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COHOMOLOGY OF ALGEBRAS OVER WEAK HOPF ALGEBRAS

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Abstract

In this paper we present the Sweedler cohomology for a cocommutative weak Hopf algebra H. We show that the second cohomology group classifies completely weak crossed products, having a common preunit, of H with a commutative left H-module algebra A.

1. Introduction

In [**15**] Sweedler introduced the cohomology of a cocommutative Hopf algebra H with coefficients in a commutative H-module algebra A. We will denote it as Sweedler cohomology $H_{\varphi_A}(H^{\bullet}, A)$, where φ_A is a fixed action of H over A. Two interesting examples are the following: If H is the group algebra kG of a group G and A is an admissible kG-module, the Sweedler cohomology $H^{\bullet}_{\varphi_A}(kG, A)$ is canonically isomorphic to the group cohomology of G in the multiplicative group of invertible elements of A. If H is the enveloping algebra UL of a Lie algebra L, for $i > 1$, the Sweedler cohomology $H^{i}_{\varphi_A}(UL, A)$ is canonically isomorphic to the Lie cohomology of L in the underlying vector space of A. Also, in [**15**] we can find an interesting interpretation of $H^{2}_{\varphi_A}(H, A)$ in terms of extensions: this cohomology group classifies the group of equivalence classes of cleft extensions, i.e., classes of equivalent crossed products determined by a 2-cocycle. This result was extended by Doi [**5**], proving that, in the non-commutative case, there exists a bijection between the isomorphism classes of H-cleft extensions B of A and equivalence classes of crossed systems for H over A. If H is cocommutative, the equivalence is described by $H^{2}_{\varphi_{\mathcal{Z}(A)}}(H, \mathcal{Z}(A))$, where $\mathcal{Z}(A)$ is the center of A.

With the recent rise of weak Hopf algebras, introduced by Böhm, Nill, and Szlachányi [**3**], the notion of crossed product can be adapted to the weak setting. In the Hopf algebra world, crossed products appear as a generalization of semi-direct products of groups to the context of Hopf algebras and are closely connected with cleft extensions and Galois extensions of Hopf algebras [**2**]. In [**4**] Brzeziński gave an interesting approach that generalizes several types of crossed products, even the ones given for braided Hopf algebras by Majid [**12**] and Guccione and Guccione [**8**]. On the other hand, in [**10**] we can find a general and categorical theory, the theory of wreath

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products, which contains as a particular instance the crossed structures presented by Brzeziński.

The key to extending the crossed product constructions presented in the previous paragraph to the weak setting is the use of idempotent morphisms combined with the ideas given in [**4**]. In [**1**] the authors defined a product on $A \otimes V$, for an algebra A and an object V both living in a strict monoidal category $\mathcal{C}$ where every idempotent splits. In order to obtain that product we must consider two morphisms $\psi_V^A : V \otimes A \to A \otimes V$ and $\sigma_V^A : V \otimes V \to A \otimes V$ that satisfy some twisted-like and cocycle-like conditions. Associated to these morphisms it is possible to define an idempotent morphism $\nabla_{A\otimes V} : A \otimes V \to A \otimes V$ and the image of $\nabla_{A\otimes V}$ inherits the associative product from $A \otimes V$. In order to define a unit for $Im(\nabla_{A\otimes V})$, and hence to obtain an algebra structure, we require the existence of a preunit $\nu : K \to A \otimes V$. In [**6**] we can find a characterization of weak crossed products with a preunit as associative products on $A \otimes V$ that are morphisms of left A-modules with preunit. Finally, it is convenient to observe that, if the preunit is a unit, the idempotent becomes the identity and we recover the classical examples of the Hopf algebra setting. The theory presented in [**1, 6**] contains as a particular instance the one developed by Brzeziński in [**4**]. There are many other examples of this theory, such as the theory of wreath products presented in [**10**] and the weak crossed products for weak bialgebras given in [**14**]. Recently, in [**7**] we proved that partial crossed products [**13**] are particular instances of weak crossed products.

Then, if in the Hopf algebra setting the second cohomology group classifies crossed products of H with a commutative left H-module algebra A, what about the weak setting? The answer to this question is the main motivation of this paper. More precisely, we show that if H is a cocommutative weak Hopf algebra and A is a commutative left H-module algebra, all the weak crossed products defined in $A \otimes H$ with a common preunit can be described by the second cohomology group of a new cohomology that we call the Sweedler cohomology of a weak Hopf algebra with coefficients in A.

The paper is organized as follows: In Section 2, after recalling the basic properties of weak Hopf algebras, we introduce the notion of weak H-module algebra and define the cosimplicial complex $Reg_{\varphi_A}(H^\bullet, A)$ for a cocommutative weak Hopf algebra H and a commutative left H-module algebra A. Then, we introduce the Sweedler cohomology of H with coefficients in A as the one defined by the associated cochain complex. In section 3 we present the results about the characterization of weak crossed products induced by morphisms $\sigma \in Reg_{\varphi_A}(H^2, A)$, proving that the twisted and the cocycle conditions of the general theory of weak crossed products can be reduced to twisted 2-cocycle conditions for the morphism σ. Also, in this section we introduce the normal condition that permits us to obtain a preunit in the weak crossed product induced by the morphism σ. Finally, in section 4 we obtain the main result of this paper that assures the following: There is a bijective correspondence between $H^2_{\varphi_A}(H, A)$ and the equivalence classes of weak crossed products of $A \otimes_\alpha H$, where $\alpha : H \otimes H \to A$ satisfies the 2-cocycle and the normal conditions.

2. The Sweedler cohomology in a weak setting

From now on $\mathcal{C}$ denotes a strict symmetric category with tensor product denoted by $\otimes$ and unit object K. With c we will denote the natural isomorphism of symmetry, and we also assume that $\mathcal{C}$ has equalizers. Then, under these conditions, every idempotent morphism $q : Y \to Y$ splits, i.e., there exist an object Z and morphisms $i : Z \to Y$ and $p : Y \to Z$ such that $q = i \circ p$ and $p \circ i = id_Z$. We denote the class of objects of $\mathcal{C}$ by $|\mathcal{C}|$, and for each object $M \in |\mathcal{C}|$, we denote the identity morphism by $id_M : M \to M$. For simplicity of notation, given objects M, N, P in $\mathcal{C}$ and a morphism $f : M \to N$, we write $P \otimes f$ for $id_P \otimes f$ and $f \otimes P$ for $f \otimes id_P$.

We want to point out that there is no loss of generality in assuming that $\mathcal{C}$ is strict, because by Theorem 3.5 of [**9**] (this result implies Mac Lane's coherence theorem) we know that every monoidal category is monoidally equivalent to a strict one usually denoted by $\mathcal{C}^{str}$. Then the results proved in this paper hold for every non-strict monoidal category with equalizers.

We assume that the reader is familiar with the notion of algebra, coalgebra, module, and comodule in a monoidal setting. For an algebra in $\mathcal{C}$, $A = (A, \eta_A, \mu_A)$, $\eta_A : K \to A$ denotes the unit and $\mu_A : A \otimes A \to A$ the product. If A, B are algebras in $\mathcal{C}$, the object $A \otimes B$ is an algebra in $\mathcal{C}$ where $\eta_{A\otimes B} = \eta_A \otimes \eta_B$ and $\mu_{A\otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$. Similarly, for a coalgebra $D = (D, \varepsilon_D, \delta_D)$, $\varepsilon_D : D \to K$ denotes the counit and $\delta_D : D \to D \otimes D$ the coproduct. When D, E are coalgebras in $\mathcal{C}$, $\delta_{D\otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)$ is the coproduct of the coalgebra $D \otimes E$ and $\varepsilon_{D\otimes E} = \varepsilon_D \otimes \varepsilon_E$ its counit. In this paper all algebras are associative and all coalgebras coassociative.

If A is an algebra, B is a coalgebra, and $\alpha : B \to A$, $\beta : B \to A$ are morphisms, we define the convolution product by $\alpha \wedge \beta = \mu_A \circ (\alpha \otimes \beta) \circ \delta_B$.

By weak Hopf algebras we understand the objects introduced in [**3**] as a generalization of ordinary Hopf algebras. Here we recall the definition of these objects in the symmetric monoidal setting.

Definition 2.1. A weak Hopf algebra H is an object in $\mathcal{C}$ with an algebra structure (H, η_H, μ_H) and a coalgebra structure $(H, \varepsilon_H, \delta_H)$ such that the following axioms hold:

(a1) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_{H\otimes H}$.

(a2) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes \delta_H \otimes H)$
$= (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H)$.

(a3) $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$
$= (H \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$.

(a4) There exists a morphism $\lambda_H : H \to H$ in $\mathcal{C}$ (called the antipode of H) satisfying

(a4-1) $id_H \wedge \lambda_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$,

(a4-2) $\lambda_H \wedge id_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$,

(a4-3) $\lambda_H \wedge id_H \wedge \lambda_H = \lambda_H$.

Remark 2.2. If H is a weak Hopf algebra in $\mathcal{C}$, the antipode λ_H is unique, antimultiplicative, and anticomultiplicative, and leaves the unit and the counit invariant:

$$\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}, \quad \delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H, \tag{1}$$

$$\lambda_H \circ \eta_H = \eta_H, \quad \varepsilon_H \circ \lambda_H = \varepsilon_H. \tag{2}$$

If we define the morphisms Π_H^L (target), Π_H^R (source), $\overline{\Pi}_H^L$, and $\overline{\Pi}_H^R$ by

$$\Pi_H^L = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H),$$

$$\Pi_H^R = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)),$$

$$\overline{\Pi}_H^L = (H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\delta_H \circ \eta_H) \otimes H),$$

$$\overline{\Pi}_H^R = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)),$$

it is straightforward to show (see [**3**]) that they are idempotent and Π_H^L, Π_H^R satisfy the equalities

$$\Pi_H^L = id_H \wedge \lambda_H, \quad \Pi_H^R = \lambda_H \wedge id_H, \quad \Pi_H^L \wedge \Pi_H^L = \Pi_H^L, \quad \Pi_H^R \wedge \Pi_H^R = \Pi_H^R. \tag{3}$$

Moreover,

$$\Pi_H^L \circ \overline{\Pi}_H^L = \Pi_H^L, \quad \Pi_H^L \circ \overline{\Pi}_H^R = \overline{\Pi}_H^R, \quad \Pi_H^R \circ \overline{\Pi}_H^L = \overline{\Pi}_H^L, \quad \Pi_H^R \circ \overline{\Pi}_H^R = \Pi_H^R, \tag{4}$$

$$\overline{\Pi}_H^L \circ \Pi_H^L = \overline{\Pi}_H^L, \quad \overline{\Pi}_H^L \circ \Pi_H^R = \Pi_H^R, \quad \overline{\Pi}_H^R \circ \Pi_H^L = \Pi_H^L, \quad \overline{\Pi}_H^R \circ \Pi_H^R = \overline{\Pi}_H^R. \tag{5}$$

For the target and source morphisms we have the following identities:

$$\Pi_H^L \circ \mu_H \circ (H \otimes \Pi_H^L) = \Pi_H^L \circ \mu_H, \quad \Pi_H^R \circ \mu_H \circ (\Pi_H^R \otimes H) = \Pi_H^R \circ \mu_H, \tag{6}$$

$$(H \otimes \Pi_H^L) \circ \delta_H \circ \Pi_H^L = \delta_H \circ \Pi_H^L, \quad (\Pi_H^R \otimes H) \circ \delta_H \circ \Pi_H^R = \delta_H \circ \Pi_H^R, \tag{7}$$

$$\mu_H \circ (H \otimes \Pi_H^L) = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H), \tag{8}$$

$$(H \otimes \Pi_H^L) \circ \delta_H = (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H), \tag{9}$$

$$\mu_H \circ (\Pi_H^R \otimes H) = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H) \tag{10}$$

$$(\Pi_H^R \otimes H) \circ \delta_H = (H \otimes \mu_H) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)) \tag{11}$$

$$\mu_H \circ (\overline{\Pi}_H^R \otimes H) = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes \delta_H), \tag{12}$$

$$\mu_H \circ (H \otimes \overline{\Pi}_H^L) = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (\delta_H \otimes H), \tag{13}$$

$$(\overline{\Pi}_H^L \otimes H) \circ \delta_H = (H \otimes \mu_H) \circ ((\delta_H \circ \eta_H) \otimes H), \tag{14}$$

$$(H \otimes \overline{\Pi}_H^R) \circ \delta_H = (\mu_H \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)), \tag{15}$$

Finally, if H is (co)commutative we have that λ_H is an isomorphism and $\lambda_H^{-1} = \lambda_H$.

Example 2.3. As group algebras and their duals are natural examples of Hopf algebras, groupoid algebras and their duals provide examples of weak Hopf algebras. Recall that a groupoid G is simply a small category in which every morphism is an isomorphism. In this example, we consider finite groupoids, i.e., groupoids with a finite number of objects. The set of objects of G will be denoted by G_0 and the set of morphisms by G_1. The identity morphism on $x \in G_0$ will also be denoted by id_x, and for a morphism $\sigma : x \to y$ in G_1 we write $s(\sigma)$ and $t(\sigma)$, respectively, for the source and the target of σ.

Let G be a groupoid, and let R be a commutative ring. The groupoid algebra is the direct product

$$RG = \bigoplus_{\sigma \in G_1} R\sigma,$$

with the product of two morphisms being equal to their composition if the latter is defined and 0 otherwise, i.e., $\sigma.\tau = \sigma \circ \tau$ if $s(\sigma) = t(\tau)$ and $\sigma.\tau = 0$ if $s(\sigma) \neq t(\tau)$. The unit element is $1_{RG} = \sum_{x \in G_0} id_x$. Then RG is a cocommutative weak Hopf algebra, with coproduct δ_{RG}, counit ε_{RG}, and antipode λ_{RG} given by the formulas $\delta_{RG}(\sigma) = \sigma \otimes \sigma$, $\varepsilon_{RG}(\sigma) = 1$, and $\lambda_{RG}(\sigma) = \sigma^{-1}$. For the weak Hopf algebra RG the target and source morphisms are, respectively, $\Pi^L_{RG}(\sigma) = id_{t(\sigma)}$, $\Pi^R_{RG}(\sigma) = id_{s(\sigma)}$.

Definition 2.4. Let H be a weak Hopf algebra. We will say that A is a weak left H-module algebra if there exists a morphism $\varphi_A : H \otimes A \to A$ satisfying.

(b1) $\varphi_A \circ (\eta_H \otimes A) = id_A$,

(b2) $\varphi_A \circ (H \otimes \mu_A) = \mu_A \circ (\varphi_A \otimes \varphi_A) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes A \otimes A)$,

(b3) $\varphi_A \circ (\mu_H \otimes \eta_A) = \varphi_A \circ (H \otimes (\varphi_A \circ (H \otimes \eta_A)))$,

and any of the following equivalent conditions holds:

(b4) $\varphi_A \circ (\Pi^L_H \otimes A) = \mu_A \circ ((\varphi_A \circ (H \otimes \eta_A) \otimes A)$.

