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# ALGEBRAIC ANALOGUE OF THE ATIYAH COMPLETION THEOREM

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(*communicated by Ulf Rehmann*)

*Abstract*

In topology there is a well-known theorem of Atiyah, Hirzebruch, and Segal which states that for a connected compact Lie group $G$ there is an isomorphism $\widehat{R(G)} \cong K^0(BG)$, where $BG$ is the classifying space of $G$. In the present paper we consider an algebraic analogue of this theorem. For a split reductive group $G$ over a field $k$, we prove that there is a natural isomorphism

$$\widehat{K_n^G(k)}_{I_G} \cong K_n(BG),$$

where $K_n^G(k)$ is Thomason's $G$-equivariant $K$-theory of Spec $k$, $BG$ is a motivic étale classifying space introduced by Voevodsky and Morel, and $I_G$ is the augmentation ideal of $K_0^G(k)$.

## 1. Introduction

The classical topological result by Atiyah and Hirzebruch [**1**] states that for a compact connected Lie group $G$ there is an isomorphism $\widehat{R(G)} \cong K^0(BG)$, where $BG$ is a topological classifying space of $G$, $K^0$ stands for the topological $K$-theory, and $\widehat{R(G)}$ denotes the representation ring of $G$ completed in the augmentation ideal $I$. This ideal is the kernel of the dimension map $R(G) \to \mathbb{Z}$. Later, this result was proved for all compact Lie groups $G$ by Atiyah and Segal in [**2**]. In the present paper we establish an algebraic analogue of the Atiyah-Hirzebruch result. In the algebraic setting we take a split reductive algebraic group $G$ and its étale classifying space $BG$ constructed by Morel and Voevodsky.

In the paper by B. Totaro [**14**] it is shown that $\varprojlim K_0(BG_i)$ is equal to $\widehat{R(G)}$ for a specially chosen sequence $BG_i$. However, to compute $K_0(BG)$ one needs to prove that $\varprojlim^1 K_1(BG_i)$ vanishes.

Two months after the present work was finished, there appeared a preprint by A. Krishna [**6**] (unpublished) where a more general result is shown. For the action of a split reductive algebraic group $G$ on a smooth projective $X$, there is established an isomorphism $\widehat{K_n^G(X)} = K_n(X/G)$, where $X/G$ is the motivic quotient space. The

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author also constructs a counterexample when the theorem does not hold for action on a non-projective variety ([**6**, Theorem 1.4]).

For a split reductive group $G$ we present the following approach: We will work over an arbitrary base field $k$. Morel and Voevodsky in [**8**, Definition 4.2.4, Proposition 4.2.6] constructed a model of the étale classifying space of a linear algebraic group $G$ in the form $BG = \bigcup BG_m$, where $BG_m = EG_m/G$ and $EG_m$ are $k$-smooth algebraic varieties with a free $G$-action, connected by a sequence of $G$-equivariant closed embeddings $i_k$

$$\cdots \xrightarrow{i_{m-1}} (EG)_m \xrightarrow{i_m} (EG)_{m+1} \xrightarrow{i_{m+1}} \cdots .$$

The motivic space $EG = \bigcup EG_m$ is $\mathbb{A}^1$-contractible with a free $G$-action ([**8**, Proposition 4.2.3]). We consider a split reductive affine algebraic group $G$. A $G$-equivariant vector bundle over the Spec $k$ is the same as a $k$-rational representation of $G$. So we will identify these two categories. Notice that this identification respects the tensor products. In particular, we will identify Thomason's $K_0^G(k)$ with the representation ring of $k$-rational representations $R(G)$ of the group $G$.

The Borel construction sends a $G$-equivariant vector bundle $V$ over the point to the vector bundle $V_m = (V \times EG_m)/G$ over $BG_m$. This construction respects tensor products. Therefore, it induces a $K_0^G(k)$-module map $\phi_m : K_n^G(k) \to K_n(BG_m)$. Obviously, $\phi_m = \bar{i}_m^{\,*} \circ \phi_{m+1}$, where $\bar{i}_m : BG_m \to BG_{m+1}$ is an embedding induced by $i_m$. As we prove below, $K_n(BG) = \varprojlim K_n(BG_m)$. Combining all these, we get a $K_0^G(k)$-module map

$$\Phi_n : K_n^G(k) \to K_n(BG).$$

We will write $Borel_n^G$ for $\Phi_n$. Let $I_G$ be the kernel of the augmentation $K_0^G(k) \to K_0(k) = \mathbb{Z}$. Our main result is the following.

**Theorem 1.1.** *In the following diagram both maps are $K_0^G(k)$-module isomorphisms:*

$$\widehat{K_n^G(k)}_{I_G} \xrightarrow{\widehat{Borel_n^G}} \widehat{K_n(BG)}_{I_G} \xleftarrow{completion_G} K_n(BG),$$

*where $\widehat{Borel_n^G}$ is the $I_G$ completion of $Borel_n^G$, and $completion_G$ is the canonical map.*

Let us mention that in the case when $char\ k = 0$ and $G$ is semi-simple, all linear $G$-representations are completely reducible, so the category of linear representations is equivalent to the direct sum of the categories of vector spaces for every irreducible representation, and so there is an isomorphism $K_n^G(k) = R(G) \otimes K_n(k)$.

The main idea of the proof is the reduction to a Borel subgroup $B$ of $G$. For the Borel subgroup $B$ the $K_0^B(k)$-modules $K_n(BB)$ and $K_n^B(k)$ can be computed explicitly. It results in the following theorem.

**Theorem 1.2.** *The Borel construction induces an isomorphism*

$$\widehat{K_n^B(k)}_{I_B} \xrightarrow{\widehat{Borel_n^B}} \widehat{K_n(BB)}_{I_B} \xleftarrow{\cong} K_n(BB).$$

To make a reduction to the latter theorem, we prove the following.

**Theorem 1.3.** *There is a commutative diagram of the form:*

$$\begin{array}{ccccc} \widehat{K_n^G(k)}_{I_G} & \xrightarrow{\widehat{Borel_n^G}} & \widehat{K_n(BG)}_{I_G} & \longleftarrow & K_n(BG) \\ \Big\downarrow{\scriptstyle res} & & \Big\downarrow{\scriptstyle \widehat{p^*}} & & \Big\downarrow{\scriptstyle p^*} \\ \widehat{K_n^B(k)}_{I_B} & \xrightarrow{\widehat{Borel_n^B}} & \widehat{K_n(BB)}_{I_B} & \xleftarrow{\cong} & K_n(BB) \\ \Big\downarrow{\scriptstyle ind} & & \Big\downarrow{\scriptstyle \widehat{p_*}} & & \Big\downarrow{\scriptstyle p_*} \\ \widehat{K_n^G(k)}_{I_G} & \xrightarrow{\widehat{Borel_n^G}} & \widehat{K_n(BG)}_{I_G} & \longleftarrow & K_n(BG), \end{array} \tag{1}$$

*with* $ind \circ res = id$, $\widehat{p_*} \circ \widehat{p^*} = id$, $p_* \circ p^* = id$.

Note that the induction-restriction facts are similar to Theorem 1.13 in [**13**]. Clearly, the main theorem follows from Theorem 1.2 and Theorem 1.3. We expect the analogous result for non-connected linear groups, as in the case of non-connected compact Lie groups established by Atiyah and Segal.

The paper is organized as follows: In Section 2 we prove some auxiliary results. The proof of the main result can be found in Section 3.

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## 2. Auxiliary results

In this section we give basic definitions of equivariant $K$-theory, which was developed by Thomason in [**12**] (c.f. [**7**]). In Subsection 2.1 we prove some properties of pullback and pushforward morphisms for the functor $K_n^G$. Some of them may be found in [**13**]. In Subsection 2.2 we prove a number of statements needed to establish the main result. Throughout this section we work in the category $\mathbf{Sch}_k$ of finite-type schemes over the base field $k$, and the direct product is understood as the direct product over $k$.

