

$$I(\Gamma) := \pi_{\Gamma*} \omega_\Gamma \in \Omega_{DR}^*(\mathcal{K}_{n,j}).$$

$I(\Gamma)$ is independent of the representative of the orientation of Γ . The degree of $I(\Gamma)$ can be given using $\text{ord}(\Gamma)$, $\text{deg}(\Gamma)$, and the first betti number of Γ (thought of as a 1-dimensional cell complex); see [25, §3].

Remark 4.1. The above integrals converge since we may replace C_Γ° with its (fiber-wise) Fulton–MacPherson compactification (see [27]) denoted by C_Γ , over which the generalized Gauss maps are smoothly extended. Thus we obtain a linear map $I: \mathcal{D}^* \rightarrow \Omega_{DR}^*(\mathcal{K}_{n,j})$.

By the generalized Stokes’ theorem, $dI(\Gamma)$ is a linear combination of integrals along the codimension 1 boundary faces of the fibers of π_Γ , which are subspaces of compactified configuration spaces (Remark 4.1). The boundary faces of the compactifications are stratified according to the “complexity of collisions of points.” The strata in which exactly two points collide are called *principal*. The author proved in [25] that, if the volume forms are (anti-)invariant, many such integrals along the non-principal boundary faces cancel out or vanish; the large part of the proof follows the arguments in [8, 9, 33]. Thus if we define the *coboundary maps* $\delta: \mathcal{D}^{p,q} \rightarrow \mathcal{D}^{p,q+1}$ as the signed sum of graphs obtained by collapsing two vertices together with the edges between them (see [25, §2.3] for signs), the map I becomes a cochain map. More precisely, the following holds.

Theorem 4.2 ([25, Theorem 1.2]). *The map I is a cochain map if*

- *restricted to the subcomplex of tree graphs and n, j are of same parity, or*

- restricted to the subcomplex of graphs of betti number not greater than 1 and both n, j are odd.

Conjecturally, the map I would always be a cochain map and a quasi-isomorphism. In fact, \mathcal{D}^* looks very similar to the graph complexes given in [2] that compute the rational homology and homotopy of $\mathcal{K}_{n,j}$ in the stable dimensions. The map I yields many non-zero cohomology classes even in the non-stable dimensions and in some dimensions not necessarily satisfying the condition in Theorem 4.2; for example, $H_{DR}^3(\mathcal{K}_{5,2}) \neq 0$. See [26, 33].

4.2. The Haefliger invariant

Let $X, Y \in \mathcal{D}^{2,0}$ be graphs in Figure 4.1, and denote $C_{4,0} := C_X$ and $C_{3,1} := C_Y$. We note that $C_{4,0} = \mathcal{K}_{6k,4k-1} \times \text{Conf}_4^\circ(\mathbb{R}^{4k-1})$ and $C_{3,1} \subset \mathcal{K}_{6k,4k-1} \times \text{Conf}_3^\circ(\mathbb{R}^{4k-1}) \times \mathbb{R}^{6k}$. In fact, $H := X/2 + Y/6 \in \mathcal{D}^{2,0}$ is a cocycle and $I(H)$ is a 0-form of $\mathcal{K}_{6k,4k-1}$. Unfortunately, Theorem 4.2 might fail for $(n, j) = (6k, 4k - 1)$; at present, it is not known whether the integral along the ‘‘anomalous boundary face’’ $\Sigma_{3,1} \subset \partial C_{3,1}$ (where all the four points $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3)$, and \mathbf{x}_4 collapse to a single point) vanishes or not. In [25] we add a correction term c (defined below) to $I(H)$ to kill the anomalous contribution and get a closed-form $\mathcal{H} := I(H) + c$.

The correction term c is defined as follows. The interior $\text{Int } \Sigma_{3,1}$ of $\Sigma_{3,1}$ can be described by the following pullback square:

$$\begin{array}{ccc} \text{Int } \Sigma_{3,1} & \longrightarrow & B \\ \downarrow & & \downarrow \rho \\ \mathbb{R}^{4k-1} \times \mathcal{K}_{6k,4k-1} & \xrightarrow{D} & \text{Inj}_{6k,4k-1} \end{array}$$

Let us explain the spaces and maps in the above diagram. $\text{Inj}_{6k,4k-1}$ is the space of linear, injective maps $\mathbb{R}^{4k-1} \hookrightarrow \mathbb{R}^{6k}$. The space B is defined as

$$B := \{(\lambda; (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3); \mathbf{x}_4) \in \text{Inj}_{6k,4k-1} \times \text{Conf}_3^\circ(\mathbb{R}^{4k-1}) \times \mathbb{R}^{6k} \mid \lambda(\mathbf{x}_i) \neq \mathbf{x}_4, 1 \leq i \leq 3\} / \mathbb{R}_+^1 \ltimes \mathbb{R}^{4k-1},$$

where $\mathbb{R}_+^1 \ltimes \mathbb{R}^{4k-1}$ acts diagonally on $\text{Conf}_3^\circ(\mathbb{R}^{4k-1}) \times \mathbb{R}^{6k}$ as the positive scalings and translations along $\lambda(\mathbb{R}^{4k-1})$. The map ρ is the natural projection, and D is the differential

$$D(x; f) := (df_x : T_x \mathbb{R}^{4k-1} = \mathbb{R}^{4k-1} \hookrightarrow \mathbb{R}^{6k} = T_{f(x)} \mathbb{R}^{6k}).$$

For $i = 1, 2, 3$, the map

$$\psi_i : B \rightarrow S^{6k-1}, \quad [\iota; (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3); \mathbf{x}_4] \mapsto \frac{\mathbf{x}_4 - \iota(\mathbf{x}_i)}{|\mathbf{x}_4 - \iota(\mathbf{x}_i)|}$$

is well defined. Put $\psi := \psi_1 \times \psi_2 \times \psi_3 : B \rightarrow (S^{6k-1})^{\times 3}$ and consider

$$\omega := \psi^* \text{vol}_{S^{6k-1}}^{\times 3} \in \Omega_{DR}^{18k-3}(B).$$

We can see that $\rho_* \omega \in \Omega_{DR}^{4k}(\text{Inj}_{6k,4k-1})$ is closed [25, Lemma 5.22]. $\text{Inj}_{6k,4k-1}$ is homeomorphic to the Stiefel manifold $V_{6k,4k-1}$ of $(4k - 1)$ -frames in \mathbb{R}^{6k} , and hence

$H_{DR}^{4k}(\text{Inj}_{6k,4k-1}) = 0$. Therefore, we can find $\mu \in \Omega_{DR}^{4k-1}(\text{Inj}_{6k,4k-1})$ such that

$$\rho_*\omega = d\mu.$$

The correction term $c: \mathcal{K}_{6k,4k-1} \rightarrow \mathbb{R}$ is defined by

$$c(f) := \frac{1}{6} \int_{\mathbb{R}^{4k-1}} (df)^* \mu \in \mathbb{R},$$

where $df: \mathbb{R}^{4k-1} \rightarrow \text{Inj}_{6k,4k-1}$ is defined by the differential $x \mapsto df_x$.

Let $C_{4,0}(f)$ and $C_{3,1}(f)$ be the fibers of $\pi_X: C_{4,0} \rightarrow \mathcal{K}_{6k,4k-1}$ and $\pi_Y: C_{3,1} \rightarrow \mathcal{K}_{6k,4k-1}$ over f . These fibers are finite-dimensional configuration spaces with $\dim C_{4,0}(f) = \deg \omega_X$ and $\dim C_{3,1}(f) = \deg \omega_Y$. $\mathcal{H}(f)$ is calculated as

$$\mathcal{H}(f) = \frac{1}{2} \int_{C_{4,0}(f)} \omega_X + \frac{1}{6} \int_{C_{3,1}(f)} \omega_Y + c(f).$$

Theorem 4.3 ([25]). *If $\text{vol}_{S^{N-1}}$ ($N = 6k, 4k - 1$) are (anti-)invariant, then the 0-form $\mathcal{H} := I(H) + c: \mathcal{K}_{6k,4k-1} \rightarrow \mathbb{R}$ is closed and induces a group isomorphism $\pi_0(\mathcal{K}_{6k,4k-1}) \cong \mathbb{Z}$. In particular, \mathcal{H} is a \mathbb{Z} -valued invariant.*

In [25] Theorem 4.3 is proved by evaluating \mathcal{H} over a generator of $\pi_0(\mathcal{K}_{6k,4k-1})$ given by the spinning construction [5, 24]. The computation in §3 gives an alternative proof.

To simplify the computations below, we will take the (anti-)invariant volume forms $\text{vol}_{S^{N-1}}$ ($N = 6k, 4k - 1$) so that their supports are contained in small neighborhoods of the poles $\pm e_N := (0, \dots, 0, \pm 1) \in \mathbb{R}^N$. We call such a volume form a *Dirac-type* volume form. The following allows us to use such volume forms.

