THE HAMMOCK LOCALIZATION PRESERVES HOMOTOPIES

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Abstract
The hammock localization provides a model for a homotopy function complex in any Quillen model category. We prove that a homotopy between a pair of morphisms induces a homotopy between the maps induced by taking the hammock localization. We also show that, under Vopěnka's principle, every homotopy idempotent functor in a cofibrantly generated model category is determined by simplicial orthogonality with respect to a set of morphisms. Finally, we give a new proof of the fact that left Bousfield localizations with respect to a class of morphisms always exist in any left proper combinatorial model category under Vopěnka's principle.

1. Introduction

The hammock localization was introduced by Dwyer and Kan in a series of articles [**DK80a, DK80c, DK80b**]. Given a category $\mathcal{C}$ with a fixed class of morphisms $\mathcal{W}$, the hammock localization $\mathcal{L}^H\mathcal{C}$ is a simplicial category such that $\pi_0(\mathcal{L}^H\mathcal{C}(X,Y))$ is the set of morphisms from X to Y in the category obtained by inverting the morphisms in $\mathcal{W}$ for every pair of objects X and Y in $\mathcal{C}$. In the case that $\mathcal{C}$ is a model category and $\mathcal{W}$ is its class of weak equivalences, then $\pi_0(\mathcal{L}^H\mathcal{C}(X,Y))$ is in natural bijection with the set of homotopy classes of morphisms $[X,Y]$ and, as a bifunctor, $\mathcal{L}^H\mathcal{C}(-,-)$ sends weak equivalences to weak homotopy equivalences. Hence, $\mathcal{L}^H\mathcal{C}(-,-)$ defines a homotopy function complex on $\mathcal{C}$. Moreover, if $\mathcal{C}$ is a simplicial model category, with simplicial mapping space $\mathrm{Map}(-,-)$, then $\mathcal{L}^H\mathcal{C}(X,Y) \simeq \mathrm{Map}(X^c,Y^f)$, where X^c is a cofibrant replacement of X and Y^f is a fibrant replacement of Y.

In Theorem 3.1 we prove that $\mathcal{L}^H\mathcal{C}(-,-)$ sends left or right homotopies to simplicial homotopies. This is applied in Section 4 to study homotopy idempotent functors. We recall that a (coaugmented) homotopy idempotent functor on a model category $\mathcal{C}$ is a functor $L\colon \mathcal{C} \to \mathcal{C}$ together with a natural transformation $\ell\colon 1 \to L$ that induces

a localization, i.e., a left adjoint of the inclusion of a reflective subcategory, in the homotopy category. An object X is L-local if it is weakly equivalent to an object of the form LY for some Y, and a morphism f is an L-equivalence if Lf is a weak equivalence. We prove in Proposition 4.3 that, in any model category, L-local objects and L-equivalences are simplicially orthogonal with respect to $\mathcal{L}^H\mathcal{C}(-,-)$. The first result of this kind was obtained by Dror Fajoun in [**Far96b**] in the category of spaces, and later it was extended to other contexts (cf. [**CSS05, CC06**]). If we assume a certain large cardinal axiom, called Vopěnka's principle, we prove in Corollary 4.10 that for each homotopy idempotent functor (L, ℓ) in any cofibrantly generated model category, the class of L-local objects correspond to the class of objects that are simplicially orthogonal to *just a set* of morphisms. This result extends a previous result in [**CC06**, Theorem 2.3] for simplicial combinatorial model categories to all cofibrantly generated model categories. In the same spirit, we extend in Theorem 4.12 the analogous result for augmented homotopy idempotent functors [**Cho07**, Theorem 2.1].

It was proved in [**RT03**, Theorem 2.3] that, under Vopěnka's principle, left Bousfield localizations with respect to a class of morphisms exist in any left proper combinatorial model category. We give a new proof of this fact in Corollary 4.6. The proof can be easily modified to give the analogous result for right Bousfield localizations, as we state in Corollary 4.11. This last result extends a previous result in [**Cho07**, Theorem 1.4] for simplicial combinatorial model categories to all combinatorial model categories.

The hammock localization $\mathcal{L}^H$ can be extended to a functor from the category of small categories with weak equivalences to the category of small simplicial categories

$$\mathcal{L}^H \colon \mathbf{wCat} \longrightarrow \mathbf{sCat},$$

as we make precise in Section 3. We prove that $\mathcal{L}^H$ can be extended so as to send natural transformations to simplicial natural transformations *up to homotopy* in Theorem 3.2. This result is used in a joint paper of the author with Casacuberta and Tonks [**CRT**] to study homotopy algebra structures preserved by localizations.

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2.   The hammock localization

The hammock localization defines one model for the homotopy function complex of a model category. It was introduced by Dwyer and Kan in a series of articles [**DK80a, DK80c, DK80b**]. We will summarize some of their results following the more recent exposition contained in [**DHKS04**, Chapters 34 and 35].

A *category with weak equivalences* is a pair $(\mathcal{C}, \mathcal{W})$ with $\mathcal{C}$ a category and $\mathcal{W}$ a fixed class of morphisms in $\mathcal{C}$ that contains all identities. The morphisms in $\mathcal{W}$ are called *weak equivalences*. Assume (just for the moment) that $\mathcal{C}$ is small. For every pair of objects X and Y in $\mathcal{C}$, and every odd natural number n, we define a category

$\mathcal{L}_n^H \mathcal{C}(X, Y)$ with objects being strings of n morphisms on $\mathcal{C}$ in alternating directions

$$C_0 \xleftarrow{d_0} C_1 \xrightarrow{d_1} \ldots \xrightarrow{d_{n-2}} C_{n-1} \xleftarrow{d_{n-1}} C_n, \tag{1}$$

with $X = C_0$, $Y = C_n$ and the arrows pointing to the left being weak equivalences. A morphism is a commutative diagram of the form

$$
\begin{array}{ccccccccc}
C_0 & \longleftarrow & C_1 & \longrightarrow & \ldots & \longrightarrow & C_{n-1} & \longleftarrow & C_n \\
\| & & \downarrow & & & & \downarrow & & \| \\
C_0' & \longleftarrow & C_1' & \longrightarrow & \ldots & \longrightarrow & C_{n-1}' & \longleftarrow & C_n'.
\end{array}
$$