**Definition 2.1.** Let $X$ be a $G$-variety. We consider an action $\mu_x : G \times X \to X$ and a projection $p_x : G \times X \to X$. Let M be an $\mathcal{O}_X$-module. Following [**7**], we will call $M$ a $G$-module if there is an isomorphism of $\mathcal{O}_{G\times X}$-modules $\alpha : \mu_X^*(M) \to p_X^*(M)$ such that the cocycle condition holds:

$$p_{23}^*(\alpha) \circ (id_G \times \mu_x)^*(\alpha) = (m \times id_X)^*(\alpha),$$

where $p_{23} : G \times G \times X \to G \times X$ is a projection and $m : G \times G \to G$ is a product morphism.

**Definition 2.2.** We denote by $\mathcal{P}(G;X)$ the exact category of locally free $G$-modules on $X$, and by $\mathcal{M}(G;X)$ we denote the abelian category of coherent $G$-modules on $X$. Following [**7**], we set $K_n(G;X) = K_n(\mathcal{P}(G;X))$ and $K'_n(G;X) = K_n(\mathcal{M}(G;X))$.

In the case when $X$ is smooth over $k$, the natural homomorphism $K_n(G;X) \to K'_n(G;X)$ is an isomorphism by [**13**, Remark 1.9(a)], and we will denote $K_n(G;X) = K'_n(G;X)$ by $K_n^G(X)$.

### 2.1. Pullback and pushforward maps in equivariant $K$-theory.

Here we recall some standard facts about pullback and pushforward maps. For any equivariant $f : X \to Y$ morphism between $G$-varieties and a $G$-equivariant vector bundle $L$ over $Y$, its pullback $f^*L$ has a natural structure of a $G$-equivariant vector bundle over $X$. According to [**7**, §2.2], this induces a pullback morphism $f^*\colon K_n^G(Y) \to K_n^G(X)$. The morphism $f\colon X \to Y$ is called $G$-projective if $f$ factors as $f\colon X \to \mathbb{P}(E) \to Y$, where $E$ is a $G$-vector bundle over $Y$, and the map $X \to \mathbb{P}(E)$ is an equivariant closed embedding. This morphism yields the pushforward homomorphism $f_*\colon K_n^G(X) \to K_n^G(Y)$ (see [**12**, 1.5] or [**7**, §2.2]). We will need the following technical facts:

**Lemma 2.3.** *Consider the following diagram:*

$$\begin{array}{ccccc} Y_3 & \xrightarrow{Q} & Y_2 & \xrightarrow{q} & Y_1 \\ {\scriptstyle f_3}\uparrow & & {\scriptstyle f_2}\uparrow & & {\scriptstyle f_1}\uparrow \\ X_3 & \xrightarrow{T} & X_2 & \xrightarrow{t} & X_1. \end{array}$$

*Here $q$ and $Q$ are flat, $X_2 = X_1 \times_{Y_1} Y_2$, $X_3 = X_2 \times_{Y_2} Y_3$. Let $M$ be an $\mathcal{O}_{X_1}$-module. Define $hh_1 : q^*R^i f_{1*} \to R^i f_{2*} t^*$, $hh_{12} : Q^*q^*R^i f_{1*} \to R^i f_{3*} T^* t^*$, $hh_2 : Q^* R^i f_{2*} \to R^i f_{3*} T^*$ to be natural isomorphisms given by Proposition 9.3 of [3]. Then the following diagram commutes:*

$$\begin{array}{ccc} Q^*q^*R^i f_{1*} M & \xrightarrow{Q^* hh_1(M)} & Q^* R^i f_{2*} t^* M \\ & {\scriptstyle hh_{12}(M)}\searrow \quad \swarrow{\scriptstyle hh_2(t^*M)} & \\ & R^i f_{3*} T^* t^* M. & \end{array}$$

*Proof.* Since the statement is local on $Y_i$, we consider the case when all $Y_i$ are affine, $Y_i =$ Spec $A_i$. If $F$ is an $R$-module, we will denote by $\widetilde{F}$ the corresponding sheaf on Spec $R$. Recall the construction of $hh_1$. Let $M$ be an $\mathcal{O}_{X_1}$-module. Then

$$R^i f_*(M) = H^i\widetilde{(X_1, M)}; q^*R^i f_{1*}M = A_2 \otimes_{A_1} \widetilde{H^i}(X_1, M); R^i f_{2*} t^* M = H^i\widetilde{(X_2, t^*M)}.$$

Let $U_i$ be an affine covering of $X_1$. Denote by $K = \check{C}(X_1, M)$ the corresponding Čech complex. Since $Y_1$ and $Y_2$ are affine, $t^{-1}(U_i)$ is the affine covering of $X_2$. For this covering we have that $A_2 \otimes_{A_1} K$ is a Čech complex of $X_2$-module $t^*M$. Then $hh_1$ is an obvious morphism

$$A_2 \otimes_{A_1} H^i(K) \to H^i(A_2 \otimes_{A_1} K),$$

which becomes an isomorphism since $A_2$ is flat over $A_1$. In a similar way, one can construct $hh_{12}$ and $hh_2$. Then one can rewrite the diagram as

$$\begin{array}{ccc} A_3 \otimes_{A_2} A_2 \otimes_{A_1} H^i(K) & \xrightarrow{id \otimes hh_1} & A_3 \otimes_{A_2} H^i(A_2 \otimes_{A_1} K) \\ & {\scriptstyle hh_{12}(M)} \searrow \quad \swarrow {\scriptstyle hh_2(t^*M)} & \\ & H^i(A_3 \otimes_{A_1} K), & \end{array}$$

which is trivially commutative. □

**Lemma 2.4** (Equivariant version of [**3**, Proposition 9.3])**.** *Consider the base change diagram*

$$\begin{CD} A @>F>> B \\ @VQVV @VVqV \\ X @>f>> Y, \end{CD}$$

*where $X, Y, A, B$ are $G$-varieties; $f, F, Q, q$ are $G$-morphisms; and $f$ is flat.*

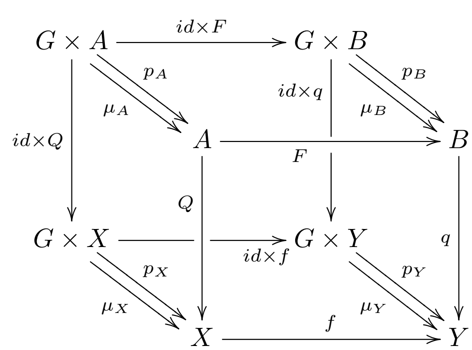
*Let $M$ be a $G$-module on $B$. Then there is a natural $G$-module isomorphism on $X$:*

$$f^* R^i q_* M \to R^i Q_* F^* M.$$

*Proof.* By Proposition 9.3 from [**3**] we have a natural isomorphism of $\mathcal{O}_X$-modules $hh_{X,Y,A,B} \colon f^* R^i q_* M \to R^i Q_* F^* M$. We need to check that this is a $G$-morphism. That means commutativity of the following diagram:

$$\begin{CD} \mu_X^* f^* R^i q_* M @>>{\text{G-structure}}> p_X^* f^* R^i q_* M \\ @V{\mu_X^* hh_{X,Y,A,B}}VV @VV{p_X^* hh_{X,Y,A,B}}V \\ \mu_X^* R^i Q_* F^* M @>>{\text{G-structure}}> p_X^* R^i Q_* F^* M. \end{CD}$$

Consider the following diagram:




For any square in this cube denote by $hh$ (with corresponding subscript) the isomorphism arising from Proposition 9.3 of [**3**], applied to this square. We rewrite the $G$-structure diagram:

$$
\begin{array}{ccc}
\mu_X^* f^* R^i q_* M & \xrightarrow[1]{\qquad} & p_X^* f^* R^i q_* M \\
\| & & \| \\
(id \times f)^* \mu_Y^* R^i q_* M & \xrightarrow[2]{\qquad} & (id \times f)^* p_Y^* R^i q_* M \\
\downarrow {\scriptstyle (id\times f)^* hh^{\mu}_{G\times Y, Y, G\times B, B}(M)} & & \downarrow {\scriptstyle (id\times f)^* hh^{p}_{G\times Y, Y, G\times B, B}(M)} \\
(id \times f)^* R^i (id \times q)_* \mu_B^* M & \xrightarrow[3]{\qquad} & (id \times f)^* R^i (id \times q)_* p_B^* M \\
\downarrow {\scriptstyle hh_{G\times X, G\times Y, G\times A, G\times B}(\mu_B^* M)} & & \downarrow {\scriptstyle hh_{G\times X, G\times Y, G\times A, G\times B}(p_B^* M)} \\
R^i (id \times Q)_* (id \times F)^* \mu_B^* M & \xrightarrow[4]{\qquad} & R^i (id \times Q)_* (id \times F)^* p_B^* M \\
\| & & \| \\
R^i (id \times Q)_* \mu_A^* F^* M & \xrightarrow[5]{\qquad} & R^i (id \times Q)_* p_A^* F^* M \\
\downarrow {\scriptstyle hh^{\mu}_{G\times X, X, G\times A, A}} & & \downarrow {\scriptstyle hh^{p}_{G\times X, X, G\times A, A}} \\
\mu_X^* R^i Q_* F^* M & \xrightarrow{\qquad} & p_X^* R^i Q_* F^* M.
\end{array}
$$

Square 1 is commutative because of the definition of the $G$-structure on pullback.

