Homology, Homotopy and Applications, vol. 16(2), 2014, pp.231–237

ANNIHILATION OF COHOMOLOGY AND DECOMPOSITIONS OF DERIVED CATEGORIES

SRIKANTH B. IYENGAR AND RYO TAKAHASHI

(*communicated by J. Daniel Christensen*)

Abstract

It is proved that an element r in the center of a coherent ring Λ annihilates $\mathrm{Ext}^n_\Lambda(M,N)$, for some positive integer n and all finitely presented Λ-modules M and N, if and only if the bounded derived category of Λ is an extension of the subcategory consisting of complexes annihilated by r and those obtained as n-fold extensions of Λ. This has applications to finiteness of dimension of derived categories.

1. Introduction

Let Λ be a right coherent ring, $\operatorname{mod}\Lambda$ the category of finitely presented right Λ-modules, and $\mathsf{D}^{\mathsf{b}}(\Lambda)$ its bounded derived category. The purpose of this note is to prove the result below that reveals a close link between the existence of uniform annihilators of Ext-modules, as modules over the center Λ^{c} of Λ, and a kind of decomposition of the derived category. In the statement, $\mathcal{G}$ is the class of morphisms in $\mathsf{D}^{\mathsf{b}}(\Lambda)$ that induce the zero map in cohomology, r is an element in Λ^{c}, and $\mathsf{D}^{\mathsf{b}}(\Lambda)_r$ consists of complexes X with $r\,\mathrm{Ext}^0_\Lambda(X,X)=0$, while $\mathsf{C}\diamond\mathsf{D}$ is the subcategory of complexes obtained as extensions of complexes in C and D; see 2.1.

Theorem 1.1. *Fix a non-negative integer n and an element r in Λ^{c}. The following conditions on $\mathsf{D}^{\mathsf{b}}(\Lambda)$ are equivalent.*

(1) $r\mathcal{G}^n=0$;
(2) $\mathsf{D}^{\mathsf{b}}(\Lambda)=\mathsf{D}^{\mathsf{b}}(\Lambda)_r\diamond\{\Lambda\}^{n\diamond}$;
(3) $\mathsf{D}^{\mathsf{b}}(\Lambda)=\{\Lambda\}^{n\diamond}\diamond\mathsf{D}^{\mathsf{b}}(\Lambda)_r$.

When they hold, $r\,\mathrm{Ext}^n_\Lambda(\operatorname{mod}\Lambda,\operatorname{mod}\Lambda)=0$. Conversely, the latter condition gives $r^3\mathcal{G}^{2n}=0$.

This result is a consequence of Theorem 2.10, which applies to abelian categories with enough projectives. In fact, the equivalence of conditions (1)–(3), and the proofs,

The first author was supported in part by NSF grant DMS-1201889.
The second author was supported in part by JSPS Grant-in-Aid for Young Scientists (B) 22740008, JSPS Grant-in-Aid for Scientific Research (C) 25400038 and JSPS Postdoctoral Fellowships for Research Abroad.
Received May 20, 2014, revised July 25, 2014; published on September 29, 2014.
2010 Mathematics Subject Classification: 16E30, 16E35, 18G25.
Key words and phrases: cohomology annihilator, derived category, projective class.
Article available at `http://dx.doi.org/10.4310/HHA.2014.v16.n2.a12`

carry over verbatim to generating projective classes in triangulated categories, in the sense of Christensen [**1**]; with Ext as in Section 4 of *op. cit.*, the entire statement carries over.