The *hammock localization* of $(\mathcal{C}, \mathcal{W})$ is a simplicial category (meaning simplicially enriched) $\mathcal{L}^H \mathcal{C}$ with the same objects as $\mathcal{C}$ and, for every pair of objects X and Y, a simplicial set

$$\mathcal{L}^H \mathcal{C}(X, Y) = \operatorname*{colim}_k N \mathcal{L}_{2k+1}^H \mathcal{C}(X, Y),$$

where the sequential colimit (that is also a homotopy colimit) is taken over the nerve of the embedding functors that send an object like (1) in $\mathcal{L}_n^H \mathcal{C}(X, Y)$ to

$$X = C_0 \longleftarrow C_1 \xrightarrow{id} C_1 \xleftarrow{id} C_1 \longrightarrow \ldots \longrightarrow C_{n-1} \longleftarrow C_n = Y \tag{2}$$

in $\mathcal{L}_{n+2}^H \mathcal{C}(X, Y)$. The composition in $\mathcal{L}^H \mathcal{C}$ is given by concatenation. More precisely, given an object

$$X \longleftarrow C_1 \longrightarrow \ldots \longrightarrow C_{n-1} \xleftarrow{d_{n-1}} Y$$

in $\mathcal{L}_n^H \mathcal{C}(X, Y)$ and an object

$$Y \xleftarrow{d_0'} C_1' \longrightarrow \ldots \longrightarrow C_{n-1}' \longleftarrow Z$$

in $\mathcal{L}_n^H \mathcal{C}(Y, Z)$, their composition is the object

$$X \longleftarrow C_1 \longrightarrow \ldots \longrightarrow C_{n-1} \xleftarrow{d_{n-1} \circ d_0'} C_1' \longrightarrow \ldots \longrightarrow C_{n-1}' \longleftarrow Z$$

in $\mathcal{L}_{2n-1}^H \mathcal{C}(X, Z)$. Hence, for every triple of objects X, Y, and Z in $\mathcal{C}$, we define a simplicial composition $\mathcal{L}^H \mathcal{C}(X, Y) \times \mathcal{L}^H \mathcal{C}(Y, Z) \to \mathcal{L}^H \mathcal{C}(X, Z)$ as

$$\operatorname*{colim}_k N \left(\mathcal{L}_{2k+1}^H \mathcal{C}(X, Y) \times \mathcal{L}_{2k+1}^H \mathcal{C}(Y, Z) \to \mathcal{L}_{4k+1}^H \mathcal{C}(X, Z) \right).$$

Here we use the well-known facts that filtered colimits commute with finite limits, that nerves commute with products, and that the category of simplicial sets is a cartesian closed model category. The identity of X is determined by fixing, for each n, the object in $\mathcal{L}_n^H(X, X)$ with only identity arrows. The associativity and identity properties can be checked levelwise.

Remark 2.1. The hammock localization was originally defined using a colimit over all natural numbers (cf. [**DK80b**]). We restrict to odd natural numbers because in this case the morphisms in the extremes are always going backwards and then we do not need to distinguish two cases in every proof. It can be seen that both definitions

coincide using a cofinality argument, as proved in [**DHKS04**, Chapter 34]. It is also worth mentioning that if $\mathcal{C}$ is a model category and $\mathcal{W}$ is its class of weak equivalences, then $\mathcal{L}^H\mathcal{C}(X,Y) \simeq \mathcal{L}_3^H\mathcal{C}(X,Y)$ for every pair of objects X and Y (cf. [**DK80b**]). Although working with $\mathcal{L}_3^H$ has certain advantages, for the purposes of this article it is more convenient to work with $\mathcal{L}^H$ even in the case of model categories.

Remark 2.2. We recall that the nerve of a category $\mathcal{D}$ is the simplicial set with n-simplices $(N\mathcal{D})_n = \mathbf{Cat}([n], \mathcal{D})$ and that this defines a fully faithful functor from **Cat** to the category of simplicial sets. We will often use the fact that natural transformations induce homotopies after taking nerves [**Qui73**, Section 1, Proposition 2]. In particular, adjunctions induce homotopy equivalences. It is also useful to know that, for any pair of objects X and Y, $\mathcal{L}^H\mathcal{C}(X,Y)$ is weakly equivalent to the nerve of the Grothendieck construction on the diagram defining the sequential colimit, as observed in [**DHKS04**, Proposition 35.7].

Definition 2.3. Let **wCat** denote the category of small categories with weak equivalences and morphisms being the functors that preserve weak equivalences. Then there is a functor

$$\mathcal{L}^H \colon \mathbf{wCat} \longrightarrow \mathbf{sCat},$$

where **sCat** is the category of small simplicial categories. The image of a morphism $F\colon \mathcal{C} \to \mathcal{C}'$ in **wCat** is defined levelwise, i.e., each object X in $\mathcal{L}^H\mathcal{C}$ is sent to FX in $\mathcal{L}^H\mathcal{C}'$ and, for each pair of objects X and Y in $\mathcal{C}$, we define a simplicial map $\mathcal{L}^H\mathcal{C}(X,Y) \to \mathcal{L}^H\mathcal{C}'(FX, FY)$ determined by sending an object like (1) in $\mathcal{L}_n^H\mathcal{C}(X,Y)$ to

$$FX \xleftarrow{Fd_0} FC_1 \xrightarrow{Fd_1} \ldots \xrightarrow{Fd_{n-2}} FC_{n-1} \xleftarrow{Fd_{n-1}} FY$$

in $\mathcal{L}_n^H\mathcal{C}'(FX, FY)$.