Square 2 is an $(id \times f)^*$ image of the $G$-structure diagram for $R^i q_* M$. Thus it commutes.

Square 3 arises from the functor isomorphism $(id \times f)^* R^i (id \times q)_* \to R^i (id \times Q)_* (id \times F)^*$ applied to the $G$-structure isomorphism $\mu_B^* M \to p_B^* M$. So it commutes.

Square 4 is commutative because of the definition of the $G$-structure on pullback.

Square 5 is commutative by the definition of the $G$-structure on $R^i Q_* F^* M$.

By Lemma 2.3 compositions of vertical arrows are equal to $\mu_X^* hh_{X,Y,A,B}$ and $p_X^* hh_{X,Y,A,B}$. This concludes the proof of the lemma. □

**Lemma 2.5.** *Let $X, Y$ be smooth $G$-varieties, let $G$ be a smooth reductive affine algebraic group, and let $\pi : X \times Y \to Y$ be a projection. Moreover, let $X$ be projective and $Y$ be connected.*

*Denote by $\mathcal{P}_\pi(G; X \times Y)$ the full subcategory of $\mathcal{P}(G; X \times Y)$ consisting of locally free $G$-modules $P$ such that $R^k \pi_* P = 0$ for $k > 0$.*

*Then any $G$-module $M$ possesses a finite-length resolution of the form*

$$M \to P^0 \to P^1 \to \cdots \to P^N \to 0,$$

*with $P^i \in OB(\mathcal{P}_\pi(G; X \times Y))$.*

*Proof.* First, we prove that for every $M$ there is an embedding $M \hookrightarrow P^0$. We will construct $P^0$ in the form of $M(n)$ for a large enough $n$. To do this, we construct a very ample $G$-equivariant sheaf $\mathcal{O}_X(1)$ and a $G$-equivariant embedding $i : X \hookrightarrow \mathbb{P}^n$ such that $\mathcal{O}_X(1) = i^* \mathcal{O}_{\mathbb{P}}(1)$. Let $L$ be a very ample line bundle. By Corollary 1.6 of [**10**] $L^{\otimes k}$ is G-equivariant for some $k$. Then it defines the action of $G$ on $V = \Gamma(X, L^{\otimes k})$ and equivariant morphism $i : X \to \mathbb{P}(V)$, which is an embedding since $L^{\otimes k}$ is very ample. Then we set $\mathcal{O}_X(1) = L^{\otimes k}$.

The standard embedding of the tautological bundle $\tau_{\mathbb{P}(V)} \hookrightarrow V \times \mathbb{P}(V)$ gives us a $G$-equivariant embedding of locally free sheaves $\mathcal{O}_{\mathbb{P}(V)}(-1) \hookrightarrow \mathcal{O}_{\mathbb{P}(V)} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}(V)}$. After twisting by $\mathcal{O}_{\mathbb{P}}(1)$, we have $\mathcal{O}_{\mathbb{P}(V)} \hookrightarrow \mathcal{O}_{\mathbb{P}(V)}(1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}(V)}(1)$. Inductively we have the $G$-equivariant embedding $\mathcal{O}_{\mathbb{P}(V)} \hookrightarrow \mathcal{O}_{\mathbb{P}(V)}(n) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}(V)}(n)$. Applying $i^*$, we get

$$\mathcal{O}_X \hookrightarrow \mathcal{O}_X(n) \oplus \cdots \oplus \mathcal{O}_X(n).$$

Define $\mathcal{O}_{X\times Y}(1) = \pi^*\mathcal{O}_X(1)$. Applying $\pi^*$ we get an equivariant embedding

$$M \hookrightarrow M(n) \oplus \cdots \oplus M(n)$$

for an arbitrary locally free $G$-module $M$. Clearly its cokernel is $G$-equivariant. It is easy to check that it is a locally free sheaf. Then for every locally free $G$-module there is a resolution consisting of direct sums of modules of the form $M(n)$.

Let us show that $M(n)$ lies in $\mathcal{P}_\pi(G; X \times Y)$ for a large enough $n$. $R^k\pi_*M(n)$ is associated to a presheaf $V \mapsto H^k(X \times V, M(n))$. Consider a finite affine covering $V_i$ of $Y$. By Serre's theorem $H^k(X \times V_i, M(n))$ equals zero for $n > n_i$. Thus, $R^k\pi_*M(n) = 0$ for $n > n_M = \max\{n_i\}$.

It remains to show that this resolution ends at some finite step. Let $N = \dim(X \times Y)$. Let $C^0$ be a cokernel of the first resolution step: $0 \to M \to P^0 \to C^0 \to 0$. Then we have the exact sequence

$$0 = R^N\pi_*P^0 \to R^N\pi_*C^0 \to R^{N+1}\pi_*M = 0.$$

So, $R^N\pi_*C^0 = 0$. For the second cokernel $C^1$ we have the exact sequence $0 \to C^0 \to P^1 \to C^1 \to 0$. Then

$$0 = R^{N-1}\pi_*P^1 \to R^{N-1}\pi_*C^1 \to R^N\pi_*C^0 = 0.$$

So, $R^{N-1}\pi_*C^{N-1} = 0$. By induction we have all $R^k\pi_*C^N = 0$. Then $C^N \in Ob(\mathcal{P}_\pi(G; X \times Y))$. □

**Corollary 2.6.** *This lemma allows us to give an explicit presentation of the push-forward map $f_*$ in the case when there is an equivariant decomposition $f\colon X \overset{i}{\hookrightarrow} Y \times W \to Y$ where $W$ is $G$-equivariant and projective. Since all $R^k i_*M = 0$ for any $G$-module $M$ and $k > 0$, we have two exact functors $i_*\mathcal{P}(G; X) \to \mathcal{P}(G; Y \times \mathbb{P}^n)$ and $\pi_{Y*} : \mathcal{P}_{\pi_Y}(G; Y \times \mathbb{P}^n) \to \mathcal{P}(G; Y)$. By Quillen's theorem, the inclusion of $\mathcal{P}_{\pi_Y}(G; Y \times \mathbb{P}^n)$ into $\mathcal{P}(G; Y \times \mathbb{P}^n)$ induces an isomorphism*

$$K_n(\mathcal{P}_{\pi_Y}(G; Y \times W)) \overset{\alpha}{\to} K_n(\mathcal{P}(G; Y \times W)) = K_n^G(Y \times \mathbb{P}^n).$$

*Then we can describe the pushforward map $f_* : K_n^G(X) \to K_n^G(Y)$ as the following composition:*

$$K_n^G(X) \xrightarrow{K_n(i_*)} K_n^G(Y \times W) \xrightarrow{\alpha^{-1}} K_n(\mathcal{P}_{\pi_Y}(G; Y \times W)) \to K_n(\mathcal{P}(G; Y)) = K_n^G(Y).$$

### 2.2. Reduction arguments.

According to [**8**, §4.2], for a given embedding $G \to \mathbf{GL_N}$ there is a sequence $EG_j$ of open subsets of corresponding linear representations on $\mathbb{A}^{Nj}$ such that $G$ acts freely on $EG_j$ and the quotient $EG_j/G$ exists as a scheme. Moreover, the codimension of the closed complement limits to infinity: $\lim_{j\to\infty} \operatorname{codim}_{\mathbb{A}^{Nj}}(\mathbb{A}^{Nj} \setminus EG_j) = \infty$. Also, we assume that we fix closed embeddings $EG_j \to EG_{j+1}$. Take $BG_j = EG_j/G$. This gives a geometric model for the étale classifying space $BG = \mathrm{colim}_j BG_j$.

