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ON THE VANISHING OF CHARACTERISTIC NUMBERS

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(*communicated by Dev P. Sinha*)

Abstract

In this article we introduce the notion of *pure type* for Killing vector fields on compact Riemannian and almost-Hermitian manifolds and present an application of the celebrated Atiyah-Bott-Singer localization formula for these Killing vector fields. Our central result is that if a $4n$-dimensional compact Riemannian manifold has a Killing vector field of pure type such that the dimension of its zero point set is less than n, then the vanishing statements for low-degree polynomials as given by the Atiyah-Bott-Singer localization formula imply the vanishing of Pontrjagin numbers of this manifold. An analogous result for the Chern numbers of compact almost-Hermitian manifolds is also established. The main strategy of our proof is to construct a family of lower-degree polynomials originating from the monomial symmetric polynomials.

1. Introduction

Unless otherwise stated, all the manifolds mentioned throughout this paper are closed, connected, and *oriented* and we use superscripts to indicate the *real* dimensions of the manifolds.

Given a smooth manifold X^{4n} (resp. an almost-complex manifold M^{2n}), the corresponding Pontrjagin numbers (resp. Chern numbers) are its basic numerical invariants [**10**]. However, it is difficult to calculate these numbers directly from their definitions. One remarkable result of Atiyah, Bott, and Singer [**3**], [**1**, §8] tells us that, if X^{4n} (resp. M^{2n}) admits a Killing vector field A (resp. a Killing vector field A which preserves the almost-complex structure), we can reduce the calculation of these characteristic numbers to the consideration of local information around zero(A), the zero point set of A. When zero(A) consists of isolated points, this formula was first established by Bott in [**3**] using a purely differential-geometric argument, which is now called Bott's residue formula. The general situation was established by Atiyah and Singer in [**1**,

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§8], which is a beautiful application of their general Lefschetz fixed point formula and is now commonly called the Atiyah-Bott-Singer localization formula.

Taking a closer look at the precise statements of these two formulae, we see that they provide *more vanishing-type information* for low-degree polynomials than just a method of calculating the characteristic numbers (further details can be found in Section 2). In fact, the author has exploited this observation in another article [**5**]. When zero(A) consists of isolated fixed points, the author used this "additional" vanishing information to derive a lower bound for the cardinality of zero(A) in [**5**], which can be in turn used to reprove and generalize some previously known results.

The starting point of the current article is to see to what extent this "additional" vanishing-type information for low-degree polynomials can be used to determine the whole structure of the localization formulae (Question 3.1). The main purpose of this article is to show that, under some very special conditions, this is the case (Theorem 3.2). Among these results, the central one is that if a compact $4n$-dimensional Riemannian manifold admits a Killing vector field of pure type (Definition 3.4) such that the dimension of its zero point set is less than n, then all the Pontrjagin numbers of this manifold vanish. Since the vector field generating a *semi-free* circle action is pure type by definition (further details can be found in Section 3.2), an immediate corollary is that all the Pontrjagin numbers of a $4n$-dimensional compact smooth manifold vanish if it is equipped with a semi-free circle action such that the dimension of the fixed point set of this action is less than n.

The rest of this article is arranged as follows. In Section 2 we review the Atiyah-Bott-Singer localization formulae for Riemannian and almost-Hermitian manifolds respectively and, through this process, introduce some notation and symbols. Section 3.1 contains the motivation and statement of our main result (Theorem 3.2), Section 3.2 is devoted to some applications in geometry and topology, and some remarks related to the main result and applications are presented in Section 3.3. We first treat in Section 4 the proof for low-dimensional cases in order to introduce the basic idea and to make the general proof in Section 6 more accessible. In Section 5 we review briefly some basic facts related to monomial symmetric polynomials and establish a key algebraic lemma (Lemma 5.1), both of which are needed later in our proof of Theorem 3.2. The proof of Theorem 3.2 itself will be presented in Section 6.

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2. Localization formulae

In this section, we recall the Atiyah-Bott-Singer localization formulae established in [**1**, §8].

Although Pontrjagin numbers only make sense for manifolds whose dimensions are divisible by 4, this localization formula established in [**1**, §8] is still valid and actually produces nontrivial results for all even-dimensional Riemannian manifolds equipped with Killing vector fields (see Remark 2.2). Our main result in this paper, Theorem 3.2, is also nontrivial for $(4k+2)$-dimensional manifolds. For this reason we consider $2n$-dimensional Riemannian manifolds, pointing out where necessary the special features which arise when n is even.

Suppose X^{2n} is a compact Riemannian manifold and A is a Killing vector field on X. This means that the actions of the one-parameter group (or flows) $\exp(tA)$ generated by A are isometries of X. As is well-known, each connected component in $\mathrm{zero}(A)$ is a compact smooth submanifold whose dimension is even. Let $F = F^{2r}$ be such a connected component. The normal bundle of F in X, which is denoted by $\mu(F)$, can be decomposed into a direct sum of $n-r$ 2-dimensional subbundles

$$\mu(F) = \bigoplus_{i=1}^{n-r} L(F,\lambda_i), \qquad \lambda_i > 0, \tag{2.1}$$

such that the eigenvalues of the skew-adjoint transformation induced by A on the 2-dimensional subbundle $L(F,\lambda_i)$ are $\pm\sqrt{-1}\lambda_i$. We can orient each $L(F,\lambda_i)$ so that, relative to an oriented basis, the skew-adjoint transformation is given by the matrix

$$\begin{pmatrix} 0 & -\lambda_i \\ \lambda_i & 0 \end{pmatrix}.$$

Accordingly, $\mu(F)$ can be oriented by the matrix

$$\bigoplus_{i=1}^{n-r} \begin{pmatrix} 0 & -\lambda_i \\ \lambda_i & 0 \end{pmatrix}.$$

The orientations of X and $\mu(F)$ induce an orientation of F and throughout the paper we will use this orientation of F. Suppose that the total Pontrjagin classes of X and F have the following *formal* decompositions:

$$\begin{aligned} p(X^{2n}) &= 1 + p_1(X) + \cdots + p_n(X) = \prod_{i=1}^{n}(1+x_i^2), \\ p(F^{2r}) &= 1 + p_1(F) + \cdots + p_r(F) = \prod_{j=1}^{r}(1+y_j^2), \end{aligned} \tag{2.2}$$

where these x_i and y_j are *virtual* 2-dimensional cohomology classes, i.e., the i-th elementary symmetric polynomial of $x_1^2,\dots,x_n^2$ (resp. $y_1^2,\dots,y_r^2$) represents $p_i(X)$ (resp. $p_i(F)$). Note that by our definition $p_i(X^{2n}) = 0$ (resp. $p_j(F^{2r}) = 0$) provided $i > \frac{n}{2}$ (resp. $j > \frac{r}{2}$). We use $z_i \in H^2(F;\mathbb{Z})$ $(1 \leqslant i \leqslant n-r)$ to denote the Euler class of the oriented 2-dimensional subbundle $L(F,\lambda_i)$.

Let $f(t_1,\dots,t_n)$ be a symmetric polynomial. We define a complex number $R_f(X,A)$

as follows:

$$R_f(X,A) := \sum_F \langle f\big(y_1^2,\ldots,y_r^2,(\sqrt{-1}\lambda_1+z_1)^2,\ldots,(\sqrt{-1}\lambda_{n-r}+z_{n-r})^2\big)\cdot\prod_{i=1}^{n-r}(\sqrt{-1}\lambda_i+z_i)^{-1},[F]\rangle, \tag{2.3}$$

where $[F]\in H_{2r}(F;\mathbb{Z})$ is the fundamental class determined by the orientation of F, $\langle\cdot,\cdot\rangle$ is the Kronecker pairing, and the sum is over all the connected components in zero(A). Here $(\sqrt{-1}\lambda_i+z_i)^{-1}$ is understood to be

$$\begin{aligned}(\sqrt{-1}\lambda_i+z_i)^{-1} = \frac{1}{\sqrt{-1}\lambda_i}(1+\frac{z_i}{\sqrt{-1}\lambda_i})^{-1} &:= \frac{1}{\sqrt{-1}\lambda_i}\sum_{j=0}^{r}(-\frac{z_i}{\sqrt{-1}\lambda_i})^j \\ &= \frac{1}{\sqrt{-1}\lambda_i}+\sum_{j=1}^{r}\frac{(-z_i)^j}{(\sqrt{-1}\lambda_i)^{j+1}}\in H^*(F;\mathbb{C}).\end{aligned} \tag{2.4}$$