Notice that for every morphism $f\colon A \to B$ in a category $\mathcal{C}$ there are induced maps of simplicial sets

$$f^*\colon \mathcal{L}^H\mathcal{C}(B,Y) \longrightarrow \mathcal{L}^H\mathcal{C}(A,Y) \text{ and } f_*\colon \mathcal{L}^H\mathcal{C}(X,A) \longrightarrow \mathcal{L}^H\mathcal{C}(X,B)$$

for all X and Y in $\mathcal{C}$. They are defined just using the composition in $\mathcal{L}^H\mathcal{C}$ and fixing one of the variables as the diagram

$$A \xleftarrow{id} A \xrightarrow{f} B \xleftarrow{id} B$$

in $\mathcal{L}_3^H\mathcal{C}(A,B)$. To be more precise, f^* is induced by the functors f_n^* that send an object like (1) in $\mathcal{L}_n^H\mathcal{C}(B,Y)$ to

$$A \xleftarrow{id} A \xrightarrow{f} B \longleftarrow C_1 \longrightarrow \ldots \longrightarrow C_{n-1} \longleftarrow Y$$

in $\mathcal{L}_{n+2}^H\mathcal{C}(A,Y)$ for every odd natural number n. If f is a weak equivalence, then f^* is a weak homotopy equivalence and a homotopy inverse is given by the functors that send an object like (1) in $\mathcal{L}_n^H\mathcal{C}(A,Y)$ to

$$B \xleftarrow{f} A \xrightarrow{id} A \longleftarrow C_1 \longrightarrow \ldots \longrightarrow C_{n-1} \longleftarrow Y$$

in $\mathcal{L}_{n+2}^H\mathcal{C}(B,Y)$ for every odd natural number n. Indeed, if f is a weak equivalence,

then f_n^* is an equivalence of categories for each n. The map f_* is defined similarly.

If $\mathcal{C}$ is a model category, we will let $\mathcal{W}$ be exactly the class of weak equivalences in $\mathcal{C}$. In this case, $\pi_0(\mathcal{L}^H\mathcal{C}(X,Y)) \cong \mathrm{Ho}(\mathcal{C})(X,Y)$ and $\mathcal{L}^H\mathcal{C}(X,Y)$ defines a *homotopy function complex* (or *homotopy mapping space*) for $\mathcal{C}$ (cf. [**Hir03**, Chapter 17]).

We would like to apply the hammock localization not only to small categories. This has some technical set theoretical issues that can be nicely handled using the axiomatization of universes. We refer to [**DHKS04**, Section 32] for a detailed explanation.

3. A property of the hammock localization

The following result asserts that the hammock localization respects homotopies. For the basic properties of homotopies in model categories, we refer to [**Hir03**, Chapter 7]. As usual, by *simplicial homotopy* in a simplicial model category we mean the equivalence relation generated by the strict homotopies.

Theorem 3.1. *Let $\mathcal{C}$ be a model category, and let $f, g\colon A \to B$ be a pair of left or right homotopic morphisms in $\mathcal{C}$. Then the induced maps $f_*, g_*\colon \mathcal{L}^H\mathcal{C}(X,A) \to \mathcal{L}^H\mathcal{C}(X,B)$ are simplicially homotopic for all X in $\mathcal{C}$ and $f^*, g^*\colon \mathcal{L}^H\mathcal{C}(B,Y) \to \mathcal{L}^H\mathcal{C}(A,Y)$ are simplicially homotopic maps for all Y in $\mathcal{C}$.*

Proof. Assume that f and g are left homotopic. Fix a cylinder object

$$A \coprod A \xrightarrow{i_0 \amalg i_1} \mathrm{Cyl}(A) \xrightarrow{\quad p \quad} A \ ,$$

where $p \circ i_0 = p \circ i_1 = id$ and i_0, i_1, and p are weak equivalences. Let $H\colon \mathrm{Cyl}(A) \to B$ be a left homotopy between f and g. Thus, $H \circ i_0 = f$ and $H \circ i_1 = g$.

Let $\tilde{H}\colon \mathcal{L}^H\mathcal{C}(X,A) \to \mathcal{L}^H\mathcal{C}(X,B)$ be the map induced by the functors $\tilde{H}^n$ that send an object like (1) in $\mathcal{L}^H\mathcal{C}_n(X,A)$ to

$$X \longleftarrow C_1 \longrightarrow \dots \longrightarrow C_{n-1} \xleftarrow{d_{n-1}\circ p} \mathrm{Cyl}(A) \xrightarrow{\ H\ } B \xleftarrow{\ id\ } B$$

in $\mathcal{L}^H\mathcal{C}_{n+2}(X,B)$. Hence, the commutative diagram

determines a zig-zag of natural transformations

$$f_*^n \xrightarrow{\ \phi^n\ } \tilde{H}^n \xleftarrow{\ \psi^n\ } g_*^n$$

between functors from $\mathcal{L}^H\mathcal{C}_n(X,A)$ to $\mathcal{L}^H\mathcal{C}_{n+2}(X,B)$ for each odd natural number n,

which are compatible with the inclusions

$$\mathcal{L}^H\mathcal{C}_n(X,A) \to \mathcal{L}^H\mathcal{C}_{n+2}(X,A).$$

Because $\tilde{H} = \operatorname{colim}_k N\tilde{H}^{2k+1}$, the nerve functor sends natural transformations to simplicial homotopies, and ϕ^n and ψ^n are compatible with the colimit, we have an induced zig-zag of homotopies of simplicial sets $f_* \simeq \tilde{H} \simeq g_*$.

It f and g are right homotopic, then the statement can be proved similarly using path objects. $\square$

The following result describes the image of a natural transformation under the hammock localization functor.