Proposition 4.4 ([25, Propositions 3.5, 3.6]). *The invariant \mathcal{H} is independent of the choice of the (anti-)invariant volume forms $\text{vol}_{S^{N-1}}$, $N = 6k, 4k - 1$.*

5. Proofs of Theorems 2.5 and 2.9

It is well known that the *linking number* of closed oriented submanifolds $M^{2k-1} \sqcup N^{2k-1} \subset \mathbb{R}^{4k-1}$ is an isotopy invariant and can be defined as

$$lk(M, N) := \int_{M \times N} \varphi^* \text{vol}_{S^{4k-2}},$$

where $\varphi: M \times N \rightarrow S^{4k-2}$ is the generalized Gauss map given by

$$\varphi(x, y) := \frac{y - x}{|y - x|}.$$

If M, N are in generic positions, then $p(M \sqcup N)$ is generically immersed in \mathbb{R}^{4k-2} and the pairs $(x, y) \in M \times N$ such that $p(\varphi(x, y)) = \mathbf{0} \in \mathbb{R}^{4k-2}$ form a 0-dimensional submanifold of $M \times N$. If, moreover, $\text{vol}_{S^{4k-2}}$ is Dirac-type (see the paragraph before

Proposition 4.4), then

$$\begin{aligned}
 lk(M, N) &= \sum_{(x,y) \in (p \circ \varphi)^{-1}(\mathbf{0})} \int_{\text{a neighborhood of } (x,y)} \varphi^* \text{vol}_{S^{4k-2}} \\
 &= \frac{1}{2} \sum_{(x,y) \in (p \circ \varphi)^{-1}(\mathbf{0})} \text{deg } \varphi|_{\text{a neighborhood of } (x,y)}
 \end{aligned} \tag{5.1}$$

and each $\text{deg } \varphi = \pm 1$. This is because the integrand $\varphi^* \text{vol}_{S^{4k-2}}$ is zero outside neighborhoods of such pairs, and the integral of $\text{vol}_{S^{4k-2}}$ over one component of $\text{supp}(\text{vol}_{S^{4k-2}})$ is $1/2$. This interpretation gives us the following, which we use in §6.

Lemma 5.1. *Let M_1, M_2 and N_1, N_2 be $(2k - 1)$ -dimensional disjoint oriented submanifolds of \mathbb{R}^{4k-1} . If the connected-sums $M_1 \# M_2$ and $N_1 \# N_2$ are taken in \mathbb{R}^{4k-1} so that $|p(M_i) \cap p(N_j)|$ ($i, j = 1, 2$) do not increase, then*

$$\sum_{i,j=1,2} lk(M_i, N_j) = lk(M_1 \# M_2, N_1 \# N_2).$$

For a single oriented submanifold $L^{2k-1} \subset \mathbb{R}^{4k-1}$, a similar formula to (5.1) does not give rise to an isotopy invariant, but if L is almost planar, then such a function can be computed by counting $(x, y) \in \text{Conf}_2^o(L)$ with $p(\varphi(x, y)) = \mathbf{0}$.

Definition 5.2. Let $L^{2k-1} \subset \mathbb{R}^{4k-1}$ be a generic closed oriented submanifold such that $p(L) \subset \mathbb{R}^{4k-2}$ is a generically immersed manifold. Define the *writhe* $w(L)$ of L by

$$\begin{aligned}
 w(L) &= \sum_{(x,y) \in (p \circ \varphi)^{-1}(\mathbf{0})} \int_{\text{a neighborhood of } (x,y)} \varphi^* \text{vol}_{S^{4k-2}} \\
 &= \frac{1}{2} \sum_{(x,y) \in (p \circ \varphi)^{-1}(\mathbf{0})} \text{deg } \varphi|_{\text{a neighborhood of } (x,y)}.
 \end{aligned} \tag{5.2}$$

For an almost planar $f \in \mathcal{K}_{6k,4k-1}$, denote by f^δ the embedding obtained by a scaling in the x_{6k} -direction so that $f^\delta(\mathbb{R}^{4k-1}) \subset \mathbb{R}^{6k-1} \times [0, \delta]$ (we often abbreviate f^δ as f). $L_i^{\epsilon'}$ s remain unchanged for any δ . We compute $I(X)(f^\delta)$, $I(Y)(f^\delta)$, and $c(f^\delta)$ in the limit $\delta \rightarrow 0$.

Proposition 5.3 (cf. [16, Proposition 4.3]). *Suppose $\text{vol}_{S^{N-1}}$ ($N = 6k, 4k - 1$) are Dirac-type. If $f \in \mathcal{K}_{6k,4k-1}$ is almost planar and generic so that $lk(L_i^\epsilon, L_j^{\epsilon'})$ and $w(L_i^\epsilon)$ can be calculated by (5.1) and (5.2), then*

$$\lim_{\delta \rightarrow 0} I(X)(f^\delta) = \frac{1}{2} \sum_{(i,\epsilon) < (j,\epsilon')} (-1)^{\epsilon+\epsilon'} lk(L_i^\epsilon, L_j^{\epsilon'}) + \frac{1}{4} \sum_{i,\epsilon} w(L_i^\epsilon).$$

Proof. A configuration $\vec{\xi} = (\xi_1, \dots, \xi_4) \in C_{4,0}(f)$ can non-trivially contribute to $I(X)$ only if $\vec{\xi} \in \varphi_X^{-1}(\text{supp}(\text{vol}_{S^{6k-1}}^{\times 2} \times \text{vol}_{S^{4k-2}}))$. Since $\text{vol}_{S^{N-1}}$ are Dirac-type, such a $\vec{\xi}$ must be in a neighborhood of $\varphi_X^{-1}(\pm e_{6k}, \pm e_{6k}, \pm e_{4k-1})$, where $e_N := (0, \dots, 0, 1) \in S^{N-1}$.

If δ is sufficiently close to 0, then no vectors tangent to $f(\mathbb{R}^{4k-1})$ point $\text{supp}(\text{vol}_{S^{6k-1}})$. Thus $\vec{\xi}$ can be in $\varphi_X^{-1}(\text{supp}(\text{vol}_{S^{6k-1}}^{\times 2} \times \text{vol}_{S^{4k-2}}))$ in the limit $\delta \rightarrow 0$

only if $(\xi_1, \xi_2) \in N(L_i^\epsilon) \times N(L_i^{\epsilon+1})$ and $(\xi_3, \xi_4) \in N(L_j^{\epsilon'}) \times N(L_j^{\epsilon'+1})$ for some i, j , possibly $i = j$ (recall that $N(L_i^\epsilon) \subset \mathbb{R}^{4k-1}$ are closed disjoint tubular neighborhoods of L_i^ϵ). For $(s, t) = (1, 2), (4, 3)$, and any $\xi_t \in N(L_i^\epsilon)$, we always find $\xi_s \in N(L_i^{\epsilon+1})$ such that $p(f(\xi_s) - f(\xi_t))$ is close to $\mathbf{0}$. Therefore, finding such a $\vec{\xi} = (\xi_1, \dots, \xi_4) \in \varphi_X^{-1}(\pm e_{6k}, \pm e_{6k}, \pm e_{4k-1})$ is equivalent to finding (ξ_2, ξ_3) satisfying $p(\varphi_{23}^\eta(\xi_2, \xi_3)) = \mathbf{0}$. By our assumption on f , the set of such $\vec{\xi}$'s is a 0-dimensional submanifold of $C_{4,0}(f)$, each component of whose neighborhood is mapped homeomorphically into a component of $\text{supp}(\text{vol}_{S^{6k-1}}^{\times 2} \times \text{vol}_{S^{4k-2}})$ via φ_X . The integral of ω_X over such a component is $\pm(1/2)^3$, where the sign is the local degree of φ_X at $\vec{\xi}$ —that is, the determinant of the Jacobian $J(\varphi_X)_{\vec{\xi}}$. The sum of these degrees would amount to linking numbers by (5.1).