This means, before being evaluated on the fundamental class $[F]$, the expression

$$f\big(y_1^2,\ldots,y_r^2,(\sqrt{-1}\lambda_1+z_1)^2,\ldots,(\sqrt{-1}\lambda_{n-r}+z_{n-r})^2\big)\cdot\prod_{i=1}^{n-r}(\sqrt{-1}\lambda_i+z_i)^{-1} \tag{2.5}$$

on the right-hand side of (2.3) can be viewed as a polynomial in the variables $y_1^2,\ldots,y_r^2$ and $z_1,\ldots,z_{n-r}$ and we are only concerned with its homogeneous part of degree r if we assume that degree(y_i) = degree$(z_i) := 1$. We remind the reader that, although $f(t_1,\ldots,t_n)$ itself is symmetric, (2.5) is only symmetric with respect to the virtual cohomology classes $y_1^2,\ldots,y_r^2$ and so can be expressed in terms of Pontrjagin classes via the formal decomposition formula (2.2). In general (2.5) is *not* symmetric with respect to $z_1,\ldots,z_{n-r}$ as the eigenvalues λ_i may be different. Note that $R_f(X,A)$ is *a priori* a complex number.

With the above-mentioned notation understood, we have the following Atiyah-Bott-Singer localization formula in the Riemannian case [**1**, p. 597, Theorem 8.11].

Theorem 2.1 (Localization formula, Riemannian case)**.** Let $f(t_1,\ldots,t_n)$ be a symmetric polynomial whose degree is no more than $\frac{n}{2}$ $\big($degree$(t_i):=1\big)$. Then we have

$$R_f(X,A)=\begin{cases}0, & \text{degree}(f)<\frac{n}{2},\\ \langle f(x_1^2,\ldots,x_n^2),[X]\rangle, & \text{degree}(f)=\frac{n}{2}.\end{cases}$$

Here $[X]\in H_{2n}(X;\mathbb{Z})$ is the fundamental class of X determined by its orientation.

Remark 2.2.

1. By definition, the Pontrjagin numbers of X^{2n} all vanish unless n is even. Thus $\langle f(x_1^2,\ldots,x_n^2),[X]\rangle$ can be nonzero only if n is even. In [**1**, p. 597, Theorem 8.11], this result was only stated for $4k$-dimensional manifolds, whose Pontrjagin numbers were the main interest. However, its proof [**1**, 595–597], based on the twisted version of the Atiyah-Singer G-signature theorem [**1**, p. 586, (6.19)], was given for all even-dimensional manifolds.
2. Note that $R_f(X,A)\equiv 0$ if degree$(f)<\frac{n}{2}$. So, as we have mentioned in Section 1, this tells us *more* vanishing-type information rather than just a method

of calculating the Pontrjagin numbers of X^{2n} when n is even. In particular, it provides nontrivial results for the eigenvalues and characteristic classes of zero(A) on X^{2n} irrespective of the parity of n.

3. If the degree of f is *larger* than $\frac{n}{2}$, the localization formula gives no information.

For almost-Hermitian manifolds, we have a similar localization formula. Suppose M is a $2n$-dimensional almost-Hermitian manifold, which means that M is an almost-complex manifold with an almost-Hermitian metric, and A is a vector field which preserves the almost-complex structure and the metric. Then each connected component in zero(A) is an almost-Hermitian submanifold. Let $F = F^{2r}$ be such a connected component. The normal bundle of F, $\mu(F)$, can be decomposed into a direct sum of $n-r$ complex line bundles

$$\mu(F) = \bigoplus_{i=1}^{n-r} L(F, \lambda_i), \qquad \lambda_i \in \mathbb{R} - \{0\},$$

such that the eigenvalue of the skew-Hermitian transformation induced by A on the complex line bundle $L(F, \lambda_i)$ is $\sqrt{-1}\lambda_i$. Note that in this case M, F, and $\mu(F)$ all have canonical orientations from their corresponding complex structures. Suppose that the total Chern classes of M and F have the following formal decompositions:

$$\begin{aligned} c(M^{2n}) &= 1 + c_1(M) + \cdots + c_n(M) = \prod_{i=1}^{n}(1 + x_i), \\ c(F^{2r}) &= 1 + c_1(F) + \cdots + c_r(F) = \prod_{j=1}^{r}(1 + y_j), \end{aligned} \tag{2.6}$$

i.e., the i-th elementary symmetric polynomial of $x_1, \ldots, x_n$ (resp. $y_1, \ldots, y_r$) represents $c_i(M)$ (resp. $c_i(F)$). We use $z_i \in H^2(F; \mathbb{Z})$ $(1 \leqslant i \leqslant n-r)$ to denote the first Chern class of the complex line bundle $L(F, \lambda_i)$.

Let $f(t_1, \ldots, t_n)$ be a symmetric polynomial. We define a complex number $H_f(M, A)$ as follows:

$$H_f(M, A) := \sum_F \langle f\big(y_1, \ldots, y_r, \sqrt{-1}\lambda_1 + z_1, \ldots, \sqrt{-1}\lambda_{n-r} + z_{n-r}\big) \cdot \prod_{i=1}^{n-r} (\sqrt{-1}\lambda_i + z_i)^{-1}, [F]\rangle. \tag{2.7}$$

Then we have the following Atiyah-Bott-Singer localization formula in the Hermitian case [**1**, Proposition 8.13].

Theorem 2.3 (Localization formula, Hermitian case)**.** Let $f(t_1, \ldots, t_n)$ be a symmetric polynomial whose degree is no more than n (degree$(t_i) := 1$). Then we have

$$H_f(X, A) = \begin{cases} 0, & \text{degree}(f) < n, \\ \langle f(x_1, \ldots, x_n), [M]\rangle, & \text{degree}(f) = n. \end{cases}$$

Remark 2.4. The statement "$H_f(X, A) = 0$ if degree$(f) < \frac{n}{2}$" gives us more vanishing-type information rather than just a method of calculating the Chern numbers of M (cf. Remark 2.2, (2)).

3. Main result and applications

3.1. Our main result

According to the localization formulae, the conclusions that

$$R_f(X, A) \equiv 0 \qquad (\text{resp. } H_f(M, A) \equiv 0)$$

for those f whose degrees are smaller than $\frac{n}{2}$ (resp. n) provide us with many restrictions on the eigenvalues and characteristic classes of $\text{zero}(A)$ and $\mu\big(\text{zero}(A)\big)$.

In this paper, we are concerned with the following question.

Question 3.1. *Under what conditions do*

$$R_f(X, A) \equiv 0 \qquad (\textit{resp. } H_f(M, A) \equiv 0)$$

for those f whose degrees are smaller than $\frac{n}{2}$ (resp. n) guarantee that

$$R_f(X, A) \equiv 0 \qquad (\textit{resp. } H_f(M, A) \equiv 0)$$

for any *symmetric polynomial f?*

The main result of this article is the following theorem, which gives a sufficient condition for a positive answer to Question 3.1.

Theorem 3.2.