Theorem 3.2. *Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be a pair of functors in **wCat** together with a natural transformation $\eta\colon F \to G$. There is a homotopy $\mathcal{L}^H\eta(X,Y)$ from $\eta_{Y*} \circ \mathcal{L}^H F(X,Y)$ to $\eta_X^* \circ \mathcal{L}^H G(X,Y)$ for each pair of objects X and Y in $\mathcal{C}$:*

$$
\begin{array}{ccc}
\mathcal{L}^H\mathcal{C}(X,Y) & \xrightarrow{\ \mathcal{L}^H F(X,Y)\ } & \mathcal{L}^H\mathcal{D}(FX,FY) \\
{\scriptstyle\mathcal{L}^H G(X,Y)}\Big\downarrow \quad \mathcal{L}^H\eta(X,Y) & & \Big\downarrow{\scriptstyle\eta_{Y*}} \\
\mathcal{L}^H\mathcal{D}(GX,GY) & \xrightarrow[\ \eta_X^*\ ]{} & \mathcal{L}^H\mathcal{D}(FX,GY).
\end{array}
$$

Notice that $\mathcal{L}^H$ fails to be a strict 2-functor in general. To be so, $\mathcal{L}^H\eta$ would have to define a simplicially enriched natural transformation, which exactly means that $\mathcal{L}^H\eta(X,Y)$ would have to be the identity for each pair of objects X and Y (cf. [**Kel82**, Section 1.2]). Nevertheless, Theorem 3.2 has already some applications. For instance, it is used in [**CRT**] to study homotopy algebras over monads in the homotopy category. Examples of such algebras include homotopy associative H-spaces or homotopy module spectra.

In the proof of Theorem 3.2, we will need the following observation that we state without a proof.

Lemma 3.3. *Let $\mathcal{C}$ be a category in **wCat**. For any pair of objects X and Y in $\mathcal{C}$, the inclusion $\mathcal{L}_n^H\mathcal{C}(X,Y) \hookrightarrow \mathcal{L}_{n+2}^H\mathcal{C}(X,Y)$ in (2) that consists in inserting two consecutive identity morphisms in C_1 is related by a zig-zag of natural transformations to the inclusion defined by inserting two consecutive identity morphisms in C_i for any $0 \leqslant i \leqslant n$.*

Proof of Theorem 3.2. Fix an object like (1) in $\mathcal{L}_n^H\mathcal{C}(X,Y)$. Then we can describe the homotopy $\mathcal{L}^H\eta(X,Y)$ by the natural transformation defined by the morphisms

$$
\begin{array}{ccccccccccccccc}
FX & \xleftarrow{id} & FX & \xrightarrow{id} & FX & \xleftarrow{Fd_0} & FC_1 & \longrightarrow & \ldots & \longrightarrow & FC_{n-1} & \xleftarrow{Fd_{n-1}} & FY & \xrightarrow{\eta_Y} & GY & \xleftarrow{id} & GY \\
\Big\| & & \Big\| & & \downarrow{\scriptstyle\eta_X} & & \downarrow{\scriptstyle\eta_{C_1}} & & & & \downarrow{\scriptstyle\eta_{C_{n-1}}} & & \downarrow{\scriptstyle\eta_Y} & & \Big\| & & \Big\| \\
FX & \xleftarrow{id} & FX & \xrightarrow{\eta_X} & GX & \xleftarrow{Gd_0} & GC_1 & \longrightarrow & \ldots & \longrightarrow & GC_{n-1} & \xleftarrow{Gd_{n-1}} & GY & \xrightarrow{id} & GY & \xleftarrow{id} & GY
\end{array}
$$

in $\mathcal{L}_{n+4}^H\mathcal{D}(FX,GY)$ for each odd number n. It is a consequence of Lemma 3.3 that the two identities we artificially introduced at the beginning of the first row will have

no effect after we take the colimit over n to produce $\eta_{Y*} \circ \mathcal{L}^H F(X, Y)$. The same applies to the two identities at the end of the second row. $\qquad\square$

4. Homotopy idempotent functors

We next define an analogue of the notion of idempotent functor (cf. [**Bor94**, Section 4.2]), in the context of model categories following [**CSS05**].

Definition 4.1. Let $\mathcal{M}$ be a model category. A functor $L\colon \mathcal{M} \to \mathcal{M}$ together with a natural transformation $\ell\colon 1 \to L$ is called *(coaugmented) homotopy idempotent* if L sends weak equivalences to weak equivalences and the natural morphisms ℓ_{LX} and $L\ell_X\colon LX \to LLX$ are equal in the homotopy category $\mathrm{Ho}(\mathcal{M})$ and both are weak equivalences for every object X in $\mathcal{M}$.

There is a notion of *augmented* homotopy idempotent functor, also called *cellularization*. All results in this section have analogues for the augmented case, and the proofs can be easily transferred. At the end of the section, we will state the analogues of the two main results.

Given a homotopy idempotent functor (L, ℓ), a morphism f in $\mathcal{M}$ is called an *L-equivalence* if Lf is a weak equivalence, and a fibrant object X in $\mathcal{M}$ is called *L-local* if $X \simeq LY$ for some Y in $\mathcal{M}$. The class of L-equivalences and L-local objects determine each other by orthogonality in the homotopy category $\mathrm{Ho}(\mathcal{M})$. This means that a morphism $g\colon X \to Y$ is an L-equivalence if and only if the morphism

$$g^*\colon \ [Y, Z] \xrightarrow{\ \cong\ } [X, Z]$$

is an isomorphism for every L-local object Z, and a fibrant object Z is L-local if and only if g^* is an isomorphism for all L-equivalences g (cf. [**Ada75**, Proposition 2.10]).

We will prove in Proposition 4.3 that L-equivalences and L-local objects are also *simplicially orthogonal* in the model category. Let us explain what this means. Fix a homotopy function complex $\mathrm{map}(-, -)$ in a model category $\mathcal{M}$, and let $\mathcal{S}$ be any class of morphisms in $\mathcal{M}$. A fibrant object X in $\mathcal{M}$ is called *$\mathcal{S}$-local* if, for every morphism $f\colon A \to B$ in $\mathcal{S}$, the induced map of homotopy function complexes

$$f^*\colon \mathrm{map}(B, X) \longrightarrow \mathrm{map}(A, X)$$

is a weak homotopy equivalence. We denote by $\mathcal{S}^{h\perp}$ the class of $\mathcal{S}$-local objects and we call it the *simplicial orthogonal complement* of $\mathcal{S}$. Similarly, for any class of objects $\mathcal{D}$ in $\mathcal{M}$, a morphism $f\colon A \to B$ is called a *$\mathcal{D}$-equivalence* if, for every $X \in \mathcal{D}$, f^* is a weak homotopy equivalence. By an abuse of notation, we also denote by $\mathcal{D}^{h\perp}$ the class of $\mathcal{D}$-equivalences and we call it the *simplicial orthogonal complement* of $\mathcal{D}$.

