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HOMOLOGY OF NON-k-OVERLAPPING DISCS

NATALYA DOBRINSKAYA AND VICTOR TURCHIN

(*communicated by Dev P. Sinha*)

Abstract

In this paper we describe the homology and cohomology of some natural bimodules over the little discs operad, whose components are configurations of non-k-overlapping discs. At the end we briefly explain how this algebraic structure intervenes in the study of spaces of non-k-equal immersions.

1. Introduction

Let $\mathcal{B}_d$ denote the operad of little d-discs. We will consider the bimodules $\mathcal{B}_d^{(k)}$, $k \geqslant 2$, over it, with the nth component $\mathcal{B}_d^{(k)}(n)$ being the configuration space of n open discs (labeled by $1, \ldots, n$) in a unit d-disc satisfying the *non-k-overlapping condition*: the intersection of any k of them is empty. It is straightforward that $\mathcal{B}_d^{(k)}$ is a bimodule over $\mathcal{B}_d$. The left action is given by the maps

$$\mathcal{B}_d(n) \times \mathcal{B}_d^{(k)}(m_1) \times \mathcal{B}_d^{(k)}(m_2) \times \cdots \times \mathcal{B}_d^{(k)}(m_n) \to \mathcal{B}_d^{(k)}(m_1 + \cdots + m_n)$$

that consist in replacing the ith disc in $\mathcal{B}_d(n)$ by a configuration of discs from $\mathcal{B}_d^{(k)}(m_i)$. The right action is given by similar maps

$$\mathcal{B}_d^{(k)}(n) \times \mathcal{B}_d(m_1) \times \mathcal{B}_d(m_2) \times \cdots \times \mathcal{B}_d(m_n) \to \mathcal{B}_d^{(k)}(m_1 + \cdots + m_n).$$

Obviously, in both cases the resulting configuration always satisfies the non-k-overlapping condition; thus both composition maps are well defined. One of our main results, Theorem 3.6, describes the homology of the spaces $\mathcal{B}_d^{(k)}(n)$ in terms of this structure.

The space $\mathcal{B}_d^{(k)}(n)$ is homotopy equivalent to the complement in $(\mathbb{R}^d)^{\times n}$ to the union of subspaces

$$A_I = \left\{ (x_1, \ldots, x_n) \in (\mathbb{R}^d)^{\times n} \mid x_{i_1} = \cdots = x_{i_k} \right\},$$

where $I = \{i_1, \ldots, i_k\}$ runs through all cardinality k subsets of $\underline{n} = \{1, \ldots, n\}$. We denote this complement by $\mathcal{M}_d^{(k)}(n)$. By taking the centers of the balls, one gets a map $\mathcal{B}_d^{(k)}(n) \to \mathcal{M}_d^{(k)}(n)$ which is a homotopy equivalence.

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The homology groups of $\mathcal{M}_d^{(k)}(n)$ were first computed by Björner and Welker [**6**] (see also [**4, 5**]). The cohomology algebra $H^*\mathcal{M}_2^{(k)}(n)$ was determined by Yuzvinsky [**48**]. The latter reference also produces a rational model for $\mathcal{M}_2^{(k)}(n)$. Based on this model, it was shown in [**27**] that the spaces $\mathcal{M}_2^{(3)}(n)$, $n \geqslant 7$, have non-trivial Massey products and thus are not formal. The cohomology algebra $H^*\mathcal{M}_1^{(k)}(n)$ was computed by Baryshnikov [**3**]. The symmetric group action on $H_*\mathcal{M}_d^{(k)}(n)$ was computed by Sundaram and Wachs [**38**]. Even though the (co)homology of $\mathcal{M}_d^{(k)}(n)$ is now well understood, few of the references give a geometric description of this (co)homology. In fact, only in the case $d = 1$ does one have a geometric description of this homology, given by the first author in [**12**], and a geometric description of cohomology, given by Baryshnikov in [**3**]. More precisely, in [**6, 4, 5**] the authors use the Goresky–MacPherson formula that describes the homology of the complement to a subspace arrangement in terms of the cohomology of certain posets (of strata in the arrangement). In the case of $\mathcal{M}_d^{(k)}(n)$ one has to study the poset $\Pi_{n,k}$ of subsets of $\{1,\ldots,n\}$ whose cardinality is either 1 or $\geqslant k$. Yuzvinsky's method is also purely combinatorial—it produces a rational model for $\mathcal{M}_2^{(k)}(n)$ and more generally for any complement to a complex arrangement based on the combinatorics of the Goresky–MacPherson complex. Another approach for the case $d = 1$ appears in [**28**], which describes the homology over a field of more general diagonal arrangements in terms of the homology of monomial rings. Applied to the case of non-k-equal arrangements, this approach produces the Betti numbers of $\mathcal{M}_1^{(k)}$. Following this idea improved to integral coefficients and using homological algebra methods, an algebraic structure similar to the one studied in this paper for $d = 1$ appeared in [**12**].

In our paper we make use of an operad structure that naturally also brings in geometry. As a left module over $H_*\mathcal{B}_d$, the sequence $H_*\mathcal{B}_d^{(k)}$ for $k \geqslant 3$ is generated by two elements $x_1 \in H_0\mathcal{B}_d^{(k)}(1)$ and $\{x_1,\ldots,x_k\} \in H_{(k-1)d-1}\mathcal{B}_d^{(k)}(k)$; see Theorem 3.4. Notice that $\mathcal{B}_d^{(k)}(1) \simeq *$ and $\mathcal{B}_d^{(k)}(k) \simeq S^{(k-1)d-1}$. The elements x_1 and $\{x_1 \ldots x_k\}$ are the generators of the corresponding homology groups. Explicitly, this result means that the homology groups $H_*\mathcal{B}_d^{(k)}(n)$ are spanned by certain products of iterated brackets. Such classes can be geometrically realized as products of spheres in $\mathcal{B}_d^{(k)}(n)$. One should mention that such a description of the homology in terms of iterated brackets is implicitly given in [**16**], where the author shows that the poset $\Pi_{n,k}$ is quasi-isomorphic to a poset of certain trees. Here one can see a connection to work of Gaiffi [**18**] that produces a general construction of a compactification of the complement to a subspace arrangement. In the case of $\mathcal{M}_d^{(k)}(n)$ the strata of the compactification are encoded exactly by the trees from [**16**]. In fact, Gaiffi's work can be used to produce geometric cycles in the homology of the complement to any arrangement.

We also give a more geometric description of the cohomology algebra $H^*\mathcal{B}_d^{(k)}(n)$. In particular, we show that for $k \geqslant 3$ and $d \geqslant 2$ this algebra is quadratic. This was shown for $k \geqslant 2$ in [**1, 9**] and for $d = 1$ in [**3**]. In Sections 2 and 4, respectively, we recall these quadratic algebras for the two cases considered earlier. In the other cases that we study in our paper, $k \geqslant 3$ and $d \geqslant 2$, a substantial difference is that the generators lie in different degrees, but as we said, all relations still follow from quadratic ones; see Theorem 7.2. Since our description is geometric we hope it will help to better understand the rational homotopy type of $\mathcal{M}_d^{(k)}(n)$, and in particular to compute the Massey products for these spaces.

The structure of a bimodule over $H_*\mathcal{B}_d$ that $H_*\mathcal{B}_d^{(k)}$ has gives a very explicit geometric description of cycles, which is also important for applications. One application is in the study of spaces of non-k-self-intersecting immersions. We describe briefly this connection in Section 11.

Another important application is in the study of the homology of iterated loop spaces of fat wedges. First examples of such computations go back to Lemaire's work [**23**] for single-loop spaces on fat wedges of spheres, where he computed its homology over a field. In [**10, 11, 12, 13**] a more general problem for loops on fat wedges of arbitrary spaces is considered, and the homology is computed via homology of diagonal arrangements with algebraic structure similar to a bimodule on $\mathcal{B}_1^{(k)}$. The long brackets $\{x_1 \dots x_k\}$ discussed above correspond to higher commutator products on loop homology induced by Samelson products. A similar description of the homology of iterated d-loops on fat wedges must exist, and we expect our results will be useful in their further study.

1.1. Notation

By a symmetric sequence we will understand a sequence of objects $M(n)$, $n \geqslant 0$, where each $M(n)$ is endowed with an action of the symmetric group Σ_n. Alternatively, and this will be useful sometimes for our arguments, we will understand a symmetric sequence as a functor from the category of finite sets whose morphisms are bijections. For example, for a finite set I the corresponding configuration space (or its homology) whose points/discs are labeled by elements from I will be denoted by $\mathcal{M}_d^{(k)}(I)$, $\mathcal{B}_d^{(k)}(I)$, $H_*\mathcal{B}_d^{(k)}(I)$. The permutation group of I will be denoted by Σ_I. The cardinality of I will be denoted by $|I|$. The set $\{1, \dots, n\}$ will be denoted by $\underline{n}$.

All the homology and cohomology groups that we consider are taken with integral coefficients.

1.2. Main results

Our main result is Theorem 3.6, where we describe the $H_*\mathcal{B}_d$-bimodule structure of $H_*\mathcal{B}_d^{(k)} = H_*\mathcal{M}_d^{(k)}$. Another important result is in Sections 6 and 7, where we give a more natural description of the cohomology algebras $H^*\mathcal{B}_d^{(k)}(n)$, $k \geqslant 3$ and $d \geqslant 2$, as spaces spanned by certain forests. Theorem 7.2 shows that each such algebra is quadratic, which was not obvious from the previous description due to Yuzvinsky [**48**]. Our description is also nicely compatible with the structure of a cobimodule that $H^*\mathcal{B}_d^{(k)}$ has, which is given in Section 9. As we have mentioned, the spaces $\mathcal{M}_d^{(k)}(n)$ were extensively studied. In particular, to prepare this paper we found [**3, 38, 48**] very useful. However, the presentation of our paper is self contained—all the arguments and proofs are not formally relying on other results or computations. For this reason we hope it will also be of educational value.

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2. Homology and cohomology of $\mathcal{B}_d$

The homology of the little discs operad is well known [**9**]:

Theorem 2.1 (F. Cohen [**9**])**.** *The homology operad $H_*\mathcal{B}_d$ is the operad of associative unital algebras in case $d = 1$ and the operad of graded unital Poisson algebras with bracket of degree $(d-1)$ in the case $d \geqslant 2$.*

Below, we briefly describe the geometric meaning of this result. We would also like to suggest the expository paper [**35**], where Cohen's theorem is explained in full detail.

In the case $d = 1$, the space $\mathcal{B}_1(n)$, $n \geqslant 0$, has $n!$ contractible components. Thus its homology is concentrated in degree 0 and has rank $n!$. We get

$$H_*\mathcal{B}_1(n) = H_0\mathcal{B}_1(n) = \mathcal{A}ssoc(n).$$

It is also obvious that the compositions agree.

In the case $d \geqslant 2$, one has $\mathcal{B}_d(0) = *$, and $\mathcal{B}_d(2) \simeq S^{d-1}$. The generators of $H_0\mathcal{B}_d(0)$, $H_0\mathcal{B}_d(2)$, and of $H_{d-1}\mathcal{B}_d(2)$ are, respectively, the elements 1, $x_1 \cdot x_2$, and $[x_1, x_2]$ of the Poisson operad. Notice that the theorem above describes $H_*\mathcal{B}_d(n)$ as a free $\mathbb{Z}$-module spanned by products of iterated brackets. Operad composition is given by substitution. The corresponding cycles are realized as products of spheres. For example, $[[x_1, x_2], x_3] \in H_*\mathcal{B}_d(3)$ can be realized as $S^{d-1} \times S^{d-1} \to \mathcal{M}_d^{(2)}(3)$, where the point 2 rotates around 1, and 3 rotates around 1 and 2. As another example $[x_1, x_2] \cdot [x_3, x_4]$ can be realized as $S^{d-1} \times S^{d-1} \to \mathcal{M}_d^{(2)}(4)$, where 2 rotates around 1, and 4 does so around 3, and, moreover, 1 and 2 do not interact with 3 and 4.

In Section 3 we give a similar description of $H_*\mathcal{B}_d^{(k)}(n)$ as a space spanned by products of iterated brackets with each such cycle realized by products of spheres.

Theorem 2.2 ([**1, 9**])**.** *The cohomology algebra $H^*\mathcal{B}_d(n)$, $d \geqslant 2$, is generated by $\alpha_{ij} \in H^{d-1}\mathcal{B}_d(n)$, $1 \leqslant i \neq j \leqslant n$; the relations are $\alpha_{ij} = (-1)^d\alpha_{ji}$, $\alpha_{ij}^2 = 0$, $\alpha_{ij}\alpha_{jk} + \alpha_{jk}\alpha_{ki} + \alpha_{ki}\alpha_{ij} = 0$.*

To any monomial of this algebra one can assign a graph on vertices $1, \ldots, n$ by putting an edge between i and j for every factor α_{ij}. It follows from the relations that a monomial is non-zero if and only if the corresponding graph is a forest.

In Section 6 we will give a similar description of $H^*\mathcal{B}_d^{(k)}(n)$ as a free $\mathbb{Z}$-module spanned by certain forests and quotiented out by relations arising naturally from the geometry of configuration spaces. The product of such forests will essentially be their superposition similarly to the case of $H^*\mathcal{B}_d(n)$.

3. $H_*\mathcal{B}_d^{(k)}$ as a left module and as a bimodule

One has a natural inclusion

$$\mathcal{B}_d(n) \subset \mathcal{B}_d^{(k)}(n),\ n \geqslant 0, \tag{3.1}$$

which is null homotopic for $k \geqslant 3$ (in the case $k = 2$ it is an identity map). To show this, we choose a basepoint in $\mathcal{B}_d(n)$. Then we homotope the inclusion (3.1) to the constant map to this basepoint. Our homotopy consists of n pieces, where at its ith step we pull the ith disc to its position in the basepoint configuration. Such a path goes through disc configurations with at most double overlaps. More generally, any inclusion $\mathcal{B}_d^{(k)}(n) \subset \mathcal{B}_d^{(k+1)}(n)$ is null by a similar argument.

