# STRICTLY COMMUTATIVE MODELS
# FOR $E_\infty$ QUASI-CATEGORIES

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(*communicated by Brooke Shipley*)

*Abstract*

In this short note we show that $E_\infty$ quasi-categories can be replaced by strictly commutative objects in the larger category of diagrams of simplicial sets indexed by finite sets and injections. This complements earlier work on diagram spaces by Christian Schlichtkrull and the second author.

## 1.  Introduction

An $E_\infty$ space is a space with a multiplicative structure encoded by the action of an $E_\infty$ operad, i.e., an operad consisting of contractible spaces with a free $\Sigma_n$-action. It is shown in joint work by Christian Schlichtkrull and the second author [**SS12**] that $E_\infty$ spaces can be rigidified to strictly commutative objects if one passes to a larger category of $\mathcal{I}$-spaces: if $\mathcal{I}$ denotes the category of finite sets $\mathbf{n} = \{1, \dots, n\}$ and injective maps, then the functor category $\mathrm{sSet}^{\mathcal{I}}$ has a symmetric monoidal convolution product and the category $\mathrm{sSet}^{\mathcal{I}}[\mathcal{C}]$ of commutative monoid objects in $\mathrm{sSet}^{\mathcal{I}}$ admits a model structure making it Quillen equivalent to the category of $E_\infty$ spaces.

The following construction, due to Mirjam Solberg [**SS**, Section 4.14], shows that symmetric monoidal categories give rise to commutative monoid objects in $\mathrm{sSet}^{\mathcal{I}}$ in a natural way.

*Example 1.1.* Let $(\mathcal{A}, \otimes)$ be a symmetric monoidal category. We consider the functor $\Phi(\mathcal{A})\colon \mathcal{I} \to \mathrm{Cat}$ with objects of $\Phi(\mathcal{A})(\mathbf{n})$ the $n$-tuples $(a_1, \dots, a_n)$ of objects in $\mathcal{A}$ and morphisms

$$\Phi(\mathcal{A})(\mathbf{n})((a_1, \dots, a_n), (b_1, \dots, b_n)) = \mathcal{A}(a_1 \otimes \dots \otimes a_n, b_1 \otimes \dots \otimes b_n).$$

Functoriality in $\mathcal{I}$ is induced by permutation of entries and insertion of the unit object of $\mathcal{A}$. Composing with the nerve functor $N$ gives an $\mathcal{I}$-simplicial set $N\Phi(\mathcal{A})$, and the symmetric monoidal structure of $\mathcal{A}$ makes $N\Phi(\mathcal{A})$ a commutative monoid object in $\mathrm{sSet}^{\mathcal{I}}$; see [**SS**, Proposition 4.16].

Equipped with the (standard or Kan) model structure, the category of simplicial sets $\mathrm{sSet}$ is Quillen equivalent to the category of topological spaces. Therefore, weak homotopy types of spaces are represented by simplicial sets. But simplicial sets also model quasi-categories up to Joyal equivalence: there is a finer Joyal model structure

on sSet whose fibrant objects are the quasi-categories and whose weak equivalences
are called Joyal equivalences (see, e.g., [**Lur09a**] or [**DS11**] for published references).
Simplicial sets with $E_\infty$ structures are also interesting from this perspective since
they model symmetric monoidal $(\infty, 1)$-categories [**Lur**]. These play a prominent role
in Lurie's work on the cobordism hypothesis [**Lur09b**].

