Homology, Homotopy and Applications, vol. 17(1), 2015, pp.191–201

COMPARING COMMUTATIVE AND ASSOCIATIVE UNBOUNDED DIFFERENTIAL GRADED ALGEBRAS OVER $\mathbb{Q}$ FROM A HOMOTOPICAL POINT OF VIEW

ILIAS AMRANI

(*communicated by Brooke Shipley*)

Abstract

In this paper we establish a faithfulness result, in a homotopical sense, between a subcategory of the model category of augmented differential graded commutative algebras CDGA and a subcategory of the model category of augmented differential graded algebras DGA over the field of rational numbers $\mathbb{Q}$.

1. Introduction

It is well known that the forgetful functor from the category of commutative k-algebras to the category of associative k-algebras is fully faithful. We have an analogue result between the category of unbounded differential graded commutative k-algebras dgCAlg_k and the category of unbounded differential graded associative algebras dgAlg_k. The question that we want to explore is the following: Suppose that $k = \mathbb{Q}$; we want to know if it is true that forgetful functor $U : \mathsf{dgCAlg}_k \to \mathsf{dgAlg}_k$ induces a fully faithful functor at the level of homotopy categories

$$\mathbf{R}U : \mathrm{Ho}(\mathsf{dgCAlg}_k) \to \mathrm{Ho}(\mathsf{dgAlg}_k).$$

The answer is **no**. A nice and easy counterexample was given by Lurie [**10**]. He has considered $k[x, y]$ the free commutative CDGA in two variables concentrated in degree 0. It follows obviously that

$$\mathrm{Ho}(\mathsf{dgCAlg}_k)(k[x, y], S) \simeq \mathrm{H}^0(S) \oplus \mathrm{H}^0(S),$$

while

$$\mathrm{Ho}(\mathsf{dgAlg}_k)(k[x, y], S) \simeq \mathrm{H}^0(S) \oplus \mathrm{H}^0(S) \oplus \mathrm{H}^{-1}(S).$$

Something nice happens if we consider the category of augmented CDGA denoted by dgCAlg_k^* and augmented DGA denoted by dgAlg_k^*.

Supported by the project CZ.1.07/2.3.00/20.0003 of the Operational Programme Education for Competitiveness of the Ministry of Education, Youth and Sports of the Czech Republic.
Received January 26, 2014, revised July 30, 2014; published on April 15, 2015.
2010 Mathematics Subject Classification: Primary 55, Secondary 14, 16, 18.
Key words and phrases: DGA, CDGA, mapping space, rational homotopy theory, derived algebraic geometry.
Article available at http://dx.doi.org/10.4310/HHA.2015.v17.n1.a9

Theorem 1.1 (Theorem 4.1)**.** *For any R and S in dgCAlg_k^*, the induced map by the forgetful functor*

$$\Omega\mathrm{Map}_{\mathsf{dgCAlg}_k^*}(R,S) \to \Omega\mathrm{Map}_{\mathsf{dgAlg}_k^*}(R,S)$$

has a retract; in particular

$$\pi_i\mathrm{Map}_{\mathsf{dgCAlg}_k^*}(R,S) \to \pi_i\mathrm{Map}_{\mathsf{dgAlg}_k^*}(R,S)$$

is injective $\forall\ i > 0$.

Definition 1.2 ([**8**],[**7**])**.** Let M be a model category and let $a,\ a'$ be cofibrant objects and $b,\ b'$ be fibrant objects. The (derived) mapping space denoted by $\mathrm{Map}_{\mathsf{M}}$ is a simplicial set having the following properties:

- $\pi_0\mathrm{Map}(a,b) \cong \mathrm{Ho}(\mathsf{M})(a,b)$, where $Ho(\mathsf{M})$ is the homotopy category of M.
- For any weak equivalence $a \to a'$, we have a weak equivalence of simplicial sets $\mathrm{Map}(a',b) \to \mathrm{Map}(a,b)$.
- For any weak equivalence $b \to b'$, we have a weak equivalence of simplicial sets $\mathrm{Map}(a,b) \to \mathrm{Map}(a,b')$.