(b5) $\varphi_A \circ (\overline{\Pi}^L_H \otimes A) = \mu_A \circ c_{A,A} \circ ((\varphi_A \circ (H \otimes \eta_A) \otimes A)$.

(b6) $\varphi_A \circ (\Pi^L_H \otimes \eta_A) = \varphi_A \circ (H \otimes \eta_A)$.

(b7) $\varphi_A \circ (\overline{\Pi}^L_H \otimes \eta_A) = \varphi_A \circ (H \otimes \eta_A)$.

If we replace (b3) by
(b3-1) $\varphi_A \circ (\mu_H \otimes A) = \varphi_A \circ (H \otimes \varphi_A)$,
we will say that (A, φ_A) is a left H-module algebra.

Remark 2.5. Let H be a weak Hopf algebra. For $n \geqslant 1$, we denote by H^n the n-fold tensor power $H \otimes \cdots \otimes H$. By H^0 we denote the unit object of $\mathcal{C}$, i.e., $H^0 = K$.

If $n \geqslant 2$, m^n_H denotes the morphism $m^n_H : H^n \to H$ defined by $m^2_H = \mu_H$ and by $m^3_H = m^2_H \circ (H \otimes \mu_H), \cdots, m^n_H = m^{n-1}_H \circ (H^{n-2} \otimes \mu_H)$ for $k > 2$. Note that by the associativity of μ_H we have $m^n_H = m^{n-1}_H \circ (\mu_H \otimes H^{n-2})$.

Let (A, φ_A) be a weak left H-module algebra and $n \geqslant 1$. With φ^n_A we will denote the morphism $\varphi^n_A : H^n \otimes A \to A$ defined as $\varphi^1_A = \varphi_A$ and $\varphi^n_A = \varphi_A \circ (H \otimes \varphi^{n-1}_A)$. If $n > 1$, we have that $\varphi_A \circ (m^n_H \otimes \eta_A) = \varphi^{n-1}_A \circ (H^{n-1} \otimes (\varphi_A \circ (H \otimes \eta_A))$ holds. In what follows, we denote the morphism $\varphi_A \circ (m^n_H \otimes \eta_A)$ by u_n and the morphism $\varphi_A \circ (H \otimes \eta_A)$ by u_1. Note that, by (b3) of Definition 2.4, for $n \geqslant 2$, the equality $u_n = \varphi^{n-1}_A \circ (H^{n-1} \otimes u_1)$ holds.

Finally, with δ_{H^n} we denote the coproduct defined for the coalgebra H^n. Then $\delta_{H^n} = \delta_{H^k \otimes H^{n-k}} = \delta_{H^{n-k} \otimes H^k}$, for $k \in \{1, \ldots, n-1\}$.

Proposition 2.6. *Let H be a cocommutative weak Hopf algebra. The following identities hold:*

(i) $\delta_H \circ \Pi^I_H = (\Pi^I_H \otimes \Pi^I_H) \circ \delta_H$ *for* $I \in \{L, R\}$.

(ii) $(\Pi^I_H \otimes H) \circ \delta_H \circ \Pi^J_H = (H \otimes \Pi^I_H) \circ \delta_H \circ \Pi^J_H = \delta_H \circ \Pi^J_H$, *for* $I, J \in \{L, R\}$.

(iii) $(\Pi^L_H \otimes H) \circ \delta_H \circ \mu_H = (\Pi^L_H \otimes \mu_H) \circ (\delta_H \otimes H)$.

(iv) $(H\otimes \Pi_H^R)\circ\delta_H\circ\mu_H=(\mu_H\otimes\Pi_H^R)\circ(H\otimes\delta_H)$.

Proof. First note that if H is cocommutative, $\Pi_H^I=\overline{\Pi}_H^I$ for $I\in\{L,R\}$. The proof for (i) with $I=L$ follows by

$$\begin{aligned}\delta_H\circ\Pi_H^L&=\mu_{H\otimes H}\circ(\delta_H\otimes(\delta_H\circ\lambda_H))\circ\delta_H\\&=\mu_{H\otimes H}\circ(\delta_H\otimes(c_{H,H}\circ(\lambda_H\otimes\lambda_H)\circ\delta_H))\circ\delta_H\\&=(\Pi_H^L\otimes\Pi_H^L)\circ\delta_H\end{aligned}$$

where the first equality follows by (a1) of Definition 2.1, the second by the anti-multiplicative property of λ_H, and the third one relies on the naturality of c, the coassociativity of δ_H, and the cocommutativity of H.

The proof for $I=R$ is similar.

Note that, by (i) and the idempotent property of Π_H^I, we have (ii) for $I=J$. If $I=L$ and $J=R$, by (4) we have

$$\begin{aligned}(\Pi_H^L\otimes H)\circ\delta_H\circ\Pi_H^R&=((\Pi_H^L\circ\Pi_H^R)\otimes\Pi_H^R)\circ\delta_H=((\Pi_H^L\circ\overline{\Pi}_H^R)\otimes\Pi_H^R)\circ\delta_H\\&=(\overline{\Pi}_H^R\otimes\Pi_H^R)\circ\delta_H=(\Pi_H^R\otimes\Pi_H^R)\circ\delta_H=\delta_H\circ\Pi_H^R.\end{aligned}$$

The proof for $I=R$ and $J=L$ is similar. On the other hand, by the usual arguments, we get (iii):

$$\begin{aligned}(\Pi_H^L\otimes H)\circ\delta_H\circ\mu_H&=(\overline{\Pi}_H^L\otimes H)\circ\delta_H\circ\mu_H=(H\otimes\mu_H)\circ((\delta_H\circ\eta_H)\otimes\mu_H)\\&=(\overline{\Pi}_H^L\otimes\mu_H)\circ(\delta_H\otimes H)=(\Pi_H^L\otimes\mu_H)\circ(\delta_H\otimes H).\end{aligned}$$

The proof of equality (iv) follows a similar pattern and we leave the details to the reader. □

Proposition 2.7. *Let H be a cocommutative weak Hopf algebra. The following identities hold:*

(i) $\delta_{H^2}\circ\delta_H=(\delta_H\otimes\delta_H)\circ\delta_H$.

(ii) $\delta_{H^{n+1}}\circ(H^i\otimes\delta_H\otimes H^{n-i-1})=(H^i\otimes\delta_H\otimes H^{n-1}\otimes\delta_H\otimes H^{n-i-1})\circ\delta_{H^n}$ *for* $n\geqslant 2$ *and* $i\in\{0,\cdots,n-1\}$.

(iii) $\delta_{H^n}\circ(H^i\otimes\Pi_H^I\otimes H^{n-i-1})=(H^i\otimes\Pi_H^I\otimes H^{n-1}\otimes\Pi_H^I\otimes H^{n-i-1})\circ\delta_{H^n}$ *for* $I\in\{L,R\}$, $n\geqslant 2$ *and* $i\in\{0,\cdots,n-1\}$.

(iv) $\delta_{H^{n+1}}\circ(H^i\otimes((\Pi_H^I\otimes H)\circ\delta_H)\otimes H^{n-i-1})$
$=(H^i\otimes((\Pi_H^I\otimes H)\circ\delta_H)\otimes H^{n-1}\otimes((\Pi_H^I\otimes H)\circ\delta_H)\otimes H^{n-i-1})\circ\delta_{H^n}$ *for* $I\in\{L,R\}$, $n\geqslant 2$ *and* $i\in\{0,\cdots,n-1\}$.

(v) $\delta_{H^{n+1}}\circ(H^i\otimes((H\otimes\Pi_H^I)\circ\delta_H)\otimes H^{n-i-1})$
$=(H^i\otimes((H\otimes\Pi_H^I)\circ\delta_H)\otimes H^{n-1}\otimes((H\otimes\Pi_H^I)\circ\delta_H)\otimes H^{n-i-1})\circ\delta_{H^n}$ *for* $I\in\{L,R\}$, $n\geqslant 2$ *and* $i\in\{0,\cdots,n-1\}$.

Proof. Assertion (i) follows by the coassociativity of δ_H and the cocommutativity of H. The proof for (ii) can be obtained using (i) and mathematical induction. Also, by this method and Proposition 2.6 we obtain (iii), (iv), and (v). □

Remark 2.8. If H is a weak Hopf algebra, we denote by H_L the object such that $p_L\circ i_L=id_{H_L}$, where i_L, p_L are the injection and the projection associated to the target morphism Π_H^L. If H is cocommutative, by Proposition 2.6(i) we have that H_L is a coalgebra and the morphisms i_L, p_L are coalgebra morphisms for $\delta_{H_L} = (p_L \otimes p_L) \circ \delta_H \circ i_L$ and $\varepsilon_{H_L} = \varepsilon_H \circ i_L$. Therefore, $\delta_{H_L} \circ p_L = (p_L \otimes p_L) \circ \delta_H$ and $\varepsilon_{H_L} \circ p_L = \varepsilon_H$.

Proposition 2.9. *Let H be a weak Hopf algebra. Then if $n \geqslant 3$ the following equality holds:*

$$(H^{i-1} \otimes \mu_H \otimes H^{n-i-1} \otimes H^{i-1} \otimes \mu_H \otimes H^{n-i-1}) \circ \delta_{H^n} = \delta_{H^{n-1}} \circ (H^{i-1} \otimes \mu_H \otimes H^{n-i-1}), \tag{16}$$

for all $i \in \{1, \cdots, n-1\}$.

Proof. First note that, by (a1) of Definition 2.1, we have that $(\mu_H \otimes \mu_H) \circ \delta_{H^2} = \delta_H \circ \mu_H$. Then, using this identity, we obtain $(\mu_H \otimes H^{n-2} \otimes \mu_H \otimes H^{n-2}) \circ \delta_{H^n} = \delta_{H^{n-1}} \circ (\mu_H \otimes H^{n-2})$ and, as a consequence, (16) holds. □

Proposition 2.10. *Let H be a weak Hopf algebra. The following identity holds for $n \geqslant 2$:*

$$\delta_H \circ m_H^n = (m_H^n \otimes m_H^n) \circ \delta_{H^n}. \tag{17}$$

Proof. As in the previous proposition we proceed by induction. Obviously the equality (17) holds for $n = 2$. If we assume that it is true for $n = k$, so it is for $n = k + 1$ because

$$\begin{aligned}(m_H^{k+1} \otimes m_H^{k+1}) \circ \delta_{H^{k+1}} &= ((\mu_H \circ (m_H^k \otimes H)) \otimes (\mu_H \circ (m_H^k \otimes H))) \circ \delta_{H^k \otimes H} \\ &= \mu_{H \otimes H} \circ (((m_H^k \otimes m_H^k) \circ \delta_{H^k}) \otimes \delta_H) = \delta_H \circ \mu_H \circ (m_H^k \otimes H) = \delta_H \circ m_H^{k+1}.\end{aligned}$$

□

Proposition 2.11. *Let H be a weak Hopf algebra, and let (A, φ_A) be a weak left H-module algebra. Then, if $n \geqslant 1$, the equality*

$$u_n \wedge u_n = u_n \tag{18}$$

holds.

Proof. If $n \geqslant 2$, by (17) and (b2) of Definition 2.4 we obtain (18). If $n = 1$, the equality follows from (b2). □

Definition 2.12. Let H be a cocommutative weak Hopf algebra, and let (A, φ_A) be a weak left H-module algebra. For $n \geqslant 1$, with $Reg_{\varphi_A}(H^n, A)$ we will denote the set of morphisms $\sigma : H^n \to A$ such that there exists a morphism $\sigma^{-1} : H^n \to A$ (the convolution inverse of σ) satisfying the following equalities:

(c1) $\sigma \wedge \sigma^{-1} = \sigma^{-1} \wedge \sigma = u_n$.
(c2) $\sigma \wedge \sigma^{-1} \wedge \sigma = \sigma$.
(c3) $\sigma^{-1} \wedge \sigma \wedge \sigma^{-1} = \sigma^{-1}$.

We denote by $Reg_{\varphi_A}(H_L, A)$ the set of morphisms $g : H_L \to A$ such that there exists a morphism $g^{-1} : H_L \to A$ (the convolution inverse of g) satisfying

$$g \wedge g^{-1} = g^{-1} \wedge g = u_0, \;\; g \wedge g^{-1} \wedge g = g, \;\; g^{-1} \wedge g \wedge g^{-1} = g^{-1},$$

where $u_0 = u_1 \circ i_L$. Then, by (b7) of the definition of weak H-module algebra, we have $u_1 = u_0 \circ p_L$.

Proposition 2.13. *Let H be a cocommutative weak Hopf algebra and let (A, φ_A) be a weak left H-module algebra. Then for all $\sigma \in Reg_{\varphi_A}(H^{n+1}, A)$ the following equalities hold:*

(i) $\sigma \circ (H^i \otimes ((\Pi_H^L \otimes H) \circ \delta_H) \otimes H^{n-i-1}) = \sigma \circ (H^i \otimes \eta_H \otimes H^{n-i})$ *for all i in the set $\{0, \dots, n-1\}$.*

(ii) $\sigma \circ (H^{n-1} \otimes ((H \otimes \Pi_H^R) \circ \delta_H)) = \sigma \circ (H^n \otimes \eta_H)$.

Proof. First note that if $\sigma \in Reg_{\varphi_A}(H^{n+1}, A)$, by Proposition 2.7(iv) and the equality $\Pi_H^L \wedge id_H = id_H$, we obtain that

$$\sigma \circ (H^i \otimes ((\Pi_H^L \otimes H) \circ \delta_H) \otimes H^{n-i-1}) \in Reg_{\varphi_A}(H^n, A)$$

with inverse $\sigma^{-1} \circ (H^i \otimes ((\Pi_H^L \otimes H) \circ \delta_H) \otimes H^{n-i-1})$. Moreover, by the naturally of c and the equality (9), we obtain (i). The proof for (ii) is similar using (11) and we leave the details to the reader. □

Remark 2.14. Let H be a cocommutative weak Hopf algebra, and let (A, φ_A) be a weak left H-module algebra. Then $u_0 \in Reg_{\varphi_A}(H_L, A)$, $u_n \in Reg_{\varphi_A}(H^n, A)$ and the sets $Reg_{\varphi_A}(H_L, A)$, $Reg_{\varphi_A}(H^n, A)$ are groups with neutral elements u_0 and u_n, respectively. Also, if A is commutative, then we have that $Reg_{\varphi_A}(H_L, A)$ and $Reg_{\varphi_A}(H^n, A)$ are abelian groups.