It is important to notice that these definitions do not depend on the choice of homotopy function complex [**Hir03**, Proposition 17.8.2]. We fix $\mathrm{map}(-, -)$ to be $\mathcal{L}^H \mathcal{C}(-, -)$.

Recall from [**Hir03**, Definition 3.3.1] that the left Bousfield localization with respect to a class of morphisms $\mathcal{S}$ on a model category $\mathcal{M}$ (if it exists) is a new model category structure $L_{\mathcal{S}}\mathcal{M}$ on the same underlying category $\mathcal{M}$ with the same cofibrations and the weak equivalences being the $\mathcal{S}^{h\perp}$-equivalences. In particular, if we consider the fibrant replacement functor in $L_{\mathcal{S}}\mathcal{M}$, then it defines a homotopy

idempotent functor on $\mathcal{M}$. We will show that, if we assume that Vopěnka's principle holds, then in any cofibrantly generated model category a homotopy idempotent functor has the same local objects as a left Bousfield localization with respect to a set of morphisms.

Lemma 4.2. *Let $\mathcal{M}$ be a model category, and let (L, ℓ) be a homotopy idempotent functor on $\mathcal{M}$. For every pair of objects X and Y,*

1. *the map $\mathcal{L}^H L(X, LY)\colon \operatorname{map}(X, LY) \to \operatorname{map}(LX, LLY)$ is a simplicial homotopy equivalence, and*

2. *the map $\ell_X^*\colon \operatorname{map}(LX, LY) \to \operatorname{map}(X, LY)$ is also a simplicial homotopy equivalence.*

Proof. For the first part, we let $h\colon \operatorname{map}(LX, LLY) \to \operatorname{map}(X, LY)$ be the map induced by the functors h^n that send an object like (1) in $\mathcal{L}_n^H \mathcal{C}(LX, LLY)$ to

$$X \xleftarrow{\ id\ } X \xrightarrow{\ \ell_X\ } LX \longleftarrow C_1 \longrightarrow \ldots \longrightarrow C_{n-1} \xleftarrow{d_{n-1} \circ \ell_{LY}} LY$$

in $\mathcal{L}_{n+2}^H \mathcal{C}(X, LY)$ for every odd natural number n. The homotopy from the identity (see Lemma 3.3) to $h \circ \mathcal{L}^H L(X, LY)$ is determined by the commutative diagram

$$
\begin{array}{ccccccccccc}
X & \xleftarrow{id} & X & \xrightarrow{id} & X & \longleftarrow & C_1 & \longrightarrow & \ldots & \longrightarrow & C_{n-1} & \xleftarrow{d_{n-1}} & LY \\
\| & & \| & & \downarrow{\scriptstyle \ell_X} & & \downarrow{\scriptstyle \ell_{C_1}} & & & & \downarrow{\scriptstyle \ell_{C_{n-1}}} & & \| \\
X & \xleftarrow{id} & X & \xrightarrow{\ell_X} & LX & \longleftarrow & LC_1 & \longrightarrow & \ldots & \longrightarrow & LC_{n-1} & \xleftarrow{Ld_{n-1} \circ \ell_{LY}} & LY
\end{array}
$$

in $\mathcal{L}_{n+2}^H \mathcal{C}(X, LY)$ for every odd natural number n. We will now define a zig-zag of homotopies between the identity and $\mathcal{L}^H L(X, LY) \circ h$ induced by a zig-zag of natural transformations

$$id^n \longrightarrow \tilde{H}^n \longleftarrow \mathcal{L}_n^H L \circ h^n \tag{3}$$

that are compatible with the inclusions

$$\mathcal{L}_n^H \mathcal{C}(LX, LLY) \longrightarrow \mathcal{L}_{n+2}^H \mathcal{C}(LX, LLY).$$

Since $\operatorname{map}(-, -)$ is homotopy invariant, we can assume that LX, LLX, LLY, and $LLLY$ are fibrant and cofibrant. Hence, there are two cylinder objects

$$LX \coprod LX \xrightarrow{i_0 \,\coprod\, i_1} \operatorname{Cyl}(LX) \xrightarrow{\ \ p\ \ } LX \qquad \text{and}$$

$$LLY \coprod LLY \xrightarrow{i_0' \,\coprod\, i_1'} \operatorname{Cyl}(LLY) \xrightarrow{\ \ p'\ \ } LLY,$$

a left homotopy $H\colon \operatorname{Cyl}(LX) \to LLX$ between $H \circ i_0 = L\ell_X$ and $H \circ i_1 = \ell_{LX}$, and a left homotopy $H'\colon \operatorname{Cyl}(LLY) \to LLLY$ between $H' \circ i_0' = L\ell_{LY}$ and $H' \circ i_1' = \ell_{LLY}$ (notice that H' is forced to be a weak equivalence by the two-out-of-three property). Let $\tilde{H}^n$ be the functor that sends an object like (1) in $\mathcal{L}_n^H \mathcal{C}(LX, LLY)$ to

$$LX \xleftarrow{\ p\ } \operatorname{Cyl}(LX) \xrightarrow{\ H\ } LLX \xleftarrow{Ld_0} LC_1 \ldots LC_{n-1} \xleftarrow{Ld_{n-1} \circ H'} \operatorname{Cyl}(LLY) \xrightarrow{\ p'\ } LLY \xleftarrow{\ id\ } LLY$$