To compute $\text{deg } \varphi_X$ at each $\vec{\xi} \in \varphi_X^{-1}(\pm e_{6k}, \pm e_{6k}, \pm e_{4k-1})$, we recall from [10, 11] the local model for two-fold self-intersection. Let $g \in \mathcal{I}_{6k-1, 4k-1}$ be a generic immersion, and let $q = g(p_1) = g(p_2)$ be a transverse two-fold self-intersection point. In some local coordinates centered at p_1, p_2 , and q , g is given by

$$\begin{aligned} g(x_1, \dots, x_{4k-1}) &= (x_1, \dots, x_{2k-1}, x_{2k}, \dots, x_{4k-1}, 0, \dots, 0) && \text{near } p_1, \\ g(y_1, \dots, y_{4k-1}) &= (y_1, \dots, y_{2k-1}, 0, \dots, 0, y_{2k}, \dots, y_{4k-1}) && \text{near } p_2. \end{aligned}$$

Now consider a configuration $\vec{\xi} \in C_{4,0}(f)$ such that

$$\begin{cases} \xi_1 \in L_i^1, \xi_2 \in L_i^0, \xi_3 \in L_j^0 \text{ and } \xi_4 \in L_j^1, \\ \varphi_X(\vec{\xi}) = (-e_{6k}, -e_{6k}, e_{4k-1}) \end{cases} \quad (5.3)$$

(Figure 5.1, left). Suppose that (ξ_2, ξ_3) is a positive crossing—that is, $\text{deg } \varphi_{23}^\eta|_{(\xi_2, \xi_3)} = +1$. Then we can choose some local coordinates \mathbf{x}, \mathbf{y} , and \mathbf{z} centered at ξ_1, ξ_2 , and ξ_4 such that $\xi_3 = (0, \dots, 0, 1)$ in the \mathbf{y} -coordinate (the same coordinate as for ξ_2), and we can also choose local coordinates in \mathbb{R}^{6k} in which f is given by

$$\begin{aligned} f(\mathbf{x}) &= (x_1, \dots, x_{2k-1}, x_{2k}, \dots, x_{4k-1}, 0, \dots, 0, 1) && \text{near } \xi_1, \\ f(\mathbf{y}) &= (y_1, \dots, y_{2k-1}, 0, \dots, 0, y_{2k}, \dots, y_{4k-1}, 0) && \text{near } \xi_2, \end{aligned}$$

and

$$\begin{aligned} f(\mathbf{y}) &= (0, \dots, 0, y_1, \dots, y_{2k-1}, y_{2k}, \dots, y_{4k-2}, y_{4k-1}, 0, 1) && \text{near } \xi_3, \\ f(\mathbf{z}) &= (z_1, \dots, z_{2k-1}, 0, \dots, 0, z_{2k}, \dots, z_{4k-2}, 1, z_{4k-1}, 0) && \text{near } \xi_4. \end{aligned}$$

Then by Lemma 2.3, as oriented manifolds, L_*^ϵ 's are given by

- $L_i^1 \cap (\mathbf{x}\text{-coordinate}) = +\mathbb{R}^{2k-1} \times \{\mathbf{0}\}^{2k}$,
- $L_i^0 \cap (\mathbf{y}\text{-coordinate}) = +\mathbb{R}^{2k-1} \times \{\mathbf{0}\}^{2k}$,
- $L_j^1 \cap (\mathbf{y}\text{-coordinate}) = \{\mathbf{0}\}^{2k-1} \times (-\mathbb{R}^{2k-1}) \times \{1\}$, and
- $L_j^0 \cap (\mathbf{z}\text{-coordinate}) = \{\mathbf{0}\}^{2k-1} \times (-\mathbb{R}^{2k-1}) \times \{0\}$

(see Figure 5.2). Using this local model, we can compute the Jacobian $J(\varphi_X)_{\vec{\xi}}$ of

$$\varphi_X: N(L_i^1) \times N(L_i^0) \times N(L_j^1) \times N(L_j^0) \rightarrow S^{6k-1} \times S^{6k-1} \times S^{4k-2}$$

at $\vec{\xi}$ explicitly. Let $e_i := (0, \dots, 0, 1, 0, \dots, 0)$ be the i th unit vector. With respect to the natural positive basis e_1, \dots, e_{16k-4} of $T_{\vec{\xi}}\text{Conf}_4(\mathbb{R}^{4k-1}) \cong T_{\vec{\xi}}\mathbb{R}^{16k-4}$ and the

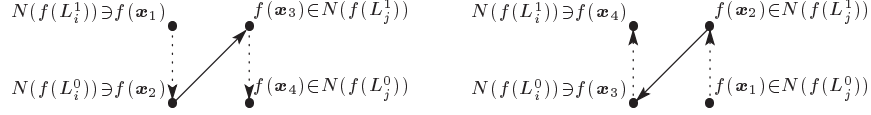


Figure 5.1: Two configurations with the same contributions to $lk(L_i^0, L_j^1)$; the left shows a neighborhood of $\vec{\xi}$ satisfying (5.3), and the right shows a neighborhood of $\vec{\xi}'$ satisfying (5.4).

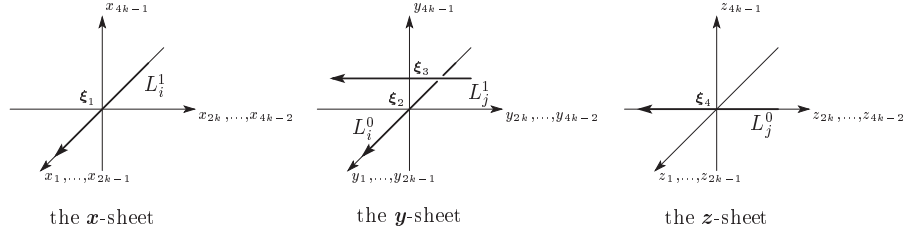


Figure 5.2: Local picture of $L_i^0 \cup L_i^1 \cup L_j^0 \cup L_j^1$

natural positive bases of the tangent spaces of spheres

$$e_1, \dots, e_{6k-1} \in T_{-e_{6k}} S^{6k-1} \quad \text{and} \quad e_1, \dots, e_{4k-2} \in T_{e_{4k-1}} S^{4k-2},$$

following the ‘‘outward normal first’’ convention, $J(\varphi_X)_{\vec{\xi}}$ is given as in Appendix A and its determinant is -1 .

A local model for a negative crossing (ξ_2, ξ_3) (namely, $\deg \varphi_{23}^\eta|_{(\xi_2, \xi_3)} = -1$) is obtained from the above model by reversing the orientation of the z -sheet, and, in this case, $J(\varphi_X)_{\vec{\xi}} = +1$. Thus the integral of ω_X over a neighborhood of $\vec{\xi}$ satisfying (5.3) with $\deg \varphi_{23}^\eta|_{(\xi_2, \xi_3)} = \pm 1$ is $\mp(1/2)^3$, and by (5.1) their sum is equal to $-lk(L_i^0, L_j^1)/4$. By symmetry of X , the configurations near $\vec{\xi}'$ satisfying

$$\begin{cases} \xi'_1 \in L_j^0, \xi'_2 \in L_j^1, \xi'_3 \in L_i^1 \text{ and } \xi'_4 \in L_i^0, \\ \varphi_X(\vec{\xi}') = (e_{6k}, e_{6k}, -e_{4k-1}) \end{cases} \quad (5.4)$$

(see Figure 5.1, right) also contribute to $I(X)(f)$ by $-lk(L_i^0, L_j^1)/4$. Thus a link $L_i^0 \sqcup L_j^1$ contributes to $I(X)(f)$ by $-lk(L_i^0, L_j^1)/2$.

A similar computation shows that L_i^0 and L_j^0 contribute to $I(X)(f)$ by $+lk(L_i^0, L_j^0)/2$; we have the same Jacobian matrix as above, but in this case $\varphi_{34}^\theta(\vec{\xi}) = e_{6k}$ and (e_1, \dots, e_{6k-1}) represent the negative orientation of $T_{e_{6k}} S^{6k-1}$ and the sign of the degree changes. This observation shows that, in general, the link $L_i^\epsilon \sqcup L_j^{\epsilon'}$, $(i, \epsilon) \neq (j, \epsilon')$, contributes to $I(X)(f)$ by $(-1)^{\epsilon+\epsilon'} lk(L_i^\epsilon, L_j^{\epsilon'})/2$.

In the case $(i, \epsilon) = (j, \epsilon')$, if f is generic so that $w(L_i^\epsilon)$ can be calculated by (5.2), then the same computation as above shows that the configurations in $N(L_i^{\epsilon+1}) \times N(L_i^\epsilon) \times N(L_i^\epsilon)$ contributes to $I(X)(f)$ by $+w(L_i^\epsilon)/4$ (no sign appears in this case since $\epsilon = \epsilon'$). \square

Lemma 5.4. *Let $f \in \mathcal{K}_{6k,4k-1}$ be almost planar, and suppose that $\text{vol}_{S^{6k-1}}$ is Dirac-type. Then*

$$\lim_{\delta \rightarrow 0} I(Y)(f^\delta) = \lim_{\delta \rightarrow 0} I(Y)(f_S^\delta) \quad \text{and} \quad \lim_{\delta \rightarrow 0} c(f^\delta) = \lim_{\delta \rightarrow 0} c(f_S^\delta).$$

Proof. The function $c: \mathcal{K}_{6k,4k-1} \rightarrow \mathbb{R}$ is defined and continuous on the space of immersions, because $c(f)$ is determined by the differential of f (see §4.2). Thus $c(f^\delta)$ continuously depends on δ including $\delta = 0$ (then f and f_S collapse down to an immersion $f^0 = f_S^0 = p \circ f$), and hence $\lim_{\delta \rightarrow 0} c(f) = c(p \circ f) = \lim_{\delta \rightarrow 0} c(f_S)$.