If (A, φ_A) is a left H-module algebra, the groups $Reg_{\varphi_A}(H_L, A)$, $Reg_{\varphi_A}(H^n, A)$, $n \geqslant 1$ are the objects of a cosimplicial complex of groups with coface operators defined by

$$\partial_{0,i} : Reg_{\varphi_A}(H_L, A) \to Reg_{\varphi_A}(H, A), \;\; i \in \{0, 1\}$$

$$\partial_{0,0}(g) = \varphi_A \circ (H \otimes (g \circ p_L \circ \Pi_H^R)) \circ \delta_H, \;\; \partial_{0,1}(g) = g \circ p_L,$$

$$\partial_{k-1,i} : Reg_{\varphi_A}(H^{k-1}, A) \to Reg_{\varphi_A}(H^k, A), \;\; k \geqslant 2, \;\; i \in \{0, 1, \cdots, k\}$$

$$\partial_{k-1,i}(\sigma) = \begin{cases} \varphi_A \circ (H \otimes \sigma), & i = 0 \\ \sigma \circ (H^{i-1} \otimes \mu_H \otimes H^{k-i-1}), & i \in \{1, \cdots, k-1\} \\ \sigma \circ (H^{k-2} \otimes (\mu_H \circ (H \otimes \Pi_H^L))), & i = k, \end{cases}$$

and codegeneracy operators defined by $s_{1,0} : Reg_{\varphi_A}(H, A) \to Reg_{\varphi_A}(H_L, A)$,

$$s_{1,0}(h) = h \circ i_L,$$

and $s_{k+1,i} : Reg_{\varphi_A}(H^{k+1}, A) \to Reg_{\varphi_A}(H^k, A), \;\; k \geqslant 1, \;\; i \in \{0, 1, \cdots, k\}$

$$s_{k+1,i}(\sigma) = \sigma \circ (H^i \otimes \eta_H \otimes H^{k-i}).$$

The morphism $\partial_{0,0}$, is a well-defined group morphism because

$$\begin{aligned} &\partial_{0,0}(g) \wedge \partial_{0,0}(f) \\ &\quad = \mu_A \circ ((\varphi_A \circ (H \otimes (g \circ p_L \circ \Pi_H^R))) \otimes (\varphi_A \circ (H \otimes (f \circ p_L \circ \Pi_H^R)))) \circ \delta_{H^2} \circ \delta_H \\ &\quad = \varphi_A \circ (H \otimes ((g \circ p_L \circ \Pi_H^R) \wedge (f \circ p_L \circ \Pi_H^R))) \circ \delta_H \\ &\quad = \varphi_A \circ (H \otimes (((g \circ p_L) \wedge (f \circ p_L)) \circ \Pi_H^R)) \circ \delta_H \\ &\quad = \partial_{0,0}(g \wedge f), \end{aligned}$$

where the first equality follows by Proposition 2.7(i), the second one by the naturality of c and (b2) of Definition 2.4, the third one by (i) of Proposition 2.6, and in the last one was used that p_L is a coalgebra morphism (see Remark 2.8).

Using that p_L is a coalgebra morphism, we obtain that $\partial_{0,1}$ is a group morphism. Moreover, by (b2) of Definition 2.4, (a1) of Definition 2.1, Proposition 2.9, and Proposition 2.6(i), we have that $\partial_{k-1,i}$ are well-defined group morphisms for $k \geqslant 1$.

On the other hand, by Proposition 2.6(i) we have that $s_{1,0}$ is a group morphism, and by Propositions 2.6 and 2.13 we obtain that $s_{k+1,i}$ are well-defined group morphisms for $k \geqslant 1$.

We have the cosimplicial identities from the following: For $j = 1$, by Proposition 2.6(iv) and the condition of left H-module algebra for A, we have $\partial_{1,1}(\partial_{0,0}(g)) = \partial_{1,0}(\partial_{0,0}(g))$. Moreover, if H is cocommutative, $\Pi_H^L = \overline{\Pi}_H^L$ and as a consequence $\Pi_H^L \circ \Pi_H^R = \Pi_H^L$. Then by Proposition 2.6(i) and (iv) and the properties of left H-module algebra we get $\partial_{1,2}(\partial_{0,0}(g)) = \partial_{1,0}(\partial_{0,1}(g))$. Also, by (6) we obtain that $\partial_{1,2}(\partial_{0,1}(g)) = \partial_{1,1}(\partial_{1,0}(g))$. In a similar way, by the associativity of μ_H, $\partial_{k,j} \circ \partial_{k-1,i} = \partial_{k,i} \circ \partial_{k-1,j-1}$ holds for $j > i$ and $k > 1$.

On the other hand, trivially $s_{k-1,j} \circ s_{k,i} = s_{k-1,i} \circ s_{k,j+1}, j \geqslant i$. Moreover, it is easy to show that $s_{1,0}(\partial_{0,0}(g)) = g = s_{1,0}(\partial_{0,1}(g))$. Also, we have $s_{2,0}(\partial_{1,0}(h)) = h = s_{2,0}(\partial_{1,1}(h))$, $s_{2,0}(\partial_{1,2}(h)) = h \circ \Pi_H^L = \partial_{0,1}(s_{1,0}(h))$,

$$s_{2,1}(\partial_{1,0}(h)) = \varphi_A \circ (H \otimes (h \circ \Pi_H^L \circ \Pi_H^R)) \circ \delta_H = \partial_{0,0}(s_{1,0}(h)),$$

and $s_{2,1}(\partial_{1,1}(h)) = h = s_{2,1}(\partial_{1,2}(h))$ because $\Pi_H^L \circ \eta_H = \eta_H$.

Finally, for $k > 2$, the identities

$$s_{k+1,j} \circ \partial_{k,i} = \begin{cases} \partial_{k-1,i} \circ s_{k,j-1}, & i < j \\ id_{Reg_{\varphi_A}(H^k,A)}, & i = j, \ i = j+1 \\ \partial_{k-1,i-1} \circ s_{k,j}, & i > j+1 \end{cases}$$

follow as in the Hopf algebra setting.

Let $D_{\varphi_A}^k = \partial_{k,0} \wedge \partial_{k,1}^{-1} \wedge \cdots \wedge \partial_{k,k+1}^{(-1)^{k+1}}$ be the coboundary morphisms of the cochain complex

$$Reg_{\varphi_A}(H_L, A) \xrightarrow{D_{\varphi_A}^0} Reg_{\varphi_A}(H, A) \xrightarrow{D_{\varphi_A}^1} Reg_{\varphi_A}(H^2, A) \xrightarrow{D_{\varphi_A}^2} \cdots$$
$$\cdots \xrightarrow{D_{\varphi_A}^{k-1}} Reg_{\varphi_A}(H^k, A) \xrightarrow{D_{\varphi_A}^k} Reg_{\varphi_A}(H^{k+1}, A) \xrightarrow{D_{\varphi_A}^{k+1}} \cdots$$

associated to the cosimplicial complex $Reg_{\varphi_A}(H^\bullet, A)$.

Then, when (A, φ_A) is a commutative left H-module algebra, $(Reg_{\varphi_A}(H^\bullet, A), D_{\varphi_A}^\bullet)$ is a cosimplicial complex in the category of abelian groups that gives the Sweedler cohomology of H in (A, φ_A). The kth group will be defined by

$$\frac{Ker(D_{\varphi_A}^k)}{Im(D_{\varphi_A}^{k-1})}$$

for $k \geqslant 1$ and $Ker(D_{\varphi_A}^0)$ for $k = 0$. We will denote it by $H_{\varphi_A}^k(H, A)$.

The normalized cochain subcomplex of $(Reg_{\varphi_A}(H^\bullet, A), D_{\varphi_A}^\bullet)$ is defined by

$$Reg_{\varphi_A}^+(H^{k+1}, A) = \bigcap_{i=0}^{k} Ker(s_{k+1,i}),$$

$$Reg^{+}_{\varphi_A}(H_L, A) = Reg_{\varphi_A}(H_L, A)$$

and $D^{k+}_{\varphi_A}$ the restriction of $D^{k}_{\varphi_A}$ to $Reg^{+}_{\varphi_A}(H^{\bullet}, A)$.

We have that $(Reg^{+}_{\varphi_A}(H^{\bullet}, A), D^{\bullet+}_{\varphi_A})$ is a subcomplex of $(Reg_{\varphi_A}(H^{\bullet}, A), D^{\bullet}_{\varphi_A})$ and the injection map induces an isomorphism of cohomology (see [**11**] for the dual result). Note that

$$Reg^{+}_{\varphi_A}(H, A) = Ker(s_{1,0}) = \{h \in Reg_{\varphi_A}(H, A)\ ;\ h \circ i_L = u_0\}$$

and

$$\begin{aligned} Reg^{+}_{\varphi_A}(H^2, A) &= Ker(s_{2,0}) \cap Ker(s_{2,1}) \\ &= \{\sigma \in Reg_{\varphi_A}(H^2, A)\ ;\ \sigma \circ (\eta_H \otimes H) = \sigma \circ (H \otimes \eta_H) = u_1\}. \end{aligned}$$

The following proposition gives another characterization of the morphisms in the group $Reg^{+}_{\varphi_A}(H, A)$.

Proposition 2.15. *Let H be a weak Hopf algebra, and let (A, φ_A) be a weak left H-module algebra. If $h : H \to A$ is a morphism satisfying*

$$h \wedge h^{-1} = h^{-1} \wedge h = u_1,\ \ h \wedge h^{-1} \wedge h = h,\ \ h^{-1} \wedge h \wedge h^{-1} = h^{-1},$$

the following equalities are equivalent:

(i) $h \circ \eta_H = \eta_A$.

(ii) $h \circ \Pi^L_H = u_1$.

Proof. The assertion (ii) $\Rightarrow$ (i) follows by $h \circ \eta_H = h \circ \Pi^L_H \circ \eta_H = u_1 \circ \eta_H = \eta_A$. Now we get (i) $\Rightarrow$ (ii) because

$$\begin{aligned} h \circ \Pi^L_H &= (u_1 \wedge h) \circ \Pi^L_H \\ &= \mu_A \circ (u_2 \otimes h) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes \Pi^L_H) \\ &= \mu_A \circ (\varphi_A \otimes h) \circ (H \otimes c_{H,A}) \circ ((\delta_H \circ \eta_H) \otimes u_1) \\ &= \mu_A \circ (\varphi_A \otimes h) \circ (\overline{\Pi}^L_H \otimes c_{H,A}) \circ ((\delta_H \circ \eta_H) \otimes u_1) \\ &= \mu_A \circ ((\mu_A \circ c_{A,A} \circ (u_1 \otimes A)) \otimes h) \circ (H \otimes c_{H,A}) \circ ((\delta_H \circ \eta_H) \otimes u_1) \\ &= u_1. \end{aligned}$$

The first equality follows by the properties of h, the second one by the naturality of c and the coassociativity of δ_H and (8), the third one by (b3) and (b6) of Definition 2.4, the fourth one by (14), the fifth one by (b5) of Definition 2.4, and the last one by the properties of h and (ii). $\square$

Remark 2.16. Note that as a consequence of Proposition 2.15:

$$Reg^{+}_{\varphi_A}(H, A) = \{h \in Reg_{\varphi_A}(H, A)\ ;\ h \circ \eta_H = \eta_A\}.$$

3. Weak crossed products for weak Hopf algebras

In the first paragraphs of this section we recall some basic facts about the general theory of weak crossed products in $\mathcal{C}$ introduced in [**6**] particularized for a weak Hopf algebra H.

Let A be an algebra, and let H be a weak Hopf algebra in $\mathcal{C}$. Suppose that there exists a morphism $\psi_H^A : H \otimes A \to A \otimes H$ such that the following equality holds:

$$(\mu_A \otimes H) \circ (A \otimes \psi_H^A) \circ (\psi_H^A \otimes A) = \psi_H^A \circ (H \otimes \mu_A). \tag{19}$$

As a consequence of (19), the morphism $\nabla_{A\otimes H} : A \otimes H \to A \otimes H$ defined by

$$\nabla_{A\otimes H} = (\mu_A \otimes H) \circ (A \otimes \psi_H^A) \circ (A \otimes H \otimes \eta_A) \tag{20}$$

is idempotent. Moreover, it satisfies $\nabla_{A\otimes H} \circ (\mu_A \otimes H) = (\mu_A \otimes H) \circ (A \otimes \nabla_{A\otimes H})$, i.e., $\nabla_{A\otimes H}$ is a left A-module morphism (see Lemma 3.1 of [**6**]) for the regular action $\varphi_{A\otimes H} = \mu_A \otimes H$. With $A \times H$, $i_{A\otimes H} : A \times H \to A \otimes H$, and $p_{A\otimes H} : A \otimes H \to A \times H$ we denote the object, the injection, and the projection associated to the factorization of $\nabla_{A\otimes H}$. Finally, if ψ_H^A satisfies (19), the following identities hold:

$$\begin{aligned}(\mu_A \otimes H) \circ (A \otimes \psi_H^A) \circ (\nabla_{A\otimes H} \otimes A) &= (\mu_A \otimes H) \circ (A \otimes \psi_H^A) \\ &= \nabla_{A\otimes H} \circ (\mu_A \otimes H) \circ (A \otimes \psi_H^A).\end{aligned} \tag{21}$$

From now on we consider quadruples $\mathbb{A}_H = (A, H, \psi_H^A, \sigma_H^A)$, where A is an algebra, H an object, $\psi_H^A : H \otimes A \to A \otimes H$ a morphism satisfiying (19), and $\sigma_H^A : H \otimes H \to A \otimes H$ a morphism in $\mathcal{C}$.

We say that $\mathbb{A}_H = (A, H, \psi_H^A, \sigma_H^A)$ satisfies the twisted condition if

$$(\mu_A \otimes H) \circ (A \otimes \psi_H^A) \circ (\sigma_H^A \otimes A) = (\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\psi_H^A \otimes H) \circ (H \otimes \psi_H^A) \tag{22}$$

and that the cocycle condition holds if

$$(\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\sigma_H^A \otimes H) = (\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\psi_H^A \otimes H) \circ (H \otimes \sigma_H^A). \tag{23}$$

Note that, if $\mathbb{A}_H = (A, H, \psi_H^A, \sigma_H^A)$ satisfies the twisted condition, in Proposition 3.4 of [**6**] we prove that

$$\begin{aligned}&(\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\psi_H^A \otimes H) \circ (H \otimes \nabla_{A\otimes H}) \\ &\qquad = \nabla_{A\otimes H} \circ (\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\psi_H^A \otimes H),\end{aligned} \tag{24}$$

$$\nabla_{A\otimes H} \circ (\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\nabla_{A\otimes H} \otimes H) = \nabla_{A\otimes H} \circ (\mu_A \otimes H) \circ (A \otimes \sigma_H^A). \tag{25}$$

Then, if $\nabla_{A\otimes H} \circ \sigma_H^A = \sigma_H^A$, we obtain

$$(\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\psi_H^A \otimes H) \circ (H \otimes \nabla_{A\otimes H}) = (\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\psi_H^A \otimes H), \tag{26}$$

$$(\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\nabla_{A\otimes H} \otimes H) = (\mu_A \otimes H) \circ (A \otimes \sigma_H^A). \tag{27}$$

In what follows, and taking into account (22) and (23), we will consider without loss of generality that $\nabla_{A\otimes H} \circ \sigma_H^A = \sigma_H^A$ holds for all quadruples $\mathbb{A}_H = (A, H, \psi_H^A, \sigma_H^A)$ (see Proposition 3.7 of [**6**]).