**Lemma 2.7.** *Under the notation of Lemma 2.5, we have a commutative up to an isomorphism diagram of exact functors:*

$$\begin{array}{ccc} \mathcal{P}_{\pi_{EG_j}}(G; EG_j \times G/B) & \xleftarrow{(i_j \times id)^*} & \mathcal{P}_{\pi_{EG_{j+1}}}(G; EG_{j+1} \times G/B) \\ \downarrow{\scriptstyle \pi_{EG_j*}} & & \downarrow{\scriptstyle \pi_{EG_{j+1}*}} \\ \mathcal{P}(G; EG_j) & \xleftarrow{i_j^*} & \mathcal{P}(G; EG_{j+1}). \end{array} \tag{2}$$

*Proof.* To simplify notation let $\pi_j = \pi_{EG_j}$ and $\mathcal{P}_j = \mathcal{P}_{\pi_{EG_j}}(G; EG_j \times G/B)$. Let us prove that $\mathcal{P}_{j+1}$ is mapped to $\mathcal{P}_j$ under $(i_j \times id)^*$. Let $M \in Ob(\mathcal{P}_{j+1})$. Let $\dim(EG_j \times G/B) = N$. Then $R^{N+1}\pi_{j*}(i_j \times id)^* M = 0$. By Corollary 2 of [**9**, §5]

$$\begin{aligned} R^N \pi_{j*}(i_j \times id)^* M \otimes_{\mathcal{O}_{EG_j}} k(y) &= H^N(EG_j \times \{y\}, (i_j \times id)^* M) \\ &= H^N(EG_j \times \{y\}, M) = 0. \end{aligned}$$

Then $R^N \pi_{j*}(i_j \times id)^* M = 0$. By induction we obtain that all $R^k \pi_{j*} i_j^* M = 0$ for $k > 0$. Then $i_j^* M \in Ob(\mathcal{P})$. By Lemma 2.3 we have a natural $G$-isomorphism $hh\colon i_j^* \pi_{j+1*} M \to \pi_{j*}(i_j \times id)^* M$, so diagram (2) is commutative up to a natural isomorphism. □

**Lemma 2.8.** *Under the notation of Lemma 2.5, for each $j \geqslant 0$ the functor*

$$\pi_j^* : \mathcal{P}(G; EG_j) \to \mathcal{P}(G; EG_j \times G/B)$$

*takes values in the subcategory $\mathcal{P}_{\pi_{EG_j}}(G; EG_j \times G/B)$. As a consequence, the following diagram of exact functors commutes up to a natural isomorphism:*

$$\begin{array}{ccc} \mathcal{P}(G; EG_j) & \xleftarrow{i_j^*} & \mathcal{P}(G; EG_{j+1}) \\ \downarrow{\scriptstyle \pi_j^*} & & \downarrow{\scriptstyle \pi_{j+1}^*} \\ \mathcal{P}_{\pi_{EG_j}}(G; EG_j \times G/B) & \xleftarrow{(i_j \times id)^*} & \mathcal{P}_{\pi_{EG_{j+1}}}(G; EG_{j+1} \times G/B). \end{array} \tag{3}$$

*Proof.* To simplify notation, let $\pi_j = \pi_{EG_j}$ and $\mathcal{P}_j = \mathcal{P}_{\pi_{EG_j}}(G; EG_j \times G/B)$. First we prove that $\pi_j^*$ maps $\mathcal{P}(G; EG_j)$ to $\mathcal{P}_j$. Let $M$ be an object of $\mathcal{P}(G; EG_j)$. Then $R^k \pi_{j*} \pi_j^* M$ is associated to the presheaf $V \mapsto H^k(V \times G/B, \pi_j^* M)$. Let $V$ be an affine open subset of $EG_j$. Let $\{U_n\}$ be an affine covering of $G/B$. For any intersection $W = U_{n_1} \cap \cdots \cap U_{n_k}$, we have

$$\pi_j^* M(V \times W) = M(V) \otimes_{\mathcal{O}_{EG_j}(V)} \mathcal{O}_{EG_j \times G/B}(V \times W) = M(V) \otimes_k \mathcal{O}_{G/B}(W).$$

Then Čech complex $\check{C}(\{V \times U_n\}, \pi_j^* M)$ equals $M(V) \otimes_k \check{C}(\{U_n\}, \mathcal{O}_{G/B})$. Consequently, $H^k(V \times G/B, \pi_j^* M) = M(V) \otimes_k H^k(G/B, \mathcal{O}_{G/B})$.

By Proposition 4.5 of [**5**], $H^k(G/B, \mathcal{O}_{G/B}) = 0$ for $k > 0$. Then $\pi_{j*} M \in Ob(\mathcal{P}_j)$. The commutativity of (3) trivially follows from the equality $\pi_{j+1} \circ (i_j \times id) = i_j \circ \pi_j$. □

**Lemma 2.9.** *Composition $\pi_{EG_j*} \circ \pi^*_{EG_j}$ is naturally isomorphic to $id_{\mathcal{P}(G;EG_j)}$:*

$$\mathcal{P}(G; EG_j) \xrightarrow{\pi^*_{EG_j}} \mathcal{P}_{\pi_{EG_j}}(EG_j \times G/B) \xrightarrow{\pi_{EG_j*}} \mathcal{P}(G; EG_j).$$

*Proof.* Let $M \in Ob(\mathcal{P}(G; EG_j))$. The sheaf $\pi_{EG_j*}\pi^*_{EG_j}M$ is associated to presheaf $V \mapsto \pi^*_{EG_j}(M)(V \times G/B)$. Since $\pi^*_{EG_j}M$ is a sheaf associated to $W \mapsto M(\pi_{EG_j}(W))$, we see that $\pi_{EG_j*}\pi^*_{EG_j}M$ is associated to the presheaf $V \mapsto M(V)$. So, in the category of presheaves, $\pi_{EG_j*}\pi^*_{EG_j} \cong id$. Applying the sheaffication functor to this isomorphism, we get a natural isomorphism $\pi_{EG_j*}\pi^*_{EG_j}M \cong M$. □

The same reasoning proves the statement for the projection $\pi_{pt} \colon G/B \to pt = \operatorname{Spec} k$.

**Lemma 2.10.** *Composition $\pi_{pt*} \circ \pi^*_{pt}$ is naturally isomorphic to $id_{\mathcal{P}(G;pt)}$:*

$$\mathcal{P}(G; \operatorname{Spec} k) \xrightarrow{\pi^*_{pt}} \mathcal{P}_{\pi_{pt}}(G/B) \xrightarrow{\pi_{pt*}} \mathcal{P}(G; \operatorname{Spec} k).$$

**Lemma 2.11.** *Using the notation of Lemma 2.5, we have a diagram of exact functors that is commutative up to an isomorphism:*

$$\begin{array}{ccc} \mathcal{P}_{\pi_{pt}}(G; G/B) & \xrightarrow{\pi^*_{G/B}} & \mathcal{P}_{\pi_{EG_j}}(G; EG_j \times G/B) \\ \downarrow{\scriptstyle \pi_{pt*}} & & \downarrow{\scriptstyle \pi_{EG_j*}} \\ \mathcal{P}(G; \operatorname{Spec} k) & \longrightarrow & \mathcal{P}(G; EG_j). \end{array} \tag{4}$$

*Proof.* Let us prove that $\mathcal{P}_{\pi_{pt}}(G; G/B)$ is mapped to $\mathcal{P}_{\pi_{EG_j}}(G; EG_j \times G/B)$ under $\pi^*_{G/B}$. Let $M \in Ob(\mathcal{P}_{\pi_{pt}}(G; G/B))$. Then, by Proposition 9.3 of [**3**], $R^k\pi_{EG_j}(\pi^*_{G/B}M)$ is isomorphic to $\pi^*_{pt}(R^k\pi_{pt*}M)$. The latter sheaf is zero by definition of $\mathcal{P}_{\pi_{pt}}(G; G/B)$ for $k > 0$. So, for $k > 0$ we have $R^k\pi_{EG_j}(\pi^*_{G/B}M) = 0$; then $\pi^*_{G/B}M \in Ob(\mathcal{P}_{\pi_{EG_j}}(G; EG_j \times G/B))$.