The above observation also implies that $\lim_{\delta \rightarrow 0} I(Y)(f^\delta)$ exists, because $I(Y)(f^\delta) = 6\mathcal{H}(f^\delta) - 3I(X)(f^\delta) - 6c(f^\delta)$ and the limit of the right-hand side exists by Proposition 5.3, the existence of $\lim c(f^\delta)$, and the fact that \mathcal{H} is an isotopy invariant. To compute $\lim_{\delta \rightarrow 0} I(Y)(f^\delta)$, we may assume that $\delta < 1$. Choose open neighborhoods $f(N(L_i^0)) \subset U_i' \subset V_i'$ in \mathbb{R}^{6k-1} (we are assuming f is almost planar) so that $\overline{U_i'} \subset V_i'$, $V_i' \cap V_j' = \emptyset$ ($i \neq j$), and $f(N(L_i^1)) \subset U_i := U_i' \times [-1, 1]$ (Figure 5.3). Let $C_{3,1}^{(1)}(f)$ be the subspace of $C_{3,1}(f)$ consisting of $\vec{x} \in C_{3,1}(f)$ with $\mathbf{x}_4 \in V_i := V_i' \times [-1, 1]$ for some $i \in S$. Now we compare the integrals of ω_Y over $C_{3,1}^{(1)}(f)$ and

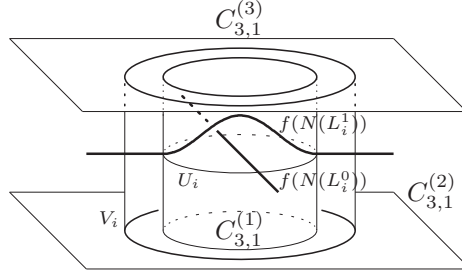


Figure 5.3: The places in which \mathbf{x}_4 is ($\vec{x} \in C_{3,1}^{(l)}$, $l = 1, 2, 3$)

$C_{3,1}^{(1)}(f_S)$. Since $\text{vol}_{S^{6k-1}}$ is Dirac-type, we only need to consider the set of \vec{x} 's with $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in f^{-1}(V_i)$ for the same i as for \mathbf{x}_4 , since ω_Y vanishes on other \vec{x} 's. The local diffeomorphism $\Phi: C_{3,1}^{(1)}(f) \rightarrow C_{3,1}^{(1)}(f_S)$,

$$\Phi((\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3); \mathbf{x}_4) := ((\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3); \iota(\mathbf{x}_4)),$$

reverses the orientation, while $\Phi^*\omega_Y = -\omega_Y$ since $\psi_j^\theta \circ \Phi = \iota \circ \psi_j^\theta$ and $f_S = \iota \circ f$ on $f^{-1}(V_i)$ ($i \in S$) and $\iota^*\text{vol}_{S^{6k-1}} = -\text{vol}_{S^{6k-1}}$ (because $\text{vol}_{S^{6k-1}}$ is anti-invariant). Thus the integrations of ω_Y over $C_{3,1}^{(1)}(f)$ and $C_{3,1}^{(1)}(f_S)$ are equal to each other.

Next, consider the subspace $C_{3,1}^{(2)}(f)$ of $C_{3,1}(f)$ consisting of \vec{x} with $\mathbf{x}_4 \in \mathbb{R}^{6k-1} \times [-1, 1]$ but $\mathbf{x}_4 \notin V_i$ for any $i \in S$. Since $\text{vol}_{S^{6k-1}}$ is Dirac-type and since $\mathbf{x}_4 \notin V_i$ for any $i \in S$, $\vec{x} \in C_{3,1}^{(2)}(f)$ can non-trivially contribute to $I(Y)$ only if \mathbf{x}_j ($j = 1, 2, 3$) is not contained in any L_i^1 ($i \in S$). Since $f = f_S$ outside L_i^1 ($i \in S$), there is no difference between ω_Y 's on $C_{3,1}^{(2)}(f)$ and $C_{3,1}^{(2)}(f_S)$, and the integrals of ω_Y over $C_{3,1}^{(2)}(f)$ and $C_{3,1}^{(2)}(f_S)$ are also the same.

Finally consider the subspace $C_{3,1}^{(3)}(f)$ of $C_{3,1}(f)$ consisting of \vec{x} with $\mathbf{x}_4 \notin \mathbb{R}^{6k-1} \times [-1, 1]$. If $\delta > 0$ is small enough, then under the diffeomorphism $C_{3,1}^{(3)}(f) \rightarrow C_{3,1}^{(3)}(f_S)$

given by $\vec{x} \mapsto \vec{x}$, the differences between the vectors $\psi_i^\theta(\vec{x})$ ($i = 1, 2, 3$) are small. This is because f differs from f_S only near $N(L_i^1)$ and the difference is small relative to $|\mathbf{x}_4|$. Thus the difference between the integrals of ω_Y over $C_{3,1}^{(3)}(f)$ and $C_{3,1}^{(3)}(f_S)$ converges to 0 in the limit $\delta \rightarrow 0$. \square

Proof of Theorem 2.5. Any f with $p \circ f$ generic can be transformed by an ambient isotopy of \mathbb{R}^{4k-1} so that f satisfies the condition in Proposition 5.3 without changing the isotopy class of $\bigcup L_i^\epsilon$. Thus we may assume that f satisfies the condition in Proposition 5.3. Notice

$$\mathcal{H}(f) - \mathcal{H}(f_S) = \frac{I(X)(f) - I(X)(f_S)}{2} + \frac{I(Y)(f) - I(Y)(f_S)}{6} + (c(f) - c(f_S)).$$

The left-hand side does not depend on δ . In the limit $\delta \rightarrow 0$ ($p \circ f$ remains unchanged), the first term of the right-hand side is computed in Proposition 5.3 and gives the right-hand side of (2.1). The second and the third terms converge to zero by Lemma 5.4. \square

Proof of Theorem 2.9. Choose three components A_1, A_2, A_3 out of $A = A_1 \sqcup \dots \sqcup A_m$, $m \geq 3$. Let $W_T(\mathcal{H})(f) := \mathcal{H}(f_T) - \mathcal{H}(f_{T \cup \{1\}})$ for any $T \subset \{2, 3\}$. Then by (2.2),

$$2W_T(\mathcal{H})(f) = \sum_{j \neq 1; \epsilon, \epsilon' = 0, 1} (-1)^{\epsilon + \epsilon'} lk(L_1^\epsilon(f_T), L_j^{\epsilon'}(f_T)).$$

Because $L_j^\epsilon(f_T) = L_i^{\epsilon+1}(f)$ if $j \in T$ and $L_j^\epsilon(f_T) = L_j^\epsilon(f)$ otherwise,

$$\begin{aligned} 2W_\emptyset(\mathcal{H})(f) &= \sum_{\epsilon, \epsilon'} (-1)^{\epsilon + \epsilon'} (lk(L_1^\epsilon, L_2^{\epsilon'}) + lk(L_1^\epsilon, L_3^{\epsilon'})) + \sum_{j \geq 4; \epsilon, \epsilon'} (-1)^{\epsilon + \epsilon'} lk(L_1^\epsilon, L_j^{\epsilon'}), \\ 2W_{\{2\}}(\mathcal{H})(f) &= \sum_{\epsilon, \epsilon'} (-1)^{\epsilon + \epsilon'} (-lk(L_1^\epsilon, L_2^{\epsilon'}) + lk(L_1^\epsilon, L_3^{\epsilon'})) + \sum_{j \geq 4; \epsilon, \epsilon'} (-1)^{\epsilon + \epsilon'} lk(L_1^\epsilon, L_j^{\epsilon'}), \\ 2W_{\{3\}}(\mathcal{H})(f) &= \sum_{\epsilon, \epsilon'} (-1)^{\epsilon + \epsilon'} (lk(L_1^\epsilon, L_2^{\epsilon'}) - lk(L_1^\epsilon, L_3^{\epsilon'})) + \sum_{j \geq 4; \epsilon, \epsilon'} (-1)^{\epsilon + \epsilon'} lk(L_1^\epsilon, L_j^{\epsilon'}), \\ 2W_{\{2,3\}}(\mathcal{H})(f) &= \sum_{\epsilon, \epsilon'} (-1)^{\epsilon + \epsilon'} (-lk(L_1^\epsilon, L_2^{\epsilon'}) - lk(L_1^\epsilon, L_3^{\epsilon'})) + \sum_{j \geq 4; \epsilon, \epsilon'} (-1)^{\epsilon + \epsilon'} lk(L_1^\epsilon, L_j^{\epsilon'}). \end{aligned}$$

Substituting into $V_3(\mathcal{H})(f) = \sum_{T \subset \{2,3\}} (-1)^{|T|} W_T(\mathcal{H})(f)$, we obtain $V_3(\mathcal{H})(f) = 0$. \square

6. Proof of Theorem 2.11

6.1. Well-definedness and invariance of E

Suppose $g \in \mathcal{I}_{6k-1, 4k-1}$ is generic and liftable, and let $f, f' \in \mathcal{K}_{6k, 4k-1}$ be lifts of g . We can transform f' by an isotopy in the x_{6k} -direction (without changing $p \circ f'$) so that $f' = f_S$ for some index set S of the self-intersection of g . Then (2.1) implies that $E(g)$ does not depend on the choice of f .