Definition 3.1. We say that M is a *left module* (respectively, *bimodule*) *under* an operad $\mathcal{O}$ if it is a left module (respectively, bimodule) over $\mathcal{O}$ and is endowed with a map of left modules (respectively, bimodules) $\mathcal{O} \to M$.

A relevant example to us is $\mathcal{B}_d^{(k)}$ which is a bimodule under $\mathcal{B}_d$ because of the inclusion (3.1). In fact, in one of the applications in Section 11 it will be important that $\mathcal{B}_d^{(k)}$ is not only a bimodule but also a bimodule *under* $\mathcal{B}_d$.

$\mathcal{C}om$ will denote the operad of commutative unital algebras over $\mathbb{Z}$.

Definition 3.2. An operad $\mathcal{O}$ in graded $\mathbb{Z}$-modules is called *augmented* if it is endowed with a surjective map of operads $\mathcal{O} \to \mathcal{C}om$.

Notice that all the operads $H_*\mathcal{B}_d$, $d \geqslant 1$, are naturally augmented since they arise as the homology of topological operads. This in particular implies that $\mathcal{C}om$ is a bimodule under $H_*\mathcal{B}_d$.

Definition 3.3. We say that M is a *pointed* left module (respectively, bimodule) under an augmented operad $\mathcal{O}$ if M is a left module (respectively, bimodule) under $\mathcal{O}$, the structure map $\mathcal{O} \to M$ factors through $\mathcal{C}om$, and, moreover, the map $\mathcal{C}om \to M$ is an inclusion.

Since all the maps (3.1) are null for any $k \geqslant 3$, the bimodules $H_*\mathcal{B}_d^{(k)}$, $k \geqslant 3$, are pointed under $H_*\mathcal{B}_d$.

One has a natural forgetful functor from the category of pointed left modules (respectively, pointed bimodules) to the category of symmetric sequences, which has a left adjoint. For a given symmetric sequence this left adjoint functor produces a free pointed left module (respectively, bimodule) generated by this sequence. Notice that the obtained left module (respectively, bimodule) is not free in the usual sense since it contains $\mathcal{C}om$, on which the Lie part of $H_*\mathcal{B}_d$ acts trivially.

Theorem 3.4. *For $k \geqslant 3$, the pointed left module $H_*\mathcal{B}_d^{(k)}$ under $H_*\mathcal{B}_d$ is generated by a single element $\{x_1,\ldots,x_k\} \in H_{(k-1)d-1}\mathcal{B}_d^{(k)}(k)$, which is symmetric or skew symmetric depending on the parity of d:*

$$\{x_{\sigma_1}\cdots x_{\sigma_k}\} = (-1)^{|\sigma|d}\{x_1\cdots x_k\},\ \sigma\in\Sigma_k. \tag{3.2}$$

The only relation that the left action has is the generalized Jacobi:

$$\sum_{i=1}^{k+1}(-1)^{(i-1)d}\,[x_i,\{x_1,\ldots,\hat{x}_i,\ldots,x_{k+1}\}] = 0. \tag{3.3}$$

The element $\{x_1,\ldots,x_k\} \in H_{(k-1)d-1}\mathcal{B}_d^{(k)}(k) \simeq \mathbb{Z}$ can be realized by a sphere in $\mathcal{M}_d^{(k)}(k) \simeq \mathcal{B}_d^{(k)}(k)$:

$$\sum_{i=1}^{k}|x_i|^2 = \varepsilon^2; \qquad \sum_{i=1}^{k}x_i = 0, \tag{3.4}$$

where x_i is the ith point in the configuration (equivalently the center of the ith disc). Notice that $\mathcal{M}_d^{(k)}(k)$ is the complement in $(\mathbb{R}^d)^k$ of the diagonal space $\mathbb{R}^d$ and therefore is homotopy equivalent to the sphere (3.4). For the theorem above one can choose any orientation of this sphere. The orientation will matter only when we will be speaking about the duality between the homology and cohomology; see Section 8.

Theorem 3.4 tells us that the left action of $H_*\mathcal{B}_d$ suffices to completely describe the homology groups $H_*\mathcal{B}_d^{(k)}(n)$ as spaces spanned by products of iterated brackets. Dually, the cohomology groups $H^*\mathcal{B}_d^{(k)}(n)$ are described in Theorem 6.1 as spaces spanned by certain forests.

Proof of equation (3.3). Our proof is inspired by Sinha's proof of the usual Jacobi relation in $H_{2(d-1)}\mathcal{M}_d^{(2)}(3)$; see [**35**, Proposition 2.7]. Consider the intersection of $\mathcal{M}_d^{(k)}(k+1)$ with the $(kd-1)$-sphere $\sum_{i=1}^{k+1}x_i = 0$, $\sum_{i=1}^{k+1}x_i^2 = 1$. The resulting space is homotopy equivalent to $\mathcal{M}_d^{(k)}(k+1)$. This space is the sphere S^{kd-1} from which one removed $(k+1)$ disjoint $(d-1)$-spheres. Now consider the $(kd-1)$-chain C that is this sphere S^{kd-1} minus small tubular neighborhoods of the aforementioned $(d-1)$-spheres. The boundary of C produces exactly relation (3.3). □

Remark 3.5. As a pointed left module under $H_*\mathcal{B}_d$, the sequence $H_*\mathcal{B}_d^{(k)}$, $k \geqslant 3$, is generated by a single element, but as a left module it is generated by two elements $x_1 \in H_0\mathcal{B}_d^{(k)}(1)$ and $\{x_1\cdots x_k\} \in H_{(k-1)d-1}\mathcal{B}_d^{(k)}(k)$. The left submodule generated by x_1 is exactly $\mathcal{C}om = H_0\mathcal{B}_d^{(k)}$. The Lie part of $H_*\mathcal{B}_d$ acts trivially on this submodule, which is equivalent to the relation

$$[x_1,x_2] = 0. \tag{3.5}$$

Geometrically, this relation says that rotating one disc around the other produces a trivial homology class in $\mathcal{B}_d^{(k)}(2) \simeq *$, $k \geqslant 3$.

The right action of $H_*\mathcal{B}_d$ on $H_*\mathcal{B}_d^{(k)}$ tells us what happens with the homology when the points in configurations get multiplied—this will be important for applications; see Section 11.

Theorem 3.6. *For $k \geqslant 3$, the pointed bimodule $H_*\mathcal{B}_d^{(k)}$ under $H_*\mathcal{B}_d$ is generated by a single element $\{x_1,\dots,x_k\} \in H_{(k-1)d-1}\mathcal{B}_d^{(k)}(k)$ satisfying the symmetry* (3.2)*, generalized Jacobi* (3.3)*, and Leibniz relations with respect to the right action:*

$$\{x_1,\dots,x_{k-1},x_k\cdot x_{k+1}\} = x_k\cdot\{x_1,\dots,x_{k-1},x_{k+1}\} + \{x_1,\dots,x_k\}\cdot x_{k+1}; \tag{3.6}$$

$$\{x_1,\dots,x_{k-1},[x_k,x_{k+1}]\} = (-1)^d\left[\{x_1,\dots,x_{k-1},x_{k+1}\},x_k\right] + \left[\{x_1,\dots,x_k\},x_{k+1}\right]. \tag{3.7}$$

The cycle on the left-hand side of (3.6) is obtained by applying the right action of $x_1\cdot x_2 \in H_0\mathcal{B}_d(2)$ on the last input of $\{x_1\cdots x_k\}$:

$$\{x_1,\dots,x_{k-1},x_k\cdot x_{k+1}\} = \{x_1\cdots x_k\}\circ_k (x_1\cdot x_2).$$

Explicitly, this cycle in $\mathcal{M}_d^{(k)}(k+1)$ can be described by the equations

$$\sum_{i=1}^{k-1} x_i + \frac{x_k+x_{k+1}}{2} = 0, \qquad \sum_{i=1}^{k-1}|x_i|^2 + \frac{|x_k+x_{k+1}|^2}{4} = \varepsilon^2, \tag{3.8}$$

and $x_{k+1}-x_k = (\delta,0,0,\dots,0)$, where $\delta \ll \varepsilon$. Similarly for the left-hand side of (3.7) one has

$$\{x_1,\dots,x_{k-1},[x_k,x_{k+1}]\} = \{x_1\cdots x_k\}\circ_k [x_1,x_2].$$

This cycle in $\mathcal{M}_d^{(k)}(k+1)$ is described by the equations (3.8) and $|x_{k+1}-x_k| = \delta$.

One can show that (3.6) implies

$$\{x_1,\dots,x_{k-1},1\} = 0,$$

where 1 is the generator of $H_0B_d(0)$. Geometrically, composition with this element forgets the corresponding disc in the configurations.

Notice also that in the case $d=1$, the second relation (3.7) follows from the first one (3.6).

Proof of Theorem 3.6. In order to prove this theorem it suffices to prove Theorem 3.4 and also relations (3.6) and (3.7). The latter relations are proved in Examples 4.1, 5.2, and 5.3. □

The following proposition is the first major step in the proof of Theorem 3.4.

Proposition 3.7. *The cycles obtained by the left action of $H_*\mathcal{B}_d$ on $H_0\mathcal{B}_d^{(k)}(1)$ and on $H_{(k-1)d-1}\mathcal{B}_d^{(k)}(k)$ span the homology of each component $H_*\mathcal{B}_d^{(k)}(n)$, $n \geqslant 0$.*

The cases $d=1$ and $d \geqslant 2$ of this proposition are proved in Sections 4 and 5, respectively. The case $d=1$ was essentially done by Baryshnikov [**3**]. For $d>1$ the argument is an easy generalization of the case $d=1$.

In order to complete the proof of Theorem 3.4, one needs to show that between the cycles produced by this left action there are no other relations besides those that follow from (3.2), (3.3), and (3.5). This is done by providing an explicit basis of $H_*\mathcal{B}_d^{(k)}(n)$.

Proposition 3.8. *The homology $H_*\mathcal{B}_1^{(k)}(n)$, $k \geqslant 3$, is torsion-free. For a basis, one can take the set whose elements are encoded by partitions $I_0,J_1,I_1,J_2,\dots,J_\ell,I_\ell$*

of $\underline{n}$, such that $\ell \geqslant 0$, $|J_s| = k$, $s = 1, \dots, \ell$, and $\max(I_s \sqcup J_{s+1}) \in J_{s+1}$ for all $s = 0, \dots, \ell - 1$. The homology class corresponding to such partition has the form

$$A_{I_0} \cdot B_{J_1} \cdot A_{I_1} \cdot B_{J_2} \cdots A_{I_{\ell-1}} \cdot B_{J_\ell} \cdot A_{I_\ell}, \tag{3.9}$$

where $A_{I_s} = x_{i_{1,s}} \cdot x_{i_{2,s}} \cdot \dots \cdot x_{i_{|I_s|,s}}$ for $I_s = \{i_{1,s} < i_{2,s} < \dots < i_{|I_s|,s}\}$ (if $I_s = \emptyset$, then $A_{I_s} = 1$ or is simply omitted), and $B_{J_s} = \{x_{j_{1,s}}, x_{j_{2,s}}, \dots, x_{j_{k,s}}\}$ for $J_s = \{j_{1,s} < j_{2,s} < \dots < j_{k,s}\}$.

It follows from Proposition 3.7 and relations (3.2), (3.3), and (3.5) that any homology class in $H_*\mathcal{B}_1^{(k)}(n)$ is a linear combination of the elements (3.9). In Section 4 we will produce an explicit set of cohomology classes described by essentially the same combinatorial data such that the pairing matrix with (3.9) is upper triangular. This proves the linear independence of the elements (3.9).

Proposition 3.9. *The homology $H_*\mathcal{B}_d^{(k)}(n)$, $d \geqslant 2$, $k \geqslant 3$, is torsion-free. For a basis, one can take the products of iterated brackets satisfying the following conditions: each factor is either x_i, $i \in \underline{n}$, or an iterated bracket of the form*

$$[\cdots[[B_1, B_2], B_3] \cdots B_\ell], \quad \ell \geqslant 1, \tag{3.10}$$

where each B_s is of the form

$$B_s = [\cdots[[\{x_{j_{1,s}}, x_{j_{2,s}}, \dots, x_{j_{k,s}}\}, x_{i_{1,s}}], x_{i_{2,s}}] \cdots x_{i_{\ell_s,s}}],$$

where $j_{1,s} < j_{2,s} < \dots < j_{k,s}$; $\ell_s \geqslant 0$; $i_{1,s} < i_{2,s} < \dots < i_{\ell_s,s} < j_{k,s}$. Also we require that the variable x_i with the smallest index i in (3.10) must appear in B_1.

Example 3.10. A basis element in $H_{10d-7}\mathcal{B}_d^{(3)}(14)$:

$$x_3 \cdot \{x_1, x_8, x_9\} \cdot \big[[\{x_{10}, x_{11}, x_{12}\}, x_2], [[[\{x_6, x_7, x_{14}\}, x_4], x_5], x_{13}]\big].$$

Again it follows from Proposition 3.7 that the elements above span $H_*\mathcal{B}_d^{(k)}(n)$. To prove that they are linearly independent, we produce an explicit dual basis in cohomology; see Section 5.

Corollary 3.11. *For any $d \geqslant 1$, $k \geqslant 2$, $n \geqslant 0$, the suspension $\Sigma\mathcal{M}_d^{(k)}(n)$ is homotopy equivalent to a wedge of spheres.*

Proof. This is always true if a space has torsion-free homology admitting a basis realized by products of spheres. □

Remark 3.12. In the case $k = 2$, the homology $H_*\mathcal{B}_d^{(2)}(\bullet) = H_*\mathcal{B}_d(\bullet)$ admits a natural decreasing filtration that respects the structure of a bimodule over $H_*\mathcal{B}_d$. The statements of Theorems 3.4, 3.6, and Propositions 3.7, 3.8, 3.9 hold if one replaces $H_*\mathcal{B}_d^{(2)}$ by the associated graded quotient. This filtration was considered in [**38**]. As Sundaram and Wachs point out, it is induced by Reutenauer's derived series filtration in the free Lie algebra [**30**].