Commutativity of diagram (4) follows immediately from Lemma 2.4. □

**Lemma 2.12.** *Under the notation of Lemma 2.5, functor*

$$\pi^*_{pt} : \mathcal{P}(G; \operatorname{Spec} k) \to \mathcal{P}(G; G/B)$$

*takes values in the subcategory $\mathcal{P}_{\pi_{pt}}(G; G/B)$. As a consequence, the following diagram of exact functors commutes up to a natural isomorphism:*

$$\begin{array}{ccc} \mathcal{P}(G; \operatorname{Spec} k) & \xrightarrow{\pi^*_{pt}} & \mathcal{P}(G; EG_j) \\ \downarrow{\scriptstyle \pi^*_{pt}} & & \downarrow{\scriptstyle \pi^*_{EG_j}} \\ \mathcal{P}_{\pi_{pt}}(G; G/B) & \xrightarrow{\pi^*_{G/B}} & \mathcal{P}_{\pi_{EG_j}}(G; EG_j \times G/B). \end{array} \tag{5}$$

*Proof.* We prove that $\pi_{pt}^*$ maps $\mathcal{P}(G; \operatorname{Spec} k)$ to $\mathcal{P}_{\pi_{pt}^*}(G; G/B)$. Let $M$ be an object of $\mathcal{P}(G; \operatorname{Spec} k)$. Then $R^k \pi_{pt*} \pi_{pt}^* M$ is a vector space $H^k(G/B, \pi_{pt}^* M)$. Let $\{U_n\}$ be an affine covering of $G/B$. For any intersection $W = U_{n_1} \cap \cdots \cap U_{n_k}$, we have

$$\pi_{pt}^* M(W) = M \otimes_k \mathcal{O}_{G/B}(W).$$

Then Čech complex $\check{C}(\{U_n\}, \pi_{pt}^* M)$ equals $M \otimes_k \check{C}(\{U_n\}, \mathcal{O}_{G/B})$. Consequently, $H^k(G/B, \pi_j^* M) = M \otimes_k H^k(G/B, \mathcal{O}_{G/B})$.

By [**5**, Proposition 4.5], $H^k(G/B, \mathcal{O}_{G/B}) = 0$ for $k > 0$. Then $\pi_{pt*} M \in Ob(\mathcal{P}_{\pi_{pt}}(G; G/B))$. The commutativity of (5) trivially follows from the equality $\pi_{pt} \circ \pi_{EG_j} = \pi_{pt} \circ \pi_{G/B}$. □

*Remark 2.13.* As we can see from proofs of Lemmas 2.7–2.12, we can replace $G/B$ by any projective $G$-variety $X$ such that $h^0(X, \mathcal{O}_X) = 1$ and $h^i(X, \mathcal{O}_X) = 0$, for $i > 0$.

**Proposition 2.14.** *There is a commutative diagram with* $\pi_{EG_i*} \circ \pi_{EG_i}^* = id_{K_n^G(EG_i)}$, $\pi_{pt*} \pi_{pt}^* = id_{K_n^G(k)}$:

$$\begin{array}{ccc}
K_n^G(k) & \xrightarrow{\pi_{pt}^*} & K_n^G(EG_i) \\
\downarrow{\scriptstyle \pi_{pt}^*} & & \downarrow{\scriptstyle \pi_{EG_i}^*} \\
K_n^G(G/B) & \xrightarrow{\pi_{G/B}^*} & K_n^G(EG_i \times G/B) \\
\downarrow{\scriptstyle \pi_{pt*}} & & \downarrow{\scriptstyle \pi_{EG_i*}} \\
K_n^G(k) & \xrightarrow{\pi_{pt}^*} & K_n^G(EG_i)
\end{array}$$

*Proof.* By Lemmas 2.11 and 2.12 we get the following commutative categories diagram with exact arrows:

$$\begin{array}{ccc}
\mathcal{P}(G; \operatorname{Spec} k) & \xrightarrow{\pi_{pt}^*} & \mathcal{P}(G; EG_j) \\
\downarrow{\scriptstyle \pi_{pt}^*} & & \downarrow{\scriptstyle \pi_{EG_j}^*} \\
\mathcal{P}_{\pi_{pt}}(G; G/B) & \xrightarrow{\pi_{G/B}^*} & \mathcal{P}_{\pi_{EG_j}}(EG_j \times G/B) \\
\downarrow{\scriptstyle \pi_{pt*}} & & \downarrow{\scriptstyle \pi_{EG_j*}} \\
\mathcal{P}(G; \operatorname{Spec} k) & \xrightarrow{\pi_{pt}^*} & \mathcal{P}(G; EG_j).
\end{array} \tag{6}$$

Recall that, by Quillen's theorem and Lemma 2.5, categories inclusion $\mathcal{P}_{\pi_{pt}}(G; G/B) \subseteq \mathcal{P}(G; G/B)$ induces an isomorphism $K_n(\mathcal{P}(G; G/B)) \to K_n(\mathcal{P}_{\pi_{pt}}(G; G/B))$. Then applying $K_n$ to diagram (6) gives us

$$\begin{array}{ccc}
K_n^G(k) & \xrightarrow{\pi_{pt}^*} & K_n^G(EG_i) \\
\downarrow{\scriptstyle \pi_{pt}^*} & & \downarrow{\scriptstyle \pi_{EG_i}^*} \\
K_n^G(G/B) & \xrightarrow{\pi_{G/B}^*} & K_n^G(EG_i \times G/B) \\
\downarrow{\scriptstyle \pi_{pt*}} & & \downarrow{\scriptstyle \pi_{EG_i*}} \\
K_n^G(k) & \xrightarrow{\pi_{pt}^*} & K_n^G(EG_i).
\end{array}$$

Equalities $\pi_{EG_i*} \circ \pi_{EG_i}^* = id_{K_n^G(EG_i)}$ and $\pi_{pt*}\pi_{pt}^* = id_{K_n^G(k)}$ immediately follow from Lemmas 2.9 and 2.10. □

*Remark 2.15.* In particular, we get a well-known fact that the natural ring map $R(G) \to R(B)$ is injective.

*Remark 2.16.* By Remark 2.13, we can replace $G/B$ in the statement of Proposition 2.14 by any projective $G$-variety $X$ such that $h^0(X, \mathcal{O}_X) = 1$ and $h^i(X, \mathcal{O}_X) = 0$ for $i > 0$.

**Lemma 2.17.** *Suppose $R$ is a commutative ring with an action of a finite group $W$. Let $S$ denote the invariant subring $S = R^W$. Let $I_R$ be an ideal of $R$, and let $I_S = S \cap I_R$. Suppose that $\mathfrak{q}$ is a prime ideal in $R$ and $\mathfrak{q} \supseteq I_S$. Then $\mathfrak{q} \supseteq I_R$.*

*Proof.* Denote $W = \{\sigma_1, \dots, \sigma_n\}$. Consider $x \in I_R$. For any symmetric polynomial $f \in R[t_1, \dots t_n]$ we have that $f(x^{\sigma_1}, \dots, x^{\sigma_n})$ is invariant under the $W$-action, and so $f(x^{\sigma_1}, \dots, x^{\sigma_n}) \in S \cap I_R = I_S \subseteq \mathfrak{q}$. Denote by $f_1, \dots, f_n$ the elementary symmetric polynomials. Then $x$ is a root of the polynomial

$$\prod_{i=1}^{n}(t - x^{\sigma_i}) = t^n - f_1(x^{\sigma_1}, \dots, x^{\sigma_n})t^{n-1} + \dots + (-1)^n f_n(x^{\sigma_1}, \dots, x^{\sigma_n}).$$

Then

$$x^n = -(-f_1(x^{\sigma_1}, \dots, x^{\sigma_n})x^{n-1} + \dots + (-1)^n f_n(x^{\sigma_1}, \dots, x^{\sigma_n})) \in \mathfrak{q}.$$

So $x^n \in \mathfrak{q}$. Since $\mathfrak{q}$ is prime, $x \in \mathfrak{q}$. Thus, $I_R \subseteq \mathfrak{q}$. □