1. In the Riemannian case, if all the eigenvalues λ_i over all the connected components F are equal, i.e., there exists a positive real number λ such that
$$\bigcup_F \{\lambda_1, \ldots, \lambda_{n-r}\} = \{\lambda\}, \tag{3.1}$$
and $\dim \text{zero}(A) < \frac{n}{2}$, then
$$R_f(X^{2n}, A) \equiv 0, \qquad \text{for any symmetric polynomial } f(t_1, \ldots, t_n).$$
2. In the Hermitian case, if all the eigenvalues λ_i over all the connected components F are equal, i.e., there exists a nonzero real number λ such that
$$\bigcup_F \{\lambda_1, \ldots, \lambda_{n-r}\} = \{\lambda\},$$
and $\dim \text{zero}(A) < \frac{2}{3}n$, then
$$H_f(M^{2n}, A) \equiv 0, \qquad \text{for any symmetric polynomial } f(t_1, \ldots, t_n).$$

Here by $\dim \text{zero}(A)$ we mean the maximal real dimension of the connected components in $\text{zero}(A)$. In particular, under the restrictions above we have

$$R_f(X, A) \equiv 0 \qquad \text{if degree}(f) = \frac{n}{2},$$

and

$$H_f(X, A) \equiv 0 \qquad \text{if degree}(f) = n.$$

Remark 3.3. Note that the polynomial $f(t_1, \ldots, t_n)$ in (2.3) and (2.7) is *symmetric* and so each summand on the right-hand side of (2.3) $\big($resp. (2.7)$\big)$ is symmetric with respect to those y_i^2 (resp. y_i), which are the characteristic classes of the connected component F via the formal decomposition formula (2.2) $\big($resp. (2.6)$\big)$. Nevertheless, this summand may *not* be symmetric with respect to those characteristic classes z_i of the normal bundle of F as the eigenvalues λ_i may be different. Partially due to this reason these two expressions are difficult to deal with. However, under our assumptions in Theorem 3.2 each summand is indeed symmetric with respect to those z_i and so we can make full use of the power of symmetric polynomial theory. This is the underlying motivation for making the assumptions in Theorem 3.2 and its advantage will be gradually clear in the next two sections.

3.2. Applications

Using the notation introduced before, we give the following definition.

Definition 3.4. A Killing vector field A on a Riemannian manifold X (resp. a Killing vector field A on an almost-Hermitian manifold M preserving the almost-complex structure) is called *pure type* if

$$\bigcup_F \{\lambda_1, \ldots, \lambda_{n-r}\} = \{\lambda\}.$$

Given this definition, a direct consequence of Theorem 3.2 is the following result, which provides an obstruction to the existence of certain pure type vector fields.

Proposition 3.5. *If a Riemannian manifold X^{2n} (resp. an almost-Hermitian manifold M^{2n}) admits a pure type Killing vector field A such that dim zero$(A) < \frac{n}{2}$ (resp. dim zero$(A) < \frac{2}{3}n$), then all the Pontrjagin numbers of X (resp. Chern numbers of M) vanish.*

Example 3.6. Let $\mathbb{C}P^n$ be the n-dimensional complex projective space with homogeneous coordinates $[z_0, z_1, \ldots, z_n]$, λ a nonzero real number, and n_1 a nonnegative integer which is less than n. Using these data we can define a one-parameter group action ψ_t on $\mathbb{C}P^n$ by

$$\psi_t \ : \ \mathbb{C}P^n \longrightarrow \mathbb{C}P^n,$$

$$[z_0, z_1, \ldots, z_n] \longmapsto [e^{\sqrt{-1}t\lambda} z_0, e^{\sqrt{-1}t\lambda} z_1, \ldots, e^{\sqrt{-1}t\lambda} z_{n_1}, z_{n_1+1}, \ldots, z_n].$$

Let A be the vector field generating this ψ_t. Then

$$\text{zero}(A) = \text{fixed point set of the action } \{\psi_t\} = F_1 \coprod F_2,$$

where

$$F_1 = \{[z_0, \ldots, z_{n_1}, 0, \ldots, 0]\} \cong \mathbb{C}P^{n_1}, \quad F_2 = \{[0, \ldots, 0, z_{n_1+1}, \ldots, z_n]\} \cong \mathbb{C}P^{n-n_1-1}.$$

This A is Killing with respect to the Fubini-Study metric and preserves the standard complex structure on $\mathbb{C}P^n$. The eigenvalues of A on F_1 (resp. F_2) are $-\sqrt{-1}\lambda$ (resp. $\sqrt{-1}\lambda$) with multiplicity $n - n_1$ (resp. $n_1 + 1$) and so it is *not* pure type as a Killing vector field on the Hermitian manifold $\mathbb{C}P^n$. However, if we ignore the complex structure and only view A as a Killing vector field on the *Riemannian* manifold $\mathbb{C}P^n$, A is indeed pure type as the unique common positive eigenvalue is $|\lambda|$.

Note that dim zero$(A) = 2 \cdot \max\{n_1, n - n_1 - 1\} \geqslant n - 1$, which is larger than $\frac{n}{2}$ if $n \geqslant 3$.

When $n = 2$, whether n_1 is 0 or 1, dim zero$(A) = 1 > \frac{1}{2}$, which does not satisfy our assumption in Proposition 3.5. Note that the unique Pontrjagin number of $\mathbb{C}P^2$ is 3, which is nonzero.

However, if we consider the 4-dimensional Riemannian manifold $\mathbb{C}P^1 \times \mathbb{C}P^1$ with the diagonal action ψ_t, the corresponding vector field consists of exactly four isolated zeros and so dim zero$(A) = 0 < \frac{1}{2}$, which does satisfy the assumption in Proposition 3.5. Note that the unique Pontrjagin number of $\mathbb{C}P^1 \times \mathbb{C}P^1$ is zero.

Clearly the example above can be extended to n copies $(\mathbb{C}P^1)^n$ to obtain a pure type vector field with 2^n isolated zeros. Note that all the Pontrjagin numbers of $(\mathbb{C}P^1)^n$ vanish.

The condition of "pure type" in Definition 3.4 seems to be very strong, but in fact any vector field on a smooth manifold which generates a *semi-free* circle action is pure type. A circle action is called *semi-free* if it is free outside the fixed point set or, equivalently, the isotropy subgroup of any point outside the fixed point set is trivial. Given any smooth circle action on a smooth manifold X, since the circle is compact, we can always choose a Riemannian metric on X such that this action is isometric and thus the vector field which generates this circle action is Killing. Therefore the fact that a vector field generating a semi-free circle action is pure type immediately follows from the well-known fact that the weights (or exponents) of the representation spaces induced by any semi-free circle action on the normal bundle of the fixed point set are all 1. Indeed, if the $n - r$ weights induced by a semi-free circle action on a $2r$-dimensional connected component of the fixed point set are $m_1, \ldots, m_{n-r}$, then in a suitable neighborhood of this connected component the circle acts as in the following model:

$$S^1 \times \mathbb{C}^n \longrightarrow \mathbb{C}^n,$$

$$\big(z, (z_1, \ldots, z_n)\big) \longmapsto (z_1, \ldots, z_r, z^{m_1} \cdot z_{r+1}, \ldots, z^{m_{n-r}} \cdot z_n).$$

Suppose some one among the weights $m_1, \ldots, m_{n-r}$, say m_1, is 2 or more. Then all the m_1-th roots of unity in S^1 act as the identity on $\{(z_1, \ldots, z_r, z_{r+1}, 0, \ldots, 0)\} \cong \mathbb{C}^{r+1}$, contradicting the semi-freeness of the action.

Let Ω_*^{SO} (resp. Ω_*^{O}) be the oriented (resp. unoriented) cobordism ring [**10**, p. 52–53, p. 201]. It is well-known that a smooth manifold X represents a torsion element in Ω_*^{SO} (resp. X bounds), i.e., $[X] = 0 \in \Omega_*^{SO} \otimes \mathbb{Q}$ (resp. $[X] = 0 \in \Omega_*^{SO}$), if and only if all the Pontrjagin numbers (resp. all the Pontrjagin numbers and Stiefel-Whitney numbers) of X vanish [**10**, p. 217]. It is also well-known that $[X] = 0 \in \Omega_*^{O}$ if and only if all the Stiefel-Whitney numbers of X vanish [**10**, p. 53]. The well-known $\frac{2}{5}$-Theorem of Boardman [**3**] states that, if X^n admits an involution such that the dimension of the fixed point set is less than $\frac{2}{5}n$, then $[X] = 0 \in \Omega_*^{O}$. By using Bott's residue formula, Pantilie and Wood [**11**, Theorem 1.1] showed that if a smooth manifold admits a semi-free circle action with isolated fixed points, then all the Pontrjagin numbers of this manifold vanish. The author and Liu strengthened this result by showing that in this case the manifold bounds [**7**, Theorem 1.6]. Now this result can be further strengthened as in the following theorem, which can be viewed as an analogue of the $\frac{2}{5}$-Theorem in the case of semi-free circle actions.

Theorem 3.7. If a smooth manifold X^{2n} admits a semi-free circle action such that the dimension of the fixed point set is less than $\frac{n}{2}$, then X bounds.