4. Case $d = 1$

First, we prove Proposition 3.7 in the case $d = 1$. This was implicitly done by Baryshnikov in [**3**]. We repeat his argument for completeness of exposition. The space

$\mathcal{B}_1^{(k)}(0)$ is a point, so $H_*\mathcal{B}_1^{(k)}(0) \simeq \mathbb{Z}$ which is obtained by the left action of the arity zero component $H_*\mathcal{B}_1(0) = H_0\mathcal{B}_1(0) \simeq \mathbb{Z}$. The generator of $H_0\mathcal{B}_1^{(k)}(0) \simeq H_0\mathcal{B}_1(0)$ is denoted by 1. We then proceed by induction over n. We use the configuration (rather than little balls) model. Consider a cycle $[\alpha] \in H_*\mathcal{M}_1^{(k)}(n)$ and a chain α representing $[\alpha]$. Consider the projection $p\colon \mathcal{M}_1^{(k)}(n) \to \mathcal{M}_1^{(k)}(n-1)$ that forgets the last point in a configuration. By a little perturbation, one can assume that each simplex of α is smooth and transversal to every fiber of p. Define a homotopy α_t, $0 \leqslant t \leqslant c$, of α in $\mathcal{M}_1^{(k)}(n-1) \times \mathbb{R}$ by adding t to the last coordinate x_n. (In other words, we pull the last point x_n to the right for every point in the cycle α.) This homotopy viewed as a chain in $\mathcal{M}_1^{(k)}(n-1) \times \mathbb{R}$ intersects transversely the "forbidden fibers"—it happens when $x_n + t$ collides with $x_{i_1} = \cdots = x_{i_{k-1}}$, $1 \leqslant i_1 < i_2 < \cdots < i_{k-1} \leqslant n-1$. To turn α_t into a chain in $\mathcal{M}_1^{(k)}(n)$, we remove from it intersections with small tubular neighborhoods of the planes $x_{i_1} = \cdots = x_{i_{k-1}} = x_n$. One gets that the boundary of such a chain is the sum of α (when $t = 0$), a cycle of the form $A \cdot x_n$, where $A \in H_*\mathcal{M}_1^{(k)}(n-1)$ (when $t = c$), and cycles of the form $A_I \cdot \{x_{i_1}, x_{i_2}, \ldots, x_{i_{k-1}}, x_n\} \cdot B_J$, where $A_I \in H_*\mathcal{M}_1^{(k)}(I)$, $B_J \in H_*\mathcal{M}_1^{(k)}(J)$, $I \sqcup J = \underline{n-1} \setminus \{i_1, i_2, \ldots, i_{k-1}\}$ (such cycles correspond to the part of the boundary appearing from the intersection of α_t with the plane $x_{i_1} = \cdots = x_{i_{k-1}} = x_n$). The set I (respectively, J) contains the indices i such that $x_i < x_n$ (respectively, $x_i > x_n$). Now, using induction, we get the result.[1] Q.E.D.

Example 4.1. Consider a natural chain representing the cycle $\{x_1, x_2, \ldots, x_{k-1}, x_k \cdot x_{k+1}\} \in H_{k-2}\mathcal{M}_1^{(k)}(k+1)$. When x_{k+1} is pulled to the right, it can only meet the plane $x_1 = x_2 = \cdots = x_{k-1} = x_{k+1}$, which produces the cycle $x_k \cdot \{x_1, \ldots, x_{k-1}, x_{k+1}\}$. At the other end of the homotopy, we get the cycle $\{x_1, \ldots, x_k\} \cdot x_{k+1}$. As a result, we get exactly relation (3.6).

Now we prove Proposition 3.8. We will exhibit an explicit dual basis in cohomology. We reiterate that it was done in [**3**] and we give it for completeness of exposition.

For a partition of $\underline{n}$ into a collection of subsets $I_0, J_1, I_1, J_2, \ldots, I_{\ell-1}, J_\ell, I_\ell$, define a subset of points in $\mathbb{R}^n$ satisfying the following (in)equalities:

$$x_i \leqslant x_j, \quad i \in I_s,\ j \in J_{s+1}; \tag{4.1}$$

$$x_j \leqslant x_i, \quad j \in J_s,\ i \in I_s; \tag{4.2}$$

$$x_{j_1} = x_{j_2}, \qquad j_1, j_2 \in J_s. \tag{4.3}$$

This set, or rather its intersection with $\mathcal{M}_1^{(k)}(n)$, will be denoted by

$$(I_0)[J_1](I_1)[J_2]\cdots(I_{\ell-1})[J_\ell](I_\ell). \tag{4.4}$$

Now let us assume that $|J_s| = k-1$ for all $s = 1, \ldots, \ell$. We get that the boundary of this set (viewed as a locally compact chain) lies in the complement of $\mathcal{M}_1^{(k)}(n)$. Thus via intersection number it defines a cocycle in $H^*\mathcal{M}_1^{(k)}(n)$. In addition, assuming the

[1] Notice that this recursive procedure shows that any cycle of $\mathcal{M}_1^{(k)}(n)$ is homologous to a linear combination of the basis elements from Proposition 3.8.

restriction

$$\max(I_s \sqcup J_{s+1}) \in I_s, \tag{4.5}$$

we get a collection of cocycles that is exactly a basis dual to (3.9). To be precise, for an appropriate order of elements the pairing is given by an upper triangular matrix with ± 1 on the diagonal. Details can be found in [**3**], or make for a pleasant exercise for a motivated reader. Without the second restriction (4.5) (but still assuming $|J_s| = k-1$ for all $s = 1, \ldots, \ell$), the cocycles (4.4) are linearly dependent in $H^*\mathcal{M}_1^{(k)}(n)$. Baryshnikov shows that all relations are spanned by the boundaries of the chains (4.4) with all J_s of cardinality $k-1$ except one of cardinality $k-2$. Moreover, Baryshnikov describes the cohomology algebra $H^*\mathcal{M}_1^{(k)}(n)$ as being generated by the elements $(I_0)[J_1](I_1)$, $|J_1| = k-1$. The relations are linear, appearing as the boundary of the elements $(I_0')[J_1'](I_1')$, $|J_1'| = k-2$, and quadratic: the square of any element $(I_0)[J_1](I_1)$ is zero and the product of two generators is zero if the intersection of the corresponding locally finite chains in $\mathcal{M}_1^{(k)}(n)$ is empty.

5. Proof of Proposition 3.7 for $d \geqslant 2$

The proof of Proposition 3.7 for $d \geqslant 2$ is similar to the case $d = 1$. Given a cycle in $\mathcal{M}_d^{(k)}(n)$, we will homotope it by pulling the last point x_n in the configuration away from the other points. This will lead to a similar recursive construction, but the recursion will be using the homology of slightly more general arrangements. Denote by $\mathcal{M}_d^{(k)}(n,m)$ the complement in $\mathbb{R}^{d(n+m)} = \{(x_1, \ldots, x_n; y_1, \ldots, y_m) | x_i \in \mathbb{R}^d,\, y_j \in \mathbb{R}^d\}$ to the union of subspaces

$$x_{i_1} = \cdots = x_{i_k},$$

for any cardinality k subset $\{i_1, \ldots, i_k\} \subset \underline{n}$,

$$x_i = y_j,\ 1 \leqslant i \leqslant n, \quad 1 \leqslant j \leqslant m;$$
$$y_{j_1} = y_{j_2}, \qquad 1 \leqslant j_1 < j_2 \leqslant m.$$

The space $\mathcal{M}_d^{(k)}(n,m)$ is homotopy equivalent to the space $\mathcal{B}_d^{(k)}(n,m)$ of configurations of n discs labeled by $1, 2, \ldots, n$, and colored by x, and of m discs labeled by $1, 2, \ldots, m$, and colored by y, in a unit disc. The *non-overlapping condition* is that no k x-colored discs have a non-trivial intersection, and all y-colored discs are disjoint one from another and from the x-discs.

We say that a family of spaces (or vector spaces) $M(n,m)$, $n \geqslant 0$, $m \geqslant 0$, is a *bi-colored left module* over an operad $\mathcal{O}$ if each $M(n,m)$ is acted on by $\Sigma_n \times \Sigma_m$ and one is given structure composition maps:

$$\mathcal{O}(\ell) \times M(n_1, m_1) \times M(n_2, m_2) \times \cdots \times M(n_\ell, m_\ell) \to M(n_1 + \cdots + n_\ell, m_1 + \cdots + m_\ell). \tag{5.1}$$

One assumes the easily guessed symmetric group equivariance, associativity, and unity conditions. For example, $\mathcal{B}_d^{(k)}(\bullet, \bullet)$ is a bi-colored left module over $\mathcal{B}_d(\bullet)$. A similar structure is induced in homology.

Theorem 5.1. *For $d \geqslant 2$, $k \geqslant 3$, the bi-colored left module $H_*\mathcal{B}_d^{(k)}(\bullet,\bullet)$ is generated by $x_1 \in H_0\mathcal{B}_d^{(k)}(1,0)$, $\{x_1 \dots x_k\} \in H_{(k-1)d-1}\mathcal{B}_d^{(k)}(k,0)$, and $y_1 \in H_0\mathcal{B}_d^{(k)}(0,1)$. The only relations are* (3.2), (3.3), *and* (3.5).

The theorem above describes the homology of each component $\mathcal{B}_d^{(k)}(n,m)$ as a space spanned by products of iterated brackets on $x_1,\dots,x_n,y_1,\dots,y_m$. The proof of this theorem is very similar to that of Theorem 3.4. We will only show that the elements obtained by the left action of $H_*\mathcal{B}_d$ on x_1, y_1, and $\{x_1 \cdots x_k\}$ do span the homology of each component $H_*\mathcal{B}_d^{(k)}(n,m)$. This will obviously imply Proposition 3.7.

For $n=0$ the statement is obvious. Indeed, $H_*\mathcal{B}_d^{(k)}(0,\bullet)$ is isomorphic to $H_*\mathcal{B}_d(\bullet)$ as a left $H_*\mathcal{B}_d$-module: it is freely generated by the single element $y_1 \in H_0\mathcal{B}_d^{(k)}(0,1)$. Now let α be a smooth generic s-dimensional chain (by this we mean each simplex is smooth and in generic position) in $\mathcal{M}_d^{(k)}(n,m)$. We consider the homotopy α_t, $0 \leqslant t \leqslant c$, of α in $\mathcal{M}_d^{(k)}(n-1,m)\times\mathbb{R}^d$ that only affects the last coordinate $x_n(t) = x_n + t\cdot v$, where the vector $v \in \mathbb{R}^d\setminus\{0\}$ is fixed. When c is big enough, $x_n(c)$ will be far away from all the other points $x_1,\dots,x_{n-1},y_1,\dots,y_m$ appearing in α. The fact that α is generic and smooth guarantees that α_t viewed as an $(s+1)$-chain in $\mathcal{M}_d^{(k)}(n-1,m)\times\mathbb{R}^d$ is transversal to the forbidden subspaces

$$x_n = y_j, \quad 1 \leqslant j \leqslant m; \tag{5.2}$$

$$x_n = x_{i_1} = x_{i_2} = \cdots = x_{i_{k-1}}, \quad 1 \leqslant i_1 < i_2 < \cdots < i_{k-1} \leqslant n-1. \tag{5.3}$$

We remove from α_t intersections with small tubular neighborhoods of the above subspaces. The part of the boundary that appears from intersection with (5.2) looks like a cycle in $\mathcal{M}_d^{(k)}(n-1,m)$ in which we replace each point y_j by a $(d-1)$-sphere obtained by making x_n orbit around y_j. The part of the boundary that appears from intersection with (5.3) looks like a cycle in $\mathcal{M}_d^{(k)}(\underline{n-1}\setminus I, m+1)$, $I=\{i_1,\dots,i_{k-1}\}$, in which we replace each point y_{m+1} by a $[(k-1)d-1]$-sphere obtained by making $x_{i_1},x_{i_2},\dots,x_{i_{k-1}}$, and x_n orbit around each other according to the equations

$$\frac{x_{i_1}+x_{i_2}+\cdots+x_{i_{k-1}}+x_n}{k} = y_{m+1};$$
$$|x_{i_1}-y_{m+1}|^2 + |x_{i_2}-y_{m+1}|^2 + \cdots + |x_{i_{k-1}}-y_{m+1}|^2 + |x_n-y_{m+1}|^2 = \epsilon^2.$$

The boundary of the resulting chain is the initial cycle α (when $t=0$), a cycle of the form $A\cdot x_n$, where $A \in H_*\mathcal{M}_d^{(k)}(n-1,m)$ (this cycle appears at the other end of the homotopy $t=c$), the cycles of the form $A|_{y_j=[y_j,x_n]}$, where $A \in H_*\mathcal{M}_d^{(k)}(n-1,m)$ (such cycles appear from intersection of α_t with (5.2)), and the cycles of the form $A|_{y_{m+1}=\{x_{i_1},\dots,x_{i_{k-2}},x_{i_{k-1}},x_n\}}$, where $A \in H_*\mathcal{M}_d^{(k)}(\underline{n-1}\setminus I, m+1)$ and $I = \{i_1,\dots,i_{k-1}\}$ (such cycles appear from intersection of α_t with (5.3)). Using induction hypothesis we express α as a linear combination of products of iterated brackets. Q.E.D.

Example 5.2. Consider the cycle $\{x_1,\dots,x_{k-1},x_k\cdot x_{k+1}\} \in H_{(k-1)d-1}\mathcal{M}_d^{(k)}(n+1,0)$.