**Proposition 2.18.** *The $I_B$-adic topology of $R(B)$ coincides with the $I_G \cdot R(B)$-adic topology.*

*Proof.* Let $T$ be a maximal torus in $G$. Then $R(B) = R(T)$ and $I_B = I_T$, where $I_T$ is the ideal of zero-dimensional representations of $T$. We will prove that $\sqrt{I_G \cdot R(T)} = I_T$. Denote by $W = N_G(T)/T$ the Weyl group of $G$. The group $W$ acts by conjugation on $R(T)$. It is known that $W$ is a finite group and $R(G)$ is the ring of invariants of $W$: $R(G) = R(T)^W$. Then taking $R = R(T)$ and $S = R(G)$ in Lemma 2.17, we have that any prime ideal of $R(T)$ containing $I_G$ contains $I_T$. Then we have equality for its radicals $\sqrt{I_G \cdot R(T)} = \sqrt{I_T}$. Since $I_T$ is prime, $I_T = \sqrt{I_T}$. So $\sqrt{I_G \cdot R(B)} = I_B$. Since $R(B)$ is noetherian, this implies that $I_B^m \subseteq I_G \cdot R(B)$ for some $m$. Then $I_B$ and $I_G \cdot R(B)$ determine the same topology on $R(B)$. □

**Proposition 2.19.** *$K_n(BG) = \varprojlim K_n(BG_i)$.*

*Proof.* By [**15**, Theorem 6.9] algebraic $K$-theory is representable in the stable $\mathbb{A}^1$-homotopy category, and then [**4**, Proposition 2.2.11(c)] implies the Milnor exact sequence:

$$0 \to \varprojlim{}^1 K_{n+1}(BG_i) \to K_n(BG) \to \varprojlim K_n(BG_i) \to 0.$$

Let us show that $\varprojlim{}^1 K_n(BG_i) = 0$, for any $n > 0$.

We prove that the sequence $K_n(BG_i)$ is a direct summand of the sequence $K_n(BB_i)$.

By Proposition 1 of [**7**] we have $K_n(BG_i) = K_n^G(EG_i)$. Since we can choose $EG_i$ as a model for $EB_i$, we obtain

$$K_n(BB_i) = K_n^B(EB_i) = K_n^B(EG_i) = K_n^G(EG_i \times G/B).$$

So, in fact, we prove that the sequence $K_n^G(EG_i)$ is a direct summand of the sequence $K_n^G(EG_i \times G/B)$.

To simplify the notation denote $\mathcal{P}_j = \mathcal{P}_{\pi_{EG_j}}(G; EG_j \times G/B)$. By Lemmas 2.11 and 2.12 we obtain a commutative diagram with exact arrows:

$$\begin{array}{ccc} \mathcal{P}(G;EG_j) & \xleftarrow[(i_j \times id)^*]{} & \mathcal{P}(G;EG_{j+1}) \\ \downarrow{\scriptstyle \pi_j^*} & & \downarrow{\scriptstyle \pi_{j+1}^*} \\ \mathcal{P}_j & \xleftarrow[(i_j \times id)^*]{} & \mathcal{P}_{j+1} \\ \downarrow{\scriptstyle \pi_{j*}} & & \downarrow{\scriptstyle \pi_{j*}} \\ \mathcal{P}(G;EG_j) & \xleftarrow[i_j^*]{} & \mathcal{P}(G;EG_{j+1}). \end{array} \tag{7}$$

By Lemma 2.10 the composition

$$\mathcal{P}(G;EG_j) \xrightarrow{\pi_j^*} \mathcal{P}_j \xrightarrow{\pi_{j*}} \mathcal{P}(G;EG_j)$$

is naturally isomorphic to $id_{\mathcal{P}(G;EG_j)}$. In the proof of Lemma 2.7 we checked that $(i_j \times id)^*(\mathcal{P}_{j+1}) \subseteq \mathcal{P}_j$. By Lemma 2.5 each $G$-module in $\mathcal{P}(G; EG_j \times G/B)$ has a finite resolution consisting of sheaves from $\mathcal{P}_j$. Then by Quillen's theorem we get the isomorphisms $\alpha_j$ such that the following diagram of groups commutes:

$$\begin{array}{ccc} K_n(\mathcal{P}_j) & \xleftarrow[(i_j \times id)^*]{} & K_n(\mathcal{P}_{j+1}) \\ \downarrow{\scriptstyle \alpha_j} & & \downarrow{\scriptstyle \alpha_{j+1}} \\ K_n^G(EG_j \times G/B) & \xleftarrow[(i_j \times id)^*]{} & K_n^G(EG_{j+1} \times G/B). \end{array} \tag{8}$$

In Corollary 2.6 we defined $\pi_{j*} : K_n^G(EG_j \times G/B) \to K_n^G(EG_j)$ as the composition of

$$K_n^G(EG_j \times G/B) \xrightarrow{\alpha_j^{-1}} K_n(\mathcal{P}_j) \xrightarrow{\pi_{j*}} K_n^G(EG_j).$$

Commutativity of the diagrams (7) and (8) gives us a commutative diagram:

$$\begin{array}{ccc} K_n^G(EG_j) & \xleftarrow[(i_j\times id)^*]{} & K_n^G(EG_{j+1}) \\ \big\downarrow{\scriptstyle \pi_j^*} & & \big\downarrow{\scriptstyle \pi_{j+1}^*} \\ K_n^G(EG_j \times G/B) & \xleftarrow[(i_j\times id)^*]{} & K_n^G(EG_{j+1}\times G/B) \\ \big\downarrow{\scriptstyle \pi_{j*}} & & \big\downarrow{\scriptstyle \pi_{j+1*}} \\ K_n^G(EG_j) & \xleftarrow[i_j^*]{} & K_n^G(EG_{j+1}). \end{array} \tag{9}$$

As we have shown, compositions of vertical arrows are identity, so $K_n^G(EG_j)$ is a direct summand of sequence $K_n^G(EG_i \times G/B) = K_n(BB_j)$. Since $\varprojlim^1(K_n(BB_j)) = 0$ we get $\varprojlim^1(K_n^G(EG_j)) = 0$. It remains to show that $\varprojlim^1(K_n(BB_j)) = 0$. Let $T$ be a maximal torus. Since $B/T$ is an affine space, we have that $BT_j \to BB_j$ is locally trivial with fibers being affine spaces. Then pullback map $K_n(BB_j) \to K_n(BT_j)$ is a natural isomorphism. Since $G$ is split, $T$ is a split torus, $T = \mathbb{G}_m \times \cdots \times \mathbb{G}_m$. Then $BT_j = \mathbb{P}^j \times \cdots \times \mathbb{P}^j$. So, $K_n(BT_j) = K_n(k)[t_1, \dots, t_n]/(t_1^{j+1}, \dots, t_n^{j+1})$. Embedding pullbacks act as follows:

$$t_k \bmod (t_1^{j+1}, \dots, t_n^{j+1}) \mapsto t_k \bmod (t_1^j, \dots, t_n^j).$$

Then all morphisms in the sequence $\cdots \to K_n(BT_j) \to K_n(BT_{j-1}) \to \cdots$ are surjective. Then $\varprojlim^1(K_n(BT_i)) = 0$, and consequently, $\varprojlim^1(K_n(BB_i)) = 0$. This concludes the proof. □

## 3. Proof of main result

**Theorem 3.1.** *The Borel construction induces an isomorphism*

$$\widehat{K_n^B(k)}_{I_B} \xrightarrow{\widehat{Borel_n^B}} \widehat{K_n(BB)}_{I_B} \xleftarrow{\cong} K_n(BB).$$

*Proof.* We define $Borel_n^B : K_n^B(k) \to K_n(BB)$ in the following way: For any $j$ we construct $(Borel_n^B)_j : K_n^B(k) \to K_n^B(EB_j)$ as a pullback of a projection $\pi_{pt} : EB_j \to \operatorname{Spec} k$. By Proposition 1 of [**7**], $K_n^B(EB_j)$ are isomorphic to $K_n(BB_j)$. So we get $(Borel_n^B)_j : K_n^B(k) \to K_n(BB_j)$. By Proposition 2.19, we obtain $Borel_n^B = \varprojlim (Borel_n^B)_j : K_n^B(k) \to K_n(BB)$.