Remark 1.3. [**4**, Theorem 2.12] In our work we use only the formal properties of the derived mapping space in a model category. For any Quillen adjuction between model categories

$$\mathsf{M} \underset{U}{\overset{G}{\rightleftarrows}} \mathsf{N}$$

and for any cofibrant object $a \in \mathsf{M}$ and any fibrant object $c \in \mathsf{N}$, we have the following (zig-zag) equivalence of simplicial sets:

$$\mathrm{Map}_{\mathsf{N}}(Ga,c) \sim \mathrm{Map}_{\mathsf{M}}(a,Uc).$$

Let S be a differential graded commutative algebra which is a "loop" of another CDGA algebra A, i.e., $S = \mathsf{Holim}(k \to A \leftarrow k)$, where the homotopy limit is taken in the model category dgCAlg_k. A direct consequence of our theorem is that the right derived functor $\mathbf{R}U$ is a faithful functor, i.e., the induced map $\mathrm{Ho}(\mathsf{dgCAlg}_k^*)(R,S) \to \mathrm{Ho}(\mathsf{dgAlg}_k^*)(R,S)$ is injective.

Interpretation of the result in the derived algebraic geometry

Rationally, any pointed topological X space can be viewed as an augmented (connective) commutative differential graded algebra via its cochain complex $C^*(X,\mathbb{Q})$. In the case where X is a simply connected rational space, the cochain complex $C^*(X,\mathbb{Q})$ carries all the homotopical information about X, by the Sullivan theorem [**6**]. Moreover, the bar construction $\mathrm{B}C^*(X,\mathbb{Q})$ is identified (as E_∞-DGA) to $C^*(\Omega X,\mathbb{Q})$ and $\Omega C^*(X,\mathbb{Q})$ is identified (as E_∞-DGA) to $C^*(\Sigma X,\mathbb{Q})$; cf. [**5**]. This interpretation allows us to make the following definition: A generalized rational pointed space is an augmented commutative differential graded $\mathbb{Q}$-algebra (possibly unbounded). In the same spirit, we define a pointed generalized **noncommutative rational space** as an augmented differential graded $\mathbb{Q}$-algebra (possibly unbounded). Let A be any augmented CDGA (resp. DGA); we will call a CDGA (resp. DGA) of the form ΩA an *op-suspended* CDGA (resp. DGA). Our Theorem 4.1 can be interpreted as follows:

The homotopy category of op-suspended generalized commutative rational spaces is a subcategory of the homotopy category of op-suspended generalized noncommutative rational spaces.

2. DGA, CDGA, and E_∞-DGA

We work in the setting of unbounded differential graded k-modules dgMod_k. This is a symmetric monoidal closed model category (k is a commutative ring). We denote the category of (reduced) operads in dgMod_k by Op_k. We follow notations and definitions of [**2**]; we say that an operad P is *admissible* if the category of $\mathsf{P}-\mathsf{dgAlg}_k$ admits a model structure where the fibrations are degree wise surjections and weak equivalences are quasi-isomorphisms. For any map of operads $\phi : \mathsf{P} \to \mathsf{Q}$ we have an induced adjunction of the corresponding categories of algebras:

$$\mathsf{P}-\mathsf{dgAlg}_k \underset{\phi^*}{\overset{\phi_!}{\rightleftarrows}} \mathsf{Q}-\mathsf{dgAlg}_k.$$

A Σ-cofibrant operad P is an operad such that $\mathsf{P}(n)$ is $k[\Sigma_n]$-cofibrant in $\mathsf{dgMod}_{k[\Sigma_n]}$. Any cofibrant operad P is a Σ-cofibrant operad [**2**, Proposition 4.3]. We denote the associative operad by Ass and the commutative operad by Com. The operad Ass is an admissible operad and Σ-cofibrant, while the operad Com is not admissible in general. In the rational case, when $k = \mathbb{Q}$ the operad Com is admissible and Σ-cofibrant. More generally any cofibrant operad P is admissible [**2**, Proposition 4.1, Remark 4.2]. We define a symmetric tensor product of operads by the formulae

$$[\mathsf{P} \otimes \mathsf{Q}](n) = \mathsf{P}(n) \otimes \mathsf{Q}(n), \qquad \forall\, n \in \mathbb{N}.$$

Lemma 2.1. *Suppose that $\phi : \mathsf{Ass} \to \mathsf{P}$ is a cofibration of operads. The operad P is admissible and the functor $\phi^* : \mathsf{P}-\mathsf{dgAlg}_k \to \mathsf{dgAlg}_k$ preserves fibrations, weak equivalences, and cofibrations with cofibrant domain in the underlying category dgMod_k.*