While pulling away x_{k+1} one can only meet the plane

$$x_{k+1} = x_1 = x_2 = \cdots = x_{k-1},$$

which produces the cycle $x_k \cdot \{x_1, \ldots, x_{k-1}, x_{k+1}\}$. At the second end we get the cycle $\{x_1, x_2, \ldots, x_k\} \cdot x_{k+1}$. This proves relation (3.6).

Example 5.3. Now let us apply the above procedure to the cycle $\{x_1, \ldots, x_{k-1}, [x_k, x_{k+1}]\} \in H_{kd-2}\mathcal{M}_d^{(k)}(n+1, 0)$. While pulling x_{k+1} away, one meets the planes

$$x_{k+1} = x_1 = \cdots = \widehat{x_i} = \cdots = x_k,$$

$i = 1, \ldots, k-1$, which produces the cycles

$$[\{x_1 \cdots \widehat{x_i} \cdots x_{k+1}\}, x_i]\,,$$

and the plane

$$x_{k+1} = x_1 = \cdots = x_{k-1},$$

which produces the cycle

$$[x_k, 1] \cdot \{x_1, \ldots, x_{k-1}, x_{k+1}\} = 0.$$

Also, at the other end of the homotopy, we get the cycle

$$\{x_1, \ldots, x_{k-1}, [x_k, 1]\} \cdot x_{k+1} = 0.$$

As a result we get

$$\{x_1, \ldots, x_{k-1}, [x_k, x_{k+1}]\} = -\sum_{i=1}^{k-1} (-1)^{(k+1-i)d} \left[\{x_1 \cdots \widehat{x_i} \cdots x_{k+1}\}, x_i\right].$$

Applying the generalized Jacobi identity (3.3), we get (3.7).

Remark 5.4. In the initial work [**6**] the (co)homology of the poset $\Pi_{n,k}$ was computed recursively by introducing auxiliary lattices $\Pi_{n,k}(\ell)$. The argument of this section gives a geometric explanation for this combinatorial recursion.

6. Cohomology $H^*\mathcal{B}_d^{(k)}(n)$ as a space of forests

Recall that the cohomology of $\mathcal{B}_d(n) \simeq \mathcal{M}_d^{(2)}(n)$ is described as a certain space of forests modulo 3-term relations; see Section 2. In this section we will give a similar description of $H^*\mathcal{B}_d^{(k)}(n) = H^*\mathcal{M}_d^{(k)}(n)$, $k \geqslant 3$, as spaces of certain *admissible k-forests* modulo narural relations. The k-forests that span $H^*\mathcal{M}_d^{(k)}(n)$ have two types of vertices: square ones that contain cardinality $(k-1)$ subsets of $\underline{n}$, and round ones that contain only one element from $\underline{n}$. Every round vertex must be either disconnected from all the other vertices or connected to a single one that must be square. Every square vertex must be connected to at least one round one. Every element from $\underline{n}$ must appear in exactly one vertex of such a k-forest. By an *orientation* of a k-forest, we mean

(a) orientation of each edge;

(b) ordering elements inside each square vertex;

(c) ordering *orientation set* consisting of all the edges (considered as elements of degree $d-1$) and all the square vertices (considered as elements of degree $k(d-2)$) in the k-forest.

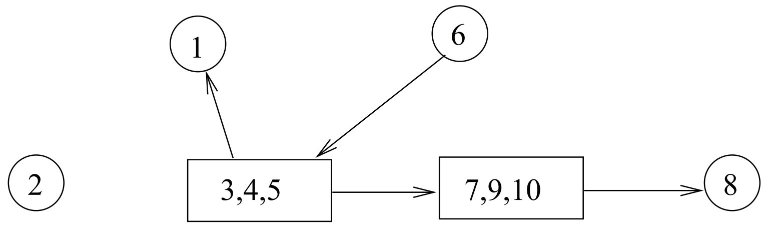


Figure 1: Example of an admissible 4-forest. This forest represents an element in $H^*\mathcal{M}_d^{(4)}(10)$.

For every such oriented forest T, we will assign a locally compact cooriented chain in $\mathcal{M}_d^{(k)}(n)$, whose boundary lies in the complement of $\mathcal{M}_d^{(k)}(n)$. Thus by Poincaré duality every such chain defines a cocycle in $H^*\mathcal{B}_d^{(k)}(n)$ (abusing notation, it will be also denoted by T), whose degree $|T|$ is the sum of degrees of the elements in the orientation set and is the codimension of the corresponding chain.

By $p_1\colon \mathbb{R}^d \to \mathbb{R}^{d-1}$ we will denote the projection $(x^1,\dots,x^d)\mapsto(x^2,\dots,x^d)$. The chain corresponding to a k-forest T is defined as a set determined by the following (in)equalities:

- If i and j from $\underline{n}$ lie in the same square vertex, then $x_i = x_j$.
- If two vertices A and B of T are connected by an edge oriented from A to B, then for all $i\in A$, $j\in B$, one has $x_i^1 \leqslant x_j^1$ and $p_1(x_i)=p_1(x_j)$.

Notice that, in particular, if i and j from $\underline{n}$ lie in the same connected component of T, then $p_1(x_i)=p_1(x_j)$. The data (b), (c) of the orientation of T determine the coorientation of this chain. Notice that each chain is a convex domain of a vector subspace of codimension $|T|$ in $\mathbb{R}^{nd}$. The coorientation will be given by an explicit map $\mathbb{R}^{nd}\to\mathbb{R}^{|T|}$, where $\mathbb{R}^{|T|}$ is the product of $\mathbb{R}^{d-1}$'s (one copy for each edge) and of $\mathbb{R}^{(k-2)d}$'s (one copy for each square vertex) appearing in the same order as the corresponding elements appear in the orientation set of T. Given an edge from a vertex A to B, we take the first elements $i\in A$ and $j\in B$ (recall that each such set is ordered either because it is a singleton or by the orientation data (b)). The projection $p_{AB}\colon\mathbb{R}^{nd}\to\mathbb{R}^{d-1}$ corresponding to this edge sends

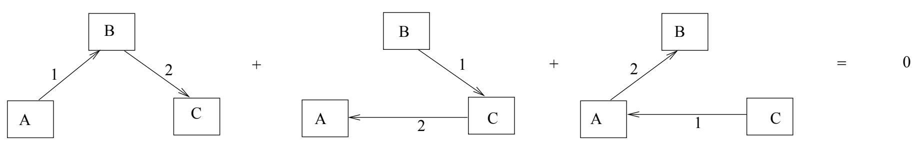
$$(x_1,\dots,x_n)\mapsto p_1(x_j-x_i).$$

Given a square vertex A, whose ordered set of elements is $(i_1,i_2,\dots,i_{k-1})$, the corresponding projection $p_A\colon\mathbb{R}^{nd}\to\mathbb{R}^{(k-2)d}$ sends

$$(x_1,\dots,x_n)\mapsto(x_{i_2}-x_{i_1},x_{i_3}-x_{i_1},\dots,x_{i_k}-x_{i_1}).$$

Theorem 6.1. *The cohomology $H^*\mathcal{B}_d^{(k)}(n)=H^*\mathcal{M}_d^{(k)}(n)$, $d\geqslant 2$, $k\geqslant 3$, $n\geqslant 0$, has no torsion and can be described as a space spanned by oriented k-forests on the index set $\underline{n}$ and quotiented out by the following relations.*

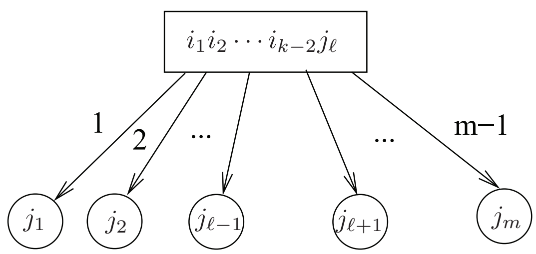
1. *Orientation relations:*
   - *1.1 Changing the order of the orientation set produces the Koszul sign of permutation.*
   - *1.2 A permutation $\sigma \in \Sigma_{k-1}$ of elements inside a square vertex produces the sign $(-1)^{|\sigma|d}$.*
   - *1.3 Changing orientation of an edge produces the sign $(-1)^d$.*
2. *3-term relations:*



   (This picture is local—we assume that the three forests are identical except for the edges going between the square vertices A, B, C. The numbers on the edges tell in which order the edges appear in the orientation set.)

3. *Relations dual to the generalized Jacobi:*

$$\sum_{\ell=1}^{m}(-1)^{\ell(d-1)}$$

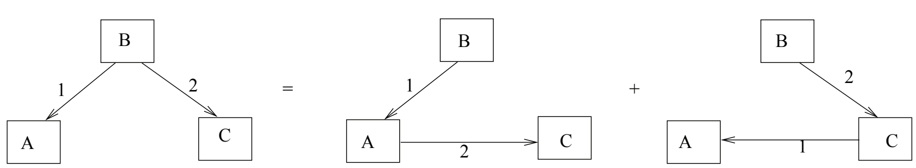


$$= 0 \qquad (6.1)$$

   (Again this picture is local. The square vertex above may be connected to other square vertices, but not to round ones.)

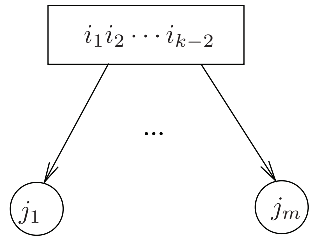
Proof. First, let us check that the cocycles corresponding to k-forests satisfy all the relations above. Relations 1.1 and 1.2 appear as a change of coorientation. To see 1.3 one notices that changing orientation of an edge produces a different chain: instead of inequality $x_i^1 \leqslant x_j^1$ one would have $x_i^1 \geqslant x_j^1$. Up to a sign (-1) these two chains are homologous—that's where we need $d \geqslant 2$. Also, their coorientation differs by $(-1)^{d-1}$. Thus the total sign contribution is $(-1) \cdot (-1)^{d-1} = (-1)^d$.

Relation 2 is equivalent to



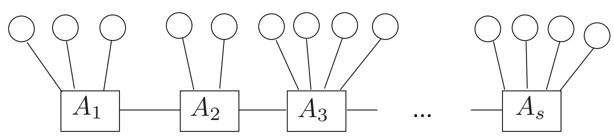
But the chain representing the left-hand side is exactly the union of the chains from the right-hand side.

Relation 3 appears as the boundary of a similar chain that can be described by a similar forest one of whose square vertices has $k-2$ elements:



Remark 6.2. Relation 3 makes sense for $m = 1$. In other words if we allow k-forests with square vertices not attached to any round vertex, then the corresponding cocycles are zero in cohomology. This will be important in the next section, where we will be studying the multiplicative structure of $H^*\mathcal{M}_d^{(k)}(n)$.

To finish the proof of Theorem 6.1, we have to show that our k-forests cocycles span the entire cohomology and that there are no other relations. We will prove this by providing an explicit basis (in the space of such forests) that will be dual to the basis in homology described by Proposition 3.9. The fact that the intersection pairing is given by an identity matrix will finish the proof of Proposition 3.9 as well. Our basis elements will be forests whose components are all either singletons or *linear k-trees*:



For a component T_0 as above, we will require the following: The elements inside each square vertex appear in their natural linear order. The round vertices attached to every square vertex also appear in their linear order. The last round vertex attached to A_i is greater than the last element inside A_i. The minimal element in T_0 appears either inside A_1 or as a round vertex attached to A_1.

We leave it as an exercise to the reader that the intersection matrix between the locally finite cycles corresponding to the aforementioned collection of k-forests and the cycles from Proposition 3.9 is identity. Otherwise, the reader might wait until Section 8, where the duality between the homology and cohomology is described more explicitly. $\square$

7. Multiplicative structure in cohomology

In the previous section we described $H^*\mathcal{B}_d^{(k)}(n) = H^*\mathcal{M}_d^{(k)}(n)$ as a space spanned by certain k-forests. We will now describe the product that is essentially given by a superposition of such forests. The theorem below makes this statement more precise.

Theorem 7.1. *The product of two k-forest cocycles $T_1, T_2 \in H^*\mathcal{M}_d^{(k)}(n)$ is zero in cases (1)–(3) below. Otherwise, it is a sum of k-forests as defined by (4) and (5).*

(1) *If there exists at least one square vertex A in T_1 and one square vertex B in T_2 such that $A \cap B \neq \emptyset$, then $T_1 \cdot T_2 = 0$.*

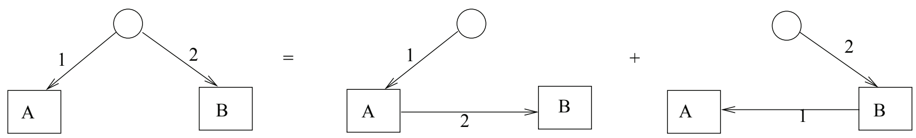
(In (2)–(5) below, we are assuming that all square vertices of T_1 are disjoint from those of T_2. In such a situation one can define a superposition of two forests denoted by $T_1 \cup T_2$.)

(2) *In the case $T_1 \cup T_2$ has cycles then $T_1 \cdot T_2 = 0$.*

(3) *In the case $T_1 \cup T_2$ has a square vertex without any round vertex attached, then $T_1 \cdot T_2 = 0$.*

(4) *If $T_1 \cup T_2$ is an admissible k-forest, then $T_1 \cdot T_2 = T_1 \cup T_2$, whose orientation set is obtained by concatenation of two orientation sets.*

(5) *It might happen that $T_1 \cup T_2$ has one or several round vertices of valence 2. In such a case one has to use the 3-term relations as follows in order to write $T_1 \cdot T_2$ as a sum of admissible k-forests:*



Proof. (1) In the case $A \neq B$, the intersection of the chains corresponding to T_1 and T_2 is empty in $\mathcal{M}_d^{(k)}(n)$. In the case $A = B$, one can move one of the chains slightly to get an empty intersection.