Let $T$ be a maximal torus of $G$. By Corollary 1 of [**7**] exact functor $\mathcal{P}(T; \operatorname{Spec} k) \to \mathcal{P}(B; B/T)$ induces an isomorphism $K_n^T(k) \cong K_n^B(B/T)$. Note that $B/T$ can be identified with a linear representation of the group $B$, so by Theorem 3 of [**7**] the pullback morphism $K_n^B(k) \to K_n^B(B/T)$ is an isomorphism. Recall that we may choose the models $BT_j$ and $BB_j$ of the form $BT_j = EB_j/T$ and $BB_j/B$. Then $BT_j \to BB_j$ is locally trivial with fibers being affine spaces. Then using the homotopy equivalence for non-equivariant $K$-theory, we have that $K_n(BB_j) \to K_n(BT_j)$ is an isomorphism.

So we get the commutative diagram

$$\begin{array}{ccc} K_n^B(k) & \xrightarrow{Borel_n^B} & K_n(BB) \\ \downarrow \cong & & \downarrow \cong \\ K_n^B(B/T) & \xrightarrow{\pi^*_{B/T}} & K_n(BB \times B/T) \\ \downarrow \cong & & \downarrow \cong \\ K_n^T(k) & \xrightarrow{Borel_n^T} & K_n(BT). \end{array}$$

Therefore, it suffices to prove our theorem for maximal torus $T$. Since $G$ is split,

$$T = \mathbb{G}_m \times \cdots \times \mathbb{G}_m \qquad (j \text{ times}).$$

Let us compute $K_n^T(k)$ and $K_n^T(k)_{I_T}$.

Since $K_n^T(k) = K_n(k) \otimes_{\mathbb{Z}} R(T)$ we have that

$$R(T) = \mathbb{Z}[\lambda_1, \dots, \lambda_j, t]/(\lambda_1 \cdot \dots \cdot \lambda_j \cdot t = 1).$$

$I_T = (1-\lambda_1, \dots, 1-\lambda_j, 1-t)$. So, we have the following:

$$\widehat{K_n^T(k)_{I_T}} = \widehat{R(T)}_{I_T} \otimes_{\mathbb{Z}} K_n(k)$$

$$\begin{aligned} \widehat{R(T)}_{I_T} &= \varprojlim \mathbb{Z}[\lambda_1, \dots, \lambda_j, t]/((\Pi\lambda_i \cdot t - 1), (1-\lambda_1)^k, \dots, (1-\lambda_j)^k, (1-t)^k) \\ &= \varprojlim \mathbb{Z}[1-\lambda_1, \dots, 1-\lambda_j, 1-t]/((\Pi\lambda_i \cdot t - 1), \\ &\qquad\qquad (1-\lambda_1)^k, \dots, (1-\lambda_j)^k, (1-t)^k) \\ &= \mathbb{Z}[[1-\lambda_1, \dots, 1-\lambda_j, 1-t]]/(\Pi\lambda_i \cdot t - 1) \\ &= \mathbb{Z}[[\mu_1, \dots, \mu_l, 1-t]]/(\Pi(1-\mu_i) \cdot t - 1). \end{aligned}$$

Since $\frac{1}{1-\mu_i} = 1 + \mu_i + \mu_i^2 + \mu_i^3 + \cdots$, it follows that $t = \prod(1 + \mu_i + \mu_i^2 + \cdots)$. Therefore we have $1 - t = 1 - (1 + \mu_1 + \cdots + \mu_j + \cdots) = -(\mu_1 + \cdots + \mu_j + \cdots)$. Then

$$\widehat{R(T)}_{I_T} = \mathbb{Z}[[\mu_1, \dots, \mu_j]].$$

Finally we get

$$\widehat{K_n^T(k)_{I_T}} = K_n(k)[[\mu_1, \dots, \mu_j]].$$

Let us compute $K_n(BT)$.

We can choose for $ET$ the space $\mathbb{A}^\infty \backslash \{0\} \times \cdots \times \mathbb{A}^\infty \backslash \{0\}$. This is a contractible space with free $T$-action. Then $ET_k = \mathbb{A}^{k+1} \backslash \{0\} \times \cdots \times \mathbb{A}^{k+1} \backslash \{0\}$ and $BT_k = \mathbb{P}^k \times \cdots \times \mathbb{P}^k$. Then $K_n(BT_k) = K_n(k)[x_1, \dots, x_n]/(x_1^k, \dots, x_n^k)$.

So we have $BT = \mathbb{P}^\infty \times \cdots \times \mathbb{P}^\infty$. And finally we get

$$K_n(BT) = \varprojlim K_n(BT_k) = K_n(k)[[x_1, \dots, x_n]].$$

The Borel construction $K_n^T(k) \to K_n(BT_k)$ works as follows:

$$\begin{aligned} \lambda_i &\mapsto 1 - x_i \\ t &\mapsto \tfrac{1}{(1-x_1)\cdots(1-x_n)} = (1 + x_1 + \cdots + x_1^{k-1}) \cdots (1 + x_1 + \cdots + x_1^{k-1}). \end{aligned}$$

Then on $\widehat{K_n^T(k)}_{I_T}$ the Borel construction induces an isomorphism $\mu_i \mapsto x_i$. Let us prove that $K_n(BT)$ is complete in the $I_T$-adic topology. The $R(T)$-module structure on $K_n(BT)$ arises from the $R(T)$-structure on $K_0(BT) = \mathbb{Z}[[x_1, \ldots, x_n]]$. Then $I_T \cdot K_n(BT) = (x_1, \ldots, x_n)$. Therefore $K_n(BT)$ is complete. This completes the proof of the theorem. □

**Theorem 3.2.** *There is a commutative diagram of the following form:*

$$\begin{array}{ccccc} \widehat{K_n^G(k)}_{I_G} & \xrightarrow{\widehat{Borel_n^G}} & \widehat{K_n(BG)}_{I_G} & \xleftarrow{completion_G} & K_n(BG) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\widehat{p^*}} & & \downarrow{\scriptstyle p^*} \\ \widehat{K_n^B(k)}_{I_B} & \xrightarrow{\widehat{Borel_n^B}} & \widehat{K_n(BB)}_{I_B} & \xleftarrow{completion_B} & K_n(BB) \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\widehat{p_*}} & & \downarrow{\scriptstyle p_*} \\ \widehat{K_n^G(k)}_{I_G} & \xrightarrow{\widehat{Borel_n^G}} & \widehat{K_n(BG)}_{I_G} & \xleftarrow{completion_G} & K_n(BG), \end{array} \tag{10}$$

*with $\beta \circ \alpha = id$, $\widehat{p_*} \circ \widehat{p^*} = id$, and $p_* \circ p^* = id$.*

*Proof.* Since $EG_i \to BG_i$ is a $G$-torsor, $K_n(BG_i) = K_n^G(EG_i)$. By Proposition 1 of [**7**] $EG$ can be chosen as a model for the contractible space $EB$. Proposition 1 of [**7**] also allows us to express all these objects in terms of $G$-equivariant $K$-theory: $K_n^B(k) \cong K_n^G(G/B)$ $K_n^B(EG_j) = K_n^G(EG_j \times G/B)$.

So, first we construct

$$\begin{array}{ccc} K_n^G(k) & \xrightarrow{\pi_{pt}^*} & K_n^G(EG_i) \\ \downarrow{\scriptstyle\pi_{pt}^*} & & \downarrow{\scriptstyle\pi_{EG_i}^*} \\ K_n^G(G/B) & \xrightarrow{\pi_{G/B}^*} & K_n^G(EG_i \times G/B) \\ \downarrow{\scriptstyle\pi_{pt*}} & & \downarrow{\scriptstyle\pi_{EG_i*}} \\ K_n^G(k) & \xrightarrow{\pi_{pt}^*} & K_n^G(EG_i). \end{array} \tag{11}$$

Proposition 2.14 proves that this diagram commutes and $\pi_{pt*} \circ \pi_{pt}^* = id$ and $\pi_{EG_i*} \circ \pi_{EG_i}^* = id$. Recall that $K_n^G(EG_j) = K_n(BG_j)$, $K_n^G(EG_j \times G/B) = K_n(BB_j)$, and $K_n^G(G/B) = K_n^B(k)$.