Let $g_t \in \mathcal{I}_{6k-1, 4k-1}$ ($t \in [0, 1]$) be a generic regular homotopy with each g_t liftable. We show that, for any $t_0 \in [0, 1]$, g_t can be lifted to an isotopy $f_t \in \mathcal{K}_{6k, 4k-1}$ —namely, $g_t = p \circ f_t$ —in an open neighborhood of t_0 . Let f_{t_0} be a lift of g_{t_0} . Then f_{t_0} can be

written as $f_{t_0} = (g_{t_0}, h)$ by using some $h: \mathbb{R}^{4k-1} \rightarrow \mathbb{R}$. Define $G_t: \text{Conf}_2^\circ(\mathbb{R}^{4k-1}) \rightarrow \mathbb{R}^{6k-1}$ by

$$G_t(\mathbf{x}, \mathbf{y}) := g_t(\mathbf{x}) - g_t(\mathbf{y}).$$

The first projection $\text{Conf}_2^\circ(\mathbb{R}^{4k-1}) \rightarrow \mathbb{R}^{4k-1}$ restricts to a diffeomorphism $G_t^{-1}(\mathbf{0}) \cong g_t^{-1}(A_t)$, where A_t is the self-intersection of g_t . Since g_t is a generic regular homotopy, $G_t^{-1}(\mathbf{0})$ gives an isotopy of a closed submanifold of $\text{Conf}_2^\circ(\mathbb{R}^{4k-1})$. Because $f_{t_0} \in \mathcal{K}_{6k,4k-1}$, there exists an open neighborhood W of $G_{t_0}^{-1}(\mathbf{0})$ such that $h(\mathbf{x}) \neq h(\mathbf{y})$ for any $(\mathbf{x}, \mathbf{y}) \in W$. The compactness of $G_t^{-1}(\mathbf{0})$ (for any t) implies that there exists $\epsilon > 0$ such that $G_t^{-1}(\mathbf{0}) \subset W$ for $|t - t_0| < \epsilon$. Then $f_t := (g_t, h): \mathbb{R}^{4k-1} \rightarrow \mathbb{R}^{6k}$ is in $\mathcal{K}_{6k,4k-1}$ for $|t - t_0| < \epsilon$ and is a lift of g_t .

Because $\mathcal{H}(f_t)$ is constant and the linking part of (2.4) is invariant unless the double point set varies, $E(g_t)$ is also constant. Thus for any t_0 there exists $\epsilon > 0$ such that $E(g_t)$ is constant on $(t_0 - \epsilon, t_0 + \epsilon)$, and hence $E(g_t)$ is constant on $[0, 1]$.

Remark 6.1. By Proposition 5.3 and (2.4), for a generic liftable $g \in \mathcal{I}_{6k-1,4k-1}$,

$$E(g) = \lim_{\delta \rightarrow 0} \left(\frac{1}{6} I(Y)(f^\delta) + c(f^\delta) + \frac{1}{8} \sum_{i,\epsilon} w(L_i^\epsilon) \right). \tag{6.1}$$

This gives a geometric interpretation of $I(Y)$ (added by c and the writhes), and is a higher-dimensional analogue to [16, Definition 5.4].

6.2. Local models of non-generic self-intersections

As explained in [10, 11], the set of generic immersions $g: \mathbb{R}^{4k-1} \looparrowright \mathbb{R}^{6k-1}$ is an open dense subspace of $\mathcal{I}_{6k-1,4k-1}$ and the complement is a stratified hypersurface. To characterize an invariant of generic immersions, we must study its jumps at non-generic strata. The codimension 1 strata (in $\mathcal{I}_{6k-1,4k-1}$) consist of immersions with a single generic self-tangency point or a single generic triple point [11, Lemma 3.4]. The local picture of the *versal deformation* [10, §5.3], [11, §3.2] of an immersion with a self-tangency or a triple point is given in [10, 11].

Proposition 6.2 ([11, Lemma 3.5]). *Let $g_0 \in \mathcal{I}_{6k-1,4k-1}$ be an immersion with a single generic self-tangency point. Then the versal deformation g_t of g_0 is constant far from the self-tangency point, and in some local coordinates near the self-tangency point, g_t is given by*

$$\begin{aligned} g_t(\mathbf{x}) &= (x_1, \dots, x_{2k}, x_{2k+1}, \dots, x_{4k-1}, 0, && 0, && \dots, 0), \\ g_t(\mathbf{y}) &= (y_1, \dots, y_{2k}, 0, && \dots, 0, && Q(y_1, \dots, y_{2k}) + t, y_{2k+1}, \dots, y_{4k-1}), \end{aligned}$$

where Q is a non-degenerate quadratic form on $2k$ variables.

We say a self-tangency point *definite* (resp. *indefinite*) if the quadratic form Q is definite (resp. indefinite).

Proposition 6.3 ([11, Lemma 3.6]). *Let $g_0 \in \mathcal{I}_{6k-1,4k-1}$ be an immersion with a single generic triple point. Then the versal deformation g_t of g_0 is constant far from*

the triple point, and in some local coordinates near the triple point, g_t is given by

$$\begin{aligned} g_t(\mathbf{x}) &= (x_1, \dots, x_{2k-1}, x_{2k}, x_{2k+1}, \dots, x_{4k-1}, 0, \dots, 0), \\ g_t(\mathbf{y}) &= (y_1, \dots, y_{2k-1}, 0, 0, \dots, 0, y_{2k}, y_{2k+1}, \dots, y_{4k-1}), \\ g_t(\mathbf{z}) &= (0, \dots, 0, z_{2k}, z_1, \dots, z_{2k-1}, z_{2k} - t, z_{2k+1}, \dots, z_{4k-1}). \end{aligned}$$

6.3. The jump of E at a non-generic liftable immersion

Suppose that $g_0 \in \mathcal{I}_{6k-1,4k-1}$ is liftable and has a single generic self-tangency point or a single generic triple point, and let g_t be its versal deformation.

We show that g_t is liftable for $|t|$ small. Let $f_0 = (g_0, h)$ be a lift of g_0 and $f_t := (g_t, h) \in \mathcal{I}_{6k-1,4k-1}$. Similarly to the argument in §6.1, choose an open neighborhood W of $G_0^{-1}(\mathbf{0})$ such that $h(\mathbf{x}) \neq h(\mathbf{y})$ for any $(\mathbf{x}, \mathbf{y}) \in W$. Then there exists $\epsilon > 0$ such that $G_t^{-1}(\mathbf{0}) \subset W$ for $|t| < \epsilon$; this follows from the explicit description of the change of the multiple point set of g_t (see below). Thus $f_t \in \mathcal{K}_{6k,4k-1}$ for $|t| < \epsilon$ and is a lift of g_t .

By the definition (2.4) of the invariant E , its jump $E(g_t) - E(g_{-t})$ ($t \neq 0$) is described by the change of linking numbers of L_i^ϵ 's because $\mathcal{H}(f_t)$ remains unchanged.

6.3.1. Definite self-tangencies

First, we study the jump of E at a positive definite self-tangency point (the argument needs no change for the negative definite case). It is clear from Proposition 6.2 that in some local coordinate near the tangency point, the double point set is given by

$$\begin{aligned} K^0 &= \{(x_1, \dots, x_{2k}, \mathbf{0}^{2k-1}) \in \mathbb{R}^{4k-1} \mid x_1^2 + \dots + x_{2k}^2 = -t\} \quad \text{in the } \mathbf{x}\text{-sheet,} \\ K^1 &= \{(y_1, \dots, y_{2k}, \mathbf{0}^{2k-1}) \in \mathbb{R}^{4k-1} \mid y_1^2 + \dots + y_{2k}^2 = -t\} \quad \text{in the } \mathbf{y}\text{-sheet,} \end{aligned}$$

which is empty when $t > 0$ and the trivial link when $t < 0$. This link $K^0 \sqcup K^1$ is separated from the other links since each K^ϵ is contained in a small open set that intersects no other components of the double point set. Thus we have the following.