For $\mathbb{A}_H = (A, H, \psi_H^A, \sigma_H^A)$ define the associated product

$$\mu_{A\otimes H} = (\mu_A \otimes H) \circ (\mu_A \otimes \sigma_H^A) \circ (A \otimes \psi_H^A \otimes H), \tag{28}$$

and let $\mu_{A\times H}$ be the restriction of $\mu_{A\otimes H}$ to $A \times H$, i.e.,

$$\mu_{A\times H} = p_{A\otimes H} \circ \mu_{A\otimes H} \circ (i_{A\otimes H} \otimes i_{A\otimes H}). \tag{29}$$

If the twisted and the cocycle conditions hold, the product $\mu_{A\otimes H}$ is associative and normalized with respect to $\nabla_{A\otimes H}$ (i.e., $\nabla_{A\otimes H}\circ\mu_{A\otimes H}=\mu_{A\otimes H}=\mu_{A\otimes H}\circ(\nabla_{A\otimes H}\otimes\nabla_{A\otimes H})$) and, by the definition of $\mu_{A\otimes H}$, the equality $\mu_{A\otimes H}\circ(\nabla_{A\otimes H}\otimes A\otimes H)=\mu_{A\otimes H}$ holds and therefore $\mu_{A\otimes H}\circ(A\otimes H\otimes\nabla_{A\otimes H})=\mu_{A\otimes H}$. Due to the normality condition, $\mu_{A\times H}$ is associative as well (Proposition 2.5 of [**6**]). Hence we have the following definition:

Definition 3.1. If $\mathbb{A}_H=(A,H,\psi_H^A,\sigma_H^A)$ satisfies (22) and (23), we say that the pair $(A\otimes H,\mu_{A\otimes H})$ is a weak crossed product.

The next natural question that arises at this point is if it is possible to endow $A\times H$ with a unit, and hence with an algebra structure. As we recall in [**6**], in order to do that we need to use the notion of preunit to obtain an unit in $A\times H$. In our setting, if A is an algebra, H an object in $\mathcal{C}$, and $m_{A\otimes H}$ is an associative product defined in $A\otimes H$, a preunit $\nu:K\to A\otimes H$ is a morphism satisfying

$$m_{A\otimes H}\circ(A\otimes H\otimes\nu)=m_{A\otimes H}\circ(\nu\otimes A\otimes H)=m_{A\otimes H}\circ(A\otimes H\otimes(m_{A\otimes H}\circ(\nu\otimes\nu))). \tag{30}$$

As we have shown in [**6**], if $(A\otimes H,\mu_{A\otimes H})$ is a weak crossed product with a preunit ν such that the equalities

$$(\mu_A\otimes H)\circ(A\otimes\sigma_H^A)\circ(\psi_H^A\otimes H)\circ(H\otimes\nu)=\nabla_{A\otimes H}\circ(\eta_A\otimes H), \tag{31}$$

$$(\mu_A\otimes H)\circ(A\otimes\sigma_H^A)\circ(\nu\otimes H)=\nabla_{A\otimes H}\circ(\eta_A\otimes H), \tag{32}$$

$$(\mu_A\otimes H)\circ(A\otimes\psi_H^A)\circ(\nu\otimes A)=(\mu_A\otimes H)\circ(A\otimes\nu) \tag{33}$$

hold, then $A\times H$ is an algebra with the product defined in (29) and unit $\eta_{A\times H}=p_{A\otimes H}\circ\nu$.

Definition 3.2. Let H be a weak Hopf algebra, let (A,φ_A) be a weak left H-module algebra, and let $\sigma:H\otimes H\to A$ be a morphism. We define the morphisms $\psi_H^A:H\otimes A\to A\otimes H$ and $\sigma_H^A:H\otimes H\to A\otimes H$ by $\psi_H^A=(\varphi_A\otimes H)\circ(H\otimes c_{H,A})\circ(\delta_H\otimes A)$ and $\sigma_H^A=(\sigma\otimes\mu_H)\circ\delta_{H^2}$.

Proposition 3.3. *Let H be a weak Hopf algebra, and let (A,φ_A) be a weak left H-module algebra. The morphism ψ_H^A defined above satisfies (19). As a consequence, the morphism $\nabla_{A\otimes H}$, defined in (20), is an idempotent and the following equalities hold:*

$$\nabla_{A\otimes H}=((\mu_A\circ(A\otimes u_1))\otimes H)\circ(A\otimes\delta_H), \tag{34}$$

$$\mu_A\circ(u_1\otimes\varphi_A)\circ(\delta_H\otimes A)=\varphi_A, \tag{35}$$

$$(\mu_A\otimes H)\circ(u_1\otimes\psi_H^A)\circ(\delta_H\otimes A)=\psi_H^A, \tag{36}$$

$$(A\otimes\varepsilon_H)\circ\psi_H^A\circ(H\otimes\eta_A)=u_1, \tag{37}$$

$$(\mu_A\otimes H)\circ(u_1\otimes c_{H,A})\circ(\delta_H\otimes A)=(\mu_A\otimes H)\circ(A\otimes c_{H,A})\circ((\psi_H^A\circ(H\otimes\eta_A))\otimes A), \tag{38}$$

$$(A\otimes\varepsilon_H)\circ\nabla_{A\otimes H}=\mu_A\circ(A\otimes u_1), \tag{39}$$

$$(A\otimes\delta_H)\circ\nabla_{A\otimes H}=(\nabla_{A\otimes H}\otimes H)\circ(A\otimes\delta_H). \tag{40}$$

Proof. First note that, by the naturality of c, the coassociativity of δ_H, and (b2) of Definition (2.4), we obtain that ψ_H^A satisfies (19). As a consequence, $\nabla_{A\otimes H}$ is an idempotent and (34), (37), and (39) follow easily from the definition of ψ_H^A. On the other hand, (35) follows by (34) and (b2) of Definition 2.4. Analogously, by (b2) of

Definition 2.4 we obtain (36). Finally, the equality (40) follows from (34) and the coassociativity of δ_H, and (38) is an easy consequence of the naturality of c. □

Proposition 3.4. *Let H be a weak Hopf algebra, let (A, φ_A) be a weak left H-module algebra, and let $\sigma : H \otimes H \to A$ be a morphism. The morphism σ_H^A introduced in Definition 3.2 satisfies the following identity:*

$$(A \otimes \delta_H) \circ \sigma_H^A = (\sigma_H^A \otimes \mu_H) \circ \delta_{H^2}. \tag{41}$$

Proof. The proof is an easy consequence of (a1) of Definition 2.1, the coassociativity of δ_H, and the naturality of c. □

Proposition 3.5. *Let H be a cocommutative weak Hopf algebra, let (A, φ_A) be a weak left H-module algebra, and let $\sigma \in Reg_{\varphi_A}(H^2, A)$. The morphism σ_H^A introduced in Definition 3.2 satisfies the following identities:*

(i) $\nabla_{A\otimes H} \circ \sigma_H^A = \sigma_H^A$.

(ii) $(A \otimes \varepsilon_H) \circ \sigma_H^A = \sigma$.

Proof. By Proposition 3.4 and the properties of σ, we have that

$$\begin{aligned}\nabla_{A\otimes H} \circ \sigma_H^A &= ((\mu_A \circ (A \otimes u_1)) \otimes H) \circ (A \otimes \delta_H) \circ \sigma_H^A \\ &= ((\mu_A \circ (A \otimes u_1)) \otimes H) \circ (\sigma_H^A \otimes \mu_H) \circ \delta_{H^2} = ((\sigma \wedge \sigma^{-1} \wedge \sigma) \otimes \mu_H) \circ \delta_{H^2} = \sigma_H^A,\end{aligned}$$

(ii) follows by (39) and (i) because

$$(A \otimes \varepsilon_H) \circ \sigma_H^A = (A \otimes \varepsilon_H) \circ \nabla_{A\otimes H} \circ \sigma_H^A = \mu_A \circ (A \otimes u_1) \circ \sigma_H^A = \sigma \wedge u_2 = \sigma. \quad □$$

Remark 3.6. Let H be a cocommutative weak Hopf algebra, let (A, φ_A) be a weak left H-module algebra, and let $\sigma \in Reg_{\varphi_A}(H^2, A)$. Note that, by Propositions 3.3, 3.4, and 3.5, we have a quadruple $\mathbb{A}_H = (A, H, \psi_H^A, \sigma_H^A)$ such that ψ_H^A satisfies (19) and $\nabla_{A\otimes H} \circ \sigma_H^A = \sigma_H^A$.

Definition 3.7. Let H be a cocommutative weak Hopf algebra, let (A, φ_A) be a weak left H-module algebra, and let $\sigma \in Reg_{\varphi_A}(H^2, A)$. We say that σ satisfies the twisted condition if

$$\begin{aligned}\mu_A \circ ((\varphi_A \circ (H \otimes \varphi_A)) \otimes A) \circ (H \otimes H \otimes c_{A,A}) \circ (((H \otimes H \otimes \sigma) \circ \delta_{H^2}) \otimes A) \\ = \mu_A \circ (A \otimes \varphi_A) \circ (\sigma_H^A \otimes A).\end{aligned} \tag{42}$$

If

$$\partial_{2,3}(\sigma) \wedge \partial_{2,1}(\sigma) = \partial_{2,0}(\sigma) \wedge \partial_{2,2}(\sigma) \tag{43}$$

holds, we will say that σ satisfies the 2-cocycle condition.

Remark 3.8. For a weak Hopf algebra H, the idempotent morphisms

$$\Omega_{H\otimes H}^L = ((\varepsilon_H \circ \mu_H) \otimes H \otimes H) \circ \delta_{H\otimes H} : H \otimes H \to H \otimes H, \tag{44}$$

$$\Omega_{H\otimes H}^R = (H \otimes H \otimes (\varepsilon_H \circ \mu_H)) \circ \delta_{H\otimes H} : H \otimes H \to H \otimes H \tag{45}$$

satisfy the identities

$$\Omega_{H\otimes H}^L = ((\mu_H \circ (H \otimes \Pi_H^L)) \otimes H) \circ (H \otimes \delta_H), \tag{46}$$

$$\Omega^R_{H\otimes H} = (H\otimes(\mu_H\circ(\Pi^R_H\otimes H)))\circ(\delta_H\otimes H). \tag{47}$$

By (a1) of Definition 2.1 we obtain that

$$\mu_H\circ\Omega^L_{H\otimes H} = \mu_H\circ\Omega^R_{H\otimes H} = \mu_H, \tag{48}$$

and it is easy to show that, if we consider the left-right H-module actions and the left-right H-comodule coactions $\varphi_{H\otimes H} = \mu_H\otimes H$, $\phi_{H\otimes H} = H\otimes\mu_H$, $\varrho_{H\otimes H} = \delta_H\otimes H$, $\rho_{H\otimes H} = H\otimes\delta_H$ on $H\otimes H$, we have that $\Omega^L_{H\otimes H}$ is a morphism of left and right H-modules and right H-comodules and $\Omega^R_{H\otimes H}$ is a morphism of left and right H-modules and left H-comodules. Moreover, if H is cocommutative it is an easy exercise to prove that $\Omega^L_{H\otimes H} = \Omega^R_{H\otimes H}$ and the following equalities hold:

$$\delta_{H\otimes H}\circ\Omega^L_{H\otimes H} = (H\otimes H\otimes\Omega^L_{H\otimes H})\circ\delta_{H\otimes H} = (\Omega^L_{H\otimes H}\otimes H\otimes H)\circ\delta_{H\otimes H}. \tag{49}$$

As a consequence,

$$\delta_{H\otimes H}\circ\Omega^L_{H\otimes H} = (\Omega^L_{H\otimes H}\otimes\Omega^L_{H\otimes H})\circ\delta_{H\otimes H}. \tag{50}$$

Then, if H is cocommutative, we will denote the morphism $\Omega^L_{H\otimes H}$ by Ω^2_H.

Proposition 3.9. *Let H be a cocommutative weak Hopf algebra, let (A,φ_A) be a weak left H-module algebra, and let $\sigma\in Reg_{\varphi_A}(H^2,A)$. The following identities hold:*

(i) $\sigma\circ\Omega^2_H = \sigma$.

(ii) $\sigma^A_H\circ\Omega^2_H = \sigma^A_H$.

(iii) $(A\otimes\Omega^2_H)\circ(\sigma^A_H\otimes H) = (\sigma^A_H\otimes H)\circ(H\otimes\Omega^2_H)$.

(iv) $\partial_{2,3}(\sigma) = (\sigma\otimes\varepsilon_H)\circ(H\otimes\Omega^2_H)$.

Proof. To prove (i) we first show that $u_2\circ\Omega^2_H = u_2$. Indeed, by (48) we have

$$u_2\circ\Omega^2_H = \varphi_A\circ((\mu_H\circ\Omega^2_H)\otimes\eta_A) = \varphi_A\circ(\mu_H\otimes\eta_A) = u_2.$$

Then (i) holds because, by (49), we obtain

$$\begin{aligned}\sigma = \sigma\wedge\sigma^{-1}\wedge\sigma &= \mu_A\circ(u_2\otimes\sigma)\circ\delta_{H^2} = \mu_A\circ((u_2\circ\Omega^2_H)\otimes\sigma)\circ\delta_{H^2}\\ &= \mu_A\circ(u_2\otimes\sigma)\circ\delta_{H^2}\circ\Omega^2_H = (\sigma\wedge\sigma^{-1}\wedge\sigma)\circ\Omega^2_H = \sigma\circ\Omega^2_H.\end{aligned}$$

By (49) and the properties of (i) we have $\sigma^A_H\circ\Omega^2_H = ((\sigma\circ\Omega^2_H)\otimes\mu_H)\circ\delta_{H^2} = \sigma^A_H$. Then (ii) holds.

Using that Ω^2_H is a morphism of left H-comodules and H-modules, we obtain (iii). Finally, (iv) is a consequence of (46). □

Proposition 3.10. *Let H be a cocommutative weak Hopf algebra, let (A,φ_A) be a weak left H-module algebra, and let $\sigma\in Reg_{\varphi_A}(H^2,A)$. Then σ satisfies the 2-cocycle condition if and only if the equality*

$$\mu_A\circ(A\otimes\sigma)\circ(\sigma^A_H\otimes H) = \mu_A\circ(A\otimes\sigma)\circ(\psi^A_H\otimes H)\circ(H\otimes\sigma^A_H) \tag{51}$$

holds.