Therefore we can rewrite the above diagram as follows:

$$\begin{array}{ccc} K_n^G(k) & \xrightarrow{\pi_{pt}^*} & K_n(BG_i) \\ \downarrow{\scriptstyle\pi_{pt}^*} & & \downarrow{\scriptstyle\pi_{EG_i}^*} \\ K_n^B(k) & \xrightarrow{\pi_{G/B}^*} & K_n(BB_i) \\ \downarrow{\scriptstyle\pi_{pt*}} & & \downarrow{\scriptstyle\pi_{EG_i*}} \\ K_n^G(k) & \xrightarrow{\pi_{pt}^*} & K_n(BG_i). \end{array} \tag{12}$$

Take the projective limit of this diagram. Recall that $\varprojlim K_n(BB_i) = K_n(BB)$ and by Proposition 2.19 we have $\varprojlim K_n(BG_i) = K_n(BG)$. So we get a commutative diagram of $K_0^G(k)$-modules

$$\begin{array}{ccc} K_n^G(k) & \xrightarrow{Borel_n^G} & K_n(BG) \\ \downarrow{\scriptstyle \pi_{pt}^*} & & \downarrow{\scriptstyle \varprojlim \pi_{EG_i}^*} \\ K_n^B(k) & \xrightarrow{Borel_n^B} & K_n(BB) \\ \downarrow{\scriptstyle \pi_{pt*}} & & \downarrow{\scriptstyle \varprojlim \pi_{EG_i*}} \\ K_n^G(k) & \xrightarrow{Borel_n^G} & K_n(BG). \end{array} \tag{13}$$

Here we still have $\pi_{pt*} \circ \pi_{pt}^* = id$ and $\varprojlim \pi_{EG_i*} \circ \varprojlim \pi_{EG_i}^* = id$. Let us denote $p_* = \varprojlim \pi_{EG_i*}$ and $p^* = \varprojlim \pi_{EG_i}^*$. Recall that $R(G)$-structures on $K_n(BB)$ and $K_n^B(k)$ are induced by the $R(G)$-structure on $R(B)$. Then Proposition 2.18 implies that $I_G$-adic completions of $K_n(BB)$ and $K_n^B(k)$ coincide with $I_B$-adic completions. So, by taking the $I_G$-adic completion of (13), we obtain the commutative diagram

$$\begin{array}{ccc} \widehat{K_n^G(k)}_{I_G} & \xrightarrow{\widehat{Borel_n^G}} & \widehat{K_n(BG)}_{I_G} \\ \downarrow{\scriptstyle \widehat{\pi_{pt}^*}} & & \downarrow{\scriptstyle \widehat{p^*}} \\ \widehat{K_n^B(k)}_{I_B} & \xrightarrow{\widehat{Borel_n^B}} & \widehat{K_n(BB)}_{I_B} \\ \downarrow{\scriptstyle \widehat{\pi_{pt*}}} & & \downarrow{\scriptstyle \widehat{p_*}} \\ \widehat{K_n^G(k)}_{I_G} & \xrightarrow{\widehat{Borel_n^G}} & \widehat{K_n(BG)}_{I_G} \end{array} \tag{14}$$

with $\widehat{\pi_{pt*}} \circ \widehat{\pi_{pt}^*} = id$ and $\widehat{p_*} \circ \widehat{p^*} = id$. Consider the commutative diagram

$$\begin{array}{ccc} \widehat{K_n(BG)}_{I_G} & \xleftarrow{completion_G} & K_n(BG) \\ \downarrow{\scriptstyle \widehat{p^*}} & & \downarrow{\scriptstyle p^*} \\ \widehat{K_n(BB)}_{I_B} & \xleftarrow{completion_B} & K_n(BB) \\ \downarrow{\scriptstyle \widehat{p_*}} & & \downarrow{\scriptstyle p_*} \\ \widehat{K_n(BG)}_{I_G} & \xleftarrow{completion_G} & K_n(BG). \end{array} \tag{15}$$

Set $\alpha = \widehat{\pi_{pt}^*}$, $\beta = \widehat{\pi_{pt*}}$, and recall that $K_n^G(G/B) = K_n^B(k)$. Then by gluing together

(15) and (14), we obtain the diagram (10):

$$\begin{array}{ccccc}
\widehat{K_n^G(k)}_{I_G} & \xrightarrow{\widehat{Borel_n^G}} & \widehat{K_n(BG)}_{I_G} & \xleftarrow{completion_G} & K_n(BG) \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \widehat{p^*}} & & \downarrow{\scriptstyle p^*} \\
\widehat{K_n^B(k)}_{I_B} & \xrightarrow{\widehat{Borel_n^B}} & \widehat{K_n(BB)}_{I_B} & \xleftarrow{completion_B} & K_n(BB) \\
\downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \widehat{p_*}} & & \downarrow{\scriptstyle p_*} \\
\widehat{K_n^G(k)}_{I_G} & \xrightarrow{\widehat{Borel_n^G}} & \widehat{K_n(BG)}_{I_G} & \xleftarrow{completion_G} & K_n(BG),
\end{array}$$

with $\beta \circ \alpha = id$, $\widehat{p_*} \circ \widehat{p^*} = id$, and $p_* \circ p^* = id$. □

These two theorems immediately imply the main result:

**Theorem 3.3.** *In the following diagram both maps are $K_0^G(k)$-module isomorphisms:*

$$\widehat{K_n^G(k)}_{I_G} \overset{\widehat{Borel_n^G}}{\longrightarrow} \widehat{K_n(BG)}_{I_G} \overset{completion_G}{\longleftarrow} K_n(BG).$$

*Proof.* Theorem 3.2 states that $\widehat{Borel_n^G}$ and $completion_G$ are retracts of $\widehat{Borel_n^B}$ and $completion_B$ which are isomorphisms by Theorem 3.1. Then $\widehat{Borel_n^G}$ and $completion_G$ are also isomorphisms. □

## References


**[1]** M.F. Atiyah and F. Hirzebruch, Vector bundles and homogeneous spaces, *Proc. Sympos. Pure Math.* **3** (1961), 7–38.

**[2]** M.F. Atiyah and G. Segal, Equivariant K-theory and completion, *J. Diff. Geom.* **3** (1969), 1–18.

**[3]** R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, 1977.

**[4]** M. Hovey, J. Palmieri, and N. Strickland, Axiomatic stable homotopy theory, *Mem. of Amer. Math. Soc.* **610** (1997).

**[5]** J.C. Jantzen, *Representations of Algebraic Groups*, Academic Press, Orlando–London, 1987.

**[6]** A. Krishna, *Completion theorem for equivariant K-theory*, preprint, http://arxiv.org/abs/1201.5766.

**[7]** A.S. Merkurjev, Equivariant K-theory, *Handbook of K-theory, Vols 1, 2*, Springer (2005), 925–954.

**[8]** F. Morel and V. Voevodsky, $\mathbb{A}^1$-homotopy theory of schemes, *Publ. Math., Inst. Hautes Étud. Sci.* **90** (1999), 45–143.

**[9]** D. Mumford, *Abelian Varieties*, Oxford University Press, 1988.

**[10]** D. Mumford, J. Fogarty, and F. Kirwan, *Geometric Invariant Theory*, Springer-Verlag, New York, 1994.

**[11]** D. Quillen, Higher algebraic K-theory I, *Lecture Notes in Math.* **341** (1973), 85–147.

**[12]** R.W. Thomason, Algebraic K-theory of group scheme actions. In Browder, W. (ed.) *Algebraic topology and algebraic K-theory* (Princeton, NJ, 1983), 539–563, *Ann. Math. Stud.* **113**, Princeton University Press, 1987.

**[13]** R.W. Thomason, Equivariant algebraic vs. topological K-homology Atiyah-Segal-style, *Duke Math. J.* **56** (1988), no. **3**, 589–636.

**[14]** B. Totaro, The Chow ring of a classifying space, *Algebraic K-theory* (Seattle, WA, 1997), 249–281, *Proc. Sympos. Pure Math.* **67**, Amer Math. Soc. (1999).

**[15]** V. Voevodsky, $\mathbb{A}^1$-homotopy theory, *Doc. Math.*, Extra Volume ICM (1998), 579–604.

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