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MOTIVES AND ORIENTED COHOMOLOGY OF GENERICALLY CELLULAR VARIETIES

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(*communicated by Ulf Rehmann*)

Abstract

For a cellular variety X over a field k of characteristic 0 and an algebraic oriented cohomology theory $\mathtt{h}$ of Levine-Morel we construct a filtration on the cohomology ring $\mathtt{h}(X)$ such that the associated graded ring is isomorphic to the Chow ring of X. Using this filtration we establish the following comparison result between Chow motives and $\mathtt{h}$-motives of generically cellular varieties: any irreducible Chow-motivic decomposition of a generically cellular variety Y gives rise to an $\mathtt{h}$-motivic decomposition of Y with the same generating function. Moreover, under some conditions on the coefficient ring of $\mathtt{h}$ the obtained $\mathtt{h}$-motivic decomposition will be irreducible. We also prove that if the Chow motives of two twisted forms of Y coincide, then their $\mathtt{h}$-motives coincide as well.

1. Introduction

We work over the base field k of characteristic 0. The notion of an algebraic oriented cohomology theory was studied by Levine-Morel [**11**] and Panin-Smirnov [**15**]. In this paper we will work with an oriented cohomology theory $\mathtt{h}$ in the sense of Levine-Morel introduced in [**11**, §1.1]. Moreover, we assume that $\mathtt{h}$ is generically constant and has the localization property (see Definition 2.7). We denote its coefficient ring $\mathtt{h}(\mathrm{Spec}\ k)$ by Λ. Let X be a cellular variety with $N = \dim X$. We construct a filtration

$$\mathtt{h}(X) = \mathtt{h}^{(0)}(X) \supseteq \mathtt{h}^{(1)}(X) \supseteq \cdots \supseteq \mathtt{h}^{(N)}(X) \supseteq 0$$

on the cohomology ring such that the associated graded ring

$$Gr^* \mathtt{h}(X) = \bigoplus_{i \geqslant 0} \mathtt{h}^{(i)}(X)/\mathtt{h}^{(i+1)}(X)$$

is isomorphic (as a graded ring) to the Chow ring $\mathrm{CH}^*(X, \Lambda)$ of algebraic cycles modulo rational equivalence relation with coefficients in the ring Λ. We exploit this filtration and isomorphism in the context of $\mathtt{h}$-motives of generically cellular varieties. The latter is a natural generalization of the notion of Chow motives to the case of an

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arbitrary algebraic oriented cohomology theory of Levine-Morel. The first construction of the category of h-motives was introduced by Manin in [**12**]. This notion was also studied by Nenashev-Zainoulline in [**14**] and Vishik-Yagita in [**18**].

Let Λ^i denote the i-th graded component of the coefficient ring Λ. We prove the following theorem that relates the $\mathtt{h}$-motive of a generically cellular variety to its Chow motive:

Theorem A. Let X be a generically cellular variety over k, i.e., cellular over the function field $k(X)$. Assume that the Chow motive of X with coefficients in Λ^0 splits as

$$M^{\mathrm{CH}}(X,\Lambda^0) = \bigoplus_{i=1}^{n} \mathcal{R}(\alpha_i),$$

for some motive $\mathcal{R}$ that splits as a direct sum of twisted Tate motives $\overline{\mathcal{R}} = \bigoplus_{j=1}^{m} \Lambda^0(\beta_j)$ over its splitting field.

Then the $\mathtt{h}$-motive of X splits as

$$M^{\mathtt{h}}(X) = \bigoplus_{i=1}^{n} \mathcal{R}_{\mathtt{h}}(\alpha_i)$$

for some motive $\mathcal{R}_{\mathtt{h}}$, and over the same splitting field $\mathcal{R}_{\mathtt{h}}$ splits as a direct sum of twisted $\mathtt{h}$-Tate motives $\overline{\mathcal{R}_{\mathtt{h}}} = \bigoplus_{j=1}^{m} \Lambda(\beta_j)$.

This result can also be derived from the arguments of [**18**] where it is proved that sets of isomorphism classes of objects of categories of Chow motives and Ω-motives coincide. However, our approach gives a more explicit correspondence between the idempotents defining the (Chow) motive $\mathcal{R}$ and the $\mathtt{h}$-motive $\mathcal{R}_{\mathtt{h}}$. The latter allows us to prove the following result concerning the indecomposability of the $\mathtt{h}$-motive $\mathcal{R}_{\mathtt{h}}$:

Theorem B. Assume that $\Lambda^1 = \cdots \Lambda^N = 0$, where $N = \dim X$.

If the Chow motive $\mathcal{R}$ is indecomposable (over Λ^0), then the $\mathtt{h}$-motive $\mathcal{R}_{\mathtt{h}}$ is indecomposable (over Λ).

and also the following comparison property:

Theorem C. Suppose that X, Y are generically cellular and Y is a twisted form of X, i.e., Y becomes isomorphic to X over some splitting field.

If $M^{\mathrm{CH}}(X,\Lambda^0) \cong M^{\mathrm{CH}}(Y,\Lambda^0)$, then $M^{\mathtt{h}}(X) \cong M^{\mathtt{h}}(Y)$.

The paper is organized as follows: In section 2 we recall concepts of an algebraic oriented cohomology theory $\mathtt{h}$ of Levine-Morel and the corresponding category of $\mathtt{h}$-motives. In section 3 we introduce the filtration on the cohomology ring $\mathtt{h}(X)$ of a cellular variety X, which plays a central role in the paper. In section 4 we apply the filtration to obtain comparison results between $\mathtt{h}$-motives and Chow-motives of generically split varieties.

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2. Preliminaries

In the present section we recall the notions of algebraic oriented cohomology theory, formal group law, and cellular variety. We recall the definition of the category of $\mathtt{h}$-motives with the inverted Tate object.

Oriented cohomology theories

In this subsection we give a definition of a generically constant oriented cohomology theory with localization property (Definition 2.7). Let $\mathbf{Sch}_k$ denote the category of separated schemes of finite type over Spec $k = pt$, and $\mathbf{Sm}_k$ its full subcategory of smooth quasiprojective schemes.

Definition 2.1. ([**11**, Definition 1.1.2])

An algebraic oriented cohomology theory $\mathtt{h}^*$ on $\mathbf{Sm}_k$ is given by

(D1) Additive functor $\mathtt{h}^*\colon \mathbf{Sm}_k^{op} \to$ Commutative graded rings;

(D2) For every projective morphism $f\colon X \to Y$ of relative codimension d, a homomorphism of graded $\mathtt{h}^*(Y)$-modules $f_{\mathtt{h}}\colon \mathtt{h}^*(X) \to \mathtt{h}^{*+d}(Y)$.

These satisfy the list of axioms $(A1), (A2), (PB), (EH)$ of [**11**, Definition 1.1.2].

Let us fix some notation. For a morphism $f\colon X \to Y$ in $\mathbf{Sm}_k$ we will denote its image $\mathtt{h}^*(f)\colon \mathtt{h}^*(Y) \to \mathtt{h}^*(X)$ by $f^{\mathtt{h}}$ and call it the pullback morphism of f. For a projective $f\colon X \to Y$ of relative codimension d the morphism $f_{\mathtt{h}}\colon \mathtt{h}^*(X) \to \mathtt{h}^{*+d}(Y)$ introduced in $(D2)$ is called the pushforward morphism of f.

We denote the coefficient ring $\mathtt{h}^*(pt)$ by Λ^*. As for the Chow groups, we will also use the lower grading notation for $\mathtt{h}$, i.e., $\mathtt{h}_i(X) = \mathtt{h}^{\dim X - i}(X)$ for an irreducible variety X.