Lemma 6.4. *If g_0 has a definite self-tangency point, then $E(g_t) = E(g_{-t})$.*

6.3.2. Indefinite self-tangencies

Next, suppose that g_0 is liftable and has an indefinite self-tangency point. Let $0 < \lambda < 2k$ be the index of Q . By Proposition 6.2, in some local coordinates the double point set of g_t near the self-tangency point is given as follows:

$$\begin{aligned} \{(x_1, \dots, x_{2k}, \mathbf{0}^{2k-1}) \in \mathbb{R}^{4k-1} \mid x_1^2 + \dots + x_\lambda^2 - x_{\lambda+1}^2 - \dots - x_{2k}^2 = t\} &\text{ in the } \mathbf{x}\text{-sheet,} \\ \{(y_1, \dots, y_{2k}, \mathbf{0}^{2k-1}) \in \mathbb{R}^{4k-1} \mid y_1^2 + \dots + y_\lambda^2 - y_{\lambda+1}^2 - \dots - y_{2k}^2 = t\} &\text{ in the } \mathbf{y}\text{-sheet.} \end{aligned}$$

The versal deformation transforms the double point set by a surgery that replaces $S^{\lambda-1} \times D^{2k-\lambda}$ with $D^\lambda \times S^{2k-(\lambda+1)}$ (see Figure 6.1). If $1 < \lambda < 2k - 1$, then this surgery transforms a single component in a small neighborhood of the self-tangency point and changes no linking numbers with other components. Thus $E(g_t) = E(g_{-t})$.

If $\lambda = 1$, then in each sheet the double point set has two components when $t > 0$:

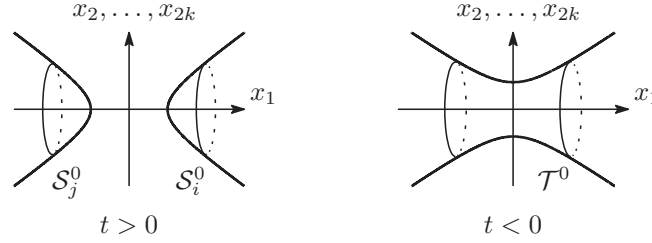


Figure 6.1: The double point set near a self-tangency point of index 1

in the \mathbf{x} -sheet

$$\mathcal{S}_i^0 := \left\{ x_1 = (x_2^2 + \dots + x_{2k}^2 + t)^{1/2} \right\} \text{ and } \mathcal{S}_j^0 := \left\{ x_1 = -(x_2^2 + \dots + x_{2k}^2 + t)^{1/2} \right\},$$

and in the \mathbf{y} -sheet

$$\mathcal{S}_i^1 := \left\{ y_1 = (y_2^2 + \dots + y_{2k}^2 + t)^{1/2} \right\} \text{ and } \mathcal{S}_j^1 := \left\{ y_1 = -(y_2^2 + \dots + y_{2k}^2 + t)^{1/2} \right\}$$

(Figure 6.1, left). Here we choose f_t so that it maps the \mathbf{x} -sheet to “below” the \mathbf{y} -sheet. In the \mathbf{x} -sheet, when $t < 0$, $\mathcal{S}_i^0 \sqcup \mathcal{S}_j^0$ is joined into a single component

$$\mathcal{T}^0 := \{x_1^2 - t = x_2^2 + \dots + x_{2k}^2\}$$

(Figure 6.1, right). Similarly, in the \mathbf{y} -sheet $\mathcal{S}_i^1 \sqcup \mathcal{S}_j^1$ is joined into a single component $\mathcal{T}^1 := \{y_1^2 - t = y_2^2 + \dots + y_{2k}^2\}$ when $t < 0$.

Let L_*^ϵ ($*$ = i, j ; $\epsilon = 0, 1$) be the components of the double point set that contains \mathcal{S}_*^ϵ when $t > 0$, and let K^ϵ be the component of the double point set containing \mathcal{T}^ϵ when $t < 0$.

Case 1. Consider the case $A_i \neq A_j$. Here the number of the components of the self-intersection decreases by 1 when t changes from $t > 0$ to $-t$.

In this case, two components $L_i^\epsilon \sqcup L_j^\epsilon$ of the double point set of g_t , $t > 0$, are joined into a connected double point set $K^\epsilon = L_i^\epsilon \# L_j^\epsilon$ of g_{-t} ($\epsilon = 0, 1$). Other components L_m^ϵ ($m \neq i, j$; $\epsilon = 0, 1$) are unchanged. The jump of the sum of linking numbers is thus

$$\begin{aligned} & \sum_{(p,\epsilon) < (q,\epsilon')} (-1)^{\epsilon+\epsilon'} lk(L_p^\epsilon(g_t), L_q^{\epsilon'}(g_t)) - \sum_{(p,\epsilon) < (q,\epsilon')} (-1)^{\epsilon+\epsilon'} lk(L_p^\epsilon(g_{-t}), L_q^{\epsilon'}(g_{-t})) \\ &= \left(lk(L_i^0, L_j^0) + lk(L_i^1, L_j^1) - \sum_{p,q=i,j} lk(L_p^0, L_q^1) + \sum_{\substack{p=i,j; m \neq i,j, \\ \epsilon, \epsilon'=0,1}} (-1)^{\epsilon+\epsilon'} lk(L_p^\epsilon, L_m^{\epsilon'}) \right) \\ & \quad - \left(-lk(K^0, K^1) + \sum_{m \neq i,j; \epsilon, \epsilon'=0,1} (-1)^{\epsilon+\epsilon'} lk(K^\epsilon, L_m^{\epsilon'}) \right). \end{aligned}$$

The connected-sums $K^\epsilon = L_i^\epsilon \# L_j^\epsilon$ are taken near the tangency point and by a small isotopy we may assume that L_*^ϵ ($*$ = i, j ; $\epsilon = 0, 1$) satisfy the condition in Lemma 5.1. Thus by Lemma 5.1

$$\sum_{p,q=i,j} lk(L_p^0, L_q^1) = lk(K^0, K^1), \quad lk(L_i^\epsilon, L_m^{\epsilon'}) + lk(L_j^\epsilon, L_m^{\epsilon'}) = lk(K^\epsilon, L_m^{\epsilon'}).$$

By the above three equations, we have

$$E(g_t) - E(g_{-t}) = \pm \frac{lk(L_i^0, L_j^0) + lk(L_i^1, L_j^1)}{4}.$$

Case 2. Consider the case when the versal deformation does not change the number of the components of the self-intersection—namely, $k > 1$ and $A_i = A_j$. Since $L_i^\epsilon = L_j^\epsilon$ turns into K^ϵ and other components L_m^ϵ ($m \neq i$; $\epsilon = 0, 1$) are unchanged, $E(g_t) = E(g_{-t})$ follows from the same argument as in the case $1 < \lambda < 2k - 1$.

Case 3. When $k = 1$ and a negative self-tangency occurs at $t = 0$ in $A_i = A_j$ (namely, two arcs get tangent to each other with opposite velocity vectors; by Lemma 2.3 no positive self-tangency occurs), the number of the components of the self-intersection increases by 1 when t changes from $t > 0$ to $-t$. This case is similar to Case 1.

The case $\lambda = 2k - 1$ is similar, and we have the following.

Lemma 6.5. *Suppose that g_0 has an indefinite self-tangency point. If the index of Q is 1 or $2k - 1$, and if the versal deformation changes the number of the components of self-intersection, then*

$$E(g_t) - E(g_{-t}) = \pm \frac{lk(L_i^0, L_j^0) + lk(L_i^1, L_j^1)}{4},$$

where $L_i^0 \sqcup L_i^1 = g_t^{-1}(A_i)$ and $L_j^0 \sqcup L_j^1 = g_t^{-1}(A_j)$ are the double point set corresponding to A_i and A_j , the distinct components of the self-intersection of g_t that are joined into a single component after the versal deformation. Otherwise, $E(g_t) = E(g_{-t})$.