Proof. The proof follows from the following facts: First, note that

$$\begin{aligned}\partial_{2,3}(\sigma)\wedge\partial_{2,1}(\sigma) &= \mu_A\circ(((\sigma\otimes\varepsilon_H)\circ(H\otimes\Omega^2_H))\otimes(\sigma\circ(\mu_H\otimes H)))\circ\delta_{H^3}\\ &= \mu_A\circ(A\otimes\sigma)\circ(\sigma^A_H\otimes H)\circ(H\otimes\Omega^2_H)\\ &= \mu_A\circ(A\otimes(\sigma\circ\Omega^2_H))\circ(\sigma^A_H\otimes H)\\ &= \mu_A\circ(A\otimes\sigma)\circ(\sigma^A_H\otimes H),\end{aligned}$$

where the first equality follows by Proposition 3.9(iv), the second one by the properties of ε_H and by Proposition 3.9(iii), and the last one by Proposition 3.9(i).

On the other hand, by the naturality of c we obtain that

$$\partial_{2,0}(\sigma)\wedge\partial_{2,2}(\sigma)=\mu_A\circ(A\otimes\sigma)\circ(\psi^A_H\otimes H)\circ(H\otimes\sigma^A_H),$$

and this finishes the proof. □

Remark 3.11. Note that, if (A,φ_A) is a commutative left H-module algebra, the 2-cocycle condition means that $\sigma\in Ker(D^2_{\varphi_A})$. Also, we have $\sigma^A_H=c_{A,H}\circ\tau^A_H$ for $\tau^A_H=(\mu_H\otimes\sigma)\circ\delta_{H^2}$. Therefore, if (A,φ_A) is a commutative left H-module algebra, the twisted condition holds for all $\sigma\in Reg_{\varphi_A}(H^2,A)$.

Theorem 3.12. *Let H be a cocommutative weak Hopf algebra, let (A,φ_A) be a weak left H-module algebra, and let $\sigma\in Reg_{\varphi_A}(H^2,A)$. The morphism σ satisfies the twisted condition (42) if and only if $\mathbb{A}_H=(A,H,\psi^A_H,\sigma^A_H)$, where ψ^A_H, σ^A_H are associated to σ as in Definition 3.2, satisfies the twisted condition (22).*

Proof. If $\mathbb{A}_H$ satisfies the twisted condition (22), composing with $A\otimes\varepsilon_H$ and using Proposition 3.5(ii), we obtain that σ satisfies the twisted condition (42). Conversely, assume that σ satisfies the twisted condition (42). Then

$$\begin{aligned}&(\mu_A\otimes H)\circ(A\otimes\sigma^A_H)\circ(\psi^A_H\otimes H)\circ(H\otimes\psi^A_H)\\ &\quad=(A\otimes\mu_H)\circ(c_{H,A}\otimes H)\circ(H\otimes(\mu_A\circ(A\otimes\varphi_A)\circ(\sigma^A_H\otimes A))\otimes H)\\ &\qquad\circ(H\otimes H\otimes H\otimes c_{H,A})\circ(\delta_H\otimes\delta_H\otimes A)\\ &\quad=(\mu_A\otimes H)\circ(A\otimes\psi^A_H)\circ(\sigma^A_H\otimes A).\end{aligned}$$

The first equality follows by the naturality of c, the cocommutativity of H, the coassociativity of δ_H, and by the twisted condition for σ. The second one is a consequence of the naturality of c and (a1) of Definition (2.1). Therefore $\mathbb{A}_H$ satisfies the twisted condition (22). □

Theorem 3.13. *Let H be a cocommutative weak Hopf algebra,let (A,φ_A) be a weak left H-module algebra, and let $\sigma\in Reg_{\varphi_A}(H^2,A)$. The morphism σ satisfies the 2-cocycle condition (51) if and only if $\mathbb{A}_H=(A,H,\psi^A_H,\sigma^A_H)$, where ψ^A_H, σ^A_H are associated to σ as in Definition 3.2, satisfies the cocycle condition (23).*

Proof. If $\mathbb{A}_H$ satisfies the cocycle condition (23), composing with $A\otimes\varepsilon_H$, and using Proposition 3.5(ii), we obtain that σ satisfies the 2-cocycle condition (51). Conversely, assume that σ satisfies the 2-cocycle condition (43). Then

$$\begin{aligned}
&(\mu_A\otimes H)\circ(A\otimes\sigma_H^A)\circ(\psi_H^A\otimes H)\circ(H\otimes\sigma_H^A)\\
&\quad=(\mu_A\otimes H)\circ(A\otimes\sigma\otimes\mu_H)\circ(\psi_H^A\otimes c_{H,H}\otimes H)\circ(H\otimes c_{H,A}\otimes H\otimes H)\\
&\qquad\circ(\delta_H\otimes((\sigma_H^A\otimes\mu_H)\circ\delta_{H^2}))\\
&\quad=((\mu_A\circ(A\otimes\sigma)\circ(\sigma_H^A\otimes H))\otimes(\mu_H\circ(H\otimes\mu_H)))\circ\delta_{H^3}\\
&\quad=(\mu_A\otimes H)\circ(A\otimes\sigma_H^A)\circ(\sigma_H^A\otimes H).
\end{aligned}$$

The first equality follows by the naturality of c, the coassociativity of δ_H, and Proposition 3.4; the second one follows by the naturality of c, the associativity of μ_H, and by the 2-cocycle condition (51). Finally, the last one follows by the naturality of c, the associativity of μ_H, and Proposition 3.4. □

Remark 3.14. By Theorems 3.12 and 3.13 and applying the general theory of weak crossed products, we have the following: If $\sigma\in Reg_{\varphi_A}(H^2,A)$ satisfies the twisted condition (42) (equivalently (51)) and the 2-cocycle condition (43), the quadruple $\mathbb{A}_H$ defined in Remark 3.6 satisfies the twisted and the cocycle conditions (22) and (23) and therefore the induced product is associative. Conversely, by Theorem 3.11 of [**6**], we obtain that, if the product induced by the quadruple $\mathbb{A}_H$ defined in Remark 3.6 is associative, $\mathbb{A}_H$ satisfies the twisted and the cocycle condition and, by Theorems 3.12 and 3.13, σ satisfies the twisted condition (42) and the 2-cocycle condition (43) (equivalently (51)).

Definition 3.15. Let H be a cocommutative weak Hopf algebra, let (A,φ_A) be a weak left H-module algebra, and let $\sigma\in Reg_{\varphi_A}(H^2,A)$. We say that σ satisfies the normal condition if

$$\sigma\circ(\eta_H\otimes H)=\sigma\circ(H\otimes\eta_H)=u_1, \tag{52}$$

i.e., $\sigma\in Reg^{+}_{\varphi_A}(H^2,A)$.

Theorem 3.16. *Let H be a cocommutative weak Hopf algebra, let (A,φ_A) be a weak left H-module algebra, and let $\sigma\in Reg_{\varphi_A}(H^2,A)$. Let $\mathbb{A}_H=(A,H,\psi_H^A,\sigma_H^A)$ be the quadruple with ψ_H^A, let σ_H^A be defined as in Definition 3.2, and assume that $\mathbb{A}_H$ satisfies the twisted and the cocycle conditions (22) and (23). Then $\nu=\nabla_{A\otimes H}\circ(\eta_A\otimes\eta_H)$ is a preunit for the weak crossed product associated to $\mathbb{A}_H$ if and only if*

$$\sigma_H^A\circ(\eta_H\otimes H)=\sigma_H^A\circ(H\otimes\eta_H)=\nabla_{A\otimes H}\circ(\eta_A\otimes H). \tag{53}$$

Proof. By Theorem 3.11 of [**6**], to prove the result we only need to show that (31), (32), and (33) hold for $\nu=\nabla_{A\otimes H}\circ(\eta_A\otimes\eta_H)$ if and only if $\sigma_H^A\circ(\eta_H\otimes H)=\sigma_H^A\circ(H\otimes\eta_H)=\nabla_{A\otimes H}\circ(\eta_A\otimes H)$. Indeed, ν satisfies (31) if and only if $\sigma_H^A\circ(H\otimes\eta_H)=\nabla_{A\otimes H}\circ(\eta_A\otimes H)$ because

$$\begin{aligned}
&(\mu_A\otimes H)\circ(A\otimes\sigma_H^A)\circ(\psi_H^A\otimes H)\circ(H\otimes\nu)\\
&\qquad=(\mu_A\otimes H)\circ(A\otimes\sigma_H^A)\circ(\psi_H^A\otimes H)\circ(H\otimes(\psi_H^A\circ(\eta_H\otimes\eta_A)))\\
&\qquad=\nabla_{A\otimes H}\circ\sigma_H^A\circ(H\otimes\eta_H)\\
&\qquad=\sigma_H^A\circ(H\otimes\eta_H).
\end{aligned}$$

The first equality follows by the definition of $\nabla_{A\otimes H}$, the second one by the twisted condition, and the last one by Proposition 3.5(ii).

Also, ν satisfies (32) if and only if $\sigma_H^A \circ (\eta_H \otimes H) = \nabla_{A\otimes H} \circ (\eta_A \otimes H)$ because by (27) we have $(\mu_A \otimes H) \circ (A \otimes \sigma_H^A) \circ (\nu \otimes H) = \sigma_H^A \circ (\eta_H \otimes H)$. Finally, (33) is always true because, by (21), we obtain $(\mu_A \otimes H) \circ (A \otimes \psi_H^A) \circ (\nu \otimes A) = \psi_H^A \circ (\eta_H \otimes A)$. □

Corollary 3.17. *Let H be a cocommutative weak Hopf algebra, let (A, φ_A) be a weak left H-module algebra, and let $\sigma \in Reg_{\varphi_A}(H^2, A)$. Let $\mathbb{A}_H = (A, H, \psi_H^A, \sigma_H^A)$ be the quadruple with ψ_H^A, σ_H^A defined as in Definition 3.2, and assume that $\mathbb{A}_H$ satisfies the twisted and the cocycle conditions (22) and (23). Then $\nu = \nabla_{A\otimes H} \circ (\eta_A \otimes \eta_H)$ is a preunit for the weak crossed product associated to $\mathbb{A}_H$ if and only if σ satisfies the normal condition (52).*

Proof. If $\nu = \nabla_{A\otimes H} \circ (\eta_A \otimes \eta_H)$ is a preunit for the weak crossed product associated to $\mathbb{A}_H$, by Theorem 3.16 we have (53). Then, composing with $(A \otimes \varepsilon_H)$ and using Proposition 3.5(ii), we obtain (52). Conversely, if (52) holds, by (14) and Proposition 2.13(i) we have $\sigma_H^A \circ (\eta_H \otimes H) = \nabla_{A\otimes H} \circ (\eta_A \otimes H)$. On the other hand by (11) and (ii) of Proposition 2.13 we obtain $\sigma_H^A \circ (H \otimes \eta_H) = \nabla_{A\otimes H} \circ (\eta_A \otimes H)$. □

Corollary 3.18. *Let H be a cocommutative weak Hopf algebra, let (A, φ_A) be a weak left H-module algebra, and let $\sigma \in Reg_{\varphi_A}(H^2, A)$. Let $\mathbb{A}_H = (A, H, \psi_H^A, \sigma_H^A)$ be the quadruple with ψ_H^A, σ_H^A defined as in Definition 3.2, and let $\mu_{A\otimes H}$ be the associated product defined in (28). Then the following statements are equivalent:*

- *(i) The product $\mu_{A\otimes H}$ is associative with preunit $\nu = \nabla_{A\otimes H} \circ (\eta_A \otimes \eta_H)$ and normalized with respect to $\nabla_{A\otimes H}$.*
- *(ii) The morphism σ satisfies the twisted condition (42), the 2-cocycle condition (43) (equivalently (51)), and the normal condition (52).*

Proof. The proof is an easy consequence of Theorem 3.11 of [**6**], Theorems 3.12, and 3.13, and Corollary 3.17. □

Remark 3.19. Let H be a cocommutative weak Hopf algebra, and let (A, φ_A) be a weak left H-module algebra. From now on we will denote by $A \otimes_\tau H = (A \otimes H, \mu_{A\otimes_\tau H})$ the weak crossed product, with preunit $\nu = \nabla_{A\otimes H} \circ (\eta_A \otimes \eta_H)$, defined by a morphism τ in $Reg_{\varphi_A}(H^2, A)$ satisfying the twisted condition, the 2-cocycle condition, and the normal condition. The associated algebra will be denoted by

$$A \times_\tau H = (A \times H, \eta_{A\times_\tau H}, \mu_{A\times_\tau H}).$$

Finally, the associated quadruple $\mathbb{A}_H$ will be denoted by $\mathbb{A}_{H,\tau}$ and σ_H^A by $\sigma_{H,\tau}^A$.

Remark 3.20. Let H be a cocommutative weak Hopf algebra, and let (A, φ_A) be a weak left H-module algebra. Let $\sigma \in Reg_{\varphi_A}(H^2, A)$ be a morphism satisfying the twisted condition (42), the 2-cocycle condition (43), and the normal condition (52). Then the weak crossed product $A \otimes_\sigma H = (A \otimes H, \mu_{A\otimes_\sigma H})$ with preunit $\nu = \nabla_{A\otimes H} \circ (\eta_A \otimes \eta_H)$ defined previously is a particular instance of the weak crossed products introduced in [**6**]. Also, it is a particular case of the ones used in [**14**], where these crossed structures were studied in a category of modules over a commutative ring without requiring cocommutativity of H and using weak measurings (see Definition 3.2 of [**14**]). To prove this assertion we will show that the conditions presented in Lemma 3.8 and Theorem 3.9 of [**14**] are completely fulfilled. First, note that, if (A, φ_A) is a weak left H-module algebra, then φ_A is a weak measuring. The idempotent morphism $\Omega_{A\otimes H}$ related with the preunit ν is the morphism $\nabla_{A\otimes H}$ because, by (27) and (53), $\Omega_{A\otimes H} = \nabla_{A\otimes H}$. Moreover, in the category of modules over an associative, commutative, unital ring, the normalized condition implies that $Im(\mu_{A\otimes_\sigma H}) \subset Im(\nabla_{A\otimes H})$.

On the other hand, the left action defined in Lemma 3.8 of [**14**] is φ_A and the morphism defined in Lemma 3.8 of [**14**] is σ. Then the equalities (a) and (b) of Lemma 3.8 of [**14**] hold because the first one is the definition of ψ_H^A and the second one is a consequence of (27) and the definition of σ_H^A. Therefore, we have that $\rho_{A\otimes H} \circ \mu_{A\otimes_\sigma H} = (\mu_{A\otimes_\sigma H} \otimes H) \circ \rho_{A\otimes H\otimes A\otimes H}$ holds where $\rho_{A\otimes H} = A \otimes \delta_H$ and

$$\rho_{A\otimes H\otimes A\otimes H} = (A \otimes H \otimes A \otimes H \otimes \mu_H) \circ (A \otimes H \otimes c_{H,A\otimes H} \otimes H) \circ (\rho_{A\otimes H} \otimes \rho_{A\otimes H}).$$

Although $\rho_{A\otimes H\otimes A\otimes H}$ is not counital, we say that $\mu_{A\otimes_\sigma H}$ is H-colinear as in Lemma 3.8 of [**14**]. Then we obtain that σ satisfies the equality (1) of [**14**]:

$$\sigma \circ ((\mu_H \circ (H \otimes \Pi_H^R)) \otimes H) = \sigma \circ (H \otimes (\mu_H \circ (\Pi_H^R \otimes H))).$$

Finally, for the preunit $\nu = \nabla_{A\otimes H} \circ (\eta_A \otimes \eta_H)$, by (34) and (9) the equality

$$(A \otimes \delta_H) \circ \nu = (A \otimes ((H \otimes \Pi_H^L) \circ \delta_H)) \circ \nu$$

holds (i.e., the equality (4) of [**14**] is true in our setting).