Here is one application (see Corollary 2.12) of the theorem above: If $r \in \Lambda^{\mathsf{c}}$ is a non-zerodivisor on Λ and satisfies $r\mathcal{G}^n = 0$, then there is an inequality

$$\dim \mathsf{D}^{\mathsf{b}}(\Lambda) \leqslant \dim \mathsf{D}^{\mathsf{b}}(\Lambda/r\Lambda) + n$$

concerning dimensions of the appropriate triangulated categories, in the sense of Rouquier [**4**]. This inequality gives a way to deduce the finiteness of the dimension of the derived category of Λ from that of the derived category of $\Lambda/r\Lambda$. The point is that the ring $\Lambda/r\Lambda$ is "smaller" than Λ; for example, the Krull dimension of $(\Lambda/r\Lambda)^{\mathsf{c}}$ is strictly smaller than that of Λ^{c}. This approach is predicated on the existence of non-zerodivisors that annihilate Ext-modules. For results in this direction, see [**2**, Section 7].

Acknowledgments

We should like to thank the referee for suggestions concerning presentation.

2. Decompositions

We deduce the statement in the Introduction from Theorem 2.10 below that concerns derived categories of abelian categories.

Definition 2.1. Let T be a triangulated category, and Σ its suspension functor; soon we will focus on the derived category of an abelian category.

Let C be a subcategory (always assumed to be full) of T. We write $\operatorname{add}(\mathsf{C})$ for the smallest subcategory of T containing C and closed under finite direct sums, retracts, and shifts. Given a subcategory D of T, the subcategory consisting of objects E that appear in exact triangles of the form

$$C \to E \to D \to \Sigma C \quad \text{with } C \in \mathsf{C} \text{ and } D \in \mathsf{D}$$

is denoted $\mathsf{C} * \mathsf{D}$. It is convenient to introduce also the following notation:

$$\mathsf{C} \diamond \mathsf{D} := \operatorname{add}(\mathsf{C} * \mathsf{D})\,.$$

It is a consequence of the octahedral axiom that there are equalities

$$(\mathsf{B} * \mathsf{C}) * \mathsf{D} = \mathsf{B} * (\mathsf{C} * \mathsf{D}) \quad \text{and} \quad (\mathsf{B} \diamond \mathsf{C}) \diamond \mathsf{D} = \mathsf{B} \diamond (\mathsf{C} \diamond \mathsf{D})\,.$$

In particular, we may denote them $\mathsf{B} * \mathsf{C} * \mathsf{D}$ and $\mathsf{B} \diamond \mathsf{C} \diamond \mathsf{D}$, respectively.

Throughout the rest of this section, R will be a commutative ring.

Definition 2.2. An additive category A is said to be *R-linear* if for each A in A there are homomorphisms of rings

$$\eta_A \colon R \to \operatorname{End}_{\mathsf{A}}(A)$$

with the property that the action of R on $\operatorname{Hom}_{\mathsf{A}}(A, B)$ induced by η_A and η_B coincide, for all A, B in A. Said otherwise, $\operatorname{Hom}_{\mathsf{A}}(A, B)$ is an R-module and this structure is compatible with compositions in A.

Let A be an R-linear Abelian category. The category of complexes over A inherits an R-linear structure, as does the bounded derived category, $\mathsf{D}^{\mathsf{b}}(\mathsf{A})$, of A. In either case, the action is compatible with the suspension, in that the morphisms $\Sigma(X \xrightarrow{r} X)$ and $\Sigma X \xrightarrow{r} \Sigma X$ coincide for all $r \in R$ and complexes X. What is used repeatedly in the sequel is that for any $r \in R$ and morphism $f\colon X \to Y$, in either category, there is an induced commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r} \\ X & \xrightarrow{f} & Y. \end{array}$$

Henceforth, we assume that A has enough projective objects, and write $\mathsf{proj}\,\mathsf{A}$ for the corresponding subcategory. For ease of notation, we abbreviate

$$\begin{aligned} \mathsf{T} &:= \mathsf{D}^{\mathsf{b}}(\mathsf{A}) \\ \mathsf{P}_n &:= \underbrace{\mathsf{proj}\,\mathsf{A} \diamond \cdots \diamond \mathsf{proj}\,\mathsf{A}}_{n \text{ copies}} \quad \text{for each } n \geqslant 0. \end{aligned}$$