6.3.3. Triple points

Suppose g_0 is liftable and has a triple point. By Proposition 6.3, in some local coordinates, the double point set near the triple point in the \mathbf{x} -sheet is given by

$$\begin{aligned} \mathcal{S}_i^1 &:= \{(x_1, \dots, x_{2k-1}, 0, 0, \dots, 0)\} = +\mathbb{R}^{2k-1} \times \{\mathbf{0}\}^{2k}, \\ \mathcal{S}_j^1 &:= \{(0, \dots, 0, t, x_{2k+1}, \dots, x_{4k-1})\} = \{\mathbf{0}\}^{2k-1} \times \{t\} \times (+\mathbb{R}^{2k-1}) \end{aligned}$$

as oriented manifolds (see Figure 6.2). The orientations are direct consequences of

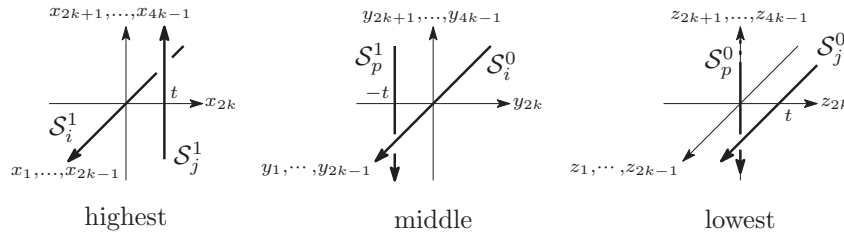


Figure 6.2: Double point sets near the triple point

Lemma 2.3. Similarly, the double point set in the \mathbf{y} -sheet is given by

$$\begin{aligned} \mathcal{S}_p^1 &:= \{(0, \dots, 0, -t, y_{2k+1}, \dots, y_{4k-1})\} = \{\mathbf{0}\}^{2k-1} \times \{-t\} \times (-\mathbb{R}^{2k-1}), \\ \mathcal{S}_i^0 &:= \{(y_1, \dots, y_{2k-1}, 0, 0, \dots, 0)\} = +\mathbb{R}^{2k-1} \times \{\mathbf{0}\}^{2k}, \end{aligned}$$

and in the z -sheet by

$$\begin{aligned} \mathcal{S}_j^0 &:= \{(z_1, \dots, z_{2k-1}, t, 0, \dots, 0)\} = +\mathbb{R}^{2k-1} \times \{t\} \times \{\mathbf{0}\}^{2k-1}, \\ \mathcal{S}_p^0 &:= \{(0, \dots, 0, z_{2k+1}, \dots, x_{4k-1})\} = \{\mathbf{0}\}^{2k} \times (-\mathbb{R}^{2k-1}). \end{aligned}$$

Here, without loss of generality, we assume that the lift f_t of g_t maps the x -sheet (in Proposition 6.3) to the “highest position,” the y -sheet to the “middle,” and the z -sheet to the “lowest.” The following holds by the above descriptions.

Lemma 6.6 (see [10, Remark 6.2.3]). *The versal deformation changes three crossing $\mathcal{S}_i^1 \sqcup \mathcal{S}_j^1$, $\mathcal{S}_i^0 \sqcup \mathcal{S}_p^1$, and $\mathcal{S}_j^0 \sqcup \mathcal{S}_p^0$, changing their linking numbers or writhes by the common value ± 1 (the signs are same for all the three crossings).*

Let L_*^ϵ ($* \in \{i, j, p\}$, $\epsilon \in \{0, 1\}$) be the component of the double point set containing \mathcal{S}_*^ϵ .

Case 1. If all the six components L_*^ϵ are different, then by Lemma 6.6 the versal deformation changes $(-1)^{1+1}lk(L_i^1, L_j^1) + (-1)^{0+1}lk(L_i^0, L_p^1) + (-1)^{0+0}lk(L_j^0, L_p^0)$ by $\pm(1 - 1 + 1) = \pm 1$. Other linking numbers do not change. Thus $E(g_t) - E(g_{-t}) = \pm 1/4$.

Case 2. If $L_i^\epsilon = L_j^\epsilon \neq L_p^\epsilon$ ($\epsilon = 0, 1$), then the crossing change at $\mathcal{S}_i^0 \sqcup \mathcal{S}_p^1$ and $\mathcal{S}_j^0 \sqcup \mathcal{S}_p^0$ changes $(-1)^{0+1}lk(L_i^0, L_p^1) + (-1)^{0+0}lk(L_i^0, L_p^0)$ by $\pm 1 \mp 1 = 0$. Other linking numbers do not change, and the change of $w(L_i^1)$ (by ± 1) arising from the crossing change at $\mathcal{S}_i^1 \sqcup \mathcal{S}_j^1$ does not change E . Thus $E(g_t) = E(g_{-t})$.

Case 3. If $L_p^\epsilon = L_i^\epsilon \neq L_j^\epsilon$ ($\epsilon = 0, 1$), then the versal deformation changes $lk(L_i^1, L_j^1) - lk(L_i^1, L_i^0) + lk(L_i^0, L_j^0)$ by $\pm(1 - 1 + 1) = \pm 1$, and $E(g_t) - E(g_{-t}) = \pm 1/4$.

Case 4. The case $L_j^\epsilon = L_p^\epsilon \neq L_i^\epsilon$ is similar to the Case 2 by symmetry, and $E(g_t) = E(g_{-t})$.

Case 5. If $L_i^\epsilon = L_j^\epsilon = L_p^\epsilon$, then the versal deformation changes $-lk(L_i^0, L_p^1)$ by ± 1 , and the changes of $w(L_i^1)$, $w(L_j^0)$ do not affect E . Thus in this case $E(g_t) - E(g_{-t}) = \pm 1/4$.

Putting them all together, we obtain the following.

Lemma 6.7. *Suppose that g_0 has a triple point. Then $E(g_t) = E(g_{-t})$ if $L_i^\epsilon = L_j^\epsilon \neq L_p^\epsilon$ or $L_j^\epsilon = L_p^\epsilon \neq L_i^\epsilon$, $\epsilon = 0, 1$ (in Figure 6.2). Otherwise, $E(g_t) - E(g_{-t}) = \pm 1/4$.*

6.4. The case $k = 1$

For $g \in \mathcal{K}_{5,3}$ the invariant E is essentially the Smale invariant;

Proposition 6.8 ([10, 30]). *If $g \in \mathcal{K}_{5,3}$ (also regarded as $g \in \mathcal{K}_{6,3}$ by composing $\mathbb{R}^5 \hookrightarrow \mathbb{R}^6$), then $\mathcal{H}(g) = E(g) = -\Omega(g)/12$, where $\Omega: \pi_0(\mathcal{I}_{5,3}) \xrightarrow{\cong} \mathbb{Z}$ is the Smale invariant.*

Proof. $\mathcal{H} = E$ follows from (2.4), since g has no self-intersection. As explained in [10], there exists a “Seifert surface” for g —that is, an embedding $V^4 \hookrightarrow \mathbb{R}^5$ that restricts to $g: \partial V = \mathbb{R}^3 \hookrightarrow \mathbb{R}^5$ —and $\Omega(g) = 3\sigma(V)/2$, where σ denotes the signature. [30, Corollary 2.4] states that $\mathcal{H}(g) = -\sigma(V)/8$ ($e_F = 0$ for $g \in \mathcal{K}_{5,3}$). \square

Remark 6.9. If $g \in \mathcal{K}_{5,3}$, then $E(g) = I(Y)(g)/6 + c(g)$ by (6.1). Thus $\Omega(g) = -2I(Y)(g) - 12c(g)$.

The double point set of generic $g \in \mathcal{I}_{5,3}$ is a classical link. A result of Ogasa [21] characterizes which link can be realized as a double point set of an immersion $\mathbb{R}^3 \looparrowright \mathbb{R}^5$.

Theorem 6.10 (A special case of [21, Theorem 1.1]). *For any link $L \subset S^3$, there exist embeddings $g^\epsilon: S^3 \hookrightarrow \mathbb{R}^5$ ($\epsilon = 0, 1$) such that $(g^\epsilon)^{-1}(g^0(S^3) \cap g^1(S^3))$ ($\epsilon = 0, 1$) are isotopic to the given L . Moreover, we can choose g^ϵ to be isotopic to the natural inclusion.*

Using this, we show the second half of Theorem 2.11 and Corollary 2.7.

Proof of Theorem 2.11, the second half. We show that an arbitrarily large jump of E at a single indefinite self-tangency can occur. Any linear combination of Ekholm invariants J and St cannot satisfy this property, because their jumps are bounded [10].

For any two-component link $L = K_1 \cup K_2$ in S^3 , choose $g^\epsilon: S^3 \hookrightarrow \mathbb{R}^5$, $\epsilon = 0, 1$ as in Theorem 6.10. Taking a suitable connected-sum of the standard inclusion $f_0: \mathbb{R}^3 \hookrightarrow \mathbb{R}^5$ with g^0 and g^1 , we obtain $g := f_0 \# g^0 \# g^1 \in \mathcal{I}_{5,3}$, which satisfies the following conditions:

- (i) The self-intersection $A = A_1 \sqcup A_2$ of g satisfies $g^{-1}(A_i) = K_i^0 \cup K_i^1$ with each $K_1^\epsilon \cup K_2^\epsilon$ included in the g^ϵ -part $(g^\epsilon)^{-1}(A)$ and isotopic to the given link L .
- (ii) A lift of g exists and can be obtained by lifting g^1 -part into $\mathbb{R}^5 \times \mathbb{R}_+$ and letting g^0 -part remain inside $\mathbb{R}^5 \times \{0\}$.