Let $\mathbf{Sch}'_k$ denote the subcategory of $\mathbf{Sch}_k$ consisting of projective morphisms between all the schemes in $\mathbf{Sch}_k$

Definition 2.2. ([**11**, Definition 2.1.2])

An oriented Borel-Moore functor H_* on the category $\mathbf{Sch}_k$ is given by

- An additive functor $H_*\colon \mathbf{Sch}'_k \to$ Graded abelian groups;
- For each smooth equidimensional morphism $f\colon X \to Y$ of relative dimension d, a homomorphism of abelian groups $f^*\colon H_*(Y) \to H_{*+d}(X)$;
- For each line bundle L on X, a homomorphism of abelian groups $\widetilde{c}_1\colon H_*(X) \to H_{*-1}(X)$.

These data satisfy the axioms (A1)–(A5) of [**11**, Definition 2.1.2].

An oriented Borel-Moore functor H_* is called an oriented Borel-Moore weak homology if it additionally satisfies the axioms of [**11**, Definition 4.1.9].

Remark 2.3. Replacing $\mathbf{Sch}_k$ by $\mathbf{Sm}_k$ in the previous definition, we get the notion of an oriented Borel-Moore weak homology on $\mathbf{Sm}_k$.

Definition 2.4. ([**11**, Definition 4.4.6]) The oriented Borel-Moore weak homology H on $\mathbf{Sch}_k$ has the localization property if the following sequence is exact:

$$H_*(Z) \xrightarrow{i_*} H_*(X) \xrightarrow{j^*} H_*(U) \to 0$$

for any closed immersion $Z \to X$ and its open complement $U \to X$ in $\mathbf{Sch}_k$.

Following the approach of [**11**, §4.4.1] for a finitely generated field extension F/k and for every scheme X over k, we define the value of the oriented Borel-Moore weak homology on the scheme $X \times_k F$ as follows:

$$H_*(X \times_k F) = \mathrm{colim}_U\, H_{*+\dim U}(X \times_k U),$$

where the colimit is taken over the category whose objects are schemes $U \in \mathbf{Sch}_k$ such that $k(U) = F$, and whose morphisms are smooth birational maps between them. Since k has characteristic zero, we may assume that all the schemes U are smooth. Then we get a canonical map $H_*(X) \to H_*(X \times_k F)$ arising from the pullbacks $H_*(X) \to H_{*+\dim U}(X \times U)$. Note that for the case $F = k(Y)$ for $Y \in \mathbf{Sm}_k$, the definition gives the identification $H_*(X \times_k k(Y)) = \mathrm{colim}_{U \subseteq Y}\, H_{*+\dim Y}(X \times_k U)$, where the colimit is taken over the category of open subsets of Y.

Definition 2.5. ([**11**, Definition 4.4.1]) The oriented Borel-Moore weak homology H_* on $\mathbf{Sch}_k$ is generically constant if, for every finitely generated separable field extension F/k, the canonical morphism $H_*(k) \to H_*(F)$ is an isomorphism.

According to [**11**, Remark 5.2.7] every oriented cohomology theory $\mathtt{h}^*$ defines an oriented Borel-Moore weak homology on the category $\mathbf{Sm}_k$.

Definition 2.6. We say that an oriented cohomology theory $\mathtt{h}^*$ is associated to a Borel-Moore weak homology H_* on $\mathbf{Sch}_k$ if the restriction of H_* to the category $\mathbf{Sm}_k$ coincides with the oriented Borel-Moore homology defined on $\mathbf{Sm}_k$ by the theory $\mathtt{h}^*$.

Definition 2.7. An oriented cohomology theory $\mathtt{h}^*$ is generically constant with a localization property if it is associated to some oriented Borel-Moore weak homology H_* that is generically constant and satisfies the localization property.

Remark 2.8. Examples of the theories $\mathtt{h}^*$ defined above include Chow groups CH^*, Grothendieck K_0, algebraic cobordism of Levine-Morel Ω^*, and the theories of the form $\Omega^* \otimes_{\mathbb{L}} \Lambda$ given by arbitrary formal group law over the base ring Λ.

Formal group law

For an oriented cohomology theory $\mathtt{h}^*$ there is a notion of the first Chern class of a line bundle. For $X \in \mathbf{Sm}_k$ and a line bundle L over X it is defined as $c_1^{\mathtt{h}}(L) = z^{\mathtt{h}} z_{\mathtt{h}}(1) \in \mathtt{h}^1(X)$, where $z\colon X \to L$ is a zero section. There is a commutative associative 1-dimensional formal group law F over Λ^* such that for any two line bundles L_1 and L_2 over X we have $c_1^{\mathtt{h}}(L_1 \otimes L_2) = F(c_1^{\mathtt{h}}(L_1), c_1^{\mathtt{h}}(L_2))$ [**11**, Lem. 1.1.3]. We will use the notation $x +_F y$ for $F(x, y)$. For any x we will denote by $-_F x$ the unique element such that $x +_F (-_F x) = 0$. For any $n \in \mathbb{Z}$ we will denote by $n \cdot_F x$ the expression $x +_F \cdots +_F x$ (n times) if n is positive, and $(-_F x) +_F \cdots +_F (-_F x)$ ($-n$ times) if n is negative.

By [**11**] there is a universal formal group law F_U over the Lazard ring $\mathbb{L}$. It corresponds to the algebraic cobordism theory Ω and there is a natural transformation $\Omega^*(-) \to \mathtt{h}^*(-)$ that commutes with pushforwards. This gives rise to a morphism

$$\nu_X : \Omega^*(X) \otimes_{\mathbb{L}^*} \Lambda^* \to \mathtt{h}^*(X),$$

where the ring homomorphism $\mathbb{L}^* \to \Lambda^*$ is obtained by specializing the coefficients of F_U to the coefficients of F. We will call ν_X the specialization homomorphism.

Cellular and generically cellular varieties

Definition 2.9. A variety $X \in \mathbf{Sm}_k$ is called cellular if there is a filtration of $X = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_m \supseteq \emptyset$ such that each $X_i \setminus X_{i+1}$ is a disjoint union of affine spaces of the same rank c_i: $X_i \setminus X_{i+1} \cong \mathbb{A}_k^{c_i} \coprod \cdots \coprod \mathbb{A}_k^{c_i}$.

Definition 2.10. A variety X is called generically cellular if $X_{k(X)}$ is a cellular variety over $k(X)$, where $k(X)$ is the function field of X.

Example 2.11. Let G be a split semisimple algebraic group, B its Borel subgroup containing a fixed maximal split torus T, and W the corresponding Weyl group. For any $w \in W$ let $l(w)$ denote its length. Let $w_0 \in W$ denote the longest element of W and $N = l(w_0)$. Then the flag variety $X = G/B$ is cellular of dimension N and the cellular structure is given by the Schubert cells X_w:

$$X = X_{w_0} \supseteq \bigcup_{l(w)=N-1} X_w \supseteq \bigcup_{l(w)=N-2} X_w \supseteq \cdots \supseteq X_e = pt,$$

where X_w is the closure of BwB/B in X.

Example 2.12. Let $\zeta \in Z^1(k, G)$ be a 1-cocycle with values in G. Then the twisted form ${}_\zeta(G/B)$ of $X = G/B$ provides an example of a generically split variety.

h-motives

The notion of h-motives for the algebraic oriented cohomology theory h first appeared in [**12**]. Later it was studied in [**14**] and [**18**]. We refer to [**18**, §2] for the definition of the category of effective h-motives. In the present paper we will deal with the category of h-motives $\mathcal{M}_{\mathtt{h}}$ with the inverted Tate object. It is constructed as follows:

Let $\mathbf{SmProj}_k$ denote the category of smooth projective varieties over k. Following [**5**], we consider the category $Corr_{\mathtt{h}}$ defined as follows: For $X, Y \in \mathbf{SmProj}_k$ with irreducible X and $m \in \mathbb{Z}$ we set

$$Corr_m(X, Y) = \mathtt{h}_{\dim X + m}(X \times Y).$$

Objects of $Corr_{\mathtt{h}}$ are pairs (X, i) with $X \in \mathbf{SmProj}_k$ and $i \in \mathbb{Z}$. For $X \in \mathbf{SmProj}_k$ with irreducible components X_l define the morphisms

$$Hom_{Corr}\big((X, i), (Y, j)\big) = \coprod_l Corr_{i-j}(X_l, Y).$$

For $\alpha \in Hom\big((X, i), (Y, j)\big)$ and $\beta \in Hom\big((Y, j), (Z, k)\big)$ the composition is given by the usual correspondence product: $\alpha \circ \beta = (p_{XZ})_{\mathtt{h}}\big((p_{YZ})^{\mathtt{h}}(\beta) \cdot (p_{XY})^{\mathtt{h}}(\alpha)\big)$, where p_{XY}, p_{YZ}, p_{XZ} denote the projections from $X \times Y \times Z$ onto the corresponding summands.