In view of these two interpretations of simplicial sets, it is an obvious question
if the above comparison of $E_\infty$ objects in sSet and strictly commutative objects in
sSet$^{\mathcal{I}}$ still holds if we regard simplicial sets as models for quasi-categories. The aim
of this note is to prove that this is indeed the case:

**Theorem 1.2.**    *(i)  The category* sSet$^{\mathcal{I}}[\mathcal{C}]$ *of commutative monoid objects in* sSet$^{\mathcal{I}}$
*admits a left proper* positive $\mathcal{I}$-model structure *where a map $f$ is a weak equiv-
alence if and only if* hocolim$_{\mathcal{I}} f$ *is a Joyal equivalence.*

*(ii)  If $\mathcal{D}$ is an $E_\infty$ operad, then there is a chain of Quillen equivalences relating*
sSet$^{\mathcal{I}}[\mathcal{C}]$ *and the category* sSet$[\mathcal{D}]$ *of $E_\infty$ simplicial sets with the model structure
lifted from the Joyal model structure.*

Here, an $E_\infty$ operad is an operad $\mathcal{D}$ in simplicial sets such that $\mathcal{D}(n)$ has a free
$\Sigma_n$-action and $\mathcal{D}(n)$ is contractible with respect to the Joyal model structure. Every
$E_\infty$ operad in this sense is an $E_\infty$ operad in the classical sense since being contractible
with respect to the Joyal model structure implies being contractible with respect to
the Kan model structure. Moreover, every operad $\mathcal{D}$ with $\mathcal{D}(n)$ a $\Sigma_n$-free Kan complex
such that $\mathcal{D}(n) \to *$ is a weak homotopy equivalence is an $E_\infty$-operad in the sense of
the theorem; for example, the Barratt–Eccles operad whose $n$th space is $E\Sigma_n$.

By [**SS**, Lemma 4.15], the object $N\Phi(\mathcal{A})$ in sSet$^{\mathcal{I}}[\mathcal{C}]$ considered in Example 1.1 is
fibrant in the model structure of Theorem 1.2(i). It models the $E_\infty$ quasi-category $N\mathcal{A}$
associated with the symmetric monoidal category $\mathcal{A}$. Therefore, Example 1.1 shows
that the nerve of a symmetric monoidal category can be rigidified to a commutative
monoid object in sSet$^{\mathcal{I}}$ in a natural way.

More generally, it follows from Theorem 1.2 that for any $E_\infty$ simplicial set $X$ in the
sense of the theorem, there is an $A \in$ sSet$^{\mathcal{I}}[\mathcal{C}]$ and a chain of maps $A \leftarrow B \rightarrow \text{const}_{\mathcal{I}} X$
of $E_\infty$ objects in sSet$^{\mathcal{I}}$ that induces a chain of Joyal equivalences when applying
hocolim$_{\mathcal{I}}$ (compare [**SS12**, Corollary 3.7]). Hence the $E_\infty$ object $X$ can be replaced
by the strictly commutative object $A$. Although this rigidification of a structure up
to homotopy by a strict one is in contrast to the philosophy of quasi-categories, we
think that it is valuable to observe that $E_\infty$ quasi-categories can be expressed this
way: when viewing simplicial sets as models for spaces, it is often easy to write down
explicit objects in sSet$^{\mathcal{I}}[\mathcal{C}]$ that model $E_\infty$ spaces. This applies, for example, to $Q(X)$
if $X$ is connected [**SS12**, Example 1.3] or to $B\text{GL}_\infty(R)^+$[**Sch04**, Remark 2.2]. It is
likely that besides Example 1.1 above, there are more instances where interesting $E_\infty$
quasi-categories arise from commutative $\mathcal{I}$-functors.

Theorem 1.2 and the corresponding statement about weak homotopy types of $E_\infty$
spaces [**SS12**, Theorem 1.2] refer to different model structures on the same categories
that have the same cofibrations. Nonetheless, several arguments from [**SS12**] do not
apply here, since [**SS12**, Theorem 1.2] was derived from a result about diagram
spaces indexed by more general categories than $\mathcal{I}$ and some of the more general
arguments were based on special features of the Kan model structure. However, the

Joyal model structure differs from the Kan model structure since it fails to be right proper and simplicial, and because it doesn't have an explicit set of generating acyclic cofibrations. In the proof of Theorem 1.2 presented here, we put emphasis on the points where new arguments are required and simply cite those parts of the proof of [**SS12**, Theorem 1.2] that also apply here.