$K_1^0 \cup K_2^0$ is separated from $K_1^1 \cup K_2^1$ by (i), above. Take $q_i \in A_i$ and $p_i^\epsilon \in K_i^\epsilon$ so that $g^\epsilon(p_i^\epsilon) = q_i$. Choose paths $\gamma^\epsilon: [0, 1] \rightarrow \mathbb{R}^3 \setminus (g^\epsilon)^{-1}(A)$ from p_1^ϵ to p_2^ϵ in the g^ϵ -part. Then $C := g(\gamma^0([0, 1]) \cup \gamma^1([0, 1]))$ is a circle in \mathbb{R}^5 , which can be seen as the image of a trivial knot. Thus C bounds an embedded 2-disk D in \mathbb{R}^5 whose interior transversely intersects g at finitely many points outside K_i^ϵ (Figure 6.3). There is a homotopy that transforms g only near $\gamma^0([0, 1])$ so that $g(\gamma^0([0, 1]))$ gets close to $g(\gamma^1([0, 1]))$ along D and eventually D does not intersect g in its interior. During this homotopy a number of non-generic self-intersections may appear, but $A_1 \sqcup A_2$ remains unchanged. Thus we get $g' \in \mathcal{I}_{5,3}$, $g' \simeq g$, for which $\text{Int } D \cap g' = \emptyset$ and we can choose a local coordinate of \mathbb{R}^5 around the tubular neighborhood of D so that

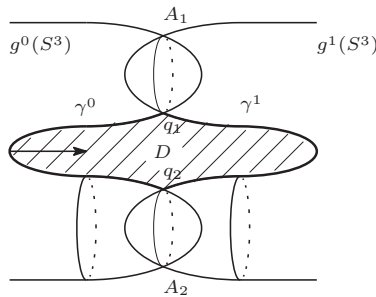


Figure 6.3: The self-intersection of g in the proof of Theorem 2.11

- (1) a tubular neighborhood of $g'(\gamma^0([0, 1]))$ in \mathbb{R}^3 corresponds to $\{(x_1, x_2, x_3, 0, 0) \mid x_1^2 + x_3^2 < r, |x_2| \leq 1\}$ ($r > 0$ small),
- (2) a tubular neighborhood of $g'(\gamma^1([0, 1]))$ in \mathbb{R}^3 corresponds to $\{(x_1, x_2, 0, x_1^2 - x_2^2 + 1, x_5) \mid x_1^2 + x_5^2 < r, |x_2| \leq 1\}$, and

where I_N is the $(N \times N)$ -identity matrix and $\mathbf{0} \in \mathbb{R}^{2k-1}$ is the zero vector. The rows correspond to the bases of $T_{(e_{6k-1}, e_{6k-1}, e_{4k-1})}(S^{6k-1} \times S^{6k-1} \times S^{4k-2})$ and the columns correspond to the natural basis of $T_{\xi}\text{Conf}_4(\mathbb{R}^{4k-1}) \cong T_{\xi}\mathbb{R}^{16k-4}$. Its determinant is -1 .

References

- [1] V. Arnold, *Plane curves, their invariants, perestroikas and classifications*, Singularities and bifurcations, Adv. Soviet Math., vol. 21, Amer. Math. Soc., Providence, RI, 1994, pp. 33–91.
- [2] G. Arone and V. Turchin, *On the rational homology of high-dimensional analogues of spaces of long knots*, Geom. Topol. **18** (2014), no. 3, 1261–1322.
- [3] J. Birman and X.-S. Lin, *Knot polynomials and Vassiliev’s invariants*, Invent. Math. **111** (1993), no. 2, 225–270.
- [4] R. Bott and C. Taubes, *On the self-linking of knots*, J. Math. Phys. **35** (1994), no. 10, 5247–5287.
- [5] R. Budney, *A family of embedding spaces*, Groups, homotopy and configuration spaces (Tokyo 2005), Geom. Topol. Monogr., vol. 13, 2008, pp. 41–84.
- [6] R. Budney, J. Conant, K. P. Scannell, and D. Sinha, *New perspectives on self-linking*, Adv. in Math. **191** (2005), no. 1, 78–113.
- [7] R. Budney, J. Conant, R. Koytcheff, and D. Sinha, *Embedding calculus knot invariants are of finite type*, arXiv:1411.1832.
- [8] A. Cattaneo, P. Cotta-Ramusino, and R. Longoni, *Configuration spaces and Vassiliev classes in any dimensions*, Algebr. Geom. Topol. **2** (2002), 949–1000.
- [9] A. Cattaneo and C. Rossi, *Wilson surfaces and higher dimensional knot invariants*, Comm. Math. Phys. **256** (2005), no. 3, 513–537.
- [10] T. Ekholm, *Differential 3-knots in 5-space with and without self-intersection*, Topology **40** (2001), 157–196.
- [11] T. Ekholm, *Invariants of generic immersions*, Pac. J. Math. **199** (2001), no. 2, 321–345.
- [12] T. Goodwillie and M. Weiss, *Embeddings from the point of view of immersion theory II*, Geom. Topol. **3** (1999), 103–118.
- [13] K. Habiro, T. Kanenobu, and A. Shima, *Finite type invariants of ribbon 2-knots*, Low-dimensional topology (Funchal, 1998), 187–196, Contemp. Math., **233**, Amer. Math. Soc., Providence, RI, 1999.
- [14] A. Haefliger, *Knotted $(4k - 1)$ -spheres in $6k$ -space*, Ann. of Math. (2) **75** (1962), 452–466.
- [15] T. Kohno, *Vassiliev invariants and de Rham complex on the space of knots*, Contemp. Math., **179** (1994), 123–138.
- [16] X.-S. Lin and Z. Wang, *Integral geometry of plane curves and knot invariants*, J. Differential Geom. **44** (1996), no. 1, 74–95.
- [17] R. Milgram, *On the Haefliger knot groups*, Bull. Amer. Math. Soc. **78** (1972), 861–865.

- [18] B. Munson, *Embeddings in the 3/4 range*, *Topology* **44** (2005), no. 6, 1133–1157.
- [19] T. Ochiai, *The combinatorial Gauss diagram formula for Kontsevich integral*, *J. Knot Theory Ramifications* **10** (2001), no. 6, 851–906.
- [20] T. Ochiai, *Invariants of plane curves and Polyak-Viro type formulas for Vassiliev invariants*, *J. Math. Sci. Univ. Tokyo* **11** (2004), no. 2, 155–175.
- [21] E. Ogasa, *The intersection of spheres in a sphere and a new geometric meaning of the Arf invariant*, *J. Knot Theory Ramifications* **11** (2002), no. 8, 1211–1231.
- [22] K. Ohba, *The unknotting numbers of knotted 3-spheres in the 6-sphere in the sense of Haefliger*, *RIMS kokyuroku* **1393** (2004), 132–139 (written in Japanese).
- [23] M. Polyak and O. Viro, *On the Casson knot invariant*, *J. Knot Theory Ramifications* **10** (2001), no. 5, 711–738.
- [24] D. Roseman and M. Takase, *High-codimensional knots spun about manifolds*, *Algebr. Geom. Topol.* **7** (2007), 359–377.
- [25] K. Sakai, *Configuration space integral for embedding spaces and the Haefliger invariant*, *J. Knot Theory Ramifications* **19** (2010), no. 12, 1597–1644.
- [26] K. Sakai and T. Watanabe, *1-loop graphs and configuration space integral for embedding spaces*, *Math. Proc. Cambridge Philos. Soc.* **152** (2012), no. 3, 497–533.
- [27] D. Sinha, *Manifold theoretic compactifications of configuration spaces*, *Selecta Math. (new ser.)* **10** (2004), no. 3, 391–428.
- [28] M. Takase, *A geometric formula for Haefliger knots*, *Topology* **43** (2004), no. 6, 1425–1447.
- [29] M. Takase, *Homology 3-spheres in codimension three*, *Internat. J. Math.* **17** (2006), no. 8, 869–885.
- [30] M. Takase, *The Hopf invariant of a Haefliger knot*, *Math. Z.* **256** (2007), no. 1, 35–44.
- [31] I. Volić, *Finite type knot invariants and calculus of functors*, *Compos. Math.* **142** (2006), no. 1, 222–250.
- [32] I. Volić, *Configuration space integrals and Taylor towers for spaces of knots*, *Topology Appl.* **153** (2006), no. 15, 2893–2904.
- [33] T. Watanabe, *Configuration space integral for long n -knots and the Alexander polynomial*, *Algebr. Geom. Topol.* **7** (2007), 47–92.

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