Taking consecutive additive and idempotent completion of $Corr_{\mathtt{h}}$ we obtain the category $\mathcal{M}_{\mathtt{h}}$ of h-motives with inverted Tate object. Objects of this category are $(\coprod_i (X_i, n_i), p)$ where p is a matrix with entries $p_{i,j} \in Corr_{n_i - n_j}(X_i, X_j)$ such that $p \circ p = p$. Morphisms between $(\coprod(X_i, n_i), p)$ and $(\coprod(Y_j, m_j), q)$ are given by the set $q \circ \bigoplus_{i,j} Corr_{n_i - m_j}(X_i, Y_j) \circ p$ considered as a subset of $\bigoplus_{i,j} Corr_{n_i - m_j}(X_i, Y_j)$. This is an additive category where each idempotent splits. There is a functor

$$M^{\mathtt{h}} \colon \mathbf{SmProj}_k \to \mathcal{M}_{\mathtt{h}}$$

that maps a variety X to the motive $M^{\mathtt{h}}(X) = ((X,0), id_X)$ and any morphism $f\colon X \to Y$ to the correspondence $(\Gamma_f)_{\mathtt{h}}(1) \in \mathtt{h}_{\dim X}(X \times Y) = Corr_0(X,Y)$, where $\Gamma_f\colon X \to X \times Y$ is the graph inclusion. We will denote by $\Delta\colon X \to X \times X$ the diagonal embedding. Then $\Delta_{\mathtt{h}}(1)$ is the identity in $Corr_0(X,X)$.

We will denote Spec k by pt and its motive $M^{\mathtt{h}}(pt)$ by Λ. The $\mathtt{h}$-Tate motive $\Lambda(1)$ is defined as $((pt,1), id_{pt})$. We write $\Lambda(n)$ for $\Lambda(1)^{\otimes n}$ and $M^{\mathtt{h}}(X)(n)$ for $M^{\mathtt{h}}(X) \otimes \Lambda(n)$. The motive $M^{\mathtt{h}}(X)(n)$ is called the n-th twist of the motive $M^{\mathtt{h}}(X)$.

By definition we have

$$\mathtt{h}^i(X) = Hom_{\mathcal{M}_{\mathtt{h}}}(M^{\mathtt{h}}(X), \Lambda(i)) \text{ and } \mathtt{h}_i(X) = Hom_{\mathcal{M}_{\mathtt{h}}}(\Lambda(i), M^{\mathtt{h}}(X)).$$

Lemma 2.13. *For $X \in \mathbf{SmProj}_k$ with the structure morphism $\pi\colon X \to pt$, a choice of an isomorphism $M^{\mathtt{h}}(X) \cong \bigoplus_{i=1}^n \Lambda(\alpha_i)$ is equivalent to a choice of two Λ-basis sets*

$$\{\tau_i \in \mathtt{h}^{\alpha_i}(X)\}_i \text{ and } \{\zeta_i \in \mathtt{h}_{\alpha_i}(X)\}_i$$

such that $\pi_{\mathtt{h}}(\tau_i\zeta_j) = \delta_{i,j}$ in Λ and $\sum_i \zeta_i \otimes \tau_i = \Delta_{\mathtt{h}}(1)$ in $\mathtt{h}(X \times X)$.

Proof. In the decomposition $M^{\mathtt{h}}(X) \cong \bigoplus_i \Lambda(\alpha_i)$ the i-th projection $p_i\colon M^{\mathtt{h}}(X) \to \Lambda(\alpha_i)$ is defined by an element $\tau_i \in \mathtt{h}^{\alpha_i}(X)$ and the i-th inclusion $\imath_i\colon \Lambda(\alpha_i) \to \mathtt{h}(X)$ is defined by an element $\zeta_i \in \mathtt{h}_{\alpha_i}(X)$. Then the identities $p_i \circ \imath_j = \delta_{i,j}$ and $\sum \imath_i \circ p_i = id_{M^{\mathtt{h}}(X)}$ are equivalent to the equalities $\pi_{\mathtt{h}}(\tau_i\zeta_j) = \delta_{i,j}$ and $\sum_{i=1}^n \zeta_i \otimes \tau_i = \Delta_{\mathtt{h}}(1)$. Let us check that $\{\zeta_i\}_i$ form a basis of $\mathtt{h}(X)$. Indeed, we have

$$\begin{aligned}\mathtt{h}^*(X) &= \bigoplus_{j\in\mathbb{Z}} \mathtt{h}^j(X) = \bigoplus_{j\in\mathbb{Z}} Hom_{\mathcal{M}_{\mathtt{h}}}(M^{\mathtt{h}}(X), \Lambda(j)) \\ &\cong \bigoplus_{j\in\mathbb{Z}} Hom_{\mathcal{M}_{\mathtt{h}}}\left(\bigoplus_{i=1}^n \Lambda(\alpha_i), \Lambda(j)\right) = \oplus_{j\in\mathbb{Z}} \bigoplus_{i=1}^n \Lambda_{\alpha_i - j} = \bigoplus_{i=1}^n \Lambda_{\alpha_i - *} = \bigoplus_{i=1}^n \Lambda^{*-\alpha_i}\end{aligned}$$

and ζ_i are the images of standard generators. So $\{\zeta_i\}_{i=1}^n$ form a Λ-basis of $\mathtt{h}(X)$. Finally, since τ_i are dual to ζ_i, $\{\tau_i\}_{i=1}^n$ is also a basis. □

3. Filtration on the cohomology ring

In the present section we construct a filtration on the oriented cohomology $\mathtt{h}(X)$ of a cellular variety X with $\dim X = N$, which will play an important role in the sequel.

Proposition 3.1. *Assume that X is a cellular variety over k. Then*

(1) the $\mathtt{h}$-motive of X splits as $M^{\mathtt{h}}(X) = \bigoplus_{i=1}^n \Lambda(\alpha_i)$;

(2) the Künneth formula holds, i.e., the natural map $\mathtt{h}(X) \otimes_\Lambda \mathtt{h}(X) \to \mathtt{h}(X \times X)$ is an isomorphism;

(3) the specialization maps $\nu_X\colon \Omega(X) \otimes \Lambda \to \mathtt{h}(X)$ and $\nu_{X\times X}\colon \Omega(X \times X) \otimes \Lambda \to \mathtt{h}(X \times X)$ are isomorphisms.