Proof. First of all, the operad P is admissible; indeed we use the cofibrant resolution $r : \mathsf{E}_\infty \to \mathsf{Com}$ and consider the following pushout in Op_k given by:

$$\begin{array}{ccc} \mathsf{Ass}_\infty & \hookrightarrow & \mathsf{E}_\infty \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\alpha} \\ \mathsf{Ass} & \xrightarrow{f} & \mathsf{E}'_\infty \end{array}$$

where Ass_∞ is a cofibrant replacement of Ass in Op_k and $\mathsf{Ass}_\infty \to \mathsf{E}_\infty$ is a cofibration which factors $\mathsf{Ass}_\infty \to \mathsf{Ass} \to \mathsf{Com}$. Since the category Op_k is left proper in the sense of [**13**, Theorem 3], we have that $\alpha : \mathsf{E}_\infty \to \mathsf{E}'_\infty$ is an equivalence. We denote by I the unit interval in the category dgMod_k which is strictly coassociative [**12**, p. 503]; i.e., there is a map of operads $\mathsf{Ass} \to \mathsf{End}^{op}(I)$. Moreover, there is a map of operads $\mathsf{E}_\infty \to \mathsf{End}^{op}(I)$ which endows I with the structure of E_∞-coalgebra [**2**, Remark 4.2]. Since I is a strict coalgebra (in particular an Ass_∞-coalgebra), the operad map $\mathsf{Ass}_\infty \to$

$\mathsf{End}^{op}(I)$ factors through Ass; i.e., we have two compatible maps of operads:

$$\begin{array}{ccccc} \mathsf{Ass}_\infty & \hookrightarrow & \mathsf{E}_\infty & & \\ \downarrow \sim & & \downarrow \alpha & \searrow & \\ \mathsf{Ass} & \overset{f}{\hookrightarrow} & \mathsf{E}'_\infty & \dashrightarrow & \\ & \searrow & & & \mathsf{End}^{op}(I) \end{array} \tag{1}$$

By the universality of the pushout, we have a map of operads $\mathsf{E}'_\infty \to \mathsf{End}^{op}(I)$. This means that the unit interval I has a structure of E'_∞-coalgebra [**2**, p. 4]. Moreover, we have a commutative diagram in Op_k given by

$$\begin{array}{ccccc} \mathsf{Ass} & \xrightarrow{\Delta} & \mathsf{Ass} \otimes \mathsf{Ass} & \xrightarrow{\phi \otimes f} & \mathsf{P} \otimes \mathsf{E}'_\infty \\ \Big\downarrow \phi & & & & \Big\downarrow id\otimes r \ \sim \\ \mathsf{P} & & \xrightarrow{\quad id \quad} & & \mathsf{P} \otimes \mathsf{Com} = \mathsf{P} \end{array}$$

where the operad map $r : \mathsf{E}'_\infty \to \mathsf{Com}$ is obtained by the universal property of the pushout (1) and the diagonal map $\Delta : \mathsf{Ass} \to \mathsf{Ass} \otimes \mathsf{Ass}$ is induced by the diagonals $\Sigma_n \to \Sigma_n \times \Sigma_n$; therefore the commutativity of the diagram is a consequence of the co-unit property of the diagonal map Δ and universal choice of r. Hence, the map $\mathsf{P} \otimes \mathsf{E}'_\infty \to \mathsf{P}$ admits a section. It implies by [**2**, Proposition 4.1] that P is admissible and Σ-cofibrant. Since all objects in $\mathsf{P} - \mathsf{dgAlg}_k$ are fibrant and ϕ^* is a right Quillen adjoint, it preserves fibrations and weak equivalences.

Since P is an admissible operad, we have a Quillen adjunction

$$\mathsf{dgAlg}_k \underset{\phi^*}{\overset{\phi_!}{\rightleftarrows}} \mathsf{P} - \mathsf{dgAlg}_k,$$

where the functor ϕ^* is identified to the forgetful functor. Moreover, the model structure on $\mathsf{P} - \mathsf{dgAlg}_k$ is the transferred model structure from the cofibrantly generated model structure dgAlg_k via the adjunction $\phi_!, \phi^*$. Suppose that $f : A \to B$ is a cofibration in $\mathsf{P} - \mathsf{dgAlg}_k$ such that A is cofibrant in dgMod_k. We factor this map as a cofibration followed by a trivial fibration

$$A \overset{i}{\hookrightarrow} P \underset{\sim}{\overset{p}{\twoheadrightarrow}} B$$

in the category dgAlg_k; therefore i is a cofibration [**14**, Proposition 2.3 (3)] (Toën's initial argument is for cofibraant objects, but it works for cofibrations, i.e., the forgetful functor $\mathsf{dgAlg}_k \to \mathsf{dgMod}_k$ preserves cofibrations) and p is obviously a trivial fibration in dgMod_k. By [**11**, Lemma 4.1.16], we have an induced map of endomorphism operads of diagrams [**11**, Section 4.1.1]:

$$\mathrm{End}_{\{A \to P \to B\}} \to \mathrm{End}_{\{A \to B\}}$$

which is a trivial fibration of operads since p is a trivial fibration. Notice that Rezk's arguments [**11**] are performed in the category of simplicial k-modules and are formally transposable in the context of differential graded k-modules. By definition of our endomorphism operads of diagrams, we have the following commutative diagram in Op_k:

$$\begin{array}{ccc} \mathsf{Ass} & \longrightarrow & \mathrm{End}_{\{A\to P\to B\}} \\ \big\downarrow & \nearrow & \big\downarrow \sim \\ \mathsf{P} & \longrightarrow & \mathrm{End}_{\{A\to B\}} \end{array}$$

where the first horizontal map of operads translates the fact that $A \to P \to B$ are maps in dgAlg_k; respectively, the second horizontal map translates the fact that $A \to B$ is a map of P-algebra. Since Op_k is a model category, it implies that we have a lifting map of operads $\mathsf{P} \to \mathrm{End}_{\{A\to P\to B\}}$; hence i and p are maps of $\mathsf{P}-\mathsf{dgAlg}_k$. Therefore, we consider the following commutative square in the category $\mathsf{P}-\mathsf{dgAlg}_k$:

$$\begin{array}{ccc} A & \xrightarrow{i} & P \\ f\big\downarrow & \overset{r}{\nearrow} & \big\downarrow p \ \sim \\ B & \xrightarrow{id} & B \end{array}$$

The lifting map r exists since $\mathsf{P}-\mathsf{dgAlg}_k$ is a model category. We conclude that $p \circ r = id$ and $r \circ f = i$, which means that f is a retract of i; hence f is a cofibration in dgAlg_k. □

Remark 2.2. With the same notation as in Lemma 2.1, if A is a cofibrant object in $\mathsf{P}-\mathsf{dgAlg}_k$ then A is a cofibrant object in dgMod_k. Indeed $k \to A$ is a cofibration in $\mathsf{P}-\mathsf{dgAlg}_k$; by the previous lemma $k \to A$ is a cofibration in dgAlg_k. Therefore, $k \to A$ is a cofibration in dgMod_k.

3. Suspension in CDGA and DGA

We denote the operad E'_∞ of the previous section by E_∞, and $k = \mathbb{Q}$.

3.1. E_∞-DGA

We have a map of operads $\mathsf{Ass} \to \mathsf{Com}$, which we factor as cofibration followed by a trivial fibration:

$$\mathsf{Ass} \hookrightarrow \mathsf{E}_\infty \overset{\sim}{\twoheadrightarrow} \mathsf{Com}.$$

As a consequence, we have the following Quillen adjunctions:

$$\mathsf{dgAlg}_k \underset{U}{\overset{Ab_\infty}{\rightleftarrows}} \mathsf{E}_\infty\mathsf{dgAlg}_k \underset{U'}{\overset{str}{\rightleftarrows}} \mathsf{dgCAlg}_k.$$

These adjunctions have the following properties:

- The functors U' and $U \circ U'$ are the forgetful functors; they are fully faithful (cf. Propositions 3.3 and 3.2).

- The functors str, U' form a Quillen equivalence since $k = \mathbb{Q}$ (cf. [**9**, Corollary 1.5, Part II]). The functor str is the strictification functor.
- The functors Ab_∞, U form a Quillen pair.
- The composition $str \circ Ab_\infty$ is the abelianization functor $Ab : \mathsf{dgAlg}_k \to \mathsf{dgCAlg}_k$.
- The functors str and Ab are idempotent functors (cf. Propositions 3.3 and 3.2).

The model categories dgCAlg_k^*, dgAlg_k^*, and $\mathsf{E}_\infty\mathsf{dgAlg}_k^*$ are pointed model categories. It is natural to introduce the suspension functors in these categories.

Definition 3.1. Let C be any pointed model category. We denote the point by 1, and let $A \in \mathsf{C}$; a suspension ΣA is defined as $\mathsf{hocolim}(1 \leftarrow A \to 1)$.