4. Equivalent weak crossed products and $H^2_{\varphi_A}(H, A)$

The aim of this section is to give necessary and sufficient conditions for two weak crossed products, $A \otimes_\alpha H$ and $A \otimes_\beta H$, to be equivalent in the cocommutative setting. To define a good notion of equivalence we need the definition of right H-comodule algebra for a weak Hopf algebra H.

Definition 4.1. Let H be a weak bialgebra, and let (B, ρ_B) be an algebra that is also a right H-comodule such that $\mu_{B\otimes H} \circ (\rho_B \otimes \rho_B) = \rho_B \circ \mu_B$ holds. The object (B, ρ_B) is called a right H-comodule algebra if one of the following equivalent conditions holds:

(d1) $(\rho_B \otimes H) \circ \rho_B \circ \eta_B = (B \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ ((\rho_B \circ \eta_B) \otimes (\delta_H \circ \eta_H))$.

(d2) $(\rho_B \otimes H) \circ \rho_B \circ \eta_B = (B \otimes \mu_H \otimes H) \circ ((\rho_B \circ \eta_B) \otimes (\delta_H \circ \eta_H))$.

(d3) $(B \otimes \overline{\Pi}_H^R) \circ \rho_B = (\mu_B \otimes H) \circ (B \otimes (\rho_B \circ \eta_B))$.

(d4) $(B \otimes \Pi_H^L) \circ \rho_B = ((\mu_B \circ c_{B,B}) \otimes H) \circ (B \otimes (\rho_B \circ \eta_B))$.

(d5) $(B \otimes \overline{\Pi}_H^R) \circ \rho_B \circ \eta_B = \rho_B \circ \eta_B$.

(d6) $(B \otimes \Pi_H^L) \circ \rho_B \circ \eta_B = \rho_B \circ \eta_B$.

Proposition 4.2. *Let H be a cocommutative weak Hopf algebra, let (A, φ_A) be a weak left H-module algebra, and let $\alpha \in Reg^+_{\varphi_A}(H^2, A)$ that satisfies the twisted condition (42) and the 2-cocycle condition (43) (equivalently (51)). Then the algebra $A \times_\alpha H = (A \times H, \eta_{A\times_\alpha H}, \mu_{A\times_\alpha H})$ is a right H-comodule algebra for the coaction*

$$\rho_{A\times_\alpha H} = (p_{A\otimes H} \otimes H) \circ (A \otimes \delta_H) \circ i_{A\otimes H}.$$

Proof. First note that $(A \times_\alpha H, \rho_{A\times_\alpha H})$ is a right H-comodule because

$$(A \times H \otimes \varepsilon_H) \circ \rho_{A\times_\alpha H} = p_{A\otimes H} \circ i_{A\otimes H} = id_{A\times H}$$

and, by (40) and the coassociativity of δ_H,

$$\begin{aligned}(\rho_{A\times_\alpha H} \otimes H) \circ \rho_{A\times_\alpha H} &= (p_{A\otimes H} \otimes H \otimes H) \circ (A \otimes ((\delta_H \otimes H) \circ \delta_H)) \circ i_{A\otimes H} \\ &= (A \times H \otimes \delta_H) \circ \rho_{A\times_\alpha H}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\mu_{(A\times_\alpha H)\otimes H} \circ (\rho_{A\times_\alpha H} \otimes \rho_{A\times_\alpha H}) \\ &\quad= (p_{A\times H} \otimes H) \circ (\mu_{A\otimes_\alpha H} \otimes \mu_H) \circ (A \otimes H \otimes A \otimes c_{H,H} \otimes H) \\ &\qquad \circ (A \otimes H \otimes c_{H,A} \otimes H \otimes H) \circ (((A \otimes \delta_H) \circ i_{A\otimes H}) \otimes ((A \otimes \delta_H) \circ i_{A\otimes H})) \\ &\quad= (p_{A\times H} \otimes H) \circ (\mu_A \otimes H \otimes H) \circ (\mu_A \otimes ((A \otimes \delta_H) \circ \sigma_H^A)) \circ (A \otimes \psi_H^A \otimes H) \\ &\qquad \circ (i_{A\otimes H} \otimes i_{A\otimes H}) \\ &\quad= \rho_{A\times_\alpha H} \circ \mu_{A\times_\alpha H},\end{aligned}$$

where the first equality follows by the normalized condition for $\mu_{A\otimes_\alpha H}$, the second one by the naturality of c, by the coassociativity of δ_H, and by (41), and the last one by (40).

Finally, by (40) and (9), we obtain that

$$\begin{aligned}(A \times_\alpha H \otimes \Pi_H^L) \circ \rho_{A\times_\alpha H} \circ \eta_{A\times_\alpha H} &= (p_{A\times H} \otimes \Pi_H^L) \circ (\eta_A \otimes (\delta_H \circ \eta_H)) \\ &= (p_{A\times H} \otimes H) \circ (\eta_A \otimes (\delta_H \circ \eta_H)) \\ &= \rho_{A\times_\alpha H} \circ \eta_{A\times_\alpha H}. \qquad \square\end{aligned}$$

Definition 4.3. Let H be a cocommutative weak Hopf algebra, let (A, φ_A) be a weak left H-module algebra, and let $\alpha, \beta \in Reg^+_{\varphi_A}(H^2, A)$ that satisfy the twisted condition (42) and the 2-cocycle condition (43) (equivalently (51)). Let $A \otimes_\alpha H$ and $A \otimes_\beta H$ be the weak crossed products associated to α and β. We say that $A \otimes_\alpha H$ and $A \otimes_\beta H$ are equivalent if there is an isomorphism of left A-modules and right H-comodule algebras $\omega_{\alpha,\beta} : A \times_\alpha H \to A \times_\beta H$.

Remark 4.4. Let H be a weak Hopf algebra, and let (A, φ_A) be a weak left H-module algebra. Let $\Gamma : A \otimes H \to A \otimes H$ be a morphism of left A-modules and right H-comodules for the regular action $\varphi_{A\otimes H} = \mu_A \otimes H$ and coaction $\rho_{A\otimes H} = A \otimes \delta_H$. Then

$$\Gamma \circ (\eta_A \otimes H) = (A \otimes \varepsilon_H \otimes H) \circ \rho_{A\otimes H} \circ \Gamma \circ (\eta_A \otimes H) = (f_\Gamma \otimes H) \circ \delta_H, \tag{54}$$

where $f_\Gamma = (A \otimes \varepsilon_H) \circ \Gamma \circ (\eta_A \otimes H)$. As a consequence,

$$\Gamma = (\mu_A \otimes H) \circ (A \otimes (\Gamma \circ (\eta_A \otimes H))) = ((\mu_A \circ (A \otimes f_\Gamma)) \otimes H) \circ (A \otimes \delta_H). \tag{55}$$

If $f : H \to A$ is a morphism and we define $\Gamma_f : A \otimes H \to A \otimes H$ by $\Gamma_f = ((\mu_A \circ (A \otimes f)) \otimes H) \circ (A \otimes \delta_H)$, it is clear that Γ_f is a morphism of left A-modules and right H-comodules such that $f_{\Gamma_f} = f$. Also, $\Gamma_{f_\Gamma} = \Gamma$ and then there is a bijection $\Phi : {}_A Hom^H_{\mathcal{C}}(A \otimes H, A \otimes H) \to Hom_{\mathcal{C}}(H, A)$ defined by $\Phi(\Gamma) = f_\Gamma$ with inverse the morphism $\Phi^{-1}(f) = \Gamma_f$. Note that $\Phi^{-1}(u_1) = \Gamma_{u_1} = \nabla_{A\otimes H}$.

Then it is easy to show that $\Gamma, \Gamma' \in {}_A Hom^H_{\mathcal{C}}(A \otimes H, A \otimes H)$ satisfy

(e1) $\Gamma \circ \Gamma' = \Gamma' \circ \Gamma = \nabla_{A\otimes H}$, (e2) $\Gamma\circ\Gamma'\circ\Gamma=\Gamma$, and

(e3) $\Gamma'\circ\Gamma\circ\Gamma'=\Gamma'$,

if and only if for the morphism f_Γ there exists a morphism f_Γ^{-1} such that

(i) $f_\Gamma\wedge f_\Gamma^{-1}=f_\Gamma^{-1}\wedge f_\Gamma=u_1$,

(ii) $f_\Gamma\wedge f_\Gamma^{-1}\wedge f_\Gamma=f_\Gamma$, and

(iii) $f_\Gamma^{-1}\wedge f_\Gamma\wedge f_\Gamma^{-1}=f_\Gamma^{-1}$.

Indeed, if $\Gamma,\Gamma'\in {}_AHom_{\mathcal{C}}^H(A\otimes H,A\otimes H)$ satisfies (e1)-(e3) define f_Γ^{-1} by $f_\Gamma^{-1}=f_{\Gamma'}$. Conversely, if for f_Γ we have a morphism f_Γ^{-1} satisfying (i)-(iii), define Γ' by $\Gamma'=\Gamma_{f_\Gamma^{-1}}$.

As a consequence, if H is cocommutative, $\Gamma\in {}_AHom_{\mathcal{C}}^H(A\otimes H,A\otimes H)$ satisfies (e1)–(e3) if and only if $\Phi(\Gamma)=f_\Gamma\in Reg_{\varphi_A}(H,A)$. Conversely, $f\in Reg_{\varphi_A}(H,A)$ if and only if $\Phi^{-1}(f)=\Gamma_f$ satisfies (e1)–(e3).

Theorem 4.5. *Let H be a cocommutative weak Hopf algebra, let (A,φ_A) be a weak left H-module algebra, and let $\alpha,\beta\in Reg_{\varphi_A}^+(H^2,A)$ that satisfy the twisted condition (42) and the 2-cocycle condition (43) (equivalently (51)). The weak crossed products associated to α and β are equivalent if and only if there exist multiplicative and preunit-preserving morphisms $\Gamma,\Gamma'\in {}_AHom_{\mathcal{C}}^H(A\otimes H,A\otimes H)$ satisfying (e1)–(e3).*

Proof. Assume that $A\otimes_\alpha H$ and $A\otimes_\beta H$ are equivalent. Thus there exists an isomorphism of left A-modules and right H-comodule algebras $\omega_{\alpha,\beta}:A\times_\alpha H\to A\times_\beta H$. Define Γ and Γ' by

$$\Gamma=i_{A\otimes H}\circ\omega_{\alpha,\beta}\circ p_{A\otimes H},\qquad \Gamma'=i_{A\otimes H}\circ\omega_{\alpha,\beta}^{-1}\circ p_{A\otimes H}.$$

Then

$$\Gamma\circ\Gamma'=i_{A\otimes H}\circ\omega_{\alpha,\beta}\circ p_{A\otimes H}\circ i_{A\otimes H}\circ\omega_{\alpha,\beta}^{-1}\circ p_{A\otimes H}=\nabla_{A\otimes H},$$

and

$$\Gamma'\circ\Gamma=i_{A\otimes H}\circ\omega_{\alpha,\beta}^{-1}\circ p_{A\otimes H}\circ i_{A\otimes H}\circ\omega_{\alpha,\beta}\circ p_{A\otimes H}=\nabla_{A\otimes H}.$$

Also,

$$\Gamma\circ\Gamma'\circ\Gamma=\nabla_{A\otimes H}\circ\Gamma=\Gamma,\quad \Gamma'\circ\Gamma\circ\Gamma'=\nabla_{A\otimes H}\circ\Gamma'=\Gamma',$$

and therefore (e1)–(e3) hold.

The morphism Γ is multiplicative because $\omega_{\alpha,\beta}$ is an algebra morphism, and, in a similar way, using that $\omega_{\alpha,\beta}^{-1}$ is multiplicative, it is possible to prove that Γ' is multiplicative.

On the other hand, Γ preserve the preunit because

$$\Gamma\circ\nu=i_{A\otimes H}\circ\omega_{\alpha,\beta}\circ\eta_{A\times_\alpha H}=i_{A\otimes H}\circ\eta_{A\times_\beta H}=\nu.$$

By the same arguments, we obtain that $\Gamma'\circ\nu=\Gamma'$.

Using (e1), (e2), and the left A-linearity of $\omega_{\alpha,\beta}$, we have

$$\begin{aligned}\varphi_{A\otimes H}\circ(A\otimes\Gamma) &= \varphi_{A\otimes H}\circ(A\otimes(\nabla_{A\otimes H}\circ\Gamma))\\ &= \nabla_{A\otimes H}\circ(\mu_A\otimes H)\circ(A\otimes\Gamma)\\ &= i_{A\otimes H}\circ\varphi_{A\times_\beta H}\circ(A\otimes\omega_{\alpha,\beta})\circ(A\otimes p_{A\otimes H})\\ &= i_{A\otimes H}\circ\omega_{\alpha,\beta}\circ\varphi_{A\times_\alpha H}\circ(A\otimes p_{A\otimes H})\\ &= \Gamma\circ(\mu_A\otimes H)\circ(A\otimes\nabla_{A\otimes H})\\ &= \Gamma\circ\nabla_{A\otimes H}\circ(\mu_A\otimes H)\\ &= \Gamma\circ\varphi_{A\otimes H}.\end{aligned}$$

Similarly, by (e1), (e3), and the left A-linearity of $\omega_{\alpha,\beta}^{-1}$ we obtain that Γ' is a morphism of left A-modules.

Finally, Γ is a morphism of right H-comodules by (40) and the right H-comodule morphism property of $\omega_{\alpha,\beta}$. Indeed,

$$\begin{aligned}\rho_{A\otimes H}\circ\Gamma &= (i_{A\otimes H}\otimes H)\circ\rho_{A\times_\beta H}\circ\omega_{\alpha,\beta}\circ p_{A\otimes H}\\ &= ((i_{A\otimes H}\circ\omega_{\alpha,\beta})\otimes H)\circ\rho_{A\times_\alpha H}\circ p_{A\otimes H} = (\Gamma\otimes H)\circ(A\otimes\delta_H)\circ\nabla_{A\otimes H}\\ &= ((\Gamma\circ\nabla_{A\otimes H})\otimes H)\circ(A\otimes\delta_H) = (\Gamma\otimes H)\circ\rho_{A\otimes H}.\end{aligned}$$

By a similar calculation we obtain that Γ' is a morphism of right H-comodules.