(2) One can choose an orientation of edges in T_1 and T_2 so that the intersection of the corresponding chains is empty.

(3) See Remark 6.2.

(4) The chains corresponding to T_1 and T_2 are transversal to one another and their intersection is exactly the chain corresponding to $T_1 \cup T_2$.

(5) Same as proof of relation 2 in Theorem 6.1. □

The theorem below describes $H^*\mathcal{B}_d^{(k)}(n)$ as a quadratic algebra. For a pair of vertices A and B of a k-forest joined by an edge, we will agree to denote this edge either by (A, B) or by (i, j), where i is any element in A and j any element in B.

Theorem 7.2. *The algebra $H^*\mathcal{M}_d^{(k)}(n)$ is generated by the forests that have only one square vertex. (Therefore, only one of their components is not a singleton.) The relations are as follows:*

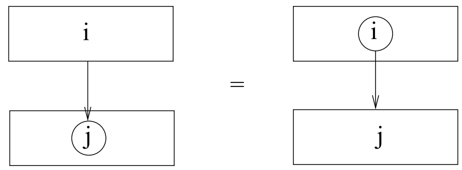
(1) *Linear relations in the space of generators are as described by (1) and (3) from Theorem 6.1.*

(2) $T_1 \cdot T_2 = 0$ *if the square vertex of T_1 is not disjoint from that of T_2 (in particular, $(T_1)^2 = 0$).*

(3) $T_1 \cdot T_2 = 0$ *if $T_1 \cup T_2$ has cycles.*

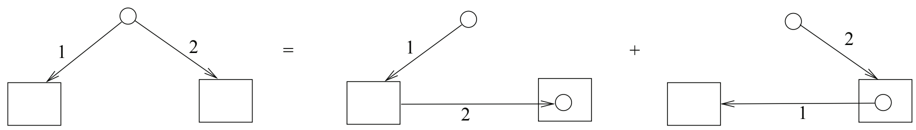
(4) $T_1 \cdot T_2 = 0$ *if $T_1 \cup T_2$ has a square vertex without any round vertex attached. (This can happen if the square vertex of one of the forests has only one round vertex attached and that belongs to the square vertex of the second forest.)*

(5) *Let i belong to a square vertex of T_1, and let j belong to a square vertex of T_2; also assume that the edge $a = (i, j)$ belongs to T_1. Then one has a relation $T_1 \cdot T_2 = (T_1 \setminus a) \cdot (T_2 \cup a)$, where $T_1 \setminus a$ is a forest obtained from T_1 by removing a, and $T_2 \cup a$ is obtained from T_2 by adding a:*



The sign is positive assuming that the edge a is the last element in the orientation set of T_1 and the first element in the orientation set of $T_2 \cup a$.

(6) *If $T_1 \cup T_2$ happens to have a bivalent round vertex, one gets a quadratic relation that one can draw as follows:*



Proof. It is clear that any admissible k-forest can be obtained as a product of generators as above. It is also straightforward that relations from Theorem 6.1 follow from relations above, and vice versa. □

Remark 7.3. For the case $k = 2$, Theorems 6.1, 7.1, and 7.2, and also the duality between the homology and cohomology described in the next section, still hold if one replaces $H^*\mathcal{M}_d^{(2)}(n)$ with a natural associated graded quotient; see Remark 3.12.

8. Duality between homology and cohomology

So far, we described the homology $H_*\mathcal{M}_d^{(k)}(n)$, $d \geqslant 2$, $k \geqslant 3$, as a certain space spanned by products of iterated brackets, where each such product of brackets is a cycle realized by products of spheres in $\mathcal{M}_d^{(k)}(n)$. We also described the cohomology $H^*\mathcal{M}_d^{(k)}(n)$, $d \geqslant 2$, $k \geqslant 3$, as a space spanned by admissible k-forests, where each forest is a cocycle realized via intersection number with a certain locally finite chain. In this section we will describe how the aforementioned cycles pair with the cocycles, or, in other words, how the cycles (realized by products of spheres) intersect with the locally finite chains described in Section 6. A similar duality for $\mathcal{M}_d^{(2)}(n)$ is well known [**41, 32, 35**]. Notice that in top degree $H_*\mathcal{B}_d(\bullet)$ is the operad of graded Lie algebras with bracket of degree $(d-1)$. Thus $H^*\mathcal{B}_d(\bullet)$ in top degree is the Lie cooperad whose components are explicitly described as spaces of trees quotiented out by 3-term relations; see Section 2. Such a description of the Lie cooperad is important in its application to rational homotopy theory [**36, 37**]. Also, it was used to prove the formality of the operad of little discs [**21, 22**].

Let $\mathcal{F}_d^{(k)}(n)$ denote the space of admissible k-forests from Theorem 6.1 modulo only orientation relations (1). Then $H^*\mathcal{M}_d^{(k)}(n)$ is $\mathcal{F}_d^{(k)}(n)$ quotiented out by the subspace $\mathcal{R}_d^{(k)}(n) \subset \mathcal{F}_d^{(k)}(n)$ spanned by relations (2) and (3):

$$H^*\mathcal{M}_d^{(k)}(n) = \mathcal{F}_d^{(k)}(n) \Big/ \mathcal{R}_d^{(k)}(n).$$

The space $\mathcal{F}_d^{(k)}(n)$ is naturally self-dual by defining its basis set (of admissible k-forests) to be orthonormal. The homology $H_*\mathcal{M}_d^{(k)}(n)$ is dual to $H^*\mathcal{M}_d^{(k)}(n)$ and can be described as the subspace $\left(\mathcal{R}_d^{(k)}(n)\right)^{\perp} \subset \mathcal{F}_d^{(k)}(n)$. We will describe explicitly this isomorphism

$$\Psi_n\colon H_*\mathcal{M}_d^{(k)}(n) \to \left(\mathcal{R}_d^{(k)}(n)\right)^{\perp},$$

which in fact encodes the pairing as

$$\Psi_n(B) = \sum_T \langle T, B\rangle \cdot T.$$

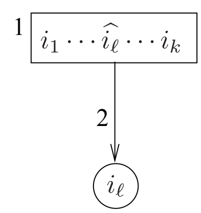
Here, the sum is taken over the basis set of $\mathcal{F}_d^{(k)}(n)$.

For simplicity of notation we will be omitting the subscript n. This map Ψ can be described recursively. First, we define $\Psi(1)$ as the empty graph and

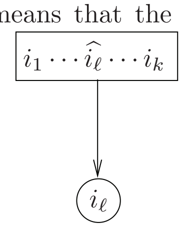
$$\Psi(x_i) = \text{ⓘ},$$

where the right-hand side is the forest with only one vertex. We also define

$$\Psi(\{x_{i_1}\cdots x_{i_k}\}) = \sum_{\ell=1}^{k}(-1)^{(\ell-1)d}$$



The numbers 1 and 2 above describe the order in which the corresponding elements appear in the orientation set. This identity means that the spherical cycle $\{x_{i_1}\cdots x_{i_k}\} \in \mathcal{M}_d^{(k)}(\{i_1\cdots i_k\})$ intersects each chain



exactly once, and $(-1)^{(\ell-1)d}$ is the sign of intersection.[2] Then if B happens to be a product $B = B_1 \cdot B_2$, we get

$$\Psi(B_1 \cdot B_2) := \Psi(B_1) \sqcup \Psi(B_2).$$

If $B = [B_1, B_2]$ and neither B_1 nor B_2 is a singleton, we get

$$\Psi([B_1, B_2]) = \sum_{\substack{A_1 \in B_1^{\square} \\ A_2 \in B_2^{\square}}} \Psi(B_1) \cup (A_1, A_2) \cup \Psi(B_2), \tag{8.1}$$

where $B_1^{\square}$ (respectively, $B_2^{\square}$) is the set of square vertices of each summand of $\Psi(B_1)$ (respectively, of $\Psi(B_2)$)[3]; (A_1, A_2) is the edge going from A_1 to A_2. The orientation

[2] At this point we need to fix orientation of the sphere $\{x_1\cdots x_k\} \in H_{(k-1)d-1}\mathcal{M}_d^{(k)}(n)$ in order to make this pairing work.

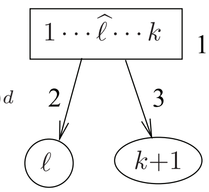
[3] Notice that the set of square vertices for each summand of $\Psi(B_1)$ (respectively, $\Psi(B_2)$) is in one-to-one correspondence with the long brackets in B_1 (respectively, B_2).

set for each summand is obtained by writing first the orientation set of a summand of $\Psi(B_1)$, then (A_1, A_2), then the orientation set for a summand of $\Psi(B_2)$.

One similarly has

$$\Psi([B, x_i]) = \sum_{A \in B^{\square}} \Psi(B) \cup (A, i) \cup \Psi(x_i). \tag{8.2}$$

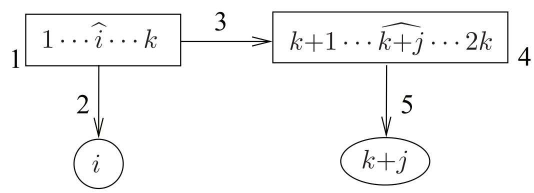
Example 8.1. (a) $\Psi\left([\{x_1 \cdots x_k\}, x_{k+1}]\right) = \sum_{\ell=1}^{k} (-1)^{(\ell-1)d}$



.

(b) $\psi\left([\{x_1 \cdots x_k\}, \{x_{k+1} \cdots x_{2k}\}]\right) =$

$$\sum_{i,j=1}^{k} (-1)^{(i+j)d}$$



.

Remark 8.2. One can consider a slightly larger class of admissible k-forests by allowing round vertices to be connected to any number of square vertices. The advantage of such a definition is that the multiplicative structure will be given simply by the superposition of forests. The downside is that the space of cohomology would be less clearly described. In any case, if one decides to do so, one will also need to take into account in the formula for pairing the intersections with the new locally finite chains. In the latter case the formula for (8.1) and (8.2) will be the same—the sum will run over all vertices A_1 in $\Psi(B_1)$ and A_2 in $\Psi(B_2)$, with the only restriction that at least one of the two is square.

9. Coproduct and cobimodule structures

9.1. Coproduct

Since $\mathcal{B}_d$ is a topological operad, its homology is an operad in coalgebras. This structure is sometimes called a *Hopf operad*. Let $B \in H_*\mathcal{B}_d(n)$, $d \geqslant 2$, be any product of iterated brackets. This cycle is realized by a product of spheres

$$(S^{d-1})^k \to \mathcal{B}_d(n).$$

Thus $\Delta B \in H_*\mathcal{B}_d(n) \otimes H_*\mathcal{B}_d(n)$ can be computed from the coproduct of the fundamental class of $(S^{d-1})^k$. For example:

$$\begin{aligned}\Delta\left([x_1, x_3] \cdot [x_2, x_4]\right) = [x_1, x_3] \cdot [x_2, x_4] \otimes x_1 \cdot x_2 \cdot x_3 \cdot x_4 + \\ [x_1, x_3] \cdot x_2 \cdot x_4 \otimes x_1 \cdot x_3 \cdot [x_2, x_4] + (-1)^{d-1} x_1 \cdot x_3 \cdot [x_2, x_4] \otimes [x_1, x_3] \cdot x_2 \cdot x_4 + \\ x_1 \cdot x_2 \cdot x_3 \cdot x_4 \otimes [x_1, x_3] \cdot [x_2, x_4],\end{aligned}$$

$$\Delta([[x_1,x_3],x_2]) = [[x_1,x_3],x_2]\otimes x_1\cdot x_2\cdot x_3 + [x_1,x_3]\cdot x_2\otimes[x_1\cdot x_3,x_2]+ (-1)^{d-1}[x_1\cdot x_3,x_2]\otimes[x_1,x_3]\cdot x_2 + x_1\cdot x_2\cdot x_3\otimes[[x_1,x_3],x_2].$$

Similarly, $H_*\mathcal{B}_d^{(k)}$ is a *Hopf bimodule*. The coproduct of any product of iterated brackets (which is also realized as a map from products of spheres) is computed in the same manner. For example,

$$\begin{aligned}\Delta[\{x_1\cdots x_k\},\{x_{k+1}\cdots x_{2k}\}] = {} & [\{x_1\cdots x_k\},\{x_{k+1}\cdots x_{2k}\}]\otimes x_1\cdots x_{2k}+\\ & [\{x_1\cdots x_k\},x_{k+1}\cdots x_{2k}]\otimes x_1\cdots x_k\cdot\{x_{k+1}\cdots x_{2k}\}+\\ & (-1)^{kd}[x_1\cdots x_k,\{x_{k+1}\cdots x_{2k}\}]\otimes\{x_1\cdots x_k\}\cdot x_{k+1}\cdots x_{2k}+\\ & \{x_1\cdots x_k\}\cdot x_{k+1}\cdots x_{2k}\otimes[x_1\cdots x_k,\{x_{k+1}\cdots x_{2k}\}]+\\ & (-1)^{kd}x_1\cdots x_k\cdot\{x_{k+1}\cdots x_{2k}\}\otimes[\{x_1\cdots x_k\},x_{k+1}\cdots x_{2k}]+\\ & x_1\cdots x_{2k}\otimes[\{x_1\cdots x_k\},\{x_{k+1}\cdots x_{2k}\}].\end{aligned}$$

(The two summands producing zero were omitted.)

Notice that the space of primitives is spanned by the elements that have exactly one long bracket. This space is dual to the space of generators; see Theorem 7.2.