Proof. First note that the fixed point set of this circle action is exactly the zero point set of the vector field generated by this circle action. Thus Theorem 3.2 and our assumption tell us that all the Pontrjagin numbers of X^{2n} vanish. So it suffices to show that all the Stiefel-Whitney numbers of X also vanish. We use g to denote the involution of X determined by the semi-free circle action, i.e., $g \longleftrightarrow -1 \in S^1$. Let X^{S^1} (resp. X^g) be the fixed point set of the whole circle action (resp. the involution g). Clearly $X^{S^1} \subset X^g$. Since this circle action is semi-free, which means the isotropy subgroup of any point outside X^{S^1} is trivial, then $X^{S^1} = X^g$. Consequently $\dim X^g = \dim X^{S^1} < \frac{n}{2}$. So the $\frac{2}{5}$-Theorem tells us that in this case $[X] = 0 \in \Omega_*^O$, i.e., all the Stiefel-Whitney numbers of X vanish. □

Remark 3.8.

1. This result indicates that in some sense examples of semi-free circle actions with *low-dimensional* fixed point sets are very rare. Indeed, according to the author's best knowledge, the only existing examples of closed orientable manifolds admitting semi-free circle actions with low-dimensional fixed point sets are homotopy spheres and their products (cf. [**4**, Chapter 6, §9] and the references therein), which obviously bound.
2. The model described in Example 3.6 can also be used to construct examples of semi-free circle actions with *high-dimensional* fixed point sets:

$$S^1 \times \mathbb{C}P^n \longrightarrow \mathbb{C}P^n,$$

$$(g, [z_0, z_1, \dots, z_n]) \longmapsto [g \cdot z_0, g \cdot z_1, \dots, g \cdot z_{n_1}, z_{n_1+1}, \dots, z_n].$$

3.3. Further remarks

Compared to the upper bound $\frac{2}{5}\dim X$ in Boardman's theorem, the upper bound $\frac{1}{4}\dim X$ in our Theorem 3.7 seems not to be sharp. So it would be interesting to find out the sharp upper bound.

In [**8**], the author and Liu, by combining the Witten-Taubes-Bott rigidity theorem and the Atiyah-Singer G-signature theorem, showed that if a *spin* manifold M^{2n} admits a *prime* circle action, which is defined by the authors in [**8**] and includes the semi-free case, such that the dimension of the fixed point set is less than n, then the indices of some twisted signature operators all vanish [**8**, Theorem 1.7, Corollary 1.8]. Note that the indices of these twisted signature operators are all linear combinations of Pontrjagin numbers. So [**8**, Corollary 1.8] is a direct consequence of the current article when the dimension of the fixed point set is less than one fourth of the manifold.

If an *almost-complex* manifold admits a semi-free circle action which preserves the almost-complex structure, the weights of the normal bundle of the fixed point set with respect to this action are 1 *or* −1. This means that, if an almost-complex manifold admits a vector field which preserves the almost-complex structure and generates a semi-free circle action, then using the notation in the previous section, we can conclude that

$$\bigcup_F \{\lambda_1, \dots, \lambda_{n-r}\} \subset \{\lambda, -\lambda\} \tag{3.2}$$

for some nonzero real number λ. Indeed, if the Euler characteristics of the connected components in zero(A) are all nonnegative (resp. nonpositive) and at least one of them is positive (resp. negative), the symbol " $\subset$ " in (3.2) is actually " $=$ ", which is a corollary of [**6**, Theorem 1.1]. Thus Theorem 3.2 *cannot* be applied to the case of almost-complex manifolds. In fact, in contrast to the smooth case, even if an almost-complex manifold admits a semi-free circle action which preserves the almost-complex structure and has only *isolated* fixed points, some Chern numbers of this almost-complex manifold are nonzero (see Lemma 3.6 and the paragraph before Theorem 1.8 in [**7**]).

Evidently the statements of Theorems 2.1 and 2.3 can be reformulated as purely algebraic results. In fact, we can replace the cohomologies of X (resp. M) and zero(A) by some abstract graded ring and replace the Kronecker pairing $\langle\cdot,\cdot\rangle$ by some abstract evaluation map on the top-dimensional vector space of this graded ring. However, as we have illustrated in the previous sections, any nontrivial answer to Question 3.1 may provide related applications to geometry and topology, and for this reason we would like to draw more mathematicians' attention to Theorems 2.1 and 2.3 and Question 3.1.

4. Warm-up for the proof

As we will see in the next two sections, the proof of Theorem 3.2 relies heavily on the manipulations of monomial symmetric polynomials and their variants defined in Section 5. As this may distract the reader from the underlying idea of the proof, in this "warm-up" section we illustrate it in the Riemannian case for dim zero$(A) \leqslant 2$ in detail and then indicate how to deal with the case of dim zero$(A) = 4$. This will also allow us to motivate the unified method which will be used in the general situation.

Lemma 4.1. Under the assumption (3.1), if $r = 0$, 1, or 2, the expression

$$\langle f\big(y_1^2,\ldots,y_r^2,(\sqrt{-1}\lambda+z_1)^2,\ldots,(\sqrt{-1}\lambda+z_{n-r})^2\big)\cdot\prod_{i=1}^{n-r}(\sqrt{-1}\lambda+z_i)^{-1},[F^{2r}]\rangle \quad (4.1)$$

has the form

1. $$c_0(f)\cdot\epsilon(F^0), \quad (4.2)$$

   where

   $$c_0(f) := (\sqrt{-1}\lambda)^{-n}\cdot f(-\lambda^2,\ldots,-\lambda^2)$$

   and $\epsilon(F^0) := 1$ if the orientation at the tangent space to F^0 agrees with that of X^{2n} and $\epsilon(F^0) := -1$ otherwise;

2. $$c_1(f)\cdot\langle\sum_{i=1}^{n-1}z_i,[F^2]\rangle, \quad (4.3)$$

   where $c_1(f)$ is a complex number depending only on f;

3.
$$c_{2,1}(f)\cdot\langle y_1^2+y_2^2,[F^4]\rangle+c_{2,2}(f)\cdot\left\langle\sum_{i=1}^{n-2}z_i^2,[F^4]\right\rangle+c_{2,3}(f)\cdot\left\langle\sum_{1\leqslant i<j\leqslant n-2}z_iz_j,[F^4]\right\rangle, \tag{4.4}$$
where $c_{2,1}(f)$, $c_{2,2}(f)$, and $c_{2,3}(f)$ are three complex numbers depending only on f and $y_1^2+y_2^2$ is the first Pontrjagin class of F^4 via the formal decomposition formula (2.2).

Proof. (4.2) is obvious. We have explained in Remark 3.3 that under the assumption (3.1) the expression $f(\cdots)\cdot\prod(\cdots)^{-1}$ in (4.1) is symmetric with respect to $z_1,\ldots,z_{n-r}$ as well as $y_1^2,\ldots,y_r^2$. Note that when $r=1$ or 2, in $f(\cdots)\cdot\prod(\cdots)^{-1}$ we are only concerned with the homogeneous part of degree 1 or 2 respectively. This gives (4.3) and (4.4). $\square$

We first treat the case of isolated zeros.

Example 4.2. The first part of Theorem 3.2 holds if $\dim \operatorname{zero}(A)=0$.

Proof. Let ρ_0 (resp. ρ_1) denote the number of isolated zeros in zero(A) such that $\epsilon(F^0)=1$ $\big($resp. $\epsilon(F^0)=-1\big)$. Then Theorem 2.1 and (4.2) tell us that
$$c_0(f)\cdot(\rho_0-\rho_1)=0,\qquad \text{if degree}(f)<\frac{n}{2}. \tag{4.5}$$
Taking $f=1$ (i.e., degree$(f)=0$) in (4.5) leads to $\rho_0-\rho_1=0$ as $c_0(f)\neq 0$, which in turn tells us that (4.5) holds for polynomials f of *all* degrees. $\square$

Remark 4.3. This case of isolated zeros has been done in [**7**, p. 444] when the vector field A is generated by a semi-free circle action.

Next we treat the case of $\dim \operatorname{zero}(A)=2$, which is slightly more complicated than the example above.

Example 4.4. The first part of Theorem 3.2 holds if $\dim \operatorname{zero}(A)=2$.