Proof. By [**5**, Cor. 66.4] the Chow motive $M^{\mathrm{CH}}(X)$ splits: $M^{\mathrm{CH}}(X) = \oplus_{i=1}^n \mathbb{Z}(\alpha_i)$. Then [**18**, Cor. 2.9] implies that the motive $M^{\Omega}(X)$ splits into a sum of twisted Tate motives $M^{\Omega}(X) = \bigoplus_{i=1}^n \mathbb{L}(\alpha_i)$. By Lemma 2.13 there are elements $\zeta_i^\Omega \in \Omega_{\alpha_i}(X)$ and $\tau_i^\Omega \in \Omega^{\alpha_i}(X)$ such that $\pi_\Omega(\zeta_i^\Omega\tau_j^\Omega) = \delta_{i,j}$ and $\Delta_\Omega(1) = \sum_i \zeta_i^\Omega \otimes \tau_i^\Omega$. Denote $\zeta_i^{\mathtt{h}} = \nu_X(\zeta_i^\Omega \otimes 1)$ and $\tau_i^{\mathtt{h}} = \nu_X(\tau_i^\Omega \otimes 1)$. Since ν_X commutes with pullbacks and pushforwards, $\pi_{\mathtt{h}}(\zeta_i^{\mathtt{h}}\tau_j^{\mathtt{h}}) = \delta_{i,j}$ and $\Delta_{\mathtt{h}}(1) = \sum_i \zeta_i^{\mathtt{h}} \otimes \tau_i^{\mathtt{h}}$. Then by Lemma 2.13 we have $M^{\mathtt{h}}(X) = \bigoplus_i \Lambda(\alpha_i)$, so (1) holds.

The Künneth map fits into the diagram

$$\begin{array}{ccc} \mathtt{h}(X) \otimes_\Lambda \mathtt{h}(X) & \longrightarrow & \mathtt{h}(X \times X) \\ \| & & \| \\ (\bigoplus_i \Lambda^{*-\alpha_i}) \otimes_\Lambda (\bigoplus_j \Lambda^{*-\alpha_j}) & \longrightarrow & \bigoplus_{i,j} \Lambda^{*-\alpha_i-\alpha_j} \end{array}$$

where the bottom arrow is an isomorphism, so the Künneth formula (2) holds.

Note that the natural map ν_X can be factored as follows

$$\Omega(X) \otimes \Lambda \cong \oplus_m Hom_{\mathcal{M}_{\Omega\otimes\Lambda}}(\bigoplus \Lambda(\alpha_i), \Lambda(m)) \xrightarrow{\cong} Hom_{\mathcal{M}_{\mathtt{h}}}(\bigoplus \Lambda(\alpha_i), \Lambda(m)) \cong \mathtt{h}(X).$$

Thus ν_X is an isomorphism. The same reasoning proves the statement for $\nu_{X\times X}$; hence, (3) holds. □

Definition 3.2. Let X be a cellular variety. Fix two basis sets $\zeta_i^\Omega \in \Omega_{\alpha_i}(X)$ and $\tau_i^\Omega \in \Omega^{\alpha_i}(X)$ provided by Proposition 3.1 and Lemma 2.13. Then $\zeta_i = \nu(\zeta_i^\Omega \otimes 1)$ and $\tau_i = \nu(\tau_i^\Omega \otimes 1)$ is a dual basis for $\mathtt{h}(X)$ as in 2.13. We define the filtration $\mathtt{h}^{(l)}(X)$ as the Λ-linear span

$$\mathtt{h}^{(l)}(X) = \bigoplus_{N-\alpha_i \geqslant l} \Lambda\zeta_i = \bigoplus_{\alpha_i \geqslant l} \Lambda\tau_i.$$

We denote $\mathtt{h}^{(l)}_{N-l}(X) = \mathtt{h}^{(l)}(X) \cap \mathtt{h}_{N-l}(X)$ and $\mathtt{h}^{(l/l+1)}(X) = \mathtt{h}^{(l)}(X)/\mathtt{h}^{(l+1)}(X)$ and $\mathtt{h}^{(l/l+1)}_{N-l}(X)$ to be the image of $\mathtt{h}^{(l)}_{N-l}(X)$ in $\mathtt{h}^{(l/l+1)}(X)$. Lemma 3.4 implies that the latter is a graded ring.

Remark 3.3. In the case when the theory $\mathtt{h}$ is generically constant and satisfies the localization property, the filtration introduced above coincides with the topological filtration on $\mathtt{h}(X)$, i.e., with the filtration where the l-th term is generated over Λ by classes $[Z \to X]$ of projective morphisms $Z \to X$ birational on its image and $\dim X - \dim Z \leqslant l$. This fact follows from the generalized degree formula [**11**, Thm. 4.4.7].

Lemma 3.4. $\mathtt{h}^{(l_1)}(X) \cdot \mathtt{h}^{(l_2)}(X) \subseteq \mathtt{h}^{(l_1+l_2)}(X)$.

Proof. We have $\tau_i^\Omega \tau_j^\Omega = \sum_l a_l \zeta_l^\Omega$ in $\Omega(X)$ for some $a_l \in \mathbb{L}$. Then $\alpha_i + \alpha_j = \deg(a_l) + \alpha_l$. Since $\deg(a_l) \leqslant 0$, $\alpha_l \geqslant \alpha_i + \alpha_j \geqslant l_1 + l_2$ for any nontrivial a_l. Since $\zeta_i = \nu(\zeta_i^\Omega \otimes 1)$ we have $\zeta_i\zeta_j = \sum_l (a_l \otimes 1)\zeta_l$ with $\alpha_l \geqslant \alpha_i + \alpha_j \geqslant l_1 + l_2$. So $\zeta_i\zeta_j \in \mathtt{h}^{(l_1+l_2)}(X)$. □

Proposition 3.5. *For a cellular X there is a graded ring isomorphism*

$$\Psi \colon \bigoplus_{i=0}^{N} \mathtt{h}^{(i/i+1)}(X) \to \mathrm{CH}^*(X, \Lambda).$$

Proof. By Proposition 3.1 it is sufficient to prove the statement for $\mathtt{h} = \Omega$. Observe that $\Omega^{(l/l+1)}(X)$ is a free $\mathbb{L}$-module with the basis $\tau_i^\Omega + \Omega^{(l+1)}(X)$ with $\alpha_i = l$ and $\mathrm{CH}^i(X, \mathbb{L})$ is a free $\mathbb{L}$-module with basis τ_i^{CH} with $\alpha_i = l$. Thus the $\mathbb{L}$-module homomorphism Ψ_l defined by

$$\Psi_l(\tau_i^\Omega + \Omega^{(i+1)}(X)) = \tau_i^{\mathrm{CH}}$$

is an isomorphism.

Let us check that $\Psi = \bigoplus \Psi_l$ preserves multiplication. For any i, j we have

$$\tau_i^{\Omega}\tau_j^{\Omega} = \sum_m a_m \tau_m^{\Omega} \qquad (*)$$

for some $a_m \in \mathbb{L}$. Then for any m we have $\deg(a_m) + \alpha_m = \alpha_i + \alpha_j$. Then in the quotient $\Omega^{(\alpha_i+\alpha_j/\alpha_i+\alpha_j+1)}(X)$ we have

$$\tau_i^{\Omega}\tau_j^{\Omega} = \sum_{\alpha_m=\alpha_i+\alpha_j} a_m \tau_m^{\Omega} \mod \Omega^{(\alpha_i+\alpha_j+1)}(X)$$

Observe that $\mathbb{L}^0 = \mathbb{Z}$ and for all $a_m \in \mathbb{L}$ such that $\deg(a_m) < 0$ we have $a_m \otimes 1_{\mathbb{Z}} = 0$ in $\mathbb{Z}$. Thus tensoring $(*)$ with $1_{\mathbb{Z}}$ we get

$$\tau_i^{\mathrm{CH}}\tau_j^{\mathrm{CH}} = \sum_{\alpha_m=\alpha_i+\alpha_j} (a_m \otimes 1)\tau_m^{\mathrm{CH}}.$$

So $\Psi_{\alpha_i+\alpha_j}(\tau_i^{\Omega} + \Omega^{(\alpha_i+1)}(X) \cdot \tau_j^{\Omega} + \Omega^{(\alpha_j+1)}(X)) = \tau_i^{\mathrm{CH}} \cdot \tau_j^{\mathrm{CH}}$. Hence Ψ is a graded ring isomorphism. □

Lemma 3.6. $\Psi(\zeta_i + \mathtt{h}^{(\alpha_i+1)}(X)) = \zeta_i^{\mathrm{CH}}$.