Recall that *ghost* in T is a morphism $f\colon X \to Y$ such that

$$\mathrm{Hom}_{\mathsf{T}}(\Sigma^n P, f) = 0 \quad \text{for all } P \text{ in } \mathsf{proj}\,\mathsf{A} \text{ and } n \in \mathbb{Z}.$$

In what follows, we write $\mathcal{G}$ for the class of ghosts; it is an ideal in T. For any integer n, the ideal $\mathcal{G}^n$ consists of morphisms that are n-fold compositions of ghosts.

Remark 2.3. For each non-negative integer n, one has

$$\mathrm{Hom}_{\mathsf{T}}(P, g) = 0 \quad \text{for all } P \in \mathsf{P}_n \text{ and } g \in \mathcal{G}^n.$$

This is the well-known Ghost Lemma; for a proof, see, for example, [**3**, Theorem 3].

Remark 2.4. For each complex X in T and integer $n \geqslant 1$, there is an exact triangle

$$P \xrightarrow{p} X \xrightarrow{q} Y \longrightarrow \Sigma P$$

with P in P_n and q in $\mathcal{G}^n$; one can get this, for instance, from the construction of an Adams resolution of X; see [**1**, Section 4]. When X is in A, such a triangle exists with $\Sigma^{-n} Y$ in A.

Definition 2.5. For $r \in R$, let T_r denote the subcategory of T consisting of complexes X such that the multiplication morphism $X \xrightarrow{r} X$ is zero in T; in other words, r is in the kernel of the natural map $R \to \mathrm{End}_{\mathsf{T}}(X)$.

Remark 2.6. Let r, s be elements of R. In any exact triangle $X \to Y \to Z \to \Sigma X$ in T, if $X \in \mathsf{T}_r$ and $Z \in \mathsf{T}_s$, then $Y \in \mathsf{T}_{rs}$ holds.

Indeed, this is a well-known argument (analogous to one for the Ghost Lemma) contained in the commutative diagram below:

$$
\begin{array}{ccccccc}
 & & Y & \xrightarrow{g} & Z & & \\
 & \swarrow & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} & & \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & \Sigma X \\
\downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} & & & & \\
X & \xrightarrow{f} & Y. & & & &
\end{array}
$$

The squares in the diagram are commutative by the definition of the R-action on T. The morphism $Y \to X$ exists because $gs = sg = 0$; the second equality holds since Z is in T_s. The morphism $Y \xrightarrow{rs} Y$ thus factors as $Y \to X \xrightarrow{r} X \xrightarrow{f} Y$ and hence is zero, since X is in T_r.

In what follows, given a morphism $f \colon X \to Y$ of complexes over A, its mapping cone is denoted $\mathsf{cone}(f)$; thus

$$\mathsf{cone}(f)^n := Y^n \bigoplus X^{n+1} \quad \text{with differential } \begin{bmatrix} d^Y & f \\ 0 & -d^X \end{bmatrix}$$

The canonical exact sequence of complexes

$$0 \longrightarrow Y \longrightarrow \mathsf{cone}(f) \longrightarrow \Sigma X \longrightarrow 0$$

gives rise to an exact triangle $X \xrightarrow{f} Y \to \mathsf{cone}(f) \to \Sigma X$ in T.

Remark 2.7. For $r \in R$ and complex X over A, set $X /\!\!/ r := \mathsf{cone}(X \xrightarrow{r} X)$. Observe that $X /\!\!/ r$ is in T_r, because the map

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} : X /\!\!/ r \longrightarrow X /\!\!/ r$$

defines a homotopy between multiplication by r and the zero morphism.