Proposition 3.2. *Any map $f : A \to S$ in $\mathsf{E}_\infty\mathsf{dgAlg}_k$, where S is in dgCAlg_k, factors in a unique way as $A \to str(A) \to S$ and the forgetful functor $U' : \mathsf{dgCAlg}_k \to \mathsf{E}_\infty\mathsf{dgAlg}_k$ is fully faithful. Moreover, the unit of the adjunction $\nu_A : A \to str(A)$ is a fibration.*

Proof. Suppose that we have a map $h : R \to S$ in $\mathsf{E}_\infty\mathsf{dgAlg}_k$ such that R and S are objects in dgCAlg_k. By definition of the operad E_∞ the map h respects the multiplication; therefore h is a morphism in dgCAlg_k since R and S are commutative differential graded algebras. The forgetful functor $U' : \mathsf{dgCAlg}_k \to \mathsf{E}_\infty\mathsf{dgAlg}_k$ is fully faithful: this implies that $str(S) = S$ for any $S \in \mathsf{dgCAlg}_k$. We have a commutative diagram induced by the unit ν of the adjunction (U', str):

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & S \\ {\scriptstyle \nu_A}\downarrow & & \downarrow{\scriptstyle \nu_S = id} \\ str(A) & \xrightarrow{str(f)} & str(S) = S \end{array}$$

We conclude that $f = str(f) \circ \nu_A$. The surjectivity of the ν_A follows from the universal property of $str(A)$. Hence, ν_A is a fibration in $\mathsf{E}_\infty\mathsf{dgAlg}_k$. ☐

Proposition 3.3. *Any map $f : A \to S$ in dgAlg_k, where S is in dgCAlg_k, factors in a unique way as $A \to Ab(A) \to S$, and the forgetful functor $U \circ U' : \mathsf{dgCAlg}_k \to \mathsf{dgAlg}_k$ is fully faithful. Moreover, the unit of the adjunction $\nu_A : A \to Ab(A)$ is a fibration.*

Proof. The proof is the same as in Proposition 3.2. ☐

Proposition 3.4. *Suppose that we have a trivial cofibration $k \to \underline{k}$ in $\mathsf{E}_\infty\mathsf{dgAlg}_k$. Then the universal map $\pi\colon Ab(\underline{k}) \to str(\underline{k})$ is a trivial fibration and admits a section in the category dgCAlg_k.*

Proof. We consider the following commutative diagram in $\mathsf{E}_\infty\mathsf{dgAlg}_k$:

$$\begin{array}{ccc} k & \xrightarrow{\quad \sim \quad} & \underline{k} \\ {\scriptstyle id}\downarrow & & \downarrow \\ k = str(k) & \xrightarrow{\sim} & str(\underline{k}) \end{array}$$

The map $\underline{k} \to str(\underline{k})$ is an equivalence since str is a left Quillen functor; the same thing holds for the abelianization functor. More precisely, the forgetful functor $\mathsf{E}_\infty\mathsf{dgAlg}_k \to$

dgAlg_k preserves cofibration (Lemma 2.1) ($\mathsf{P} = \mathsf{E}_\infty$); therefore the map $Ab(k) \to Ab(\underline{k})$ is a weak equivalence in dgCAlg_k. It follows that we have a commutative diagram in dgAlg_k:

$$\begin{array}{ccc} k & \xrightarrow{\sim} & \underline{k} \\ {\scriptstyle id}\downarrow & & \downarrow \\ k = Ab(k) & \xrightarrow{\sim} & Ab(\underline{k}) \end{array}$$

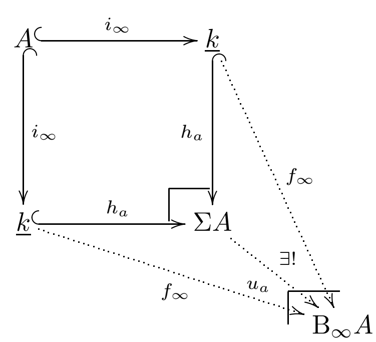
i.e., $\underline{k} \to Ab(\underline{k})$ is a trivial fibration, since $\underline{k} \to Ab(\underline{k})$ is surjective by definition of the abelianization functor. On the other hand the map $\underline{k} \to str(\underline{k})$ is a trivial fibration in $\mathsf{E}_\infty\mathsf{dgAlg}_k$ (Proposition 3.2) and hence in dgAlg_k; therefore it can be factored (cf. Proposition 3.3) as $\underline{k} \to Ab(\underline{k}) \to str(\underline{k})$, where $Ab(\underline{k}) \to str(\underline{k})$ is a trivial fibration between cofibrant objects in dgCAlg_k. It follows that we have a retract $l : str(\underline{k}) \to Ab(\underline{k})$. □

Definition 3.5. The suspension functor in the pointed model categories dgCAlg^*_k, dgAlg^*_k, and $\mathsf{E}_\infty\mathsf{dgAlg}^*_k$ are denoted by B (resp. Σ and resp. B_∞).

Remark 3.6. The notation Σ is a generic notation for the suspension functor in a pointed model category. In the case of dgCAlg^*_k and dgCAlg^*_k we have used the notation B and B_∞ to make a link with the Bar construction for commutative (E_∞) differential graded algebra; this coincides with the generic suspension functor.