Proof. Let ρ_0 (resp. ρ_1) still denote the number of isolated zeros in zero(A) such that $\epsilon(F^0)=1$ $\big($resp. $\epsilon(F^0)=-1\big)$. Then Theorem 2.1, (4.2), and (4.3) say that
$$c_0(f)\cdot(\rho_0-\rho_1)+c_1(f)\cdot\sum_{F^2}\langle\sum_{i=1}^{n-1}z_i,[F^2]\rangle=0,\qquad \text{if degree}(f)<\frac{n}{2}. \tag{4.6}$$
Now it suffices to show, under the assumptions (4.6) and $n\geqslant 5$ (recall that in Theorem 3.2 we need $\dim \operatorname{zero}(A)<\frac{n}{2}$), that
$$\rho_0-\rho_1=\sum_{F^2}\langle\sum_{i=1}^{n-1}z_i,[F^2]\rangle=0.$$
Define
$$f_0(t_1,\ldots,t_n)=1,\qquad f_1(t_1,\ldots,t_n)=\sum_{i=1}^{n}t_i(t_i+\lambda^2).$$
Note that degree$(f_0)=0<\frac{n}{2}$ and degree$(f_1)=2<\frac{n}{2}$ as $n\geqslant 5$, which means f_0 and

f_1 satisfy (4.6). Replacing f in (4.6) with f_0 we have

$$c_0(f_0)\cdot(\rho_0-\rho_1)+c_1(f_0)\cdot\sum_{F^2}\langle\sum_{i=1}^{n-1}z_i,[F^2]\rangle=0,\qquad c_0(f_0)\neq 0. \tag{4.7}$$

For f_1 it is direct to verify that $f_1(-\lambda^2,\dots,-\lambda^2)=0$ and thus $c_0(f_1)=0$, and

$$\begin{aligned} f_1\big(y_1^2,(\sqrt{-1}\lambda+z_1)^2,\dots,(\sqrt{-1}\lambda+z_{n-1})^2\big)\cdot\prod_{i=1}^{n-1}(\sqrt{-1}\lambda+z_i)^{-1}\\ =\big[2(\sqrt{-1}\lambda)^{4-n}\cdot\sum_{i}^{n-1}z_i\big]+\text{higher degree monomials.}\end{aligned}$$

This means that the coefficient $c_1(f_1)=2(\sqrt{-1}\lambda)^{4-n}\neq 0$. Replacing f in (4.6) with f_1 yields

$$c_1(f_1)\cdot\sum_{F^2}\langle\sum_{i=1}^{n-1}z_i,[F^2]\rangle=0,\qquad c_1(f_1)\neq 0. \tag{4.8}$$

Combining (4.7) with (4.8) establishes the desired property. □

From the example above we can see that the key point is to choose carefully a symmetric polynomial f_1 such that degree$(f_1)<\frac{n}{2}$, $c_0(f_1)=0$, and $c_1(f_1)\neq 0$.

Similarly, for the case of dim zero$(A)=4$, we are able to choose five symmetric polynomials $f_0=1$, f_1, $f_{2,1}$, $f_{2,2}$, and $f_{2,3}$ such that all their degrees are less than $\frac{n}{2}$ and the 5×5 matrix

$$\begin{pmatrix} c_0(f_0) & c_0(f_1) & c_0(f_{2,1}) & c_0(f_{2,2}) & c_0(f_{2,3})\\ c_1(f_0) & c_1(f_1) & c_1(f_{2,1}) & c_1(f_{2,2}) & c_1(f_{2,3})\\ c_{2,1}(f_0) & c_{2,1}(f_1) & c_{2,1}(f_{2,1}) & c_{2,1}(f_{2,2}) & c_{2,1}(f_{2,3})\\ c_{2,2}(f_0) & c_{2,2}(f_1) & c_{2,2}(f_{2,2}) & c_{2,2}(f_{2,2}) & c_{2,2}(f_{2,3})\\ c_{2,3}(f_0) & c_{2,3}(f_1) & c_{2,3}(f_{2,3}) & c_{2,3}(f_{2,2}) & c_{2,3}(f_{2,3})\end{pmatrix}=\begin{pmatrix} * & 0 & 0 & 0 & 0\\ * & * & 0 & 0 & 0\\ * & * & * & 0 & 0\\ * & * & * & * & 0\\ * & * & * & * & *\end{pmatrix}$$

has *nonzero* diagonal entries. The former implies that these five polynomials satisfy the equation

$$c_0(f)\cdot(\rho_0-\rho_1)+c_1(f)\cdot\sum_{F^2}\langle\sum_{i=1}^{n-1}z_i,[F^2]\rangle+\sum_{i=1}^{3}\big[c_{2,i}(f)\cdot\sum_{F^4}\langle\cdots,[F^4]\rangle\big]=0. \tag{4.9}$$

The latter means that, for these polynomials, the 5×5 coefficient matrix in (4.9) is nonsingular, which leads to what we need: if (4.2), (4.3), and (4.4) vanish for five symmetric polynomials of low degrees, then they vanish for all symmetric polynomials.

As dim zero(A) increases, the number of terms we need to consider in $R_f(X,A)$ also increases, and so the problem becomes more and more complicated. For instance, if $r=3$ in (4.1), the symmetric polynomials involved are

$$(y_1^2+y_2^2+y_3^2)\sum_{i=1}^{n-3}z_i,\qquad \sum_{i=1}^{n-3}z_i^3,\qquad \sum z_i^2z_j,\qquad \sum z_iz_jz_k.$$

An efficient method will be developed in the next section.

5. Algebraic preliminaries

5.1. Partition and monomial symmetric polynomial

In this subsection we review briefly some basic facts on partitions and monomial symmetric polynomials, which will be used in Section 6. A standard reference for this material is [**9**, Chapter 1].

A *partition* μ is a finite sequence of positive integers $(\mu_1, \mu_2, \ldots)$ in non-increasing order: $\mu_1 \geqslant \mu_2 \geqslant \cdots$. The total number of parts in μ is called the *length* of μ and is denoted by $l(\mu)$. Thus

$$\mu = (\mu_1, \mu_2, \ldots, \mu_{l(\mu)}), \qquad \mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{l(\mu)} > 0.$$

The *weight* of μ, denoted by $|\mu|$, is defined to be

$$|\mu| := \sum_{i=1}^{l(\mu)} \mu_i.$$

Let $t_1, \ldots, t_n$ be n variables and μ a partition of length $l(\mu) \leqslant n$. We use $m_\mu(t_1, \ldots, t_n)$ to denote the *smallest* symmetric polynomial in the variables $t_1, \ldots, t_n$ that contains

$$t_1^{\mu_1} t_2^{\mu_2} \cdots t_{l(\mu)}^{\mu_{l(\mu)}}.$$

Example 5.1.

1.

$$m_{(1)}(t_1, t_2, t_3) = t_1 + t_2 + t_3, \qquad m_{(2)}(t_1, t_2, t_3) = t_1^2 + t_2^2 + t_3^2,$$

$$m_{(11)}(t_1, t_2, t_3) = t_1 t_2 + t_1 t_3 + t_2 t_3,$$

$$m_{(3)}(t_1, t_2, t_3) = t_1^3 + t_2^3 + t_3^3, \qquad m_{(111)}(t_1, t_2, t_3) = t_1 t_2 t_3,$$

$$m_{(21)}(t_1, t_2, t_3) = t_1^2(t_2 + t_3) + t_2^2(t_1 + t_3) + t_3^2(t_1 + t_2).$$

2. The i-th elementary symmetric polynomial of $t_1, \ldots, t_n$ is $m_{(1,\ldots,1)}(t_1, \ldots, t_n)$.

Using the language of symmetric polynomial theory, $m_\mu(t_1, \ldots, t_n)$ is the *monomial symmetric polynomial* with respect to the partition μ [**9**, p. 18]. If we set the degrees of $t_1, \ldots, t_n$ to be all 1, then $m_\mu(t_1, \ldots, t_n)$ is a homogeneous symmetric polynomial of degree $|\mu|$ and it is well-known that the set

$$\{m_\mu(t_1, \ldots, t_n) \mid l(\mu) \leqslant n, \ |\mu| = k\}$$

forms an additive basis of the vector space of homogeneous symmetric polynomials of degree k

$$\{f(t_1, \ldots, t_n) \mid f \text{ is a homogeneous symmetric polynomial and degree}(f) = k\}.$$

More generally, given any *generic* polynomial $g(t_1, \ldots, t_n) \in \mathbb{C}[t_1, \ldots, t_n]$, we can construct a related symmetric polynomial $\sum g(t_1, \ldots, t_n)$ in $\mathbb{C}[t_1, \ldots, t_n]$ as follows:

$$\sum g(t_1, \ldots, t_n) := \text{the } \textit{smallest} \text{ symmetric polynomial containing } g(t_1, \ldots, t_n).$$

Here by "generic" we mean that for some very special cases this definition may be ill-defined. For example, $\sum(t_1 - t_2)$ and $\sum(t_1 + t_2 - t_3)$ are ill-defined unless we define them to be zero.