in $\mathcal{L}^H_{n+4}\mathcal{C}(LX, LLX)$. The diagram

$$
\begin{array}{ccccccccccc}
LX & \xleftarrow{id} & LX & \xrightarrow{id} & LX & \xleftarrow{d_0} & C_1 \ldots C_{n-1} & \xleftarrow{d_{n-1}} & LLY & \xrightarrow{id} & LLY & \xleftarrow{id} & LLY \\
\| & & \downarrow{i_1} & & \downarrow{\ell_{LX}} & & \downarrow{\ell_{C_1}} \quad \downarrow{\ell_{C_{n-1}}} & & \downarrow{i'_1} & & \| & & \| \\
LX & \xleftarrow{p} & \mathrm{Cyl}(LX) & \xrightarrow{H} & LLX & \xleftarrow{Ld_0} & LC_1 \ldots LC_{n-1} & \xleftarrow{Ld_{n-1}\circ H'} & \mathrm{Cyl}(LLY) & \xrightarrow{p'} & LLY & \xleftarrow{id} & LLY \\
\| & & \uparrow{i_0} & & \| & & \| \quad \| & & \uparrow{i'_0} & & \| & & \| \\
LX & \xleftarrow{id} & LX & \xrightarrow{L\ell_X} & LLX & \xleftarrow{Ld_0} & LC_1 \ldots LC_{n-1} & \xleftarrow{Ld_{n-1}\circ L\ell_{LY}} & LLY & \xrightarrow{id} & LLY & \xleftarrow{id} & LLY
\end{array}
$$

in $\mathcal{L}^H_{n+4}\mathcal{C}(LX, LLY)$ defines the zig-zag of natural transformations (3) inducing the homotopy equivalence between id and $\mathcal{L}^H L \circ h$.

The second part of the statement follows from Theorem 3.2, because ℓ induces a homotopy $\ell_{LY*} \simeq \ell_X^* \circ \mathcal{L}^H L(X, LY)$, and ℓ_{LY*} and $\mathcal{L}^H L(X, LY)$ are weak homotopy equivalences. $\qquad\square$

Proposition 4.3. *Let $\mathcal{M}$ be a model category, and let (L, ℓ) be a homotopy idempotent functor on $\mathcal{M}$. Then the class of L-equivalences coincides with the simplicial orthogonal complement of the class of L-local objects.*

Proof. We first prove that L-local objects are simplicially orthogonal to L-equivalences: Fix an object LY and a morphism $f\colon A \to B$ such that Lf is a weak equivalence. We want to prove that $\mathrm{map}(f, LY)$ is a weak homotopy equivalence. In the commutative diagram

$$
\begin{array}{ccc}
\mathrm{map}(LB, LY) & \xrightarrow{Lf^*} & \mathrm{map}(LA, LY) \\
\ell_B^* \downarrow & & \downarrow \ell_A^* \\
\mathrm{map}(B, LY) & \xrightarrow{f^*} & \mathrm{map}(A, LY),
\end{array}
$$

the vertical arrows are weak homotopy equivalences by Lemma 4.2 and the top arrow is also a weak homotopy equivalence because Lf is a weak equivalence. Hence, the bottom map has to be a weak homotopy equivalence.

If $f\colon A \to B$ is such that $\mathrm{map}(f, X)$ is a weak homotopy equivalence for each L-local object X, then, using Lemma 4.2, we deduce that $\mathrm{map}(Lf, LA)$ and $\mathrm{map}(Lf, LB)$ are weak homotopy equivalences. Hence, Lf must be a weak equivalence by [**Hir03**, Proposition 17.7.6].

Finally, let X be fibrant and such that $\mathrm{map}(f, X)$ is a weak homotopy equivalence for each L-equivalence f. In particular, $\mathrm{map}(\ell_X, X)$ is a weak homotopy equivalence. On the other hand, $\mathrm{map}(\ell_X, LX)$ is a weak homotopy equivalence by Lemma 4.2. Hence, $\ell_X\colon X \to LX$ must be a weak equivalence by [**Hir03**, Proposition 17.7.6]. $\quad\square$
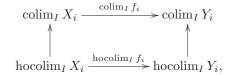
In what follows, we specialize to combinatorial model categories, i.e., cofibrantly generated model categories whose underlying category is locally presentable. We refer to [**Dug01**] or [**Bar10**] for the definition and properties. In a left proper combinatorial model category left Bousfield localizations with respect to a set always exist (cf. [**Bar10**, Theorem 4.7]). The analogue for cellular model categories is proved in [**Hir03**, Theorem 4.1.1].

The next two results correspond to [**CC06**, Lemma 1.2] and [**CC06**, Lemma 1.3], but we drop the hypothesis that the model category be simplicial.

Lemma 4.4. *Let $\mathcal{M}$ be a combinatorial model category. Then there is regular cardinal μ such that, for every class of objects $\mathcal{D}$ in $\mathcal{M}$, the class of $\mathcal{D}$-equivalences $\mathcal{D}^{h\perp}$ is closed under μ-filtered colimits.*

Proof. Since $\mathrm{map}(-,-)$ is homotopy invariant, we can assume that each object in $\mathcal{D}$ is fibrant. Since we are assuming that $\mathcal{M}$ is combinatorial, there is a regular cardinal μ such that weak equivalences are preserved by μ-filtered colimits and there are cofibrant and fibrant replacement functors that preserve μ-filtered colimits (cf. [**Dug01**, Proposition 2.3]). Let $f_i \colon X_i \to Y_i$ be $\mathcal{D}$-equivalences for all $i \in I$, where I is a μ-filtered category. Since we are assuming that cofibrant replacement preserves μ-filtered colimits, we can assume that X_i and Y_i are cofibrant for all $i \in I$.