9.2. Cobimodule structure

The cooperad structure of $H^*\mathcal{B}_d$ is given by the maps

$$H^*\mathcal{B}_d(m_1+\cdots+m_n)\to H^*\mathcal{B}_d(n)\otimes H^*\mathcal{B}_d(m_1)\otimes\cdots\otimes H^*\mathcal{B}_d(m_n) \tag{9.1}$$

induced by the composition maps in $\mathcal{B}_d$. Explicitly, given a forest $T\in H^*\mathcal{B}_d(m_1+\cdots+m_n)$, $d\geqslant 2$, the map (9.1) sends it to

$$T\mapsto \pm(T/\sim)\otimes T_1\otimes\cdots\otimes T_n, \tag{9.2}$$

where T_s is the restriction of T on the set

$$M_s = \left\{\sum_{i=1}^{s-1}m_i+1,\sum_{i=1}^{s-1}m_i+2,\ldots,\sum_{i=1}^{s-1}m_i+m_s\right\}$$

and $T/\sim$ is the quotient of T by the subgraphs T_s, $s=1,\ldots,\ell$. In particular if $T/\sim$ has cycles, the result is zero. The sign in (9.2) is the Koszul sign due to reordering of the edges of T. This cooperad structure was used for example in [**22, 36, 37**].

The coaction maps

$$H^*\mathcal{B}_d^{(k)}(m_1+\cdots+m_n)\to H^*\mathcal{B}_d(n)\otimes H^*\mathcal{B}_d^{(k)}(m_1)\otimes\cdots\otimes H^*\mathcal{B}_d^{(k)}(m_n), \tag{9.3}$$

$$H^*\mathcal{B}_d^{(k)}(m_1+\cdots+m_n)\to H^*\mathcal{B}_d^{(k)}(n)\otimes H^*\mathcal{B}_d(m_1)\otimes\cdots\otimes H^*\mathcal{B}_d(m_n) \tag{9.4}$$

are described by the same formula, (9.2). In the case of left coaction (9.3), to get non-zero each square vertex of T must be entirely inside one of the M_s's. For the right coaction (9.4), one obtains non-zero only if at most one element of each square vertex A of T is contained in each of the M_s:

$$|A\cap M_s|\leqslant 1,\qquad s=1,\ldots,n.$$

Remark 9.1. In the case $d=1$ the coaction has a different description. In fact, Baryshnikov's description of $H^*\mathcal{B}_1^{(k)}(\bullet)$—see Section 4—is also nicely compatible with the cobimodule structure over the associative cooperad $H^*\mathcal{B}_1$.

10. Symmetric group action and generating function of dimensions

The symmetric group action on the (co)homology of the poset $\Pi_{n,k}$ and on $H_*\mathcal{M}_2^{(k)}(n)$ was computed in [**38**]. The results can be generalized without any difficulty to any ambient dimension d; see Theorem 10.3, below. Our operadic approach of studying this homology makes the results of [**38**] more transparent. Also, the symmetric group action helps to produce an explicit generating function of the Betti numbers (see Corollary 10.5), which seems to be overlooked in the literature and is given here for completeness of exposition.

The symmetric sequences of graded vector spaces form a monoidal category with respect to the composition operation $\circ$ and unit $\mathtt{1}$ [**24**]. If we are working over a field any symmetric sequence $M(n)$, $n \geqslant 0$, defines a functor $M\colon Vect \to Vect$ that sends a vector space $V \mapsto \oplus_{n=0}^{\infty} M(n) \otimes_{\Sigma_n} V^{\otimes n}$. The composition is defined in such a way that $(M \circ N)(V) = M(N(V))$. In fact, one does not need the base ring to be a field in order to define this composition. The unit $\mathtt{1}$ for this operation is the sequence that is zero in all arities except one, and is the base ring in arity one. Notice that $\mathtt{1}\colon Vect \to Vect$ is the identity functor. The construction works nicely over integers: in the case where M and N are torsion-free and $N(0) = 0$, the composition $M \circ N$ is also torsion-free. For a graded vector space $V = \oplus_{n\in\mathbb{Z}} V_n$, we will define its *graded dimension* as a formal power series in q:

$$\dim V = \sum_n \dim V_n \cdot q^n.$$

For a symmetric sequence M of graded vector spaces, we define the *exponential generating function* of its components

$$F_M(x) = \sum_{j=0}^{+\infty} \dim M(j) \frac{x^j}{j!}.$$

One has

$$F_{M\circ N}(x) = F_M(F_N(x)). \tag{10.1}$$

For a symmetric sequence M, denote by $M\{d\}$ its operadic d-suspension. As a vector space $M\{d\}(n)$ is $d(n-1)$-times suspended space $M(n)$. As a Σ_n-module $M\{d\}(n) \simeq M(n) \otimes (sign_n)^{\otimes d}$, where $sign_n$ is the sign representation of Σ_n. It is straightforward that

$$F_{M\{d\}}(x) = \frac{1}{q^d} F_M(q^d x). \tag{10.2}$$

Notice also that

$$(M \circ N)\{d\} = (M\{d\}) \circ (N\{d\}). \tag{10.3}$$

Recall that $\mathcal{C}om$ denotes the operad of commutative unital algebras and that $\mathcal{L}ie$ denotes the operad of Lie algebras—both viewed as symmetric sequences over $\mathbb{Z}$. One has

$$F_{\mathcal{C}om}(x) = e^x, \tag{10.4}$$

$$F_{\mathcal{L}ie}(x) = -\ln(1-x). \tag{10.5}$$

Let $\mathcal{H}_d^{(k)}(n) \subset H_*\mathcal{B}_d^{(k)}(n)$ be the subspace spanned by elements of the form $[\cdots[\{x_{\sigma_1}\cdots x_{\sigma_k}\}, x_{\sigma_{k+1}}]\cdots x_{\sigma_n}]$ (in other words, spanned by the iterated brackets that have only one long bracket). The operadic $(d-1)$-desuspension $\mathcal{H}_d^{(k)}\{1-d\}$ of this symmetric sequence does not depend on d and will be denoted by $\mathcal{H}_1^{(k)}$. It follows from Proposition 3.9 that $\mathcal{H}_1^{(k)}(n)$ is concentrated in grading $(k-2)$ and has dimension $\binom{n-1}{k-1}$. One has

$$F_{\mathcal{H}_1^{(k)}}(x) = \frac{q^{k-2}x^k}{(k-1)!}\sum_{j=0}^{+\infty}\frac{x^j}{(j+k)\cdot j!} = (-q)^{k-2} - (-q)^{k-2}\left(\sum_{j=0}^{k-1}\frac{(-x)^j}{j!}\right)e^x. \quad (10.6)$$

The last equality was obtained by noticing that

$$F'_{\mathcal{H}_1^{(k)}}(x) = \frac{q^{k-2}}{(k-1)!}x^{k-1}e^x \quad (10.7)$$

and then integrating.

Lemma 10.1. *For any $n \geqslant k \geqslant 2$, one has an isomorphism of $\mathbb{Z}[\Sigma_n]$-modules*

$$\mathcal{H}_1^{(k)}(n) \simeq \mathbb{Z}[\Sigma_n]\cdot a\cdot b,$$

where $a = \sum_{\sigma\in\Sigma_k}(-1)^{|\sigma|}\sigma$ and $b = \sum_{\sigma\in\Sigma_{\{1,k+1,k+2,\dots,n\}}}\sigma$.

In particular this lemma says that $\mathcal{H}_1^{(k)}(n)\otimes\mathbb{Q}$ is the irreducible representation of *hook type* $(n-k+1,k)$; see [**17**].

Proof. We define a map $\mathcal{H}_1^{(k)}(n)\to\mathbb{Z}[\Sigma_n]\cdot a\cdot b$ by sending $[\cdots[\{x_1\cdots x_k\}, x_{k+1}],\cdots x_n]\mapsto e\cdot a\cdot b$, where $e\in\Sigma_n$ is the unit element. One has to check that this map is correctly defined. First, we notice that any element $\sigma\in\Sigma_k$ acts both on $[\cdots[\{x_1\cdots x_k\}, x_{k+1}]\cdots x_n]$ and on $e\cdot a\cdot b$ as multiplication by $(-1)^\sigma$. Also, any $\sigma\in\Sigma_{\{k+1,k+1,\dots,n\}}$ acts as the identity on both of them. And finally, an easy verification shows that relation (3.3) is also satisfied. On the other hand, the map is obviously surjective. The fact that the target has the same dimension $\binom{n-1}{k-1}$ as the source ensures that the map is an isomorphism. ☐

Remark 10.2. Let $\mathcal{H}_1^{(k)}(n)^\vee$ denote the dual Σ_n-module that we described as a space of k-trees with a single square vertex and quotiented out by relations (6.1). Looking at the generalized Jacobi (3.3) and the relations (6.1), it is easy to see that one has an obvious isomorphism of Σ_n-modules $\mathcal{H}_1^{(k)}(n)^\vee \simeq \mathcal{H}_1^{(n-k+1)}(n)\otimes sign_n$. This implies that one has a $\mathbb{Z}[\Sigma_n]$-module isomorphism $(\mathcal{H}_1^{(k)}(n))^\vee\simeq\mathbb{Z}[\Sigma_n]\cdot b\cdot a$, where a and b are from Lemma 10.1.[4]

Theorem 10.3 ([**38**]). *For $d\geqslant 2$, $k\geqslant 3$, one has a natural isomorphism of symmetric sequences*

$$H_*\mathcal{B}_d^{(k)} \simeq \mathcal{C}om\circ\left(\mathbb{1}\oplus(\mathcal{L}ie\circ\mathcal{H}_1^{(k)})\{d-1\}\right). \quad (10.8)$$

For $d=1$ and/or $k=2$, this isomorphism holds over $\mathbb{Q}$.

[4]Of course, rationally, a Σ_n-module is always isomorphic to its dual: $\mathbb{Q}[\Sigma_n]\cdot a\cdot b\simeq\mathbb{Q}[\Sigma_n]\cdot b\cdot a$.

Proof. In the case $d \geqslant 2$, $k \geqslant 3$, one has that $H_*\mathcal{B}_d^{(k)}$ is a left module over $H_*\mathcal{B}_d = \mathcal{C}om \circ (\mathcal{L}ie\{d-1\})$ and $\mathcal{H}_d^{(k)}(\bullet) \simeq \mathcal{H}_1^{(k)}\{d-1\}(\bullet)$ is a sequence of subobjects in $H_*\mathcal{B}_d^{(k)}(\bullet)$. This left action defines a map

$$\mathcal{C}om \circ \left(\mathtt{1} \oplus (\mathcal{L}ie\{d-1\}) \circ \mathcal{H}_d^{(k)}\right) \to H_*\mathcal{B}_d^{(k)},$$

where $\mathtt{1}$ corresponds to $H_0\mathcal{B}_d^{(k)}(1) \simeq \mathbb{Z}$. Proposition 3.9 ensures that this map is an isomorphism.

In the case $k = 2$, the right-hand side of (10.8) is isomorphic to the associated graded quotient of $H_*\mathcal{B}_d^{(2)}$ by a similar argument and by Remark 3.12; see also Remark 7.3. Since over $\mathbb{Q}$ any filtration of Σ_n-modules splits, we get the result.

Similarly, for $d = 1$, the operad $H_*\mathcal{B}_1 = \mathcal{A}ssoc$ admits a natural increasing (Poincaré–Birkhoff–Witt) filtration, whose associated graded quotient is the Poisson operad. The aforementioned filtration is compatible with a filtration in the left module $H_*\mathcal{B}_1^{(k)}$. The associated graded quotient of the latter symmetric sequence is the right-hand side of (10.8).

In the case $k = 2$ and $d = 1$ one has to take the associated graded quotient twice. □

Remark 10.4. In particular, we get an isomorphism of symmetric sequences

$$\mathtt{1} \oplus \mathcal{L}ie \circ \mathcal{H}_1^{(2)} \simeq_{\mathbb{Q}} \mathcal{L}ie, \tag{10.9}$$

which at first might appear surprising, but it simply means that for any (graded) vector space V the Lie subalgebra $\mathcal{L}ie_{\geqslant 2}(V)$ (spanned by Lie monomials of degree $\geqslant 2$) of the free Lie algebra $\mathcal{L}ie(V)$ (generated by V) is isomorphic to the free Lie algebra generated by $\mathcal{H}_1^{(2)}(V) = \oplus_{n \geqslant 2} \mathcal{H}_1^{(2)}(n) \otimes_{\Sigma_n} V^{\otimes n}$. This is a particular occurence of a general fact that a Lie subalgebra of a free Lie algebra is always free [**29**]. The isomorphism (10.9) is actually also due to Reutenauer [**30**].

Corollary 10.5. *The exponential generating function of graded dimensions for the symmetric sequence $H_*\mathcal{B}_d^{(k)}(\bullet)$ is as follows:*

$$F_{H_*\mathcal{B}_d^{(k)}}(x) = e^x \left(1 - (-q)^{k-2} + (-q)^{k-2}\left(\sum_{j=0}^{k-1} \frac{(-q^{d-1}x)^j}{j!}\right) e^{q^{d-1}x}\right)^{-\frac{1}{q^{d-1}}}. \tag{10.10}$$

Remark 10.6. For explicit computations of the Betti numbers, it is more convenient to use the formula

$$F_{H_*\mathcal{B}_d^{(k)}}(x) = e^x \left(1 - \frac{q^{kd-2}x^k}{(k-1)!} \sum_{j=0}^{+\infty} \frac{(q^{d-1}x)^j}{(j+k)\cdot j!}\right)^{-\frac{1}{q^{d-1}}}. \tag{10.11}$$

Proof of Corollary 10.5. It is a consequence of Theorem 10.3 together with (10.1), (10.2), (10.4), (10.5), and (10.6). □

Remark 10.7. The Betti numbers for $\mathcal{M}_d^{(k)}(n)$ were computed in [**6**]; see also [**28**]. The formulae (10.10), (10.11) provide a more compact way to keep track of these data.