Example 5.2.

1. $\sum g(t_1,\dots,t_n)$ is well-defined if $g(t_1,\dots,t_n)\in\mathbb{R}[t_1,\dots,t_n]$ and all the signs before the monomials in $g(t_1,\dots,t_n)$ are positive. In particular, $f_\mu(t_1,\dots,t_n)$ and $g_\mu(t_1,\dots,t_n)$ in (5.1) below are well-defined.
2. $$\sum(t_1-\sqrt{-1}t_2)=m_{(1)}(t_1,\dots,t_n)-\sqrt{-1}m_{(1)}(t_1,\dots,t_n)$$
$$\sum(t_1^2+2t_2)=m_{(2)}(t_1,\dots,t_n)+2m_{(1)}(t_1,\dots,t_n),$$
$$\sum(t_1^2t_2+t_1^2t_3)=m_{(21)}(t_1,\dots,t_n).$$
3. $\sum g(t_1,\dots,t_n)=g(t_1,\dots,t_n)$ if $g(t_1,\dots,t_n)$ itself is a symmetric polynomial.
4. If μ is a partition of length $l(\mu)\leqslant n$, then
$$\sum t_1^{\mu_1}t_2^{\mu_2}\cdots t_{l(\mu)}^{\mu_{l(\mu)}}=m_\mu(t_1,\dots,t_n).$$

5.2. A key lemma

With notation and symbols introduced in the subsection above understood, we now establish the following key lemma related to monomial symmetric polynomials, on which our proof of Theorem 3.2 relies.

Lemma 5.3. Let μ be a partition whose length $l(\mu)\leqslant n$ and λ a fixed nonzero positive constant. Using the notation introduced before Example 5.2, we define two symmetric polynomials in the variables $t_1,\dots,t_n$ related to the partition μ as follows:

$$\begin{aligned} f_\mu(t_1,\dots,t_n)&:=\sum\Big(\prod_{i=1}^{l(\mu)}t_i^{\mu_i}(t_i+\lambda^2)^{2\mu_i+1}\Big),\\ g_\mu(t_1,\dots,t_n)&:=\sum\Big(\prod_{i=1}^{l(\mu)}t_i^{[\frac{\mu_i}{2}]+1}(t_i+\lambda^2)^{\mu_i}\Big),\end{aligned}\tag{5.1}$$

where $[x]$ denotes the largest integer less than or equal to x. We suppose that all the variables $y_1,\dots,y_n$ and $z_1,\dots,z_n$ mentioned below have degree 1. Then for arbitrary nonnegative integer $r\leqslant n$, these f_μ and g_μ satisfy

$$\begin{aligned} &f_\mu\big(y_1^2,\dots,y_r^2,(\sqrt{-1}\lambda+z_1)^2,\dots,(\sqrt{-1}\lambda+z_{n-r})^2\big)\\ &\quad=\begin{cases} c_1(\mu)\cdot m_\mu(y_1^2,\dots,y_r^2)+\text{higher degree monomials}, & \text{if } l(\mu)\leqslant r,\\ \text{sum of some monomials whose degrees are all larger than } 2|\mu|, & \text{if } l(\mu)>r,\end{cases}\end{aligned}\tag{5.2}$$

and

$$\begin{aligned} &g_\mu\big(y_1^2,\dots,y_r^2,(\sqrt{-1}\lambda+z_1)^2,\dots,(\sqrt{-1}\lambda+z_{n-r})^2\big)\\ &\quad=\begin{cases} c_2(\mu)\cdot m_\mu(z_1,\dots,z_{n-r})+\text{higher degree monomials}, & \text{if } l(\mu)\leqslant n-r,\\ \text{sum of some monomials whose degrees are all larger than } |\mu|, & \text{if } l(\mu)>n-r,\end{cases}\end{aligned}\tag{5.3}$$

where $c_1(\mu)$ and $c_2(\mu)$ are two nonzero constants depending on μ. In particular, we have

$$\text{the minimum of the degrees of the monomials in } f_\mu\big(y_1^2,\dots,(\sqrt{-1}\lambda+z_{n-r})^2\big)\geqslant 2|\mu|,$$
$$\text{the minimum of the degrees of the monomials in } g_\mu\big(y_1^2,\dots,(\sqrt{-1}\lambda+z_{n-r})^2\big)\geqslant |\mu|. \tag{5.4}$$

Remark 5.4.

1. If $r=0$ or n, (5.2) is understood to be $f_\mu\big((\sqrt{-1}\lambda+z_1)^2,\dots,(\sqrt{-1}\lambda+z_n)^2\big)$ or $f_\mu(y_1^2,\dots,y_n^2)$. (5.3) is treated similarly.
2. The significance of the definitions of f_μ and f_ν, and of properties (5.2) and (5.3), will become clear in Lemma 6.1 in the next section.

Proof. Suppose y and z are two variables with the same degree 1. First note that

$$t_i^{\mu_i}(t_i+\lambda^2)^{2\mu_i+1}\big|_{t_i=y^2}=\lambda^{4\mu_i+2}\cdot y^{2\mu_i}+\text{higher degree monomials,} \tag{5.5}$$

$$\begin{aligned} t_i^{\mu_i}(t_i+\lambda^2)^{2\mu_i+1}\big|_{t_i=(\sqrt{-1}\lambda+z)^2}&=\sqrt{-1}\cdot 2^{2\mu_i+1}\cdot\lambda^{4\mu_i+1}\cdot z^{2\mu_i+1}\\ &\quad+\text{higher degree monomials.}\end{aligned} \tag{5.6}$$

This means that the lowest degree of the monomials in (5.5) is $2\mu_i$ and therefore is strictly smaller than $2\mu_i+1$, the lowest degree of the monomials in (5.6). So among all the choices

$$\{t_1,\dots,t_{l(u)}\}\subset\{y_1^2,\dots,y_r^2,(\sqrt{-1}\lambda+z_1)^2,\dots,(\sqrt{-1}\lambda+z_{n-r})^2\},$$

the minimum of the lowest degrees in $\prod_{i=1}^{l(\mu)}t_i^{\mu_i}(t_i+\lambda^2)^{2\mu_i+1}$ can be attained exactly when we choose as many $y_1^2,\dots,y_r^2$ for $t_1,\dots,t_{l(\mu)}$ as possible.

If $l(\mu)\leqslant r$, we can choose $\{t_1,\dots,t_{l(u)}\}=\{y_1^2,\dots,y_{l(\mu)}^2\}$ and thus

$$\begin{aligned} f_\mu\big(y_1^2,\dots,y_r^2,(\sqrt{-1}\lambda+z_1)^2,\dots,(\sqrt{-1}\lambda+z_{n-r})^2\big)\\ &=\sum\left(\prod_{i=1}^{l(\mu)}\lambda^{4\mu_i+2}\cdot y^{2\mu_i}+\text{higher degree monomials}\right)\\ &=\lambda^{4|\mu|+2l(\mu)}\cdot m_\mu(y_1^2,\dots,y_r^2)+\text{higher degree monomials.}\end{aligned}$$

If $l(\mu)>r$, at least one of $t_1,\dots,t_{l(\mu)}$ must belong to $\{(\sqrt{-1}\lambda+z_1)^2,\dots,(\sqrt{-1}\lambda+z_{n-r})^2\}$ and so the lowest degree is larger than $2l(\mu)$. This completes the proof of (5.2).