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## 2. Model structures on $\mathcal{I}$-simplicial sets

The category of simplicial sets sSet admits a *Joyal model structure* with cofibrations the monomorphisms and fibrant objects the quasi-categories, i.e., the *weak* or *inner* Kan complexes. See [**Lur09a**, Theorem 2.2.5.1] or [**DS11**, Theorem 2.13]. The Joyal model structure is cofibrantly generated with generating cofibrations $I = \{\partial\Delta^n \to \Delta^n \,|\, n \geqslant 0\}$. We let $J$ be a set of generating acyclic cofibrations. (There is no known explicit description of such a set $J$.)

Let $\mathcal{I}$ be the category with objects the finite sets $\mathbf{n} = \{1, \ldots, n\}$ for $n \geqslant 0$ and morphisms the injections. Concatenation of ordered sets $\sqcup$ makes $\mathcal{I}$ a symmetric monoidal category with unit $\mathbf{0}$ and symmetry isomorphism the obvious shuffle map.

Let sSet$^{\mathcal{I}}$ be the functor category of $\mathcal{I}$-diagrams of simplicial sets. For every object $\mathbf{n}$ of $\mathcal{I}$, there is a free/forgetful adjunction $F_{\mathbf{n}}^{\mathcal{I}}\colon \text{sSet} \rightleftarrows \text{sSet}^{\mathcal{I}} \colon \text{Ev}_{\mathbf{n}}$ with $F_{\mathbf{n}}^{\mathcal{I}}(K) = \mathcal{I}(\mathbf{n}, -) \times K$ and $\text{Ev}_{\mathbf{n}}(X) = X(\mathbf{n})$. For $X$ and $Y$ in sSet$^{\mathcal{I}}$, the left Kan extension of the $\mathcal{I} \times \mathcal{I}$-diagram $X(-) \times Y(-)$ along $\sqcup\colon \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ defines an object $X \boxtimes Y$ in sSet$^{\mathcal{I}}$. This construction defines a symmetric monoidal product $\boxtimes\colon \text{sSet}^{\mathcal{I}} \times \text{sSet}^{\mathcal{I}} \to \text{sSet}^{\mathcal{I}}$ with unit $F_{\mathbf{0}}^{\mathcal{I}}(*)$.

We now start to consider model structures on sSet$^{\mathcal{I}}$. Let $\mathcal{I}_+$ be the full subcategory of $\mathcal{I}$ on the objects $\mathbf{n}$ with $|\mathbf{n}| \geqslant 1$. We say that a map $f\colon X \to Y$ in sSet$^{\mathcal{I}}$ is an absolute (resp. positive) level equivalence if $f\colon X(\mathbf{n}) \to Y(\mathbf{n})$ is a Joyal equivalence for all $\mathbf{n}$ in $\mathcal{I}$ (resp. all $\mathbf{n}$ in $\mathcal{I}_+$), and an absolute (resp. positive) level fibration if $f\colon X(\mathbf{n}) \to Y(\mathbf{n})$ is a fibration in the Joyal model structure for all $\mathbf{n}$ in $\mathcal{I}$ (resp. all $\mathbf{n}$ in $\mathcal{I}_+$). A map is an absolute (resp. positive) level cofibration if it has the left lifting property with respect to any map that is both an absolute (resp. positive) level fibration and level equivalence.

**Lemma 2.1.** *These classes of maps define two cofibrantly generated left proper model structures on* sSet$^{\mathcal{I}}$, *called the* absolute *and the* positive *level model structures.*

*Proof of Lemma 2.1.* The absolute case follows from [**Hir03**, Theorem 11.6.1], and the positive case works as in [**SS12**, Proposition 6.7]. The sets

$$I_{\text{abs}}^{\text{level}} = \{F_{\mathbf{n}}^{\mathcal{I}}(i) \,|\, i \in I, \mathbf{n} \in \mathcal{I}\} \text{ and } I_{\text{pos}}^{\text{level}} = \{F_{\mathbf{n}}^{\mathcal{I}}(i) \,|\, i \in I, \mathbf{n} \in \mathcal{I}_+\} \qquad (1)$$

provide the generating cofibrations. The generating acyclic cofibrations $J_{\text{abs}}^{\text{level}}$ and $J_{\text{pos}}^{\text{level}}$ are defined similarly with $J$ in place of $I$. $\qquad\square$

