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A USEFUL LEMMA ON EQUIVARIANT MAPS

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(*communicated by Alexander Mischenko*)

Abstract

We present a short proof of the following known result. *Suppose X, Y are finite connected CW-complexes with free involutions, $f\colon X \to Y$ is an equivariant map, and l is a non-negative integer. If $f^*\colon H^i(Y) \to H^i(X)$ is an isomorphism for each $i > l$ and is onto for $i = l$, then $f^\sharp\colon \pi^i_{eq}(Y) \to \pi^i_{eq}(X)$ is a 1-1 correspondence for $i > l$ and is onto for $i = l$.*

The purpose of this note is to present a short proof of the following known result.

Lemma 1. *Suppose X, Y are finite connected CW-complexes with free involutions, $f\colon X \to Y$ is an equivariant map, and l is a non-negative integer. If $f^*\colon H^i(Y) \to H^i(X)$ is an isomorphism for each $i > l$ and is onto for $i = l$, then*

(a_l) $f^\sharp\colon \pi^i_{eq}(Y) \to \pi^i_{eq}(X)$ is a 1-1 correspondence for $i > l$ and is onto for $i = l$.

Here $H^i(X)$ is (non-equivariant) singular cohomology with integral coefficients and $\pi^i_{eq}(X)$ is the set of equivariant homotopy classes of equivariant maps from X to S^i with the antipodal involution.

Some motivation to the lemma and its proof. This motivating remark is not used in the proof below.

The lemma is interesting because it is an equivariant analogue of a cohomological version of the Hurewicz Theorem.

The lemma was used in [BG71, 3.2] (without proof), in [GS06] (the proof was given for $l = 0$ and it was erroneously stated that generalization to $l > 0$ is obvious; see a corrected version [GS12]), in [Me09] (the proof of a weaker Lemma 8.1 required there was presented and needs a note that the twisted coefficient system $\mathbb{Z}_p$ corresponds not only to double covering p but also to the involution a on $\pi_k(S^{m-1})$ induced by the antipodal involution on S^{m-1}; cf. footnote 7), and probably in other papers. The three cited papers studied relations between embeddings and equivariant maps. The lemma allowed the construction of equivariant maps using only cohomology calculations.

It would be interesting to know if the lemma holds for infinite or even infinite-dimensional complexes.

We could not find a proof of the lemma in standard textbooks and references; see e.g. references of [GS06, GS12, Sk08]. So we feel obliged to present a short proof of such a basic and useful result.

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Proof of Lemma 1. The proof uses a standard argument; see [HH62, pp. 236–237], cf. [Me09, Proof of Lemma 8.1]. We may assume that $f\colon X \to Y$ is an inclusion. Consider the following assertion:

(b_L) $H^i(Y', X'; G_\varphi) = 0$ for each $i > L$, finitely generated abelian group G, involution $\varphi : G \to G$, and local coefficient system G_φ associated to φ and double cover $(Y, X) \to (Y', X')$.

(The local coefficient system G_φ is defined by the following action of $\pi_1(Y')$ on G. Take a representative $\alpha'\colon [0,1] \to Y'$, $\alpha'(0) = \alpha'(1)$, of $[\alpha'] \in \pi_1(Y')$. Take a lift $\alpha\colon [0,1] \to Y$ of α'. If $\alpha(0) = \alpha(1)$, then $[\alpha']$ acts identically on G. If $\alpha(0) \neq \alpha(1)$, then $[\alpha']$ acts as φ. Clearly this action is well-defined.)

Since Y is finite-dimensional, (b_L) holds for large enough L. Consider the following part of the Smith-Richardson-Thom-Gysin sequences associated to the double cover $(Y, X) \to (Y', X')$ (see [GS06] and references there):

$$0 = H^i(Y, X; G) \to H^i(Y', X'; G_\varphi) \to H^{i+1}(Y', X'; G_{-\varphi}),$$

where the equality $0 = H^i(Y, X; G)$ for each $i > L$ follows from the Universal Coefficient Formula because by hypothesis we have $H^i(Y, X) = 0$ for each $i > L$. Then by downward induction on L we get (b_L).

Denote by a the involution on $\pi_k(S^i)$ induced by the antipodal involution on S^i.[1] The obstructions to the extension to Y of an equivariant map $X \to S^i$, and to the homotopy uniqueness of such an extension, take values in $H^{k+1}(Y', X'; \pi_k(S^i)_a)$ and $H^k(Y', X'; \pi_k(S^i)_a)$.[2] These groups are trivial for $k < i$ because $\pi_k(S^i) = 0$, and for $k \geqslant i > l$ by (b_l). So (a_l) holds. □

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[1] Note that $a = \mathrm{id}$ for i odd and $a = -\mathrm{id}$ for i even and $k \leqslant 2i - 2$.

[2] This can be deduced as follows [Di87], analogously to [CF60, beginning of §2, Ad93, 7.1]. Denote by t the involution on Y and its restriction to X. Define a bundle $g : \dfrac{Y \times S^i}{(x,s) \sim (tx, -s)} \overset{S^i}{\to} Y'$ by $g[(x,s)] = [x]$. Equivariant maps $Y \to S^i$ up to equivariant homotopy are in 1-1 correspondence with cross-sections of g up to equivalence. So the required obstructions are obstructions to

(*) extendability of a section on X' to a section on Y' for each $i \geqslant l$, and to
(**) uniqueness of such an extension (up to equivalence) for $i > l$.

The action of $\pi_1(Y')$ on homotopy groups of the fiber S^i gives rise to the local coefficient system $\pi_k(S^i)_a$.

There is an alternative method of extension of equivariant maps [KB96, Sections 1.2, 1.3, and 5.3].

References

[Ad93] M. Adachi, Embeddings and Immersions, *Translations of Mathematical Monographs* **124** (1993) Amer. Math. Soc., Providence.

[BG71] J.C. Becker and H. H. Glover, Note on the Embedding of Manifolds in Euclidean Space, *Proc. Amer. Math. Soc.* **27**:2 (1971) 405-410, doi:10.2307/2036329.

[CF60] P.E. Conner and E. E. Floyd, Fixed point free involutions and equivariant maps, *Bull. Amer. Math. Soc.* **66** (1960), 416–441.

[Di87] T. tom Dieck, *Transformation Groups,* W. de Gruyter, Berlin, New York, 1987.

[GS06] D. Gonçalves and A. Skopenkov, Embeddings of homology equivalent manifolds with boundary, *Topol. Appl.* **153**:12 (2006) 2026–2034.

[GS12] D. Gonçalves and A. Skopenkov, Embeddings of homology equivalent manifolds with boundary, http://arxiv.org/abs/1207.1326.

[HH62] A. Haefliger and M.W. Hirsch, Immersions in the stable range, *Ann. of Math.* **75**:2 (1962) 231–241.

[KB96] A. Kushkuley and Z. Balanov, Geometric Methods in Degree Theory for Equivariant Maps, *Lect. Notes in Math.* **1632**, Springer-Verlag, Berlin, 1996.

[Me09] S. Melikhov, The van Kampen obstruction and its relatives, *Tr. Mat. Inst. Steklova* **266** (2009) 149-183. English translation: *Proc. Steklov Inst. Math.* **266** (2009), 142-176. arXiv: math.GT/0612082.

[Sk08] A. Skopenkov, Embedding and knotting of manifolds in Euclidean spaces, in: *Surveys in Contemporary Mathematics,* Ed. N. Young and Y. Choi, *London Math. Soc. Lect. Notes* **347** (2008) 248–342, arxiv:math/0604045.

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