Lemma 3.7. *Suppose that A is a cofibrant object in $\mathsf{E}_\infty\mathsf{dgAlg}^*_k$. Then $str(\mathrm{B}_\infty A)$ is a retract of $Ab(\Sigma A)$ in the category dgCAlg_k.*

Proof. First of all, if a map f is associative (or commutative, or an E_∞ map) we put index f_a (or f_c, or f_∞, respectively). Notice that by definition of the operad E_∞ any E_∞-map is a strictly associative map. Suppose that A is a cofibrant object in $\mathsf{E}_\infty\mathsf{dgAlg}_k$. Consider the following commutative square:



where ΣA is the (homotopy in Lemma 2.1) pushout in dgAlg_k and $\mathrm{B}_\infty A$ is the (homotopy) pushout in $\mathsf{E}_\infty\mathsf{dgAlg}_k$. By Proposition 3.2 and Proposition 3.3 we have the following commutative square in dgAlg_k:

$$\begin{array}{ccc} \Sigma A & \xrightarrow{\quad u_a \quad} & \mathrm{B}_\infty A \\ \downarrow & & \downarrow \\ Ab(\Sigma A) & \xrightarrow{\quad x_c \quad} & str[\mathrm{B}_\infty A] = \mathrm{B}[str(A)]. \end{array}$$

By Proposition 3.4 we have an inclusion of commutative differential graded algebras $l_c : str(\underline{k}) \to Ab(\underline{k})$ and after strictification we obtain another homotopy pushout (inner) square in dgCAlg_k given by

$$\begin{array}{ccccc} str(A) & \xhookrightarrow{\quad i_c \quad} & str(\underline{k}) & \xrightarrow{\quad l_c \quad} & Ab(\underline{k}) \\ {\scriptstyle i_c}\downarrow & & \downarrow{\scriptstyle f_c} & & \\ str(\underline{k}) & \xhookrightarrow{\quad f_c \quad} & \mathrm{B}[str(A)] & & \downarrow{\scriptstyle h_c} \\ {\scriptstyle l_c}\downarrow & & & \overset{\exists!}{\underset{u_c}{\searrow}} & \\ Ab(\underline{k}) & & \xhookrightarrow{\qquad h_c \qquad} & & Ab(\Sigma(A)) \end{array}$$

In order to prove that $\mathrm{B}[str(A)]$ is a retract of $Ab(\Sigma(A))$ it is sufficient to prove that

$$x_c \circ h_c \circ l_c = f_c.$$

By Proposition 3.2 and Proposition 3.3, the composition E_∞-maps

$$\underline{k} \xrightarrow{f_\infty} \mathrm{B}_\infty A \longrightarrow str[\mathrm{B}_\infty A]$$

can be factored in a unique way as

$$\underline{k} \longrightarrow Ab(\underline{k}) \xrightarrow{\pi} str(\underline{k}) \xrightarrow{\alpha_c} str[\mathrm{B}_\infty A] = \mathrm{B}[str(A)].$$

By unicity, $\alpha_c = f_c$. On the other hand, using the first pushout in $\mathsf{E}_\infty\mathsf{dgAlg}_k$, the previous composition $\underline{k} \to str[\mathrm{B}_\infty A]$ is factored as

$$\underline{k} \xrightarrow{h_a} \Sigma A \longrightarrow Ab(\Sigma A) \xrightarrow{x_c} str[\mathrm{B}_\infty A].$$

We summarize the previous remarks in the following commutative diagram:

$$\begin{array}{ccccccc} \underline{k} & \xrightarrow{pr} & Ab(\underline{k}) & \xrightarrow{\pi} & str(\underline{k}) & \xrightarrow{f_c} & str[\mathrm{B}_\infty A] \\ \downarrow{\scriptstyle id} & & \vdots{\scriptstyle h_c = Ab(h_a)} & & & & \downarrow{\scriptstyle id} \\ \underline{k} & \longrightarrow & Ab(\Sigma A) & & \xrightarrow{\qquad x_c \qquad} & & str[\mathrm{B}_\infty A] \end{array}$$

By definition of h_a, the dotted map h_c makes the left square commutative. Since the whole square is commutative and the map pr is surjective, we conclude that