We have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{colim}_I X_i & \xrightarrow{\ \mathrm{colim}_I f_i\ } & \mathrm{colim}_I Y_i \\
\uparrow & & \uparrow \\
\mathrm{hocolim}_I X_i & \xrightarrow{\ \mathrm{hocolim}_I f_i\ } & \mathrm{hocolim}_I Y_i,
\end{array}
$$

where the vertical arrows are weak equivalences since μ-filtered colimits are homotopy colimits, due to the fact that μ-filtered colimits of weak equivalences are weak equivalences. To finish the proof it is enough to prove that the bottom arrow is a $\mathcal{D}$-equivalence. But now, for every object $Z \in \mathcal{D}$, we have a commutative square

$$
\begin{array}{ccc}
\mathrm{map}(\mathrm{hocolim}_I X_i, Z) & \longrightarrow & \mathrm{map}(\mathrm{hocolim}_I Y_i, Z) \\
\downarrow & & \downarrow \\
\mathrm{holim}_I \mathrm{map}(X_i, Z) & \longrightarrow & \mathrm{holim}_I \mathrm{map}(Y_i, Z),
\end{array}
$$

where the vertical arrows are weak homotopy equivalences by [**Hir03**, Theorem 19.4.4], and the bottom arrow is a weak homotopy equivalence since every f_i is a $\mathcal{D}$-equivalence. This proves that $\mathrm{hocolim}_I f_i$ is a $\mathcal{D}$-equivalence. $\square$

In the following statement we will need to assume *Vopěnka's principle.* This is a set-theoretical axiom equivalent to the fact that every full subcategory of a locally presentable category that is closed under colimits is a locally presentable coreflective subcategory, i.e., the inclusion has a right adjoint (cf. [**AR94**, Theorem 6.28 and Theorem 6.29]).

Lemma 4.5. *Assume that Vopěnka's principle holds. Let $\mathcal{M}$ be a combinatorial model category and let $\mathcal{D}$ be any class of objects in $\mathcal{M}$. Then there is a set of morphisms S such that the class of $S^{h\perp}$-equivalences, $(S^{h\perp})^{h\perp}$, is equal to the class of $\mathcal{D}$-equivalences, $\mathcal{D}^{h\perp}$. Hence, $(\mathcal{D}^{h\perp})^{h\perp}$ equals the class of S-locals.*

Proof. By Lemma 4.4, there is a regular cardinal μ' such that, for every class of objects $\mathcal{E}$ in $\mathcal{M}$ the class of $\mathcal{E}$-equivalences $\mathcal{E}^{h\perp}$ is closed under μ'-filtered colimits. On the other hand, $\mathcal{M}$ is λ-presentable for some regular cardinal λ and so is the category of arrows of $\mathcal{M}$ [**AR94**, Corollary 1.54]. Since we are under Vopěnka's principle, by [**AR94**, Theorem 6.24 and Theorem 6.29] there exists a regular cardinal

λ' and a set of λ'-presentable $\mathcal{D}$-equivalences S' such that every morphism in $\mathcal{D}^{h\perp}$ is a λ'-filtered colimit of morphisms in S'. It then follows that there exists a cardinal $\mu \geqslant \max\{\lambda', \mu'\}$ and a set of $\mathcal{D}$-equivalences S such that every morphism in $\mathcal{D}^{h\perp}$ is a μ-filtered colimit of morphisms in S and $\mathcal{D}^{h\perp}$ is closed under μ-filtered colimits [**AR94**, Corollary 2.14].

Since every object in $\mathcal{D}$ is S-local, every S-equivalence is in $\mathcal{D}^{h\perp}$. Conversely, every g in $\mathcal{D}^{h\perp}$ is a μ-filtered colimit of morphisms in S. But now $S \subset (S^{h\perp})^{h\perp}$ and $(S^{h\perp})^{h\perp}$ is closed under μ'-filtered colimits by the first comment in the proof. In particular, $(S^{h\perp})^{h\perp}$ is also closed under μ-filtered colimits. This implies that g is in $(S^{h\perp})^{h\perp}$. $\square$

As a direct consequence of Lemma 4.5, we obtain an alternative proof of [**CC06**, Theorem 2.1] that avoids the assumption of the model category being simplicial. A different proof was given in [**RT03**, Theorem 2.3].

Corollary 4.6. *Assume that Vopěnka's principle holds. Let $\mathcal{M}$ be a left proper combinatorial model category. Then the left Bousfield localization with respect to any class of morphisms $\mathcal{S}$ in $\mathcal{M}$ exists.*

Proof. By Lemma 4.5, the class $\mathcal{S}^{h\perp}$ coincides with the class $D^{h\perp}$ with respect to a set of morphisms D. By [**Bar10**, Theorem 4.7], the Bousfield localization $L_D\mathcal{M}$ with respect to D exists in $\mathcal{M}$. Since the $D^{h\perp}$-equivalences coincide with the $\mathcal{S}^{h\perp}$-equivalences, $L_D\mathcal{M}$ is also the left Bousfield localization with respect to $\mathcal{S}$. $\square$

As noticed in [**CC06**], in general we cannot take S in the conclusion of Lemma 4.5 to consist of a single morphism. However, it is possible to reduce S to a single morphism if we assume, for instance, that we work in a pointed category. In particular, the next result applies to stable combinatorial model categories.

Corollary 4.7. *Assume that Vopěnka's principle holds. Let $\mathcal{M}$ be a pointed combinatorial model category, and let $\mathcal{D}$ be any class of objects in $\mathcal{M}$. Then there is a morphism f such that the class of f-equivalences is equal to the class of $\mathcal{D}$-equivalences.*

Proof. Let S be the set of morphisms and μ the regular cardinal as in the proof of Lemma 4.5, and let $f = \coprod s$ for all $s\colon A \to B$ in S. It is enough to prove that $S^{h\perp} = f^{h\perp}$. If X is an S-local object, then every component in the product

$$\prod \mathrm{map}(s, X) \simeq \mathrm{map}\left(\coprod s, X\right) = \mathrm{map}(f, X)$$

is a weak homotopy equivalence. Hence, X is f-local.

Conversely, if X is f-local, then $\prod \mathrm{map}(s, X)$ is a weak homotopy equivalence. Since $\mathcal{M}$ is pointed, for each s in S there is a retraction map r such that the composition

$$\mathrm{map}(s, X) \xrightarrow{\ r\ } \mathrm{map}(\coprod s, X) \simeq \prod \mathrm{map}(s, X) \longrightarrow \mathrm{map}(s, X)$$

is the identity. Hence, $\mathrm{map}(s, X)$ is a weak homotopy equivalence for each s in S. Thus, X is S-local. $\square$

The following result is a direct consequence of Proposition 4.3 and Lemma 4.5.