Proof. It is sufficient to show the statement for $\mathtt{h} = \Omega^*$. Consider the expansion $\zeta_i^{\Omega} = \sum a_j \tau_j^{\Omega}$ for some $a_j \in \mathbb{L}$ with $\deg a_j + \alpha_j = N - \alpha_i$. Since $\deg a_j \leqslant 0$ we have

$$\zeta_i^{\Omega} = \sum_{\deg a_j = 0} a_j \tau_j^{\Omega} \mod \Omega^{(N-\alpha_i+1)}(X).$$

Therefore $\Psi(\zeta_i^{\Omega} + \Omega^{(N-\alpha_i+1)}(X)) = \zeta_i^{\mathrm{CH}}$. □

Corollary 3.7. *The restriction gives rise to an isomorphism* $\oplus_{i=1}^{N}\Psi^i\colon \mathtt{h}_{N-i}^{(i/i+1)}(X) \to \mathrm{CH}^*(X, \Lambda^0)$.

Proof. This follows from the previous lemma and the fact that $\mathrm{CH}^*(X,\Lambda^0) = \oplus\Lambda^0\zeta_i^{\mathrm{CH}}$. □

4. Applications to h-motivic decompositions

Throughout this section we consider a generically cellular variety X of dimension N and an oriented cohomology theory $\mathtt{h}^*$ that is generically constant and is associated with weak Borel-Moore homology $\mathtt{h}_*$ that satisfies the localization property. These assumptions imply that the generalized degree formula of Levine-Morel hold [**11**, Theorem 4.4.7]. The aim of this section is to prove theorems A, B, and C of the introduction, which provide a comparison between the Chow motive $M(X)$ and the h-motive $M^{\mathtt{h}}(X)$ of X.

Let L be a splitting field of X and $\overline{X} = X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, L$. Let p denote the projection $p\colon \overline{X} \times \overline{X} \to X \times X$. For any k-scheme Z we denote by Z_L the product $Z_L = Z \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, L$. Since $\overline{X}$ is cellular, we may consider the filtration on $\mathtt{h}(\overline{X})$ introduced in Definition 3.2. It gives rise to a filtration on $\mathtt{h}(\overline{X} \times \overline{X}) = \mathtt{h}(\overline{X}) \otimes_{\Lambda} \mathtt{h}(\overline{X})$. Namely, we set

$$\mathtt{h}^{(l)}(\overline{X} \times \overline{X}) = \sum_{i+j=l} \mathtt{h}^{(i)}(\overline{X}) \otimes_{\Lambda} \mathtt{h}^{(j)}(\overline{X}).$$

On $\mathtt{h}(X \times X)$ we consider the induced filtration

$$\mathtt{h}^{(l)}(X \times X) = (p^{\mathtt{h}})^{-1}(\mathtt{h}^{(l)}(\overline{X} \times \overline{X})).$$

Denote the quotient $\mathtt{h}^{(l)}(\overline{X} \times \overline{X})/\mathtt{h}^{(l+1)}(\overline{X} \times \overline{X})$ by $\mathtt{h}^{(l/l+1)}(\overline{X} \times \overline{X})$ and denote by $pr_l\colon \mathtt{h}^{(l)}(\overline{X} \times \overline{X}) \to \mathtt{h}^{(l/l+1)}(\overline{X} \times \overline{X})$, the usual projection. Denote

$$\mathtt{h}^{(i)}_{2N-i}(X \times X) = \mathtt{h}^{(i)}(X \times X) \cap \mathtt{h}_{2N-i}(X \times X) \text{ and}$$

$$\mathtt{h}^{(i)}_{2N-i}(\overline{X} \times \overline{X}) = \mathtt{h}^{(i)}(\overline{X} \times \overline{X}) \cap \mathtt{h}_{2N-i}(\overline{X} \times \overline{X}).$$

Lemma 4.1. *There is a graded ring isomorphism*

$$\Phi\colon \bigoplus_{i=0}^{2N} \mathtt{h}^{(i/i+1)}(\overline{X} \times \overline{X}) \to \mathrm{CH}^*(\overline{X} \times \overline{X}, \Lambda).$$

Proof. By Künneth formula 3.1 we may take $\Phi = \Psi \otimes \Psi$, where Ψ is defined in Proposition 3.5. Then the lemma follows from Proposition 3.5. $\square$

Analogous to Corollary 3.7 we get

Remark 4.2. The restriction of Φ^i gives an isomorphism $\Phi^i\colon \mathtt{h}^{(i/i+1)}_{2N-i}(\overline{X} \times \overline{X}) \to \mathrm{CH}^i(\overline{X} \times \overline{X}, \Lambda^0)$.

The following lemma provides an $\mathtt{h}$-version of the Rost Nilpotence Theorem:

Lemma 4.3. *The kernel of the pullback map $p^{\mathtt{h}}\colon End(M^{\mathtt{h}}(X)) \to End(M^{\mathtt{h}}(\overline{X}))$ consists of nilpotents.*

Proof. Let $\mu_{-}\colon \Omega(-) \to \mathrm{CH}(-)$ denote the natural transformation arising from the universality of Ω. Then we have that $\mu_{\overline{X} \times \overline{X}} \circ p^{\Omega} = p^{\mathrm{CH}} \circ \mu_{X \times X}$. The kernel of p^{CH} consists of nilpotents by [**19**, Prop 3.1]. By [**18**, Prop. 2.7] the maps $\mu_{X\times X}$ and $\mu_{\overline{X}\times\overline{X}}$ are surjective and their kernels consist of nilpotents (according to composition product). Then $\ker p^{\Omega}$ also consists of nilpotents. Note that $\ker(p^{\Omega}) \otimes \Lambda$ covers $\ker p^{\Omega} \otimes id$, so the latter consists of nilpotents. Further, $p^{\mathtt{h}} \circ \nu_{X\times X} = \nu_{\overline{X}\times\overline{X}} \circ (p^{\Omega} \otimes id)$. Since $\nu_{X\times X}$ is surjective (by the generalized degree formula) and $\nu_{\overline{X}\times\overline{X}}$ is an isomorphism (by Proposition 3.1), we have that $\ker p^{\mathtt{h}}$ is covered by $\ker(p^{\Omega} \otimes id)$ and thus consists of nilpotents. $\square$

Lemma 4.4. *We have $\mathtt{h}^{(N+i)}(\overline{X} \times \overline{X}) \circ \mathtt{h}^{(N+j)}(\overline{X} \times \overline{X}) \subseteq \mathtt{h}^{(N+i+j)}(\overline{X} \times \overline{X})$.*

Proof. Consider a generator $\zeta_m \otimes \tau_n \in \mathtt{h}^{(N+i)}(\overline{X} \times \overline{X})$, where $N - \alpha_m + \alpha_n \geqslant N + i$ and $\zeta_{m'} \otimes \tau_{n'} \in \mathtt{h}^{(N+j)}(\overline{X} \times \overline{X})$ where $N - \alpha_{m'} + \alpha_{n'} \geqslant N + j$. The composition $(\zeta_m \otimes \tau_n) \circ (\zeta_{m'} \otimes \tau_{n'}) = \pi_{\mathtt{h}}(\tau_n \zeta_{m'})(\zeta_m \otimes \tau_{n'})$ is nonzero iff $n = m'$. In this case $N - \alpha_m + \alpha_{n'} = (N - \alpha_m + \alpha_n) + (N - \alpha_{m'} + \alpha_{n'}) - N \geqslant N + i + j$. Thus $\zeta_m \otimes \tau_{n'}$ lies in the filtration term $\mathtt{h}^{(N+i+j)}(\overline{X} \times \overline{X})$. $\square$

Lemma 4.5. *The isomorphism $\Phi^N\colon \mathtt{h}^{(N/N+1)}_N(\overline{X} \times \overline{X}) \to \mathrm{CH}^N(\overline{X} \times \overline{X}, \Lambda^0)$ is a ring homomorphism with respect to the composition product.*