$x_c \circ h_c = f_c \circ \pi$. Since the map $l_c : Str(\underline{k}) \to Ab(\underline{k})$ is a retract of π (cf. 3.4), i.e., $\pi \circ l_c = id$, we conclude that $x_c \circ h_c \circ l_c = f_c$. Finally, by unicity of the pushout, we deduce that the following composition

$$\mathrm{B}[str(A)] \xrightarrow{u_c} Ab(\Sigma A) \xrightarrow{x_c} \mathrm{B}[str(A)]$$

is identity. □

4. Main result and applications

Theorem 4.1. *For any R and S in dgCAlg_k^*, the induced map by the forgetful functor*

$$\Omega\mathrm{Map}_{\mathsf{dgCAlg}_k^*}(R,S) \to \Omega\mathrm{Map}_{\mathsf{dgAlg}_k^*}(R,S)$$

has a retract; in particular

$$\pi_i\mathrm{Map}_{\mathsf{dgCAlg}_k^*}(R,S) \to \pi_i\mathrm{Map}_{\mathsf{dgAlg}_k^*}(R,S)$$

is injective $\forall\ i > 0$.

Proof. Suppose that R is a cofibrant object in $\mathsf{E}_\infty\mathsf{dgAlg}_k$ and S any object in dgCAlg_k. By adjunction, we have that

$$\begin{aligned}\Omega\mathrm{Map}_{\mathsf{dgCAlg}_k^*}(str(R),S) &\sim \mathrm{Map}_{\mathsf{dgCAlg}_k^*}(\mathrm{B}[str(R)],S) &(2)\\ &\sim \mathrm{Map}_{\mathsf{dgCAlg}_k^*}(str[\mathrm{B}_\infty R],S) &(3)\\ &\sim \mathrm{Map}_{\mathsf{E}_\infty\mathsf{dgAlg}_k^*}(\mathrm{B}_\infty R,S) &(4)\\ &\sim \Omega\mathrm{Map}_{\mathsf{E}_\infty\mathsf{dgAlg}_k^*}(R,S). &(5)\end{aligned}$$

By Lemma 3.7, we have a retract

$$\mathrm{Map}_{\mathsf{dgCAlg}_k^*}(\mathrm{B}[str(R)],S) \to \mathrm{Map}_{\mathsf{dgCAlg}_k^*}(Ab(\Sigma R),S) \to \mathrm{Map}_{\mathsf{dgCAlg}_k^*}(\mathrm{B}[str(R)],S).$$

Again by adjunction:

$$\mathrm{Map}_{\mathsf{dgCAlg}_k^*}(Ab(\Sigma R),S) \sim \mathrm{Map}_{\mathsf{dgAlg}_k^*}(\Sigma R,S) \sim \Omega\mathrm{Map}_{\mathsf{dgAlg}_k^*}(R,S).$$

We conclude that

$$\Omega\mathrm{Map}_{\mathsf{E}_\infty\mathsf{dgAlg}_k^*}(R,S) \xrightarrow{U} \Omega\mathrm{Map}_{\mathsf{dgAlg}_k^*}(R,S) \longrightarrow \Omega\mathrm{Map}_{\mathsf{E}_\infty\mathsf{dgAlg}_k^*}(R,S)$$

is a retract. Hence, the forgetful functor U induces an injective map on homotopy groups, i.e.,

$$\pi_i\mathrm{Map}_{\mathsf{dgCAlg}_k^*}(str(R),S) \simeq \pi_i\mathrm{Map}_{\mathsf{E}_\infty\mathsf{dgAlg}_k^*}(R,S) \to \pi_i\mathrm{Map}_{\mathsf{dgAlg}_k^*}(R,S)$$

is injective $\forall\ i > 0$. □

4.1. Rational homotopy theory

We give an application of our Theorem 4.1 in the context of rational homotopy theory. Let X be a simply connected rational space such that π_iX is finite dimensional $\mathbb{Q}$-vector space for each $i > 0$. Let $C^*(X)$ be the differential graded $\mathbb{Q}$-algebra cochain associated to X which is a connective $\mathsf{E}_\infty\mathsf{dgAlg}_k$. If $R = C^*(X)$ and $S = \mathbb{Q}$ then by the Sullivan theorem $\pi_iX \simeq \pi_i\mathrm{Map}_{\mathsf{dgCAlg}_k}(C^*(X),\mathbb{Q})$. By Theorem 4.1, we have that

$\pi_i X$ is a sub $\mathbb{Q}$-vector space of $\pi_i \mathrm{Map}_{\mathsf{dgAlg}_k}(R,S)$. On the other hand [**1**], since $C^*(X)$ is connective, we have that for any $i > 1$

$$\pi_i \mathrm{Map}_{\mathsf{dgAlg}_k}(C^*(X), \mathbb{Q}) \simeq \mathrm{HH}^{1-i}(C^*(X), \mathbb{Q}),$$

where HH^* is the Hochschild cohomology. Since we have assumed finiteness condition on X, we have that

$$\mathrm{HH}^{1-i}(C^*(X), \mathbb{Q}) \simeq \mathrm{HH}_{i-1}(C^*(X), \mathbb{Q}).$$