Similarly,

$$t_i^{[\frac{\nu_i}{2}]+1}(t_i+\lambda^2)^{\nu_i}\big|_{t_i=y^2}=\lambda^{2\nu_i}\cdot y^{2[\frac{\nu_i}{2}]+2}+\text{higher degree monomials,}$$

$$t_i^{[\frac{\nu_i}{2}]+1}(t_i+\lambda^2)^{\nu_i}\big|_{t_i=(\sqrt{-1}\lambda+z)^2}=2^{\nu_i}\cdot(\sqrt{-1}\lambda)^{\nu_i+2[\frac{\nu_i}{2}]+2}\cdot z^{\nu_i}+\text{higher degree monomials.}$$

For any positive integer ν_i, we have $2[\frac{\nu_i}{2}]+2>\nu_i$. Thus a similar analysis to that above yields (5.3). □

6. Proof of the main result

Here we only give a detailed proof for the first part of Theorem 3.2. The proof of the second part is similar and technically easier and so we indicate only the minor differences after the proof of the first one.

We divide the proof into two steps.

The first step is to simplify the expression $R_f(X,A)$ under the assumption (3.1).

6.1. Simplification of the expression $R_f(X,A)$

Suppose we have the assumption (3.1). Then, for any symmetric polynomial $f(t_1,\ldots,t_n)$ and any connected component F^{2r} in zero(A), the corresponding expression

$$f\big(y_1^2,\ldots,y_r^2,(\sqrt{-1}\lambda+z_1)^2,\ldots,(\sqrt{-1}\lambda+z_{n-r})^2\big)\cdot\prod_{i=1}^{n-r}(\sqrt{-1}\lambda+z_i)^{-1}$$

in $R_f(X,A)$ can be viewed as a polynomial function of the variables y_i^2 and z_j, which is symmetric with respect to both $y_1^2,\ldots,y_r^2$ and $z_1,\ldots,z_{n-r}$, and so can be written in terms of monomial symmetric polynomials introduced in Section 5 as follows:

$$\begin{aligned}
&f\big(y_1^2,\ldots,y_r^2,(\sqrt{-1}\lambda+z_1)^2,\ldots,(\sqrt{-1}\lambda+z_{n-r})^2\big)\cdot\prod_{i=1}^{n-r}(\sqrt{-1}\lambda+z_i)^{-1}\\
&\qquad=\sum_{\substack{2|\mu|+|\nu|=r\\ l(\mu)\leqslant r,l(\nu)\leqslant n-r}}c(f,r,\mu,\nu)\cdot m_\mu(y_1^2,\ldots,y_r^2)\cdot m_\nu(z_1,\ldots,z_{n-r})\\
&\qquad\qquad+(\text{sum of monomials whose degrees are not equal to } r),
\end{aligned}$$

where $c(f,r,\mu,\nu)$ are constants depending on f, r, and the partitions μ and ν.

Therefore, for arbitrarily fixed f and r, we have

$$\begin{aligned}
&\sum_{\substack{F\subset\text{zero}(A)\\ \dim F=2r}}\langle f\big(y_1^2,\ldots,y_r^2,(\sqrt{-1}\lambda+z_1)^2,\ldots,(\sqrt{-1}\lambda+z_{n-r})^2\big)\cdot\prod_{i=1}^{n-r}(\sqrt{-1}\lambda+z_i)^{-1},\ [F]\rangle\\
&=\sum_{\substack{F\subset\text{zero}(A)\\ \dim F=2r}}\sum_{\substack{2|\mu|+|\nu|=r\\ l(\mu)\leqslant r,l(\nu)\leqslant n-r}}\langle c(f,r,\mu,\nu)\cdot m_\mu(y_1^2,\ldots,y_r^2)\cdot m_\nu(z_1,\ldots,z_{n-r}),\ [F]\rangle\\
&=\sum_{\substack{2|\mu|+|\nu|=r\\ l(\mu)\leqslant r,l(\nu)\leqslant n-r}}\sum_{\substack{F\subset\text{zero}(A)\\ \dim F=2r}}\langle c(f,r,\mu,\nu)\cdot m_\mu(y_1^2,\ldots,y_r^2)\cdot m_\nu(z_1,\ldots,z_{n-r}),\ [F]\rangle\\
&=:\sum_{\substack{2|\mu|+|\nu|=r\\ l(\mu)\leqslant r,l(\nu)\leqslant n-r}}c(f,r,\mu,\nu)\cdot m_\mu m_\nu[\mathfrak{F}^{2r}].
\end{aligned}$$

Here we simplify the notation by setting

$$\sum_{\substack{F\subset\text{zero}(A)\\ \dim F=2r}}\langle m_\mu(y_1^2,\ldots,y_r^2)\cdot m_\nu(z_1,\ldots,z_{n-r}),\ [F]\rangle=:m_\mu m_\nu[\mathfrak{F}^{2r}].$$

Set

$$2r_0 := \min_{F\subset \mathrm{zero}(A)} \dim F, \qquad 2r_1 := \max_{F\subset \mathrm{zero}(A)} \dim F = \dim \mathrm{zero}(A)$$

and, for each $r_0 \leqslant r \leqslant r_1$, we define

$$T(r) := \{(\mu,\nu) \mid 2|\mu| + |\nu| = r,\ l(\mu) \leqslant r,\ l(\nu) \leqslant n-r\}. \tag{6.1}$$

So $R_f(X,A)$ can now be written as follows:

$$R_f(X,A) = \sum_{r=r_0}^{r_1} \sum_{(\mu,\nu)\in T(r)} c(f,r,\mu,\nu)\cdot m_\mu m_\nu[\mathfrak{F}^{2r}]. \tag{6.2}$$

Therefore, in order to establish Theorem 3.2, it suffices to show that

$$m_\mu m_\nu[\mathfrak{F}^{2r}] = 0, \qquad \text{for all pairs } (\mu,\nu) \in \bigcup_{r_0\leqslant r\leqslant r_1} T(r). \tag{6.3}$$

This completes the first step.

6.2. Completion of the proof

Our second step is to make full use of the key Lemma 5.3 established in Section 5. More precisely, we have the following lemma, which explains the reason for defining the two symmetric polynomials f_μ and g_ν and proving the facts (5.2) and (5.3) in Lemma 5.3.

Lemma 6.1. We associate to each pair $(\mu,\nu)\in T(r)$ $(r_0 \leqslant r \leqslant r_1)$ a symmetric polynomial $f_{(\mu,\nu)}(t_1,\ldots,t_n)$ as follows:

$$f_{(\mu,\nu)}(t_1,\ldots,t_n) := f_\mu(t_1,\ldots,t_n)\cdot g_\nu(t_1,\ldots,t_n),$$

where f_μ and g_ν are defined in (5.1). Then this $f_{(\mu,\nu)}(t_1,\ldots,t_n)$ has the following two properties.

1. If each of $t_1,\ldots,t_n$ has the same degree 1, then
$$\mathrm{degree}\big(f_{(\mu,\nu)}(t_1,\ldots,t_n)\big) \leqslant 4|\mu| + 2|\nu| = 2r. \tag{6.4}$$
2. For an arbitrarily chosen pair $(\mu,\nu)\in T(r)$ and connected component $F^{2\tilde r}\subset \mathrm{zero}(A)$, if (as before) we denote by $y_1,\ldots,y_{\tilde r}$ and $z_1,\ldots,z_{n-\tilde r}$ the corresponding characteristic classes of $F^{2\tilde r}$, then we have
$$\begin{aligned}&\left\langle \frac{f_{(\mu,\nu)}(y_1^2,\ldots,y_{\tilde r}^2,(\sqrt{-1}\lambda+z_1)^2,\ldots,(\sqrt{-1}\lambda+z_{n-\tilde r})^2)}{\prod_{i=1}^{n-\tilde r}(\sqrt{-1}\lambda+z_i)},\ [F^{2\tilde r}]\right\rangle \\ &\quad= \begin{cases}0, & \text{if } \tilde r < r,\\ c(\mu,\nu)\cdot\langle m_\mu(y_1^2,\ldots,y_r^2)\cdot m_\nu(z_1,\ldots,z_{n-r}),\ [F^{2r}]\rangle, & \text{if } \tilde r = r,\end{cases}\end{aligned} \tag{6.5}$$
where $c(\mu,\nu)$ is a *nonzero* complex number depending only on μ and ν.