Theorem 4.8. *Assume that Vopěnka's principle holds. Let $\mathcal{M}$ be a combinatorial model category. If (L, ℓ) is any homotopy idempotent functor on $\mathcal{M}$, then there is a set of morphisms S such that the class of S-local objects coincides with the class*

of L-local objects. Furthermore, if $\mathcal{M}$ is pointed, then we can take S to consist of a single morphism.

Proof. Let $\mathcal{D}$ be the class of L-local objects. It follows from Proposition 4.3 that the class of $\mathcal{D}$-equivalences coincides with the class of L-equivalences. Then Lemma 4.5 and Corollary 4.7 finish the proof. $\square$

We next extend Theorem 4.8 to any cofibrantly generated model category, in particular, to any cellular model category [**Hir03**, Definition 12.1.1].

We remind the reader that a Quillen pair $F\colon \mathcal{N} \rightleftarrows \mathcal{M} : G$ is *homotopically surjective* if, for every fibrant object X in $\mathcal{M}$ and every cofibrant replacement $(GX)^c$ of GX, the induced morphism $F(GX)^c \to X$ is a weak equivalence [**Dug01**, Definition 3.1].

Proposition 4.9. *Assume that Vopěnka's principle holds. Let $F\colon \mathcal{N} \rightleftarrows \mathcal{M} : G$ be a homotopically surjective Quillen pair, and let $\mathcal{N}$ be combinatorial. If (L, ℓ) is a homotopy idempotent functor on $\mathcal{M}$, then there is a set of morphisms S in $\mathcal{M}$ such that the class of S-local objects coincides with the class of L-local objects. Furthermore, if $\mathcal{M}$ is pointed, then we can take S to consist of a single morphism.*

Proof. Let $\mathcal{D}$ be the class of objects of the form GX with X L-local. Notice that they are fibrant because G preserves fibrant objects. By Lemma 4.5, there is a set of morphisms S' in $\mathcal{N}$ such that the class of S'-locals coincide with $(\mathcal{D}^{h\perp})^{h\perp}$. Let $S = \{Ff^c \mid f \in S'\}$. We claim that the L-locals coincide with the S-locals.

Let X be L-local (thus fibrant). By hypothesis, the morphism $F(GX)^c \to X$ is a weak equivalence. By definition, GX is S'-local. Hence,

$$\mathrm{map}(f, GX) \simeq \mathrm{map}(f, G(F(GX)^c)^f) \simeq \mathrm{map}(F(f^c), F(GX)^c)$$

are weak homotopy equivalences for any f in S'. In particular, $X \simeq F(GX)^c$ is S-local.

Now let X be S-local. By definition, $\mathrm{map}(Ff^c, X) \simeq \mathrm{map}(f, GX)$ are weak homotopy equivalences for every f in S'. Hence, GX is S'-local; i.e., GX is in $(\mathcal{D}^{h\perp})^{h\perp}$.

By Proposition 4.3, to prove that GX is L-local it is enough to prove that, for every L-equivalence g, $\mathrm{map}(g, GX) \simeq \mathrm{map}(Fg^c, X)$ is a weak equivalence. Since we have already proved that GX is $D^{h\perp}$-local, the proof will be finished if we can show that g is a $\mathcal{D}$-equivalence if and only if Fg^c is an L-equivalence. But, by Proposition 4.3 again, both conditions are equivalent to the fact that $\mathrm{map}(g, GY) \simeq \mathrm{map}(Fg^c, Y)$ is a weak equivalence for all L-local objects Y. $\square$

The following result generalizes [**CC06**, Theorem 2.3] to cofibrantly generated model categories that are not necessarily locally presentable nor simplicial. It also gives a positive answer to a question by Farjoun in [**Far96**] for a broad family of model categories.

Corollary 4.10. *Assume that Vopěnka's principle holds. Let $\mathcal{M}$ be a cofibrantly generated model category. If (L, ℓ) is a homotopy idempotent functor on $\mathcal{M}$, then there is a set of morphisms S such that the class of S-local objects coincides with the class of L-local objects. Furthermore, if $\mathcal{M}$ is pointed, then we can take S to consist of a single morphism.*

Proof. Since we are assuming Vopěnka's principle, [**Rap09**, Theorem 1.1] implies that there is a Quillen equivalence (in particular, homotopically surjective) $\mathcal{N} \rightleftarrows \mathcal{M}$ where $\mathcal{N}$ is combinatorial. Hence, the result follows from Proposition 4.9. $\square$

The cofibrantly generated condition in Corollary 4.10 is necessary. In [**Cho05**] an example is given of a left Bousfield localization with respect to a class of morphisms in a (non cofibrantly generated) model category that cannot be a left Bousfield localization with respect to any set.

We next state the analogues of the main results in this section but for *augmented* homotopy idempotent functors. We omit the proofs since they are easily reproduced following the proofs for the coaugmented case. An *augmented homotopy idempotent* functor in a model category $\mathcal{M}$ is a functor $C\colon \mathcal{M} \to \mathcal{M}$ together with a natural transformation $\varepsilon\colon C \to 1$ such that C sends weak equivalences to weak equivalences and the natural morphisms $\varepsilon_{CX}, C\varepsilon_X\colon CCX \to CX$ are equal in the homotopy category $\mathrm{Ho}(\mathcal{M})$ and both are weak equivalences for every object X in $\mathcal{M}$. The following result generalizes [**Cho07**, Theorem 1.4] to combinatorial model categories not necessarily simplicial.

Corollary 4.11. *Assume that Vopěnka's principle holds. Let $\mathcal{M}$ be a right proper combinatorial model category. Then the right Bousfield localization with respect to any class of objects in $\mathcal{M}$ exists.* $\square$

The following result generalizes [**Cho07**, Theorem 2.1] to cofibrantly generated model categories not necessarily locally presentable nor simplicial.

Theorem 4.12. *Assume that Vopěnka's principle holds. Let $\mathcal{M}$ be a cofibrantly generated model category. If (C, ε) is a homotopy augmented idempotent functor on $\mathcal{M}$, then there is a set of objects D such that the class of D-cellular equivalences coincides with the class of C-cellular equivalences. Furthermore, if $\mathcal{M}$ is pointed, then we can take D to consist of a single object.* $\square$

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