Lemma 2.8. *For each subcategory C of T and element $r \in R$ there are inclusions*

$$\mathsf{T}_r * \mathsf{C} \subseteq \mathsf{C} * \mathsf{T}_{r^2} \quad \textit{and} \quad \mathsf{C} * \mathsf{T}_r \subseteq \mathsf{T}_{r^2} * \mathsf{C}\,.$$

Proof. We verify the first inclusion; the second one can be checked along the same lines.

Fix an X in $\mathsf{T}_r * \mathsf{C}$. Thus, there exist $T \in \mathsf{T}_r$ and $C \in \mathsf{C}$ and an exact triangle in the top row of the following diagram:

$$
\begin{array}{ccccccc}
T & \longrightarrow & X & \xrightarrow{f} & C & \xrightarrow{g} & \Sigma T \\
 & & & {\scriptstyle h}\swarrow & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} \\
 & & X & \xrightarrow[f]{} & C & \xrightarrow[g]{} & \Sigma T.
\end{array}
$$

The map h exists because $gr = rg = 0$, where the second equality holds because T is in T_r. By the octahedral axiom, the factorization $r = fh$ gives rise to an exact triangle

$$T \longrightarrow \mathsf{cone}(h) \longrightarrow C/\!\!/r \longrightarrow$$

It follows from Remarks 2.6 and 2.7 that r^2 annihilates $\mathsf{cone}(h)$. It remains to notice the exact triangle $C \longrightarrow X \longrightarrow \mathsf{cone}(h) \to \Sigma C$. □

Definition 2.9. For an element $r \in R$ and an integer $n \geqslant 0$ we consider the following four conditions on the triangulated category $\mathsf{T} := \mathsf{D^b}(\mathsf{A})$.

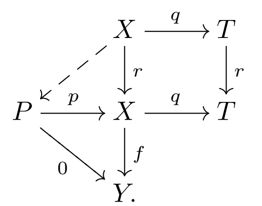
$$\begin{array}{llcll} \mathrm{D}_{r,n} & \mathsf{T} = \mathsf{T}_r \diamond \mathsf{P}_n\,, & \text{and} & \mathrm{E}_{r,n} & r\,\mathrm{Ext}^n_{\mathsf{A}}(\mathsf{A},\mathsf{A}) = 0\,, \\ \mathrm{D}'_{r,n} & \mathsf{T} = \mathsf{P}_n \diamond \mathsf{T}_r\,, & \text{and} & \mathrm{G}_{r,n} & r\mathcal{G}^n = 0\,. \end{array}$$

The statement from the introduction is a consequence of the following theorem.

Theorem 2.10. *The following implications hold*

$$\mathrm{D}'_{r,n} \iff \mathrm{D}_{r,n} \iff \mathrm{G}_{r,n} \Longrightarrow \mathrm{E}_{r,n} \Longrightarrow \mathrm{D}_{r^3,2n}$$

Proof. ($\mathrm{D}'_{r,n} \Rightarrow \mathrm{G}_{r,n}$): Fix $f\colon X \to Y$ to be in $\mathcal{G}^n$, and $P \xrightarrow{p} X \xrightarrow{q} T \to \Sigma P$ the exact triangle provided by the hypothesis. Consider the commutative diagram below where the morphism $X \to P$ is induced by the fact the $qr = rq = 0$, since T is in T_r.



It remains to note that the composition $fp = 0$, by Remark 2.3.

($\mathrm{D}_{r,n} \Rightarrow \mathrm{G}_{r,n}$) can be verified by an argument analogous to the one above.

($\mathrm{G}_{r,n} \Rightarrow \mathrm{D}'_{r,n}$) and ($\mathrm{G}_{r,n} \Rightarrow \mathrm{D}_{r,n}$): Fix X in T and $P \xrightarrow{p} X \xrightarrow{q} Y \to \Sigma P$ the exact triangle from Remark 2.4. By hypothesis $rq = 0$, so the octahedral axiom applied to the composition rq gives rises to an exact triangle

$$\Sigma P \longrightarrow Y \bigoplus \Sigma X \longrightarrow Y/\!\!/r \to \Sigma^2 P\,.$$

It remains to recall that $Y/\!\!/r$ is in T_r, by Remark 2.7, so that property $\mathrm{D}'_{r,n}$ holds. Applying the octahedral axiom to the map qr, which is also zero, shows that $\mathrm{D}_{r,n}$ holds as well.