The functor $C^*(-,\mathbb{Q}) : \mathsf{Top}^{op} \to \mathsf{E}_\infty\mathsf{dgAlg}_k$ commutes with finite homotopy limits, where Top is the category of simply connected spaces. Hence,

$$\mathrm{HH}_{-1+i}(C^*(X), \mathbb{Q}) = \mathrm{H}^{i-1}[C^*(X) \otimes^{\mathbf{L}}_{C^*(X\times X)} \mathbb{Q}] \simeq \mathrm{H}^{i-1}(\Omega X, \mathbb{Q}).$$

We conclude that $\pi_i X$ is a sub $\mathbb{Q}$-vector space of $\mathrm{H}^{i-1}(\Omega X, \mathbb{Q})$.

More generally by the Block-Lazarev result [**3**] on rational homotopy theory and [**1**], we have an injective map of $\mathbb{Q}$-vector spaces

$$\mathrm{AQ}^{-i}(C^*(X), C^*(Y)) \to \mathrm{HH}^{-i+1}(C^*(X), C^*(Y)),$$

where the $C^*(X)$-(bi)modules structure on $C^*(Y)$ is given by $C^*(X) \to \mathbb{Q} \to C^*(Y)$, and AQ^* is the André-Quillen cohomology. We also assume that X and Y are simply connected and $i > 1$.

More generally,

$$\pi_i \mathrm{Map}_{\mathsf{E}_\infty\mathsf{dgAlg}_k}(R,S) = \mathrm{AQ}^{-i}(R,S) \to \mathrm{HH}^{-i+1}(R,S) = \pi_i \mathrm{Map}_{\mathsf{dgAlg}_k}(R,S)$$

is an injective map of $\mathbb{Q}$-vector spaces for all $i > 1$ and any augmented E_∞-differential graded connective $\mathbb{Q}$-algebras R and S, where the action of S on R is given by $S \to \mathbb{Q} \to R$.

Acknowledgments

I'm grateful to Benoît Fresse for his nice explanation of Lemma 2.1; the key point of the proof is due to him. I would like to thank the referee for many useful suggestions.

References

[**1**] I. Amrani, The mapping space of unbounded differential graded algebras, *arXiv preprint arXiv:1303.6895*, 2013.

[**2**] C. Berger and I. Moerdijk, Axiomatic homotopy theory for operads, *Commentarii Mathematici Helvetici*, **78**(4):805–831, 2003.

[**3**] J. Block and A. Lazarev, André–Quillen cohomology and rational homotopy of function spaces, *Advances in Mathematics*, **193**(1):18–39, 2005.

[**4**] W. Dwyer and K. Hess, Long knots and maps between operads, *Geometry & Topology*, **16**:919–955, 2012.

[**5**] B. Fresse, The bar complex of an E-infinity algebra, *Advances in Mathematics*, **223**(6):2049–2096, 2010.

[**6**] K. Hess, Rational homotopy theory: a brief introduction, *Contemporary Mathematics*, **436**:175, 2007.

[**7**] P. Hirschhorn, Model categories and their localizations, *Mathematical Surveys and Monographs*, **99**, 2002.

[**8**] M. Hovey, Model categories, *Mathematical Surveys and Monographs*, **63**, 1999.

[**9**] I. Kriz and J.P. May, Operads, algebras, modules and motives, *Société mathématique de France*, 1995.

[**10**] J. Lurie, Model structure of commutative dg-algebras inside all dg-algebras. `http://mathoverflow.net/questions/5031/`

[**11**] C.W. Rezk, Spaces of algebra structures and cohomology of operads, *PhD thesis, Massachusetts Institute of Technology*, 1996.

[**12**] S. Schwede and B. Shipley, Algebras and modules in monoidal model categories, *Proceedings of the London Mathematical Society*, **80**(2):491–511, 2000.

[**13**] M. Spitzweck, Operads, algebras and modules in general model categories, arXiv preprint math/0101102, 2001.

[**14**] B. Toën, The homotopy theory of dg-categories and derived Morita theory, *Inventiones mathematicae*, **167**(3):615–667, 2007.

Ilias Amrani `ilias.amranifedotov@gmail.com`

Department of Mathematics, Masaryk University, Kotlarska 2, Brno, Czech Republic