Proof. This immediately follows from the fact that Φ maps residue classes of $\zeta_i^{\mathtt{h}} \otimes \tau_j^{\mathtt{h}}$ to $\zeta_i^{\mathrm{CH}} \otimes \tau_j^{\mathrm{CH}}$. $\square$

Lemma 4.6. *Let Y be a twisted form of X, i.e., $Y_L \cong X_L = \overline{X}$. Let $p\colon \overline{X} \times \overline{X} \to X \times Y$ denote the projection. Then for every m in the diagram*

$$\mathrm{CH}^m(X \times Y, \Lambda^0) \xrightarrow{p^{\mathrm{CH}}} \mathrm{CH}^m(\overline{X} \times \overline{X}, \Lambda^0) \xleftarrow{\Phi^m} \mathtt{h}^{(m/m+1)}_{2N-m}(\overline{X} \times \overline{X}) \xleftarrow{pr_m} \mathtt{h}^{(m)}_{2N-m}(X \times Y)$$

we have $\operatorname{im} p^{\mathrm{CH}} \subseteq \operatorname{im} \Phi^m \circ pr_m \circ p^{\mathtt{h}}$.

Proof. Note that $\mathrm{CH}^m(X \times Y, \Lambda^0)$ is generated over Λ^0 by classes $i_{\mathrm{CH}}(1)$ where $i\colon Z \to X \times Y$, where $Z \in \mathbf{Sm}_k$, the morphism i is projective, and $i(Z)$ is a closed integral subscheme of codimension m and $i\colon Z \to i(Z)$ is birational. Consider the Cartesian diagram

$$\begin{array}{ccc} Z & \xleftarrow{\;q\;} & Z_L \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ X \times_k Y & \xleftarrow{\;p\;} & \overline{X} \times_L \overline{X} \end{array}$$

Since this diagram is transversal, we have

$$j_{\mathtt{h}} \circ q^{\mathtt{h}} = p^{\mathtt{h}} \circ i_{\mathtt{h}} \quad \text{and} \quad j_{\mathrm{CH}} \circ q^{\mathrm{CH}} = p^{\mathrm{CH}} \circ i_{\mathrm{CH}}.$$

By lemma 4.7 we have $j_{\mathrm{CH}}(1) = \Phi^m \circ pr_m(j_{\mathtt{h}}(1))$. Then $p^{\mathrm{CH}}(i_{\mathrm{CH}}(1)) = \Phi^m \circ pr_m \circ p^{\mathtt{h}}(i_{\mathtt{h}}(1)) \in \operatorname{im} \Phi^m \circ pr_m \circ p^{\mathtt{h}}$. □

Lemma 4.7. *Consider a morphism $j\colon Z \to \overline{X} \times_L \overline{X}$, where $Z \in \mathbf{Sm}_k$, the morphism j is projective, and $j(Z)$ is a closed integral subscheme of codimension m and $j\colon Z \to j(Z)$ is birational. Then $j_{\mathtt{h}}(1) \in \mathtt{h}^{(m)}_{2N-m}(\overline{X} \times \overline{X})$ and $j_{\mathrm{CH}}(1) = \Phi^m \circ pr_m(j_{\mathtt{h}}(1))$.*

Proof. Observe that $j_{\mathtt{h}}(1) = j_\Omega(1) \otimes_{\mathbb{L}} 1_\Lambda$ and $j_{\mathrm{CH}}(1) = j_\Omega(1) \otimes_{\mathbb{L}} 1_{\mathbb{Z}}$. Expanding in the basis we obtain

$$j_\Omega(1) = \sum_{i_1, i_2} r_{i_1,i_2} \tau^\Omega_{i_1} \otimes \tau^\Omega_{i_2} \quad \text{for some } r_{i_1,i_2} \in \mathbb{L}. \tag{$*$}$$

Since $j_\Omega(1)$ is homogeneous of degree m, we have $r_{i_1,i_2} \in \mathbb{L}^{m-\alpha_{i_1}-\alpha_{i_2}}$. Then for every nonzero r_{i_1,i_2} we have $\alpha_{i_1} + \alpha_{i_2} \geqslant m$. Then we have

$$j_\Omega(1) \equiv \sum_{\alpha_{i_1}+\alpha_{i_2}=m} r_{i_1,i_2} \tau^\Omega_{i_1} \otimes \tau^\Omega_{i_2} \bmod \Omega^{(m+1)}(\overline{X} \times \overline{X}).$$

If $\alpha_{i_1} + \alpha_{i_2} = m$ then $r_{i_1,i_2} \in \mathbb{L}^0 = \mathbb{Z}$. Thus taking $j_{\mathtt{h}}(1) = j_\Omega(1) \otimes_{\mathbb{L}} 1_\Lambda$ and $j_{\mathrm{CH}}(1) = j_\Omega(1) \otimes_{\mathbb{L}} 1_{\mathbb{Z}}$ we get $\Phi^m \circ pr_m(j_{\mathtt{h}}(1)) = j_{\mathrm{CH}}(1)$, since $\tau^{\mathrm{CH}}_{i_1} \otimes \tau^{\mathrm{CH}}_{i_2} = \Phi^m \circ pr_m(\tau^{\mathtt{h}}_{i_1} \otimes \tau^{\mathtt{h}}_{i_2})$. □

Lemma 4.8. *The kernel of the composition*

$$pr_N \circ p^{\mathtt{h}}\colon \mathtt{h}^{(N)}_N(X \times X) \to \mathtt{h}^{(N)}_N(\overline{X} \times \overline{X}) \to \mathtt{h}^{(N/N+1)}_N(\overline{X} \times \overline{X})$$

consists of nilpotents.

Proof. This follows from Lemma 4.3 and the fact that $\mathtt{h}^{(N+1)}(\overline{X} \times \overline{X})$ is nilpotent by Lemma 4.4. □

Lemma 4.9. *Let $\mathcal{C}$ be an additive category, $A, B \in Ob(\mathcal{C})$. Let $f \in Hom_{\mathcal{C}}(A, B)$ and $g \in Hom_{\mathcal{C}}(B, A)$such that $f \circ g - id_B$ is nilpotent in the ring $End_{\mathcal{C}}(B)$ and $g \circ f - id_A$ is nilpotent in the ring $End_{\mathcal{C}}(A)$. Then A is isomorphic to B.*

Proof. Since fg is a sum of identity and a nilpotent, then fg is invertible in $End(B)$, so fg is an isomorphism, so then f has a right inverse. Symmetrically, gf is invertible, so then f has a left inverse. Then f is an isomorphism. □

We are now ready to prove theorems A, B, and C of the introduction.

Theorem A. Suppose X is generically cellular. Assume that there is a decomposition of the Chow motive with coefficients in Λ^0

$$M^{\mathrm{CH}}(X,\Lambda^0)=\bigoplus_{i=0}^{n}\mathcal{R}(\alpha_i) \qquad (*)$$

such that over the splitting field L the motive $\mathcal{R}$ equals the sum of twisted Tate motives: $\overline{\mathcal{R}}=\bigoplus_{j=0}^{m}\Lambda^0(\beta_j)$.

Then there is a $\mathtt{h}$-motive $\mathcal{R}_{\mathtt{h}}$ such that

$$M^{\mathtt{h}}(X)=\bigoplus_{i=0}^{n}\mathcal{R}_{\mathtt{h}}(\alpha_i),$$

such that over the splitting field $\mathcal{R}_{\mathtt{h}}$ splits into the $\mathtt{h}$-Tate motives $\overline{\mathcal{R}_{\mathtt{h}}}=\bigoplus_{j=0}^{m}\Lambda(\beta_j)$.