11. Application to spaces of non-k-equal immersions

This section is quite separate from the rest of the paper. Its goal is to show that the considered bimodules appear very naturally in topology, and what we explain here is just one of its applications. We use our calculations to study the structure—but not the convergence properties—of the Goodwillie–Weiss and Vassiliev approaches. Indeed, these spaces of k-equal immersions are conjectured to play a role in connecting these two different successful approaches to embedding spaces. Theorems 11.1, and 11.2–11.3, below, were proved for embedding spaces in [**33, 39**] and [**2**], respectively. We just want to point out that the proofs are completely analogous for spaces of non-k-equal immersions.

Let M be an open subset of $\mathbb{R}^m$, and let $n > m$. Consider the space $\mathrm{Imm}^{(k)}(M, \mathbb{R}^n)$ of smooth immersions $f\colon M \looparrowright \mathbb{R}^n$ such that for any cardinality k subset $K \subset M$, one has that $f|_K$ is non-constant. We call such maps *non-k-equal immersions.* For example the space $\mathrm{Imm}^{(2)}(M, \mathbb{R}^n)$ is the space of embeddings $\mathrm{Emb}(M, \mathbb{R}^n)$.

Let $\mathrm{Imm}(M, \mathbb{R}^n)$ denote the space of smooth immersions, and let $\overline{\mathrm{Imm}}^{(k)}(M, \mathbb{R}^n)$ be the homotopy fiber of the natural inclusion $\mathrm{Imm}^{(k)}(M, \mathbb{R}^n) \hookrightarrow \mathrm{Imm}(M, \mathbb{R}^n)$ over the composition $M \subset \mathbb{R}^m \subset \mathbb{R}^n$.

We will also consider spaces $\mathrm{Imm}_c^{(k)}(\mathbb{R}^m, \mathbb{R}^n)$ of *long* non-k-equal smooth immersions, where the subscript c stays for *compact support.* Points of this space are non-k-equal immersions $\mathbb{R}^m \looparrowright \mathbb{R}^n$ coinciding with the fixed linear inclusion $\mathbb{R}^m \subset \mathbb{R}^n$ outside a compact subset of $\mathbb{R}^m$. One gets a similar fiber sequence

$$\overline{\mathrm{Imm}}_c^{(k)}(\mathbb{R}^m, \mathbb{R}^n) \to \mathrm{Imm}_c^{(k)}(\mathbb{R}^m, \mathbb{R}^n) \to \mathrm{Imm}_c(\mathbb{R}^m, \mathbb{R}^n). \tag{11.1}$$

The Smale–Hirsch principle [**20**] provides us with natural equivalences

$$\mathrm{Imm}_c(\mathbb{R}^m, \mathbb{R}^n) \simeq \Omega^m V_{m,n}, \tag{11.2}$$

$$\mathrm{Imm}(M, \mathbb{R}^n) \simeq \mathrm{Maps}(M, V_{m,n}), \tag{11.3}$$

where $V_{m,n}$ is the Stiefel manifold of isometric linear maps $\mathbb{R}^m \hookrightarrow \mathbb{R}^n$.

The reason we study $\overline{\mathrm{Imm}}^{(k)}(M, \mathbb{R}^n)$ and $\overline{\mathrm{Imm}}_c^{(k)}(\mathbb{R}^m, \mathbb{R}^n)$ is that their homotopy type and homology have nice properties in comparison with the initial spaces of non-k-equal immersions. But, at the same time, they differ from $\mathrm{Imm}^{(k)}(M, \mathbb{R}^n)$ and $\mathrm{Imm}_c^{(k)}(\mathbb{R}^m, \mathbb{R}^n)$ by an easily controllable term (11.2), (11.3).

There are two main approaches to study such functional spaces. The first approach, due to Vassiliev and usually called the *theory of discriminants* [**45**], considers the space of all smooth maps from our manifold to $\mathbb{R}^n$. This space is an affine space of infinite dimension and thus contractible. The cohomology classes of the space of maps that avoid any given types of singularities are described via linking number with cycles (of finite codimension) in the complement space called *discriminant* that consists of singular maps. The discriminant is a semi-algebraic set whose stratification provides the necessary combinatorial information to compute the homology of the complement. The second approach, called *manifold calculus*, was developed by Goodwillie and Weiss [**19, 47**]. This second approach was mostly used to study spaces of embeddings, but it can also be used to study more general functional spaces. For this approach instead of looking on maps from M to N (avoiding given multi-singularities), one varies the source to be any open subset $U \subset M$. This produces a presheaf on M in topological spaces. In some cases, the obtained presheaf is a homotopy sheaf (for example, it is the case for spaces of immersions), but in general it is not true. Homotopy sheaves are *linear* functors from the point of view of *manifold calculus*. But there are also quadratic, cubical, and, more generally, polynomial of any degree k presheaves, which also means that they have some nice "from local to global" properties. The manifold calculus assigns to any topological presheaf on M a *Taylor tower* of its polynomial approximations: $T_0F \leftarrow T_1F \leftarrow T_2F \leftarrow T_3F \leftarrow \cdots$. In good cases the limit of the tower $T_\infty F$ is equivalent to F.

We believe that Vassiliev's theory of discriminants can also be expressed in terms of the manifold calculus by describing the discriminant set as a spectrum Spanier–Whitehead dual to the given space of non-singular maps. (Here, one has to consider the copresheaf that assigns to U the corresponding spectrum. Notice that one will need to use the covariant version of the calculus instead of the contravariant one usually used.) This construction would prove an equivalence of two approaches. There is work in this direction [**31**] that establishes the equivalence between the two approaches in many cases—in particular, it covers the case of spaces $\mathrm{Imm}^{(k)}(M,\mathbb{R}^n)$, where M is any compact smooth m-manifold (in particular, it can be closed) $k \geqslant 3$, $n \geqslant 2m+1$, implying $T_\infty C_* \,\mathrm{Imm}^{(k)}(M,\mathbb{R}^n) \simeq C_* \,\mathrm{Imm}^{(k)}(M,\mathbb{R}^n)$. Unfortunately, the results of [**31**] cannot be applied to the spaces we consider. Some technicalities appear, but we believe it is feasible that they can be resolved.

Both methods produce spectral sequences computing the homology, and the first term of the Vassiliev spectral sequence is isomorphic to the second term of the manifold calculus homology spectral sequence. Manifold calculus also produces the homotopy spectral sequence that in the case of embeddings converges to the homotopy of the underlying space in the case of codimension $n-m \geqslant 3$, while the homology spectral sequence has this property only in the range $n \geqslant 2m+2$. For $\overline{\mathrm{Imm}}^{(k)}(M,\mathbb{R}^n)$ and $\overline{\mathrm{Imm}}_c^{(k)}(\mathbb{R}^m,\mathbb{R}^n)$, Vassiliev's spectral sequence can be easily shown, by the techniques developed in [**45**], to converge to the homology of those spaces for $n \geqslant 2m+2$, and for $\mathrm{Imm}^{(k)}(M,\mathbb{R}^n)$ and $\mathrm{Imm}_c^{(k)}(\mathbb{R}^m,\mathbb{R}^n)$, $k \geqslant 3$, for $n \geqslant 2m+1$. The convergence of the homotopy spectral sequences for those spaces arising from the Goodwillie–Weiss calculus approach has not been studied yet in the case $k \geqslant 3$ and seems to be a difficult question. The results of [**31**] prove the convergence of the homology Goodwillie–Weiss spectral sequence for $\mathrm{Imm}^{(k)}(M,\mathbb{R}^n)$, $k \geqslant 3$, $n \geqslant 2m+1$ (which is equivalent to saying $T_\infty C_* \,\mathrm{Imm}^{(k)}(M,\mathbb{R}^n) \simeq C_* \,\mathrm{Imm}^{(k)}(M,\mathbb{R}^n)$), but not for $\overline{\mathrm{Imm}}^{(k)}(M,\mathbb{R}^n)$, $\mathrm{Imm}_c^{(k)}(\mathbb{R}^m,\mathbb{R}^n)$, $\overline{\mathrm{Imm}}_c^{(k)}(\mathbb{R}^m,\mathbb{R}^n)$.

On the other hand, the manifold calculus can be translated into operadic language [**2, 7, 43**]. We explain below how this interpretation is applied to the spaces $\overline{\mathrm{Imm}}_c^{(k)}(\mathbb{R}^m,\mathbb{R}^n)$, $\overline{\mathrm{Imm}}^{(k)}(M,\mathbb{R}^n)$.

As we have seen in Section 3, $H_*\mathcal{B}_n^{(k)}$ is a bimodule under $H_*\mathcal{B}_n$. Inclusion $\mathbb{R}^1 \subset \mathbb{R}^n$ induces inclusion of operads $\mathcal{B}_1 \hookrightarrow \mathcal{B}_n$, which produces a map of operads in homology: $\mathcal{A}ssoc \to H_*\mathcal{B}_n$. Due to this morphism, $H_*\mathcal{B}_n^{(k)}$ is also a bimodule under $\mathcal{A}ssoc$, which endows $H_*\mathcal{B}_n^{(k)}$ with a cosimplicial structure.[5]

[5]One uses compositions with the product $x_1x_2 \in \mathcal{A}ssoc(2)$ to get coface maps, and compositions with the unit $1 \in \mathcal{A}ssoc(0)$ to get codegeneracies.

Theorem 11.1. *For $n \geqslant 3$, the first term of the Vassiliev spectral sequence and the second term of the manifold calculus homology spectral sequence computing $H_*\overline{\mathrm{Imm}}_c^{(k)}(\mathbb{R}^1, \mathbb{R}^n)$ is isomorphic to the homology of the total cosimplicial complex $\mathrm{Tot}\, H_*\mathcal{B}_n^{(k)}(\bullet)$. In the case $n \geqslant 4$, the Vassiliev spectral sequence converges to the homology of this space.*

For the Vassiliev method, one has to consider the space $\overline{\mathrm{Imm}}_c^{(k)}(\mathbb{R}^1, \mathbb{R}^n)$ as an open subset in the space of all smooth maps $[0,1] \times \mathbb{R} \to \mathbb{R}^n$ with the restriction $f(t,x) = (x, 0, 0, \dots, 0)$ outside a compact subset of $[0,1) \times \mathbb{R}$, as in [**40**]. We reiterate that the convergence of the manifold calculus spectral sequence to the homology of the underlying space still needs to be studied. One way to produce the manifold calculus spectral sequence is to use the idea and techniques described by D. Sinha [**33, 34**] and construct a cosimplicial space whose totalization is $T_\infty\overline{\mathrm{Imm}}_c^{(k)}(\mathbb{R}^1, \mathbb{R}^n)$ (D. Sinha does it for the space of long embeddings). Thus the spectral sequence in question appears as the Bousfield–Kan spectral sequence associated to the corresponding cosimplicial space. The only difference is that one does not have a natural compactifiation of configuration spaces $\mathcal{M}_n^{(k)}(\bullet)$ that would turn them into a cosimplicial object. So instead one can first notice that $\mathcal{B}_n^{(k)}$ is a bimodule under $\mathcal{B}_1$ by restriction and due to the inclusion $\mathcal{B}_1 \hookrightarrow \mathcal{B}_n$. Therefore, $\mathcal{B}_n^{(k)}$ is an *infinitesimal bimodule over* $\mathcal{B}_1$; see below. Using an obvious projection $\mathcal{B}_1 \to \mathcal{A}ssoc$ (which is an equivalence of operads) we get a restriction functor from infinitesimal bimodules over $\mathcal{A}ssoc$ to infinitesimal bimodules over $\mathcal{B}_1$. Its left adjoint induction functor applied to a cofibrant replacement of $\mathcal{B}_n^{(k)}$ (as $\mathcal{B}_1$ infinitesimal bimodule) produces an infinitesimal $\mathcal{A}ssoc$ bimodule. That's exactly what we need, as the structure of an infinitesimal bimodule over $\mathcal{A}ssoc$ is essentially the same thing as the structure of a cosimplicial object [**42**, Lemma 4.2].

For the manifold calculus approach, Theorem 11.1 is a particular instance of Theorem 11.2, below. In order to formulate a higher-dimensional analogue, we need to recall some terminology from the theory of operads.

An *infinitesimal bimodule* over an operad $\mathcal{O}$ is a sequence of objects $M = \{M(n), n \geqslant 0\}$ (symmetric sequence in case $\mathcal{O}$ is a Σ-operad, or just a seqence in case $\mathcal{O}$ is non-Σ), endowed with composition maps:

$$\circ_i\colon \mathcal{O}(n) \otimes M(k) \to M(n+k-1) \text{ (infinitesimal left action)},$$
$$\circ_i\colon M(n) \otimes \mathcal{O}(k) \to M(n+k-1) \text{ (infinitesimal right action)}.$$

These composition maps have to satisfy natural unity, associativity, and Σ-compatibility conditions [**24, 26, 42**]. For example, an infinitesimal bimodule over the non-Σ associative operad is exactly the same thing as a cosimplicial object.

Notice that infinitesimal right action is equivalent to the usual right action since one can use the identity element $\mathrm{id} \in \mathcal{O}(1)$ to mimic empty insertions. But infinitesimal left action is essentially different from the usual left action. Moreover, neither of them can be obtained one from another. However, in the case where M is a bimodule under $\mathcal{O}$, i.e., M is a bimodule over $\mathcal{O}$ endowed with a map of $\mathcal{O}$-bimodules $\rho\colon \mathcal{O} \to M$, then M inherits the structure of an infinitesimal bimodule.[6] Thus $\mathcal{B}_n^{(k)}$ is an infinitesimal bimodule over $\mathcal{B}_n$ and also over $\mathcal{B}_m$, $m < n$, by restriction.

[6]One uses $\rho(\mathrm{id})$ to mimic empty insertions.

Theorem 11.2 appeared in [**2**] for spaces of embeddings. The proof works also in our situation.