($\mathrm{G}_{r,n} \Rightarrow \mathrm{E}_{r,n}$): This holds because any morphism $f\colon A \to \Sigma^n B$, with A, B in A, is in $\mathcal{G}^n$; see Remarks 2.3 and 2.4.

($\mathrm{E}_{r,n} \Longrightarrow \mathrm{D}_{r^3,2n}$): For a start observe that $\mathsf{A} \subseteq \mathsf{T}_r \diamond \mathsf{P}_n$; this follows by an argument along the lines of the one for $\mathrm{G}_{r,n} \Rightarrow \mathrm{D}'_{r,n}$ above. For a complex X over A let $Z^*(X)$ and $B^*(X)$ denote the cycles and boundaries of X, respectively. There are canonical exact triangles

$$Z^*(X) \longrightarrow X \longrightarrow \Sigma B^*(X) \longrightarrow \Sigma Z^*(X)$$
$$B^*(X) \longrightarrow Z^*(X) \longrightarrow H^*(X) \longrightarrow \Sigma B^*(X).$$

As $Z^*(X)$ and $B^*(X)$ are in $\operatorname{add}(\mathsf{A})$, one gets the first of the following chain of inclusions:

$$\begin{aligned}\mathsf{T} &\subseteq \mathsf{A} \diamond \mathsf{A}\\ &\subseteq (\mathsf{T}_r \diamond \mathsf{P}_n) \diamond (\mathsf{T}_r \diamond \mathsf{P}_n)\\ &\subseteq \mathsf{T}_r \diamond \mathsf{T}_{r^2} \diamond \mathsf{P}_n \diamond \mathsf{P}_n\\ &\subseteq \mathsf{T}_{r^3} \diamond \mathsf{P}_{2n}.\end{aligned}$$

The third inclusion holds by the associativity of $\diamond$ and Lemma 2.8. The last one holds by Remark 2.6 and the definition of the P_n. This is the desired implication. □

Non-zerodivisors

Now let Λ be a right coherent ring and $r \in \Lambda^{\mathsf{c}}$ a non-unit element in the center of Λ. The homomorphism of rings $\Lambda \to \Lambda/r\Lambda$ then induces, by restriction of scalars, an exact functor of triangulated categories

$$\mathsf{D}^{\mathsf{b}}(\Lambda/r\Lambda) \longrightarrow \mathsf{D}^{\mathsf{b}}(\Lambda).$$

Evidently, its image lies in the subcategory $\mathsf{D}^{\mathsf{b}}(\Lambda)_r$.

Lemma 2.11. *When r is a non-zerodivisor on Λ, the functor $\mathsf{D}^{\mathsf{b}}(\Lambda/r\Lambda) \to \mathsf{D}^{\mathsf{b}}(\Lambda)_r$ is dense up to direct summands.*

Proof. Since r is a non-zerodivisor on Λ, the canonical map $\Lambda/\!\!/r \to H^0(\Lambda/\!\!/r) \cong \Lambda/r\Lambda$ is a quasi-isomorphism in $\mathsf{D}^{\mathsf{b}}(\Lambda)$. This gives rise to an exact triangle

$$\Lambda \xrightarrow{\ r\ } \Lambda \longrightarrow \Lambda/r\Lambda \longrightarrow \Sigma\Lambda\,.$$

For any $X \in \mathsf{D}^{\mathsf{b}}(\Lambda)_r$, applying $X \otimes^{\mathbf{L}}_{\Lambda} -$ yields an exact triangle

$$X \xrightarrow{\ r\ } X \longrightarrow X \otimes^{\mathbf{L}}_{\Lambda} (\Lambda/r\Lambda) \longrightarrow \Sigma X\,.$$