Proof. We may assume that $\alpha_0=0$ in $(*)$. Then each summand $\mathcal{R}(\alpha_i)$ equals (X,p_i) for some idempotent p_i, and there are mutually inverse isomorphisms ϕ_i and ψ_i of degree α_i between (X,p_0) and (X,p_i). So we have

- idempotents $p_i\in\mathrm{CH}^N(X\times X)$, $\sum p_i=\Delta^X_{\mathrm{CH}}(1)$, $p_i\circ p_j=0$ for $i\neq j$;
- isomorphisms $\phi_i\in p_0\circ\mathrm{CH}^{N+\alpha_i}(X\times X)\circ p_i$ and $\psi_i\in p_i\circ\mathrm{CH}^{N-\alpha_i}(X\times X)\circ p_0$;
- such that $\phi_i\circ\psi_i=p_0$ and $\psi_i\circ\phi_i=p_i$.

Consider the homomorphisms

$$\mathrm{CH}^m(X\times X,\Lambda^0)\xrightarrow{p^{\mathrm{CH}}}\mathrm{CH}^m(\overline{X}\times\overline{X},\Lambda^0)\xleftarrow{\Phi^m\circ pr_m}\mathtt{h}^{(m)}_{2N-m}(\overline{X}\times\overline{X})\xleftarrow{p^{\mathtt{h}}}\mathtt{h}^{(m)}_{2N-m}(X\times X).$$

For $m=N$ this is a ring homomorphism by Lemma 4.5. By Lemma 4.6 the elements $p^{\mathrm{CH}}(p_i)$, $p^{\mathrm{CH}}(\phi_i)$, and $p^{\mathrm{CH}}(\psi_i)$ lie in the images $\operatorname{im}\Phi^n\circ pr_N\circ p^{\mathtt{h}}$, $\operatorname{im}\Phi^{N+\alpha_i}\circ pr_{N+\alpha_i}\circ p^{\mathtt{h}}$, and $\operatorname{im}\Phi^{N-\alpha_i}\circ pr_{N-\alpha_i}\circ p^{\mathtt{h}}$, respectively.

By Lemma 4.8 the kernel of $pr_N\circ p^{\mathtt{h}}\colon\ \mathtt{h}^{(N)}_N(X\times X)\to\mathtt{h}^{(N/N+1)}_N(\overline{X}\times\overline{X})$ is nilpotent. Then by [**1**, Prop. 27.4] there is a set of idempotents r_i such that $\Phi^N\circ pr_N\circ p^{\mathtt{h}}(r_i)=p^{\mathrm{CH}}(p_i)$ and r_i form a full system of orthogonal idempotents, i.e., $\sum r_i=\Delta^X_{\mathtt{h}}(1)$, $r_i\circ r_j=0$ for $i\neq j$.

Let us construct the isomorphisms between $\mathtt{h}$-motives (X,r_i) and $(X,r_0)(\alpha_i)$. Let ϕ'_i and ψ'_i be some elements such that $\Phi^{N+\alpha_i}\circ pr_{N+\alpha_i}\circ p^{\mathtt{h}}(\phi'_i)=p^{\mathrm{CH}}(\phi_i)$ and $\Phi^{N-\alpha_i}\circ pr_{N-\alpha_i}\circ p^{\mathtt{h}}(\psi'_i)=p^{\mathrm{CH}}(\psi_i)$. Then [**17**, Lem. 2.5] implies that there are elements $\phi''_i\in r_0\,\mathtt{h}^{(N)}_N(X\times X)r_i$ and $\psi''_i\in r_i\,\mathtt{h}^{(N)}_N(X\times X)r_0$ such that $\phi''_i\psi''_i=r_0$ and $\psi''_i\phi''_i=r_i$. So the $\mathtt{h}$-motives (X,r_i) and $(X,r_0)(\alpha_i)$ are isomorphic. Taking $\mathcal{R}_{\mathtt{h}}=(X,r_0)$ we obtain a decomposition

$$M^{\mathtt{h}}(X)=\bigoplus_{i=0}^{n}(X,r_i)=\bigoplus_{i=0}^{n}(X,r_0)(\alpha_i)=\bigoplus_{i=0}^{n}\mathcal{R}_{\mathtt{h}}(\alpha_i).$$

Over the splitting field the motive $\overline{\mathcal{R}_{\mathtt{h}}}$ becomes isomorphic to $(\overline{X},p^{\mathtt{h}}(r_0))$ Let us construct an isomorphism between $(\overline{X}, p^{\mathtt{h}}(r_0))$ and $\bigoplus_{j=0}^{m} \Lambda(\beta_j)$. Consider an isomorphism f between Chow motives $f\colon (\overline{X}, p_0) \to \bigoplus_{j=0}^{m} \Lambda^0(\beta_j)$. Then f is an element $f \in \oplus_{j=0}^{m} \mathrm{CH}^{\beta_j}(\overline{X})$. Since the map $\Psi^{N-\beta_j} \circ pr_{N-\beta_j}$ is surjective, we can lift f to $\phi \in \bigoplus_{j=0}^{m} \mathtt{h}_{N-\beta_j}^{\beta_j}(\overline{X}) \subseteq Hom((\overline{X}, r_0), \bigoplus_{j=1}^{m} \Lambda(\beta_j))$. Analogously, lift the inverse f^{-1} to $\gamma \in \oplus_{j=1}^{m} \mathtt{h}_{\beta_j}^{N-\beta_j}(\overline{X})$. Then $\gamma \circ \phi \in \mathtt{h}_N^{(N)}(\overline{X} \times \overline{X})$ is mapped to identity in $\mathrm{CH}^N(\overline{X} \times \overline{X}, \Lambda^0)$, so $\gamma \circ \phi - id$ is nilpotent according to the composition law, since it lies in the kernel of $\Phi^N \circ pr_N$. The element $\phi \circ \gamma$ lies in $End(\oplus_{j=0}^{m} \Lambda(\beta_j))$ so it is represented by a matrix $(a_{k,l})_{k,l \in \{0 \dots m\}}$ such that $a_{k,l} \in \Lambda^{\beta_k - \beta_l}$. Since $\mathtt{h}^{(N+1)}(\overline{X}) = 0$, we have that the composition of multiplication and pushforward

$$\mathtt{h}^{(d_1)}(\overline{X}) \otimes \mathtt{h}^{(d_2)}(\overline{X}) \to \mathtt{h}^{(d_1+d_2)}(\overline{X}) \xrightarrow{\pi_{\mathtt{h}}} \mathtt{h}(\mathrm{Spec}\ k)$$

is zero if $d_1 + d_2 > N$. Therefore $a_{k,l} = 0$ if $\beta_k - \beta_l > 0$. Note that for k, l such that $\beta_k = \beta_l$, we have $a_{k,l} = \pi_{\mathtt{h}}(\phi_k \gamma_l) \in \mathtt{h}_0^{(N)}(\overline{X})$. Since $\mathtt{h}^{(N+1)}(\overline{X}) = 0$ we have that $\Psi^{(N)}\colon \mathtt{h}_0^{(N)}(\overline{X}) \to \mathrm{CH}^N(\overline{X}, \Lambda^0)$ is an isomorphism and $\pi_{\mathtt{h}}(\phi_k \gamma_l) = \pi_{\mathrm{CH}}(f_k g_l) = \delta_{k,l}$. Thus $(a_{k,l})$ is a triangular matrix with identity on the diagonal, so it is the sum of the identity matrix and a nilpotent matrix.