Since the Joyal model structure fails to be simplicial, the usual Bousfield–Kan formula does not provide a homotopy invariant homotopy colimit functor. In the following, hocolim$_{\mathcal{I}}\colon \text{sSet}^{\mathcal{I}} \to \text{sSet}$ denotes the functor constructed in [**Hir03**, §19]

using cosimplicial frames. We recall from [**Hir03**, Example 19.2.10] that there is a natural map $\mathrm{hocolim}_{\mathcal{I}} X \to \mathrm{colim}_{\mathcal{I}} X$.

**Lemma 2.2.** *If $X$ is an absolute or positive level cofibrant object in $\mathrm{sSet}^{\mathcal{I}}$, then the map $\mathrm{hocolim}_{\mathcal{I}} X \to \mathrm{colim}_{\mathcal{I}} X$ is a Joyal equivalence.*

*Proof.* This is analogous to [**Hir03**, Theorem 19.9.1], with the absolute level model structure replacing the Reedy model structure in that reference. □

We say that a map $f\colon X \to Y$ in $\mathrm{sSet}^{\mathcal{I}}$ is an $\mathcal{I}$-*equivalence* if $\mathrm{hocolim}_{\mathcal{I}} f$ is a Joyal equivalence of simplicial sets, and an absolute (resp. positive) $\mathcal{I}$-*cofibration* if it is an absolute (resp. positive) level cofibration. A map is an absolute (resp. positive) $\mathcal{I}$-*fibration* if it has the right lifting property with respect to any map that is both an absolute (resp. positive) $\mathcal{I}$-cofibration and an $\mathcal{I}$-equivalence.

**Proposition 2.3.** *These classes of maps define two cofibrantly generated left proper model structures on $\mathrm{sSet}^{\mathcal{I}}$, called the* absolute *and the* positive $\mathcal{I}$-*model structures.*

We write $\mathrm{sSet}^{\mathcal{I}}_{\mathrm{abs}}$ and $\mathrm{sSet}^{\mathcal{I}}_{\mathrm{pos}}$ for these model categories. These (Joyal) $\mathcal{I}$-model structures have the same cofibrations as the corresponding (Kan) $\mathcal{I}$-model structures constructed in [**SS12**, Proposition 6.16] by a different technique.

*Proof.* Since $\mathcal{I}$ has an initial object, its classifying space is contractible. Hence the existence of the absolute $\mathcal{I}$-model structure follows from [**Dug01**, Theorem 5.2], and we recall from [**Dug01**] that it is constructed as the left Bousfield localization of the absolute level model structure at

$$S = \{\alpha^*\colon F^{\mathcal{I}}_{\mathbf{n}}(K) \to F^{\mathcal{I}}_{\mathbf{m}}(K) \,|\, \alpha\colon \mathbf{m} \to \mathbf{n} \in \mathcal{I}, K = \Delta^q \text{ or } K = \partial\Delta^q, q \geqslant 0\}.$$

The positive $\mathcal{I}$-model structure is defined to be the left Bousfield localization of the positive level model structure with respect to

$$T = \{\alpha^*\colon F^{\mathcal{I}}_{\mathbf{n}}(K) \to F^{\mathcal{I}}_{\mathbf{m}}(K) \,|\, \alpha\colon \mathbf{m} \to \mathbf{n} \in \mathcal{I}_+, K = \Delta^q \text{ or } K = \partial\Delta^q, q \geqslant 0\}.$$