Proof. We first recall from (6.1) that $(\mu,\nu)\in T(r)$ means $2|\mu|+|\nu| = r$, $l(\mu)\leqslant r$, and $l(\nu)\leqslant n-r$.

For (6.4), by definition (5.1) we have

$$\text{degree}\big(f_{(\mu,\nu)}(t_1,\ldots,t_n)\big)=3|\mu|+l(\mu)+|\nu|+l(\nu)+\sum_{i=1}^{l(\nu)}[\frac{\nu_i}{2}].$$

Note that

$$l(\mu)\leqslant|\mu|\qquad\text{and}\qquad l(\nu)+\sum_{i=1}^{l(\nu)}[\frac{\nu_i}{2}]\leqslant|\nu|.$$

Hence

$$\text{degree}\big(f_{(\mu,\nu)}(t_1,\ldots,t_n)\big)\leqslant 4|\mu|+2|\nu|=2r.$$

(6.5) is a direct consequence of Lemma 5.3. Indeed, from (5.4) we obtain

$$f_{\mu}(y_1^2,\ldots,y_{\tilde r}^2,(\sqrt{-1}\lambda+z_1)^2,\ldots,(\sqrt{-1}\lambda+z_{n-\tilde r})^2)\in\bigoplus_{j\geqslant 2|\mu|}H^{2j}(F^{2\tilde r};\mathbb{C}),$$

$$g_{\nu}(y_1^2,\ldots,y_{\tilde r}^2,(\sqrt{-1}\lambda+z_1)^2,\ldots,(\sqrt{-1}\lambda+z_{n-\tilde r})^2)\in\bigoplus_{j\geqslant |\nu|}H^{2j}(F^{2\tilde r};\mathbb{C}),$$

and thus

$$\begin{aligned}&f_{(\mu,\nu)}(y_1^2,\ldots,y_{\tilde r}^2,(\sqrt{-1}\lambda+z_1)^2,\ldots,(\sqrt{-1}\lambda+z_{n-\tilde r})^2)\cdot\prod_{i=1}^{n-\tilde r}(\sqrt{-1}\lambda+z_i)^{-1}\\&\qquad\in\bigoplus_{j\geqslant(2|\mu|+|\nu|)}H^{2j}(F^{2\tilde r};\mathbb{C})=\bigoplus_{j\geqslant r}H^{2j}(F^{2\tilde r};\mathbb{C}),\qquad(2|\mu|+|\nu|=r),\end{aligned}$$

which vanishes if $\tilde r<r$. If $\tilde r=r$, applying the facts that $l(\mu)\leqslant r$ and $l(\nu)\leqslant n-r$ to (5.2) and (5.3) yields the desired result. □

We can now complete the proof of the first part in Theorem 3.2 by establishing (6.3).

Lemma 6.2. Under the assumptions of (1) in Theorem 3.2 we have

$$m_{\mu}m_{\nu}[\mathfrak{F}^{2r}]=0,\qquad\text{for all pairs }\ (\mu,\nu)\in\bigcup_{r_0\leqslant r\leqslant r_1}T(r).$$

Consequently the first part of Theorem 3.2 holds.

Proof. Suppose dim zero$(A)<\frac{n}{2}$ as in the assumptions of Theorem 3.2. For each pair $(\mu,\nu)\in T(r)$ $(r_0\leqslant r\leqslant r_1)$, we know from (6.4) that

$$\text{degree}\big(f_{(\mu,\nu)}(t_1,\ldots,t_n)\big)\leqslant 2r\leqslant 2r_1=\dim\text{zero}(A)<\frac{n}{2}.$$

This, together with the localization theorem 2.1, yields

$$R_{f_{(\mu,\nu)}}(X,A)=0,\qquad\text{for all pairs }\ (\mu,\nu)\in\bigcup_{r_0\leqslant r\leqslant r_1}T(r).\tag{6.6}$$

Applying each pair $(\mu,\nu)\in T(r_1)$ to (6.6), together with (6.5), we deduce that

$$c(\mu,\nu)\cdot m_{\mu}m_{\nu}[\mathfrak{F}^{2r_1}]=0,\qquad\text{for all pairs }\ (\mu,\nu)\in T(r_1).\tag{6.7}$$

Applying each pair $(\mu,\nu)\in T(r)$ with $r_0\leqslant r\leqslant r_1-1$ to (6.6), together with (6.5), we deduce that

$$c(\mu,\nu)\cdot m_\mu m_\nu[\mathfrak{F}^{2r}] + (\text{linear combination of the forms } m_\mu m_\nu[\mathfrak{F}^{2\tilde{r}}] \text{ with } \tilde{r} > r) = 0, \tag{6.8}$$

for all pairs $(\mu,\nu) \in \bigcup_{r_0 \leqslant r \leqslant r_1 - 1} T(r)$.

The facts $c(\mu,\nu) \neq 0$ for all pairs (μ,ν) imply that, under some suitable order, the coefficient matrix of the equations (6.7) and (6.8) is lower triangular and with nonzero diagonal entries and so is nonsingular. This means

$$m_\mu m_\nu[\mathfrak{F}^{2r}] = 0, \qquad \text{for all pairs } (\mu,\nu) \in \bigcup_{r_0 \leqslant r \leqslant r_1} T(r).$$

□

6.3. The proof of the second part in Theorem 3.2

The proof of the second part in Theorem 3.2 is similar but easier to deal with due to the simpler form of its expression. We know from the above analysis that its proof can be reduced to showing that

$$\sum_{\substack{F \subset \mathrm{zero}(A) \\ \dim F = 2r}} \langle m_\mu(y_1,\ldots,y_r)\cdot m_\nu(z_1,\ldots,z_{n-r}),\ [F]\rangle = 0$$

for any pair $(\mu,\nu) \in \bigcup_{r_0 \leqslant r \leqslant r_1} \{(\mu,\nu) \mid |\mu| + |\nu| = r,\ l(\mu) \leqslant r,\ l(\nu) \leqslant n - r\}$.

Similarly, we need to construct a symmetric polynomial $g_{(\mu,\nu)}(t_1,\ldots,t_n)$ as follows:

$$\begin{aligned} g_{(\mu,\nu)}&(t_1,\ldots,t_n) \\ &:= \sum t_1^{\mu_1}(t_1 - \sqrt{-1}\lambda)^{\mu_1+1} t_2^{\mu_2}(t_2 - \sqrt{-1}\lambda)^{\mu_2+1} \cdots t_{l(\mu)}^{\mu_{l(\mu)}}(t_{l(\mu)} - \sqrt{-1}\lambda)^{\mu_{l(\mu)}+1} \\ &\quad \cdot \sum t_1^{\nu_1+1}(t_1 - \sqrt{-1}\lambda)^{\nu_1} t_2^{\nu_2+1}(t_2 - \sqrt{-1}\lambda)^{\nu_2} \cdots t_{l(\nu)}^{\nu_{l(\nu)}+1}(t_{l(\nu)} - \sqrt{-1}\lambda)^{\mu_{l(\nu)}}, \end{aligned}$$

which satisfies

$$\begin{aligned} g_{(\mu,\nu)}&(y_1,\ldots,y_r,\sqrt{-1}\lambda + z_1,\ldots,\sqrt{-1}\lambda + z_{n-r}) \\ &= c(\mu,\nu)\cdot m_\mu(y_1,\ldots,y_r)\cdot m_\nu(z_1,\ldots,z_{n-r}) + \text{higher degree terms}, \end{aligned}$$

with $c(\mu,\nu) \neq 0$.

Also under our assumption that $\dim \mathrm{zero}(A) < \frac{2}{3}n$, we have

$$\begin{aligned} \deg\big(g_{(\mu,\nu)}(t_1,\ldots,t_n)\big) &= 2|\mu| + l(\mu) + 2|\nu| + l(\nu) \\ &\leqslant 3(|\mu| + |\nu|) = 3r \leqslant \frac{3}{2}\dim \mathrm{zero}(A) < n. \end{aligned}$$

The resulting proof is exactly the same as that in the first part.

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