Theorem 11.2 ([**2**])**.** *The limit of the Goodwillie–Weiss tower for the space* $\overline{\mathrm{Imm}}_c^{(k)}(\mathbb{R}^m,\mathbb{R}^n)$*,* $n>m$*, is weakly equivalent to the space of derived maps of infinitesimal bimodules over* $\mathcal{B}_m$*:*

$$T_\infty\overline{\mathrm{Imm}}_c^{(k)}(\mathbb{R}^m,\mathbb{R}^n)\simeq \mathrm{hIBim}_{\mathcal{B}_m}(\mathcal{B}_m,\mathcal{B}_n^{(k)}). \tag{11.4}$$

The same is true for singular chains:

$$T_\infty C_*\overline{\mathrm{Imm}}_c^{(k)}(\mathbb{R}^m,\mathbb{R}^n)\simeq \mathrm{hIBim}_{C_*\mathcal{B}_m}(C_*\mathcal{B}_m,C_*\mathcal{B}_n^{(k)}). \tag{11.5}$$

For a codimesnion zero submanifold $M\subset\mathbb{R}^m$, denote by $\mathrm{sEmb}(\bullet,M)$ the symmetric sequence $\mathrm{sEmb}(\sqcup_n D^m,M)$, $n\geqslant 0$, where sEmb stands for the space of *standard* embeddings which on each connected component are compositions of translations and rescalings. Notice that $sEmb(\bullet,M)$ is naturally a right module over $\mathcal{B}_m$. The theorem below is a particular case of the enriched version of the manifold calculus.

Theorem 11.3 ([**2, 7, 43**])**.** *For any open* $M\subset\mathbb{R}^m$ *the limit for the Goodwillie–Weiss tower for the space* $\overline{\mathrm{Imm}}^{(k)}(M,\mathbb{R}^n)$*,* $n>m$*, is weakly equivalent to the space of derived maps of right modules over* $\mathcal{B}_m$*:*

$$T_\infty\overline{\mathrm{Imm}}^{(k)}(M,\mathbb{R}^n)\simeq \mathrm{hRmod}_{\mathcal{B}_m}(\mathrm{sEmb}(\bullet,M),\mathcal{B}_n^{(k)}). \tag{11.6}$$

The same is true for singular chains:

$$T_\infty C_*\overline{\mathrm{Imm}}^{(k)}(M,\mathbb{R}^n)\simeq \mathrm{hRmod}_{C_*\mathcal{B}_m}(C_*\,\mathrm{sEmb}(\bullet,M),C_*\mathcal{B}_n^{(k)}). \tag{11.7}$$

The second parts of Theorems 11.2 and 11.3 imply that there are natural spectral sequences computing $H_*\overline{\mathrm{Imm}}_c^{(k)}(\mathbb{R}^m,\mathbb{R}^n)$, $H_*\overline{\mathrm{Imm}}^{(k)}(M,\mathbb{R}^n)$ (manifold calculus homology spectral sequences) whose first terms together with their differentials are described using the infinitesimal $H_*\mathcal{B}_m$-bimodule structure of $H_*\mathcal{B}_n^{(k)}$.

Theorem 11.3 has a version where M is any manifold and not necessarily an open subset of $\mathbb{R}^m$. In the latter case, one has to use the framed discs operad instead, as well as the framed version of $\mathcal{B}_n^{(k)}$; see [**7, 43**].

We finish this paper by mentioning that the fact that $\mathcal{B}_n^{(k)}$ is a bimodule under $\mathcal{B}_m$ (and not only an infinitesimal bimodule) governs the $\mathcal{B}_m$-algebra structure on $T_\infty\overline{\mathrm{Imm}}_c^{(k)}(\mathbb{R}^m,\mathbb{R}^n)$. The following result was announced by Dwyer and Hess [**15**]:

Theorem 11.4 (Dwyer and Hess [**15**])**.** *Let* $\mathcal{M}$ *be a bimodule under* $\mathcal{B}_m$ *satisfying* $\mathcal{M}(0)\simeq\mathcal{M}(1)\simeq *$*; then*

$$\mathrm{hIBim}_{\mathcal{B}_m}(\mathcal{B}_m,\mathcal{M})\simeq\Omega^m\,\mathrm{hBim}_{\mathcal{B}_m}(\mathcal{B}_m,\mathcal{M}).$$

The right-hand side $\mathrm{hBim}(-,-)$ above denotes the space of derived maps of bimodules, and Ω^m denotes as usual the m-iterated loop space, where for a base point one takes the structure map $\mathcal{B}_m\to M$. In the case $m=1$ this theorem was proved in [**14, 44**]. In the case where $\mathcal{M}$ is an operad and the map $\mathcal{B}_m\to\mathcal{M}$ is a morphism of operads (which still enables $\mathcal{M}$ with a structure of a bimodule under $\mathcal{B}_m$), one can get an extra delooping

$$\mathrm{hIBim}_{\mathcal{B}_m}(\mathcal{B}_m, \mathcal{M}) \simeq \Omega^{m+1}\,\mathrm{hOperad}(\mathcal{B}_m, \mathcal{M}).$$

This equivalence for $\mathcal{M} = \mathcal{B}_n$ corresponds to the fact that the space $\overline{\mathrm{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n)$ has a structure of a $\mathcal{B}_{m+1}$-algebra thanks to the fact that one can pull one knot through the other [**8**, Corollary 7]; [**42**, Proposition 1.1]. But the space $\overline{\mathrm{Imm}}_c^{(k)}(\mathbb{R}^m, \mathbb{R}^n)$, $k \geqslant 3$, is only a $\mathcal{B}_m$-algebra—given two long non-k-equal immersions, pulling one such map through the other is impossible in general since it might produce forbidden singularities.

References

[**1**] V. Arnol'd, The cohomology ring of the group of colored braids, (Russian) *Mat. Zametki* **5** (1969), 227–231.

[**2**] G. Arone and V. Turchin, On the rational homology of high dimensional analogues of spaces of long knots, *Geom. Topol.* **18** (2014), no. 3, 1261–1322.

[**3**] Yu. Baryshnikov, On the cohomology ring of no k-equal manifolds, Preprint.

[**4**] A. Björner and M. Wachs, Shellable nonpure complexes and posets. I, *Trans. Amer. Math. Soc.* **348** (1996), no. 4, 1299–1327.

[**5**] A. Björner and M. Wachs, Shellable nonpure complexes and posets. II, *Trans. Amer. Math. Soc.* **349** (1997), no. 10, 3945–3975.

[**6**] A. Björner and V. Welker, The homology of "k-equal" manifolds and related partition lattices, *Adv. Math.* **110** (1995), no. 2, 277–313.

[**7**] P. Boavida de Brito and M. Weiss, Manifold calculus and homotopy sheaves, *Homology Homotopy Appl.* **15** (2013), no. 2, 361–383.

[**8**] R. Budney, Little cubes and long knots, *Topology* **46** (2007), no. 1, 1–27.

[**9**] F.R. Cohen, The homology of C_{n+1} spaces, In *Lect. Notes Math.*, Vol. 533, 1976.

[**10**] N.E. Dobrinskaya, Configuration spaces with labels and loop spaces on K-products, (Russian) *Uspekhi Mat. Nauk* **63** (2008), no. 6(384), 161–162, translation in *Russian Math. Surveys* **63** (2008), no. 6, 1141–1143.

[**11**] N.E. Dobrinskaya, Higher commutators in the loop homology of K-products, (Russian, Russian summary) *Tr. Mat. Inst. Steklova* **266** (2009), *Geometriya, Topologiya i Matematicheskaya Fizika.* II, 97–111, translation in *Proc. Steklov Inst. Math.* **266** (2009), no. 1, 91–104.

[**12**] N.E. Dobrinskaya, Loops on polyhedral products and diagonal arrangements, Preprint, http://arxiv.org/abs/0901.2871.

[**13**] N.E. Dobrinskaya, Moment-angle complexes, golodness and sequentially Cohen–Macauley property, Preprint.

[**14**] W. Dwyer and K. Hess, Long knots and maps between operads, *Geom. Topol.* **16** (2012), no. 2, 919–955.

[**15**] W. Dwyer and K. Hess, Delooping the space of long embeddings, to appear.

[16] E.M. Feichtner, Complexes of trees and nested set complexes, *Pacific J. Math.* **227** (2006), no. 2, 271–286.

[17] W. Fulton and J. Harris, Representation theory, a first course, *Graduate Texts in Mathematics*, Vol. 129, Readings in Mathematics, Springer-Verlag, New York, 1991.

[18] G. Gaiffi, Models for real subspace arrangements and stratified manifolds. *Int. Math. Res. Not.* **12**, (2003) 627–656.

[19] T.G. Goodwillie, and M. Weiss, Embeddings from the point of view of immersion theory. II, *Geom. Topol.* **3** (1999), 103–118.

[20] M. Hirsch, Immersions of manifolds, *Trans. Amer. Math. Soc.* **93** (1959), 242–276.

[21] M. Kontsevich and Ya. Soibelman, Deformations of algebras over operads and the Deligne conjecture, in *Conférence Moshé Flato 1999, Vol. I* (Dijon), volume 21 of *Math. Phys. Stud.*, 255–307. Kluwer Acad. Publ., Dordrecht, 2000.

[22] P. Lambrechts and I. Volić, Formality of the little N-disks operad, *Mem. Amer. Math. Soc.* **230** (2014), no. 1079.

[23] J.-M. Lemaire, Algèbres connexes et homologie des espaces de lacets, (Springer, Berlin, 1974), *Lect. Notes Math.* Vol. 422.

[24] J-L. Loday and B. Vallette, Algebraic Operads, *Grundlehren der Mathematischen Wissenschaften*, Vol. 346, Springer, Heidelberg, 2012. xxiv+634 pp.

[25] A.B. Merkov, Vassiliev invariants classify plane curves and sets of curves without triple intersections, (Russian, Russian summary) *Mat. Sb.* **194** (2003), no. 9, 31–62, translation in *Sb. Math.* **194** (2003), no. 9–10, 1301–1330.

[26] V. Merkulov and B. Vallette, Deformation theory of representation of prop(erad)s I. *J. Reine Angew. Math.* **634** (2009), 51–106.

[27] M. Miller, Massey products and k-equal manifolds, *Int. Math. Res. Not.* IMRN 2012, no. 8, 1805–1821.

[28] I. Peeva, V. Reiner and V. Welker, Cohomology of real diagonal subspace arrangements via resolutions, *Compositio Mathematica* **117** (1999) no. 01, 107–123.

[29] C. Reutenauer, Free Lie algebras, *London Mathematical Society Monographs*, new series, 7, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1993.

[30] C. Reutenauer, Dimensions and characters of the derived series of the free Lie algebra, in Mots, *Lang. Raison. Calc.*, 171–184, Hermès, Paris, 1990.

[31] R. Reis and M. Weiss, Functor calculus and the discriminant method, *Q. J. Math.* **65** (2014), no. 3, 1069–1110.

[32] D. Sinha, A pairing between graphs and trees, preprint, arXiv:math/0502547.

[33] D. Sinha, The topology of spaces of knots: cosimplicial models, *Amer. J. Math.* **131** (2009), no. 4, 945–980.

[34] D. Sinha, Operads and knot spaces, *J. Amer. Math. Soc.* 19 (2006), no. 2, 461–486.

[**35**] D. Sinha, The homology of the little disks operad, Séminaire et Congrès de Société Mathématique de France, **26** (2011) 255–281, preprint, http://arxiv.org/abs/math.GT/0610236.pdf.

[**36**] D. Sinha and B. Walter, Lie coalgebras and rational homotopy theory, I: Graph coalgebras, *Homology Homotopy Appl.* **13** (2011), no. 2, 263–292.

[**37**] D. Sinha and B. Walter, Lie coalgebras and rational homotopy theory II: Hopf invariants, *Trans. Amer. Math. Soc.* **365** (2013), no. 2, 861–883.

[**38**] S. Sundaram and M. Wachs, The homology representations of the k-equal partition lattice, *Trans. Amer. Math. Soc.* **349** (1997), no. 3, 935–954.

[**39**] V. Turchin (Tourtchine), Sur les questions combinatoires de la théorie spectrale des noeuds. Ph.D. thesis. Univ. Paris 7, 2002, http://www.pdmi.ras.ru/~arnsem/papers/.

[**40**] V. Turchin (Tourtchine), What is one-term relation for higher homology of long knots. *Mosc. Math. J.* **6** (2006), no. 1, 169–194, 223.

[**41**] V. Turchin (Tourtchine), On the other side of the bialgebra of chord diagrams, *J. Knot Theory Ramifications* **16** (2007), no. 5, 575–629.

[**42**] V. Turchin, Hodge-type decomposition in the homology of long knots, *J. Topol.* **3** (2010), no. 3, 487–534.

[**43**] V. Turchin, Context-free manifold calculus and the Fulton-MacPherson operad, *Algebr. Geom. Topol.* **13** (2013), no. 3, 1243–1271.

[**44**] V. Turchin, Delooping totalization of a multiplicative operad, *J. Homotopy Relat. Struct.* **9** (2014), no. 2, 349–418.

[**45**] V. Vassiliev, Resolutions of discriminants and topology of their complements. In *New developments in singularity theory* (Cambridge, 2000), 87–115, *NATO Sci. Ser. II Math. Phys. Chem.*, 21, Kluwer Acad. Publ., Dordrecht, 2001.

[**46**] V. Vassiliev, Invariants of ornaments, (English summary) Singularities and bifurcations, 225–262, *Adv. Soviet Math.* **21**, Amer. Math. Soc., Providence, RI, 1994.

[**47**] M. Weiss, Embeddings from the point of view of immersion theory. I. *Geom. Topol.* **3** (1999), 67–101.

[**48**] S. Yuzvinsky, Small rational model of subspace complement, *Trans. Amer. Math. Soc.* **354** (2002), no. 5, 1921–1945.

Natalya Dobrinskaya `ne.dobrinsk@gmail.com`

Université Catholique de Louvain and Vrije Universiteit, Amsterdam

Victor Turchin `turchin@ksu.edu`

Department of Mathematics, Kansas State University, Manhattan, KS 66506, USA