It exists and is left proper by [**Hir03**, Theorem 4.1.1]. Hence it remains to show that its weak equivalences, the $T$-*local equivalences*, are the $\mathcal{I}$-equivalences. Since $T \subset S$, every $T$-local equivalence is an $\mathcal{I}$-equivalence. Let $f$ be an $\mathcal{I}$-equivalence. Passing to fibrant replacements, we may assume that $f$ is a map of $T$-local objects. Restricting $f$ along the inclusion $\mathcal{I}_+ \to \mathcal{I}$ and applying [**Dug01**, Theorem 5.2] to $\mathrm{sSet}^{\mathcal{I}_+}$, it follows that $\mathrm{hocolim}_{\mathcal{I}_+} f$ is a Joyal equivalence. Since $\mathcal{I}_+ \to \mathcal{I}$ is homotopy cofinal [**SS12**, Proof of Corollary 5.9], this implies the claim. □

**Corollary 2.4.** *There is a chain of Quillen equivalences*

$$\mathrm{sSet}^{\mathcal{I}}_{\mathrm{pos}} \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightleftarrows}} \mathrm{sSet}^{\mathcal{I}}_{\mathrm{abs}} \underset{\mathrm{const}_{\mathcal{I}}}{\overset{\mathrm{colim}_{\mathcal{I}}}{\rightleftarrows}} \mathrm{sSet}$$

*relating $\mathrm{sSet}^{\mathcal{I}}$ equipped with the positive and absolute $\mathcal{I}$-model structures and $\mathrm{sSet}$ equipped with the Joyal model structure.*

*Proof.* It is clear that $(\mathrm{id}, \mathrm{id})$ is a Quillen equivalence. The adjunction $(\mathrm{colim}_{\mathcal{I}}, \mathrm{const}_{\mathcal{I}})$ is a Quillen equivalence by [**Dug01**, Theorem 5.2(b)]. □

The next lemma and the subsequent proposition are analogous to [**SS12**, Proposition 7.1(iii)–(v) and Proposition 8.2]. The proofs given here avoid using features of the Bousfield–Kan formula for homotopy colimits.

**Lemma 2.5.** *(i) The gluing lemma for levelwise monomorphisms and level equivalences holds.*

*(ii) The gluing lemma for levelwise monomorphisms and $\mathcal{I}$-equivalences holds.*

*(iii) For any ordinal $\lambda$ and any $\lambda$-sequence $(X_\alpha)_{\alpha<\lambda}$ of levelwise monomorphisms, the canonical map $\mathrm{hocolim}_{\alpha<\lambda} X_\alpha \to \mathrm{colim}_{\alpha<\lambda} X_\alpha$ is a level equivalence.*

*Proof.* Part (i) follows from the gluing lemma in left proper model categories [**Hir03**, Proposition 13.5.4]. Using (i) and the absolute level cofibrant replacement, it is enough to show (ii) for a diagram of absolute cofibrant objects. This special case follows from the gluing lemma in the Joyal model structure by applying $\mathrm{colim}_\mathcal{I}$. Part (iii) follows from [**Hir03**, Theorem 19.9.1]. $\square$

**Proposition 2.6.** *If $X$ is absolute cofibrant in $\mathrm{sSet}^\mathcal{I}$, then the functor $X \boxtimes -$ preserves $\mathcal{I}$-equivalences between not necessarily cofibrant objects.*

*Proof.* We first assume that $X = F_\mathbf{k}^\mathcal{I}(L)$ with $L \in \mathrm{sSet}$ and $\mathbf{k} \in \mathcal{I}$. Let $Y \to Z$ be an $\mathcal{I}$-equivalence. If $Y$ and $Z$ are absolute cofibrant, then the claim follows by applying the strong symmetric monoidal functor $\mathrm{colim}_\mathcal{I}$ and using Corollary 2.4 and the pushout-product axiom for the Joyal model structure [**DS11**, 2.15 Proposition]. If $Y^c \to Y$ is an absolute level cofibrant replacement, then [**SS12**, Lemma 5.6] implies that the map $(X \boxtimes (Y^c \to Y))(\mathbf{m})$ is isomorphic to

$$L \times \mathrm{colim}_{\mathbf{k} \sqcup \mathbf{l} \to \mathbf{m}} Y^c(\mathbf{l}) \to L \times \mathrm{colim}_{\mathbf{k} \sqcup \mathbf{l} \to \mathbf{m}} Y(\mathbf{l}). \tag{2}$$