So $\gamma \circ \phi - id$ is nilpotent and $\phi \circ \gamma - id$ is nilpotent. Then by lemma 4.9 $\overline{\mathcal{R}_{\mathtt{h}}} \cong \bigoplus_{j=0}^{m} \Lambda(\beta_j)$. ☐

Lemma 4.10. *Assume that* $\Lambda^1 = \cdots = \Lambda^N = 0$. *Then* $\mathtt{h}_N(\overline{X} \times \overline{X}) \subseteq \mathtt{h}^{(N)}(\overline{X} \times \overline{X})$ *and for the diagram of Lemma 4.6,*

$$\mathrm{CH}^N(X \times X, \Lambda^0) \xrightarrow{p^{\mathrm{CH}}} \mathrm{CH}^N(\overline{X} \times \overline{X}, \Lambda^0) \xleftarrow{\Phi^N} \mathtt{h}_N^{(N/N+1)}(\overline{X} \times \overline{X}) \xleftarrow{pr_N \circ p^{\mathtt{h}}} \mathtt{h}_N^{(N)}(X \times X),$$

the equality holds: $\mathrm{im}\, p^{\mathrm{CH}} = \mathrm{im}\, \Phi^N \circ pr_N \circ p^{\mathtt{h}}$.

Proof. Note that $\mathtt{h}_N(\overline{X} \times \overline{X})$ is additively generated by elements of the form $x = \lambda \tau_i \otimes \tau_j$, where $\deg \lambda + \alpha_i + \alpha_j = N$. Then $\alpha_i + \alpha_j \geqslant N$, since $\Lambda^1 = \cdots \Lambda^N = 0$. Thus $x \in \mathtt{h}^{(N)}(\overline{X} \times \overline{X})$.

Let us prove the equality $\mathrm{im}\, p^{\mathrm{CH}} = \mathrm{im}\, \Phi^N \circ pr_N \circ p^{\mathtt{h}}$. One inclusion is established in Lemma 4.6. It remains to check that $\mathrm{im}\, \Phi^N \circ pr_N \circ p^{\mathtt{h}} \subseteq \mathrm{im}\, p^{\mathrm{CH}}$. By the degree formula [**11**, Thm 4.4.7] $\mathtt{h}(X \times X)$ is generated as a Λ-module by pushforwards $i_{\mathtt{h}}(1)$, where $i\colon Z \to X \times X$ is projective, $Z \in \mathbf{Sm}_k$, and $i\colon Z \to i(Z)$ is birational. Following [**11**] we will denote such classes by $[Z \to X \times X]_{\mathtt{h}}$. Then $\mathtt{h}_N(X \times X)$ is additively generated by elements $\lambda[Z \to X \times X]_{\mathtt{h}}$, where λ is homogeneous such that $\deg \lambda + \mathrm{codim}\ Z = N$. Since $\Lambda^1 = \cdots \Lambda^N = 0$, we have $\mathrm{codim}\ Z \geqslant N$.

Note that if $\mathrm{codim}\ Z > N$ then by Lemma 4.7 $p^{\mathtt{h}}[Z \to X \times X]_{\mathtt{h}} = [Z_L \to \overline{X} \times \overline{X}]_{\mathtt{h}} \in \mathtt{h}^{(N+1)}(\overline{X} \times \overline{X})$. So $\mathrm{im}\, pr_N \circ p^{\mathtt{h}}$ is generated over Λ^0 by classes of $[Z_L \to \overline{X} \times \overline{X}]_{\mathtt{h}}$, where $Z \to X \times X$ has codimension N.

By Lemma 4.7 for any $Z \to X \times X$ of codimension N we have $\Phi^N \circ pr_N \circ p^{\mathtt{h}}([Z \to X \times X]_{\mathtt{h}}) = p^{\mathrm{CH}}([Z \to X \times X]_{\mathrm{CH}})$. Then $\mathrm{im}\, \Phi^N \circ pr_N \circ p^{\mathtt{h}} \subseteq p^{\mathrm{CH}}$ and the lemma is proven. ☐

Theorem B. Let $\mathtt{h}$ be an oriented cohomology theory with coefficient ring Λ. Assume that the Chow motive $\mathcal{R}$ is indecomposable over Λ^0 and $\Lambda^1 = \cdots = \Lambda^N = 0$. Then the $\mathtt{h}$-motive $\mathcal{R}_{\mathtt{h}}$ from theorem A is indecomposable.

Proof. By definition, $\mathcal{R}_{\mathtt{h}} = (X, r_0)$, where r_0 is an idempotent in $\mathtt{h}_N^{(N)}(X \times X)$. If $\mathcal{R}_{\mathtt{h}}$ is decomposable, then $r_0 = r_1 + r_2$ for some nontrivial orthogonal idempotents $r_1, r_2 \in \mathtt{h}_N(X \times X)$. Then by Lemma 4.10 $r_1, r_2 \in \mathtt{h}_N^{(N)}(X \times X)$ and $p_1 = \Phi^N \circ pr_N \circ p^{\mathtt{h}}(r_1)$ and $p_2 = \Phi^N \circ pr_N \circ p^{\mathtt{h}}(r_2)$ are rational idempotents and $p^{\mathrm{CH}}(p_0) = p_1 + p_2$. These idempotents are orthogonal and nontrivial, since $\ker \Phi^N \circ pr_N \circ p^{\mathtt{h}}$ is nilpotent. Hence, the Chow motive $\mathcal{R} = (X, p_0)$ is decomposable, a contradiction. □

Example 4.11. If $\mathtt{h}$ is Ω or connective K-theory, all the elements in the coefficient ring have negative degree. Then Theorems A and B prove that the $\mathtt{h}$-motivic irreducible decomposition coincides with the integral Chow-motivic decomposition. This gives another proof of the result by Vishik-Yagita [**18**, Cor. 2.8].

Example 4.12. Take $\mathtt{h}$ to be Morava K-theory $\mathtt{h} = K(n)^*$. Its coefficient ring is $\mathbb{F}_p[v_n, v_n^{-1}]$, where $\deg(v_n) = -2(p^n - 1)$. In the case $n > \log_p(\frac{N}{2} + 1)$ Theorems A and B prove that $M^{K(n)}(X)$ has the same irreducible decomposition as the Chow motive modulo p.

Theorem C. Suppose that X, Y are generically cellular and Y is a twisted form of X, i.e., $\overline{Y} \cong \overline{X}$.

If $M^{\mathrm{CH}}(X, \Lambda^0) \cong M^{\mathrm{CH}}(Y, \Lambda^0)$, then $M^{\mathtt{h}}(X) \cong M^{\mathtt{h}}(Y)$.

Proof. Let $f \in \mathrm{CH}^N(X \times Y)$ and $g \in \mathrm{CH}^N(Y \times X)$ be correspondences that give mutually inverse isomorphisms between $M^{\mathrm{CH}}(X)$ and $M^{\mathrm{CH}}(Y)$. Consider the diagram

$$\mathrm{CH}^N(X \times Y, \Lambda^0) \xrightarrow{p^{\mathrm{CH}}} \mathrm{CH}^N(\overline{X} \times \overline{X}, \Lambda^0) \xleftarrow{\Phi^N} \mathtt{h}_N^{(N/N+1)}(\overline{X} \times \overline{X}) \xleftarrow{pr_N \circ p^{\mathtt{h}}} \mathtt{h}_N^{(N)}(X \times Y).$$

Then by Lemma 4.6 we can find $f_1 \in \mathtt{h}_N^{(N)}(X \times Y)$ and $g_1 \in \mathtt{h}_N^{(N)}(Y \times X)$ such that $\Phi^N \circ pr_N \circ p^{\mathtt{h}}(f_1) = f$ and $\Phi^N \circ pr_N \circ p^{\mathtt{h}}(g_1) = g$. Then $g_1 \circ f_1 - id_X$ lies in the kernel of the map

$$\mathtt{h}_N^{(N)}(X \times X) \xrightarrow{pr_N \circ p^{\mathtt{h}}} \mathtt{h}_N^{(N/N+1)}(\overline{X} \times \overline{X}),$$

which consists of nilpotents by Lemma 4.8. So $g_1 \circ f_1 - id_X$ is nilpotent. By the same reasons $f_1 \circ g_1 - id_Y$ is nilpotent. Then $M^{\mathtt{h}}(X)$ and $M^{\mathtt{h}}(Y)$ are isomorphic by Lemma 4.9. □

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