Conversely, assume that there exist multiplicative and preunit-preserving morphisms $\Gamma,\Gamma'\in {}_AHom_{\mathcal{C}}^H(A\otimes H, A\otimes H)$ satisfying (e1)–(e3) of the previous remark. Define

$$\omega_{\alpha,\beta} = p_{A\otimes H}\circ\Gamma\circ i_{A\otimes H}, \qquad \omega_{\alpha,\beta}^{-1} = p_{A\otimes H}\circ\Gamma'\circ i_{A\otimes H}.$$

Then, by (e1), (e2), and (e3), we have

$$\begin{aligned}\omega_{\alpha,\beta}^{-1}\circ\omega_{\alpha,\beta} &= p_{A\otimes H}\circ\Gamma'\circ\nabla_{A\otimes H}\circ\Gamma\circ i_{A\otimes H}\\ &= p_{A\otimes H}\circ\Gamma'\circ\Gamma\circ i_{A\otimes H} = p_{A\otimes H}\circ\nabla_{A\otimes H}\circ i_{A\otimes H} = id_{A\times H}\end{aligned}$$

and

$$\begin{aligned}\omega_{\alpha,\beta}\circ\omega_{\alpha,\beta}^{-1} &= p_{A\otimes H}\circ\Gamma\circ\nabla_{A\otimes H}\circ\Gamma'\circ i_{A\otimes H}\\ &= p_{A\otimes H}\circ\Gamma\circ\Gamma'\circ i_{A\otimes H} = p_{A\otimes H}\circ\nabla_{A\otimes H}\circ i_{A\otimes H} = id_{A\times H},\end{aligned}$$

which proves that $\omega_{\alpha,\beta}$ is an isomorphism.

Moreover, using that Γ preserves the preunit $\nu = \nabla_{A\otimes H}\circ(\eta_A\otimes\eta_H)$, we have

$$\omega_{\alpha,\beta}\circ\eta_{A\times_\alpha H} = p_{A\otimes H}\circ\Gamma\circ\nu = p_{A\otimes H}\circ\nu = \eta_{A\times_\beta H}$$

and, by the multiplicative property of Γ, we obtain

$$\begin{aligned}\mu_{A\times_\beta H}\circ(\omega_{\alpha,\beta}\otimes\omega_{\alpha,\beta}) &= p_{A\otimes H}\circ\mu_{A\otimes_\beta H}\circ(\Gamma\otimes\Gamma)\circ(i_{A\otimes H}\otimes i_{A\otimes H})\\ &= p_{A\otimes H}\circ\Gamma\circ\mu_{A\otimes_\alpha H}\circ(i_{A\otimes H}\otimes i_{A\otimes H}) = \omega_{\alpha,\beta}\circ\mu_{A\times_\alpha H}.\end{aligned}$$

Therefore, $\omega_{\alpha,\beta}$ is an isomorphism of algebras.

On the other hand, using (e1), (e2), and the property of left A-module morphism of Γ, we have

$$\begin{aligned}\varphi_{A\times_\beta H}\circ(A\otimes\omega_{\alpha,\beta}) &= p_{A\otimes H}\circ(\mu_A\otimes H)\circ(A\otimes(\nabla_{A\otimes H}\circ\Gamma\circ i_{A\otimes H}))\\ &= p_{A\otimes H}\circ(\mu_A\otimes H)\circ(A\otimes(\Gamma\circ i_{A\otimes H}))\\ &= p_{A\otimes H}\circ\Gamma\circ(\mu_A\otimes H)\circ(A\otimes i_{A\otimes H})\\ &= p_{A\otimes H}\circ\Gamma\circ\nabla_{A\otimes H}\circ(\mu_A\otimes H)\circ(A\otimes i_{A\otimes H}) = \omega_{\alpha,\beta}\circ\varphi_{A\times_\alpha H},\end{aligned}$$

and this proves that $\omega_{\alpha,\beta}$ is a morphism of left A-modules.

Finally, using similar arguments and the property of right H-comodule morphism of Γ, we obtain that $\omega_{\alpha,\beta}$ is a morphism of right H-comodules because

$$\begin{aligned}\rho_{A\times_\beta H}\circ\omega_{\alpha,\beta} &= (p_{A\otimes H}\otimes H)\circ(A\otimes\delta_H)\circ\nabla_{A\otimes H}\circ\Gamma\circ i_{A\otimes H}\\ &= (p_{A\otimes H}\otimes H)\circ\rho_{A\otimes_\beta H}\circ\Gamma\circ i_{A\otimes H}\\ &= (((p_{A\otimes H}\circ\Gamma)\otimes H)\circ\rho_{A\otimes_\alpha H}\circ i_{A\otimes H}\\ &= (((p_{A\otimes H}\circ\Gamma\circ\nabla_{A\otimes H})\otimes H)\circ\rho_{A\otimes_\alpha H}\circ i_{A\otimes H}\\ &= (\omega_{\alpha,\beta}\otimes H)\circ\rho_{A\times_\alpha H}.\end{aligned}$$

□

Remark 4.6. By the previous theorem, the notion of equivalent crossed products is the one used in [**14**] in a category of modules over a commutative ring. Following the terminology used in [**14**], the pair of morphisms f_Γ and f_Γ^{-1} is an example of gauge transformation. Also, this notion is a generalization of the one that we can find in the Hopf algebra world (see [**5, 8**]).

The following results, Theorem 4.7, and Corollary 4.8 will be used in Theorem 4.9 to obtain the meaning of the notion of equivalence between two weak crossed products in terms of morphisms of $Reg_{\varphi_A}(H,A)$. Note that this characterization is the key to prove the main result of this section, i.e., Theorem 4.11.

Theorem 4.7. *Let Γ and f_Γ be as in Remark 4.4 and such that*

$$\Gamma\circ\nabla_{A\otimes H}=\nabla_{A\otimes H}\circ\Gamma=\Gamma. \tag{56}$$

Under the hypothesis of Theorem 4.5, Γ is a multiplicative morphism that preserves the preunit $\nu=\nabla_{A\otimes H}\circ(\eta_A\otimes\eta_H)$ if and only if the following equalities hold:

$$p_{A\otimes H}\circ\Gamma\circ\nu=p_{A\otimes H}\circ\nu \tag{57}$$

$$\mu_A\circ(A\otimes f_\Gamma)\circ\psi_H^A=\mu_A\circ(f_\Gamma\otimes\varphi_A)\circ(\delta_H\otimes A) \tag{58}$$

$$\mu_A\circ(A\otimes f_\Gamma)\circ\sigma_{H,\alpha}^A=\mu_A\circ(\mu_A\otimes\beta)\circ(A\otimes\psi_H^A\otimes H)\circ(((f_\Gamma\otimes H)\circ\delta_H)\otimes((f_\Gamma\otimes H)\circ\delta_H)). \tag{59}$$

Moreover, if Γ preserves the preunit, we have that

$$f_\Gamma\circ\eta_H=\eta_A. \tag{60}$$

Proof. Assume that Γ is a multiplicative morphism that preserves the preunit. Then (57) follows easily and, by (56), we have

$$\Gamma\circ(A\otimes\eta_H)=\nabla_{A\otimes H}\circ(A\otimes\eta_H). \tag{61}$$

On the other hand, the multiplicative condition for Γ implies that

$$\Gamma\circ\mu_{A\otimes_\alpha H}\circ(\eta_A\otimes H\otimes A\otimes\eta_H)=\mu_{A\otimes_\beta H}\circ(\Gamma\otimes\Gamma)\circ(\eta_A\otimes H\otimes A\otimes\eta_H).$$

Equivalently,

$$\begin{aligned}&\Gamma\circ(\mu_A\otimes H)\circ(A\otimes(\sigma^A_{H,\alpha}\circ(H\otimes\eta_H)))\circ\psi^A_H\\&\quad=(\mu_A\otimes H)\circ(\mu_A\otimes\sigma^A_{H,\beta})\circ(A\otimes\psi^A_H\otimes H)\circ((\Gamma\circ(\eta_A\otimes H))\otimes(\Gamma\circ(A\otimes\eta_H)).\end{aligned}\tag{62}$$

By the normal condition for α we have

$$\sigma^A_{H,\alpha}\circ(H\otimes\eta_H)=\nabla_{A\otimes H}\circ(\eta_A\otimes H),\tag{63}$$

and then the upper side of (62) is equal to $\Gamma\circ\psi^A_H$. For the lower side of (62),

$$\begin{aligned}&(\mu_A\otimes H)\circ(\mu_A\otimes\sigma^A_{H,\beta})\circ(A\otimes\psi^A_H\otimes H)\circ((\Gamma\circ(\eta_A\otimes H))\otimes(\Gamma\circ(A\otimes\eta_H))\\&\quad=(\mu_A\otimes H)\circ(f_\Gamma\otimes((\mu_A\otimes H)\circ(A\otimes\sigma^A_{H,\beta})\circ(\psi^A_H\otimes H)\\&\qquad\circ(H\otimes(\nabla_{A\otimes H}\circ(A\otimes\eta_H)))))\circ(\delta_H\otimes A)\\&\quad=(\mu_A\otimes H)\circ(f_\Gamma\otimes((\mu_A\otimes H)\circ(A\otimes\sigma^A_{H,\beta})\circ(\psi^A_H\otimes H)\circ(H\otimes A\otimes\eta_H)))\\&\qquad\circ(\delta_H\otimes A)\\&\quad=(\mu_A\otimes H)\circ(f_\Gamma\otimes((\mu_A\otimes H)\circ(A\otimes(\nabla_{A\otimes H}\circ(\eta_A\otimes H)))))\circ(H\otimes\psi^A_H)\\&\qquad\circ(\delta_H\otimes A)\\&\quad=(\mu_A\otimes H)\circ(f_\Gamma\otimes\psi^A_H)\circ(\delta_H\otimes A),\end{aligned}$$

where the first equality follows by (54) and (61), the second one by (26), the third one by (63), and the fourth one by the properties of $\nabla_{A\otimes H}$.

Thus, (62) is equivalent to

$$\Gamma\circ\psi^A_H=(\mu_A\otimes H)\circ(f_\Gamma\otimes\psi^A_H)\circ(\delta_H\otimes A),\tag{64}$$

and then composing in both sides with $A\otimes\varepsilon_H$, we get (58).

Also, the multiplicative condition for Γ implies the following:

$$\Gamma\circ\mu_{A\otimes_\alpha H}\circ(\eta_A\otimes H\otimes\eta_A\otimes H)=\mu_{A\otimes_\beta H}\circ(\Gamma\otimes\Gamma)\circ(\eta_A\otimes H\otimes\eta_A\otimes H).$$

Equivalently,

$$\begin{aligned}&\Gamma\circ(\mu_A\otimes H)\circ(A\otimes\sigma^A_{H,\alpha})\circ((\nabla_{A\otimes H}\circ(\eta_A\otimes H))\otimes H)\\&\quad=(\mu_A\otimes H)\circ(\mu_A\otimes\sigma^A_{H,\beta})\circ(A\otimes\psi^A_H\otimes H)\circ((\Gamma\circ(\eta_A\otimes H))\otimes(\Gamma\circ(\eta_A\otimes H)).\end{aligned}$$

Therefore, by (27) and (54) we obtain that the previous equality is equivalent to

$$\Gamma\circ\sigma^A_{H,\alpha}=(\mu_A\otimes H)\circ(\mu_A\otimes\sigma^A_{H,\beta})\circ(A\otimes\psi^A_H\otimes H)\circ((f_\Gamma\otimes H)\circ\delta_H)\otimes((f_\Gamma\otimes H)\circ\delta_H)).\tag{65}$$

Composing in both sides with $A\otimes\varepsilon_H$ and using (iii) of Proposition 3.5, we obtain (59).

Conversely, assume that (57), (58) and (59) hold. Then $\Gamma\circ\nu=\nabla_{A\otimes H}\circ\Gamma\circ\nu=\nabla_{A\otimes H}\circ\nu=\nu$ and Γ preserves the preunit. Moreover, to prove that Γ is multiplicative we first show that, if (58) holds, then (64) holds and similarly for (59) and (65).

Indeed:

$$\begin{aligned}\Gamma\circ\psi^A_H&=((\mu_A\circ(A\otimes f_\Gamma))\otimes H)\circ(A\otimes\delta_H)\circ\psi^A_H\\&=((\mu_A\circ(A\otimes f_\Gamma)\circ\psi^A_H)\otimes H)\circ(H\otimes c_{H,A})\circ(\delta_H\otimes A)\\&=((\mu_A\circ(f_\Gamma\otimes\varphi_A)\circ(\delta_H\otimes A))\otimes H)\circ(H\otimes c_{H,A})\circ(\delta_H\otimes A)\\&=(\mu_A\otimes H)\circ(f_\Gamma\otimes\psi^A_H)\circ(\delta_H\otimes A).\end{aligned}$$

The first equality follows by (55), the second and the last ones by the coassociativity of δ_H, and the third one by (58).

$$\begin{aligned}\Gamma\circ\sigma^A_{H,\alpha} &= ((\mu_A\circ(A\otimes f_\Gamma))\otimes H)\circ(A\otimes\delta_H)\circ\sigma^A_{H,\alpha}\\ &= ((\mu_A\circ(A\otimes f_\Gamma)\circ\sigma^A_{H,\alpha})\otimes\mu_H)\circ\delta_{H^2}\\ &= ((\mu_A\circ(\mu_A\otimes\beta)\circ(A\otimes\psi^A_H\otimes H)\circ(((f_\Gamma\otimes H)\circ\delta_H)\otimes((f_\Gamma\otimes H)\circ\delta_H)))\otimes\mu_H)\circ\delta_{H^2}\\ &= (\mu_A\otimes H)\circ(\mu_A\otimes\sigma^A_{H,\beta})\circ(A\otimes\psi^A_H\otimes H)\circ(((f_\Gamma\otimes H)\circ\delta_H)\otimes((f_\Gamma\otimes H)\circ\delta_H)).\end{aligned}$$

The first equality follows by (55), the second one by (41), the third one by (59) and the last one by the definition of ψ^A_H, the naturality of c and the coassociativity of δ_H.