Since each connected component of the comma category $\mathbf{k} \sqcup - \downarrow \mathbf{m}$ has a terminal object [**SS12**, Corollary 5.9], the colimits in (2) are Joyal equivalent to the corresponding homotopy colimits and (2) is a level equivalence. It follows that $X \boxtimes (Y \to Z)$ is an $\mathcal{I}$-equivalence since $X \boxtimes (Y^c \to Z^c)$ is. With Lemma 2.5 replacing those parts of [**SS12**, Proposition 7.1] that involve weak equivalences, the case of general $X$ follows as in the proof of [**SS12**, Proposition 8.2]. $\square$

**Corollary 2.7.** *The absolute and positive $\mathcal{I}$-model structures on $\mathrm{sSet}^\mathcal{I}$ satisfy the pushout-product axiom and the monoid axiom.*

*Proof.* The part of the pushout-product axiom involving only cofibrations results from [**SS12**, Proposition 8.4]. As in [**SS12**, §8], Proposition 2.6 implies the rest. $\square$

The following lemma is analogous to [**SS12**, Lemma 8.1].

**Lemma 2.8.** *Let $G$ be a finite group, and let $f\colon X \to Y$ and $Y \to E$ be morphisms in $(\mathrm{sSet}^\mathcal{I})^G$ such that $\mathrm{hocolim}_\mathcal{I} f$ is a Joyal equivalence. If $G$ acts freely on $E(\mathbf{m})$ for every object $\mathbf{m}$ in $\mathcal{I}$, then $f/G\colon X/G \to Y/G$ is an $\mathcal{I}$-equivalence.*

*Proof.* Since there is a $G$-map $Y \to E$, the $G$-action on $Y(\mathbf{m})$ is also free. Hence $\mathrm{hocolim}_G Y(\mathbf{m}) \to \mathrm{colim}_G Y(\mathbf{m}) \cong (Y/G)(\mathbf{m})$ is a Joyal equivalence. Using the same argument for $X$, it follows that

$$\mathrm{hocolim}_G \mathrm{hocolim}_\mathcal{I} f \simeq \mathrm{hocolim}_\mathcal{I} \mathrm{hocolim}_G f \simeq \mathrm{hocolim}_\mathcal{I}(f/G)$$

is a Joyal equivalence. $\square$

The use of the positive model structure is motivated by the positive model structure for symmetric spectra discovered by Jeff Smith. The next lemma highlights one of its key features.

**Lemma 2.9.** *If $X$ is positive $\mathcal{I}$-cofibrant, then the $\Sigma_n$-action on the simplicial set $(X^{\boxtimes n})(\mathbf{m})$ is free for every object $\mathbf{m}$ of $\mathcal{I}$.*

*Proof.* Let $f\colon U \to V$ and $U \to Y$ be maps in $\mathrm{sSet}^{\mathcal{I}}$. By a cell induction argument, it is enough to show that if $f$ is a generating cofibration and $\Sigma_n$ acts freely on $(Y^{\boxtimes n})(\mathbf{m})$ for every $\mathbf{m}$ in $\mathcal{I}$, then $Z = Y \coprod_U V$ has this property. By [**SS12**, Lemma A.8], $Y^{\boxtimes n} \to Z^{\boxtimes n}$ has a filtration by maps that are cobase changes of maps of the form $\Sigma_n \times_{\Sigma_{n-i} \times \Sigma_i} Y^{\boxtimes n-i} \boxtimes f^{\square i}$, where $f^{\square i}$ is the $i$-fold iterated pushout product map in $(\mathrm{sSet}^{\mathcal{I}}, \boxtimes)$. Hence it suffices to show that $(Y^{\boxtimes n-i} \boxtimes f^{\square i})(\mathbf{m})$ is a $(\Sigma_{n-i} \times \Sigma_i)$-projective cofibration of simplicial sets with $(\Sigma_{n-i} \times \Sigma_i)$-action. Since $f = F^{\mathcal{I}}_{\mathbf{k}}(*) \times g$ with $g$ a generating cofibration for $\mathrm{sSet}$ and $\mathbf{k} \in \mathcal{I}_+$, it follows from [**SS12**, Lemma 5.6] that there is an isomorphism