Since the first morphism in this triangle is zero, one gets an isomorphism

$$X \otimes^{\mathbf{L}}_{\Lambda} (\Lambda/r\Lambda) \cong X \oplus \Sigma X\,.$$

Note that $X \otimes^{\mathbf{L}}_{\Lambda} (\Lambda/r\Lambda)$ is in the image of the functor $\mathsf{D}^{\mathsf{b}}(\Lambda/r\Lambda) \to \mathsf{D}^{\mathsf{b}}(\Lambda)$. □

Dimension

Recall that the *dimension* of a triangulated category T, denoted $\dim \mathsf{T}$, is the least non-negative integer d for which there exists an object G such that $\{G\}^{(d+1)\diamond} = \mathsf{T}$; see [**4**, Definition 3.2].

The result below justifies the inequality stated in the introduction. Recall that $\mathcal{G}$ denotes the class of ghosts in $\mathsf{D}^{\mathsf{b}}(\Lambda)$.

Corollary 2.12. *Let Λ be a right coherent ring. If $r \in \Lambda^{\mathsf{c}}$ is a non-zerodivisor on Λ and satisfies $r\mathcal{G}^n = 0$ for some non-negative integer n, then there is an inequality*

$$\dim \mathsf{D}^{\mathsf{b}}(\Lambda) \leqslant \dim \mathsf{D}^{\mathsf{b}}(\Lambda/r\Lambda) + n$$

Proof. Part of the hypothesis is that $\mathsf{D}^{\mathsf{b}}(\Lambda)$ satisfies condition $\mathrm{G}_{r,n}$, in the notation of Theorem 2.10. Keeping in mind Lemma 2.11 and that $\mathsf{proj}\,\Lambda = \operatorname{add}\Lambda$, *op. cit.* yields

$$\mathsf{D}^{\mathsf{b}}(\Lambda) = \mathsf{D}^{\mathsf{b}}(\Lambda/r\Lambda) \diamond \{\Lambda\}^{n\,\diamond}\ .$$

We have identified $\mathsf{D}^{\mathsf{b}}(\Lambda/r\Lambda)$ with its image in $\mathsf{D}^{\mathsf{b}}(\Lambda)$. If for some complex F and integer d one has $\mathsf{D}^{\mathsf{b}}(\Lambda/r\Lambda) = \{F\}^{(d+1)\,\diamond}$, then the equality above yields

$$\mathsf{D}^{\mathsf{b}}(\Lambda) = \{F \bigoplus \Lambda\}^{(d+n+1)\,\diamond}\ .$$

This implies the desired inequality. □

References

[1] J. Daniel Christensen, *Ideals in triangulated categories: phantoms, ghosts and skeleta*, Adv. Math. **136** (1998), no. 2, 284–339, DOI 10.1006/aima.1998.1735. MR1626856 (99g:18007)

[2] S. B. Iyengar and R. Takahashi, *Annihilation of cohomology and strong generators for module categories*, available at `arXiv:1404.1476`.

[3] G. M. Kelly, *Chain maps inducing zero homology maps*, Proc. Cambridge Philos. Soc. **61** (1965), 847–854. MR0188273 (32 #5712)

[4] R. Rouquier, *Dimensions of triangulated categories*, J. K-Theory **1** (2008), no. 2, 193–256, DOI 10.1017/is007011012jkt010. MR2434186 (2009i:18008)

Srikanth B. Iyengar `s.b.iyengar@unl.edu`

Department of Mathematics, University of Nebraska, Lincoln, NE 68588-0130, USA

Ryo Takahashi `takahashi@math.nagoya-u.ac.jp`

Graduate School of Mathematics, Nagoya University, Furocho, Chikusaku, Nagoya 464-8602, Japan