$$(Y^{\boxtimes n-i} \boxtimes f^{\square i})(\mathbf{m}) \cong (\mathrm{colim}_{\mathbf{k}^{\sqcup i} \sqcup \mathbf{l} \to \mathbf{m}} Y^{\boxtimes n-i}(\mathbf{l})) \times g^{\square i}, \tag{3}$$

where $g^{\square i}$ is the $i$-fold iterated pushout-product map of $g$ in $(\mathrm{sSet}, \times)$. By [**SS12**, Corollary 5.9], each connected component of the indexing category $\mathbf{k}^{\sqcup i} \sqcup - \downarrow \mathbf{m}$ has a terminal object, and $\Sigma_i$ acts freely on the set of connected components. Hence $\mathrm{colim}_{\mathbf{k}^{\sqcup i} \sqcup \mathbf{l} \to \mathbf{m}} Y^{\boxtimes n-i}(\mathbf{l})$ is a $(\Sigma_{n-i} \times \Sigma_i)$-free simplicial set, and (3) is a $(\Sigma_{n-i} \times \Sigma_i)$-projective cofibration. $\qquad\square$

## 3.  Model structures on structured diagrams of simplicial sets

In the following, an operad $\mathcal{D}$ denotes a sequence of simplicial sets $\mathcal{D}(n)$ with $\Sigma_n$-action such that $\mathcal{D}(0) = *$, there is a unit map $* \to \mathcal{D}(1)$, and there are structure maps $\mathcal{D}(n) \times \mathcal{D}(i_1) \times \cdots \times \mathcal{D}(i_n) \to \mathcal{D}(i_1 + \cdots + i_n)$ satisfying the usual associativity, unit, and equivariance relations. It is called $\Sigma$-*free* if $\Sigma_n$ acts freely on $\mathcal{D}(n)$ for all $n$.

Let $\mathrm{sSet}^{\mathcal{I}}[\mathcal{D}]$ be the category of $\mathcal{D}$-algebras in $(\mathrm{sSet}^{\mathcal{I}}, \boxtimes)$. We say that a model structure on $\mathrm{sSet}^{\mathcal{I}}$ lifts to $\mathrm{sSet}^{\mathcal{I}}[\mathcal{D}]$ if $\mathrm{sSet}^{\mathcal{I}}[\mathcal{D}]$ admits a model structure where a map is a weak equivalence or fibration if the underlying map in $\mathrm{sSet}^{\mathcal{I}}$ is.

**Theorem 3.1.** *Let $\mathcal{D}$ be an operad. The positive $\mathcal{I}$-model structure lifts to $\mathrm{sSet}^{\mathcal{I}}[\mathcal{D}]$, and the absolute $\mathcal{I}$-model structure lifts to $\mathrm{sSet}^{\mathcal{I}}[\mathcal{D}]$ if $\mathcal{D}$ is $\Sigma$-free.*

Since the generating cofibrations coincide, these model structures have the same cofibrations as the corresponding Kan $\mathcal{I}$-model structures [**SS12**, Proposition 9.3].

*Proof.* As in the analogous statement about the Kan $\mathcal{I}$-model structure [**SS12**, Proposition 9.3], the claim reduces to showing that for a generating acyclic cofibration $f\colon U \to V$ in $\mathrm{sSet}^{\mathcal{I}}$, the bottom map in a pushout square

$$
\begin{array}{ccc}
\coprod_{n\geqslant 0} \mathcal{D}(n) \times_{\Sigma_n} U^{\boxtimes n} & \longrightarrow & \coprod_{n\geqslant 0} \mathcal{D}(n) \times_{\Sigma_n} V^{\boxtimes n} \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y
\end{array}
$$

in $\mathrm{sSet}^{\mathcal{I}}[\mathcal{D}]$ is an $\mathcal{I}$-equivalence. Replacing [**SS12**, Propositions 8.4 and 8.6] by Corollary 2.7 and [**SS12**, Lemma 8.1] by Lemma 2.8, the argument given in the proof

of [**SS12**, Lemma 9.5] applies verbatim with one exception: we need to show that for any $n \geqslant 0$ and any $\mathbf{m}$ in $\mathcal{I}$, the group $\Sigma_n$ acts freely on $V^{\boxtimes n}(\mathbf{m})$. Using that the generating cofibrations have cofibrant domains and codomains, we may assume that this also holds for the generating acyclic cofibrations [**Bar10**, Corollaries 2.7 and 2.8]. Hence the last claim follows from Lemma 2.9. □

We recall that a morphism of operads $\Phi \colon \mathcal{D} \to \mathcal{E}$ induces an adjunction

$$\Phi_* \colon \mathrm{sSet}^{\mathcal{I}}[\mathcal{D}] \rightleftarrows \mathrm{sSet}^{\mathcal{I}}[\mathcal{E}] \colon \Phi^*.$$

**Proposition 3.2.** *Let* $\Phi \colon \mathcal{D} \to \mathcal{E}$ *be a morphism of operads with* $\Phi_n \colon \mathcal{D}(n) \to \mathcal{E}(n)$ *a Joyal equivalence for each* $n \geqslant 0$. *Then* $(\Phi_*, \Phi^*)$ *is a Quillen equivalence with respect to the positive $\mathcal{I}$-model structures. If $\mathcal{D}$ and $\mathcal{E}$ are $\Sigma$-free, then it is also a Quillen equivalence with respect to the absolute $\mathcal{I}$-model structures.*

*Proof.* Again the proof of the analogous statement about the Kan $\mathcal{I}$-model structure [**SS12**, Proposition 9.12] applies almost verbatim: in the key ingredient [**SS12**, Lemma 9.13], Lemma 2.5 replaces those parts of [**SS12**, Proposition 7.1] that involve weak equivalences, Corollary 2.7 replaces [**SS12**, Proposition 8.4], Proposition 2.6 replaces [**SS12**, Proposition 8.2], and Lemma 2.8 replaces [**SS12**, Lemma 8.1]. □

*Proof of Theorem 1.2.* Part (i) follows from Theorem 3.1 applied to the commutativity operad $\mathcal{C}$ with $\mathcal{C}(n) = *$ for every $n$. Left properness follows by the arguments from [**SS12**, Lemma 11.8 and Proposition 11.9], where again the results from Section 2 replace the corresponding statements in [**SS12**].

If $\mathcal{D}$ is an $E_\infty$ operad, then there is a canonical morphism $\Phi \colon \mathcal{D} \to \mathcal{C}$, and we obtain a chain of Quillen adjunctions

$$\mathrm{sSet}^{\mathcal{I}}_{\mathrm{pos}}[\mathcal{C}] \xtofrom[\Phi^*]{\Phi_*} \mathrm{sSet}^{\mathcal{I}}_{\mathrm{pos}}[\mathcal{D}] \xtofrom[\mathrm{id}]{\mathrm{id}} \mathrm{sSet}^{\mathcal{I}}_{\mathrm{abs}}[\mathcal{D}] \xtofrom[\mathrm{const}_{\mathcal{I}}]{\mathrm{colim}_{\mathcal{I}}} \mathrm{sSet}[\mathcal{D}] \ .$$

The first adjunction is a Quillen equivalence by Proposition 3.2. The last two adjunctions are Quillen equivalences by Corollary 2.4 and the fact that cofibrant objects in $\mathrm{sSet}^{\mathcal{I}}_{\mathrm{abs}}[\mathcal{D}]$ are cofibrant in $\mathrm{sSet}^{\mathcal{I}}_{\mathrm{abs}}$ if $\mathcal{D}$ is $\Sigma$-free [**SS12**, Corollary 12.3]. □

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