Then,

$$\begin{aligned}\Gamma\circ\mu_{A\otimes_\alpha H} &= ((\mu_A\circ(A\otimes f_\Gamma))\otimes H)\circ(\mu_A\otimes\delta_H)\circ(\mu_A\otimes\sigma^A_{H,\alpha})\circ(A\otimes\psi^A_H\otimes H)\\ &= (\mu_A\otimes H)\circ(\mu_A\otimes\sigma^A_{H,\beta})\circ(\mu_A\otimes((\mu_A\otimes H)\circ(f_\Gamma\otimes\psi^A_H)\\ &\quad\circ(\delta_H\otimes A))\otimes H)\circ(A\otimes\psi^A_H\otimes((f_\Gamma\otimes H)\circ\delta_H))\\ &= (\mu_A\otimes H)\circ(\mu_A\otimes\sigma^A_{H,\beta})\circ(A\otimes(\Gamma\circ((\mu_A\otimes H)\circ(A\otimes\psi^A_H)\\ &\quad\circ(\psi^A_H\otimes A)))\otimes H)\circ(A\otimes H\otimes A\otimes((f_\Gamma\otimes H)\circ\delta_H))\\ &= (\mu_A\otimes H)\circ(A\otimes\mu_A\otimes H)\circ(A\otimes A\otimes\sigma^A_{H,\beta})\circ(A\otimes(\Gamma\circ\psi^A_H)\otimes H)\\ &\quad\circ(A\otimes H\otimes\Gamma)\\ &= (\mu_A\otimes H)\circ(A\otimes\mu_A\otimes H)\circ(A\otimes A\otimes\sigma^A_{H,\beta})\\ &\quad\circ(A\otimes((\mu_A\otimes H)\circ(f_\Gamma\otimes\psi^A_H)\circ(\delta_H\otimes A))\otimes H)\circ(A\otimes H\otimes\Gamma)\\ &= \mu_{A\otimes_\beta H}\circ(\Gamma\otimes\Gamma).\end{aligned}$$

The first equality follows by (55), the second one by the associativity of μ_A and by (65), the third and the fifth ones by (64) and the left A-linearity of Γ, the fourth one by (19), and the last one by the associativity of μ_A.

Finally, (60) follows by

$$\begin{aligned}f_\Gamma\circ\eta_H &= (A\otimes\varepsilon_H)\circ\Gamma\circ(\eta_A\otimes\eta_H) = (A\otimes\varepsilon_H)\circ\Gamma\circ\nabla_{A\otimes H}\circ(\eta_A\otimes\eta_H)\\ &= (A\otimes\varepsilon_H)\circ\Gamma\circ\nu = (A\otimes\varepsilon_H)\circ\nu = \eta_A.\end{aligned}$$ □

Corollary 4.8. *Under the hypothesis of Theorem 4.7, if (58) holds, (59) is equivalent to*

$$\mu_A\circ(A\otimes f_\Gamma)\circ\sigma^A_{H,\alpha} = [\mu_A\circ((\varphi_A\circ(H\otimes f_\Gamma))\otimes f_\Gamma)\circ(H\otimes c_{H,H})\circ(\delta_H\otimes H)]\wedge\beta. \tag{66}$$

Then, if $f_\Gamma\in Reg_{\varphi_A}(H,A)$, we obtain that (59) is equivalent to

$$\alpha\wedge\partial_{1,1}(f_\Gamma) = \partial_{1,0}(f_\Gamma)\wedge\partial_{1,2}(f_\Gamma)\wedge\beta. \tag{67}$$

Proof. Trivially, if (66) holds, by (58), the naturality of c, and the coassociativity of δ_H, we obtain (59). On the other hand, if (58) holds, we have that (64) holds and then if we assume (59), using (55), the definition of ψ^A_H, the naturality of c, and the coassociativity of δ_H, then

$$\begin{aligned}\mu_A\circ(A\otimes f_\Gamma)\circ\sigma^A_{H,\alpha} &= \mu_A\circ(\mu_A\otimes\beta)\circ(A\otimes\psi^A_H\otimes H)\circ(((f_\Gamma\otimes H)\circ\delta_H)\otimes((f_\Gamma\otimes H)\circ\delta_H))\\ &= \mu_A\circ(A\otimes\beta)\circ((\Gamma\circ\psi^A_H)\otimes H)\circ(H\otimes((f_\Gamma\otimes H)\circ\delta_H))\\ &= \mu_A\circ(A\otimes\beta)\circ(((((\mu_A\circ(A\otimes f_\Gamma))\otimes H)\circ(A\otimes\delta_H))\circ\psi^A_H)\otimes H)\\ &\quad\circ(H\otimes((f_\Gamma\otimes H)\circ\delta_H))\\ &= [\mu_A\circ((\varphi_A\circ(H\otimes f_\Gamma))\otimes f_\Gamma)\circ(H\otimes c_{H,H})\circ(\delta_H\otimes H)]\wedge\beta.\end{aligned}$$

Finally, it is obvious that $\mu_A \circ (A \otimes f_\Gamma) \circ \sigma^A_{H,\alpha} = \alpha \wedge \partial_{1,1}(f_\Gamma)$ and, by (49) and $\beta \circ \Omega^2_H = \beta$, we have

$$\partial_{1,0}(f_\Gamma) \wedge \partial_{1,2}(f_\Gamma) \wedge \beta = [\mu_A \circ ((\varphi_A \circ (H \otimes f_\Gamma)) \otimes f_\Gamma) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)] \wedge \beta. \tag{68}$$

□

Theorem 4.9. *Under the hypothesis of Theorem 4.5, the weak crossed products $A \otimes_\alpha H$ and $A \otimes_\beta H$, associated to α and β, are equivalent if and only if there exists $f \in Reg^+_{\varphi_A}(H, A)$ such that the equalities (58) and (67) hold.*

Proof. If the weak crossed products $A \otimes_\alpha H$ and $A \otimes_\beta H$ are equivalent, then by Theorem 4.5 there exist multiplicative and preunit-preserving morphisms $\Gamma, \Gamma' \in {}_A Hom^H_C(A \otimes H, A \otimes H)$ satisfying (e1)–(e3). Then, by Remark 4.4, $f_\Gamma \in Reg_{\varphi_A}(H, A)$, and by Theorem 4.7, the equalities (58) and $f_\Gamma \circ \eta_H = \eta_A$ hold. Finally, by Corollary 4.8 we get (67). Conversely, let $f \in Reg^+_{\varphi_A}(H, A)$, with inverse f^{-1}. Then, Γ_f and $\Gamma_{f^{-1}}$ are morphisms of left A-modules and right H-comodules satisfying (e1)–(e3) and preserving the preunit $\nu = \nabla_{A\otimes H} \circ (\eta_A \otimes \eta_H)$. Indeed, by (14) and Proposition 2.15(iii), we have $\Gamma_f \circ \nu = \nu$. Similarly, $\Gamma_{f^{-1}} \circ \nu = \nu$. By Theorem 4.7 and Corollary 4.8, Γ_f is multiplicative and $\omega_{\alpha,\beta} = p_{A\otimes H} \circ \Gamma_f \circ i_{A\otimes H}$ is an H-comodule algebra isomorphism with inverse $\omega^{-1}_{\alpha,\beta} = p_{A\otimes H} \circ \Gamma_{f^{-1}} \circ i_{A\otimes H}$. Then, $\Gamma_{f^{-1}}$ is multiplicative and, by Theorem 4.5, $A \otimes_\alpha H$ and $A \otimes_\beta H$ are equivalent. □

Remark 4.10. Note that, if H is a cocommutative weak Hopf algebra, (A, φ_A) is a weak left H-module algebra, and $f : H \to A$ a morphism, the equality (58) is always true if A is commutative. Then, if (A, φ_A) is a left H-module algebra, the equivalence between two weak crossed products $A \otimes_\alpha H$ and $A \otimes_\beta H$ is determined by the inclusion of f in $Reg^+_{\varphi_A}(H, A)$ and the equality (67). In this case, (67) is equivalent to saying that $\alpha \wedge \beta^{-1} \in Im(D^{1+}_{\varphi_A})$.

Theorem 4.11. *Let H be a cocommutative weak Hopf algebra, and let (A, φ_A) be a commutative left H-module algebra. Then there is a bijective correspondence between $H^2_{\varphi_A}(H, A)$ and the equivalence classes of weak crossed products of $A \otimes_\alpha H$ where $\alpha : H \otimes H \to A$ satisfies the 2-cocycle condition (43)(equivalently (51)) and the normal condition (52).*

Proof. First note that $H^2_{\varphi_A}(H, A)$ is isomorphic to $H^{2+}_{\varphi_A}(H, A)$. Then it suffices to prove the result for $H^{2+}_{\varphi_A}(H, A)$. Let $\alpha, \beta \in Reg^+_{\varphi_A}(H^2, A)$ satisfy the 2-cocycle condition (43) (in the commutative case the twisted condition is always satisfied). If $A \otimes_\alpha H$ and $A \otimes_\beta H$ are equivalent, by the previous Remark, $\alpha \wedge \beta^{-1} \in Im(D^{1+}_{\varphi_A})$. Then α and β are in the same class in $H^{2+}_{\varphi_A}(H, A)$. Conversely, if $[\alpha] = [\beta]$ in $H^{2+}_{\varphi_A}(H, A)$, α and β satisfy (67), i.e., $\alpha \wedge \beta^{-1} = D^{1+}_{\varphi_A}(f)$, for $f \in Reg^+_{\varphi_A}(H, A)$. Then, if Γ_f is the morphism defined in Remark 4.4, we have that Γ_f satisfies (57), because, using that $f \in Reg^+_{\varphi_A}(H, A)$,

$$\begin{aligned} p_{A\otimes H} \circ \Gamma_f \circ \nu &= p_{A\otimes H} \circ (f \otimes H) \circ \delta_H \circ \eta_H = p_{A\otimes H} \circ ((f \circ \overline{\Pi}^L_H) \otimes H) \circ \delta_H \circ \eta_H \\ &= p_{A\otimes H} \circ ((f \circ \Pi^L_H) \otimes H) \circ \delta_H \circ \eta_H = p_{A\otimes H} \circ (u_1 \otimes H) \circ \delta_H \circ \eta_H = p_{A\otimes H} \circ \nu. \end{aligned}$$

In a similar way, $\beta \wedge \alpha^{-1} = D^{1+}_{\varphi_A}(f^{-1})$ and $\Gamma_{f^{-1}}$ satisfies (57). Then, by Theorem 4.7, Γ_f and $\Gamma_{f^{-1}}$ are multiplicative morphisms of left A-modules and right H-comodules preserving the preunit and satisfying (e1)–(e3). Therefore, by Theorem 4.5, we obtain that $A \otimes_\alpha H$ and $A \otimes_\beta H$ are equivalent weak crossed products. $\square$

Example 4.12. Let G be a finite groupoid, and let $\mathbb{F}$ be a field. Let

$$\mathbb{F}G = \bigoplus_{\sigma \in G_1} \mathbb{F}\sigma$$

be the groupoid algebra defined in Example 2.3. If G_1 is finite, $\mathbb{F}G$ is a finite cocommutative weak Hopf algebra. Then $G\mathbb{F} = (\mathbb{F}G)^*$ is a commutative weak Hopf algebra defined by

$$G\mathbb{F} = \bigoplus_{\sigma \in G_1} \mathbb{F}f_\sigma$$

with $< f_\sigma, \tau >= \delta_{\sigma,\tau}$. The algebra structure is given by the formulas

$$f_\sigma . f_\tau = \delta_{\sigma,\tau} f_\sigma, \qquad 1_{G\mathbb{F}} = \eta_{G\mathbb{F}}(1_\mathbb{F}) = \sum_{\sigma \in G_1} f_\sigma,$$

and the coalgebra structure is

$$\delta_{G\mathbb{F}}(f_\sigma) = \sum_{t(\sigma)=t(\tau)} f_\tau \otimes f_{\tau^{-1}\sigma}, \quad \varepsilon_{G\mathbb{F}}(\sum_{\sigma \in G_1} a_\sigma f_\sigma) = \sum_{x \in G_0} a_x f_x,$$

where f_x denotes the morphism f_{id_x} and a_x denotes its coefficient. The antipode is giving by $\lambda_{G\mathbb{F}}(f_\sigma) = f_{\sigma^{-1}}$. The algebra $G\mathbb{F}$ is a left $\mathbb{F}G$-module algebra with action

$$\varphi_{G\mathbb{F}}(\omega \otimes f_\sigma) = \delta_{s(\omega),s(\sigma)} f_{\sigma\omega^{-1}}.$$

Let G be the groupoid with $G_0 = \{x, y\}$ and

$$G_1 = \{id_x, id_y, \sigma : x \to y, \sigma^{-1} : y \to x\}.$$

Then $\{id_x, id_y, \sigma, \sigma^{-1}\}$ is a basis for $\mathbb{F}G$ and $\{f_x, f_y, f_\sigma, f_{\sigma^{-1}}\}$ is a basis for $G\mathbb{F}$. The neutral element of $Reg^+_{\varphi_{G\mathbb{F}}}(\mathbb{F}G, G\mathbb{F})$ is the morphism $u_1 = \varphi_{G\mathbb{F}} \circ (\mathbb{F}G \otimes \eta_{G\mathbb{F}})$ such that

$$u_1(id_x) = u_1(\sigma^{-1}) = f_x + f_\sigma, \quad u_1(id_y) = u_1(\sigma) = f_y + f_{\sigma^{-1}}.$$

Moreover, $h \in Reg^+_{\varphi_{G\mathbb{F}}}(\mathbb{F}G, G\mathbb{F})$ if and only if

$$h(id_x) = f_x + f_\sigma, \quad h(id_y) = f_y + f_{\sigma^{-1}}, \quad h(\sigma) = af_y + bf_{\sigma^{-1}}, \quad h(\sigma^{-1}) = cf_x + df_\sigma$$

with $a, b, c, d \in \mathbb{F}^*$. Therefore, the inverse of h is the morphism h^{-1} such that

$$h^{-1}(id_x) = f_x + f_\sigma, \quad h^{-1}(id_y) = f_y + f_{\sigma^{-1}}, \quad h^{-1}(\sigma) = a^{-1}f_y + b^{-1}f_{\sigma^{-1}},$$

$$h^{-1}(\sigma^{-1}) = c^{-1}f_x + d^{-1}f_\sigma,$$

and, as a consequence, $Reg^+_{\varphi_{G\mathbb{F}}}(\mathbb{F}G, G\mathbb{F})$ is isomorphic to the group $\mathbb{F}^* \times \mathbb{F}^* \times \mathbb{F}^* \times \mathbb{F}^*$.

To compute $D^{1+}_{\varphi_{G\mathbb{F}}}(h)$ it is enough to obtain the images of all elements $\omega \otimes \tau$ such that $t(\omega) = s(\tau)$ because, if $t(\omega) \neq s(\tau)$, $\phi(\omega \otimes \tau) = 0$ for all $\phi \in Reg^+_{\varphi_{G\mathbb{F}}}(\mathbb{F}G^2, G\mathbb{F})$.

Then we have the following:

$$D^{1+}_{\varphi_{G\mathbb{F}}}(h)(id_x\otimes id_x)=D^{1+}_{\varphi_{G\mathbb{F}}}(h)(id_x\otimes \sigma^{-1})=D^{1+}_{\varphi_{G\mathbb{F}}}(h)(\sigma^{-1}\otimes id_y)=f_x+f_\sigma,$$
$$D^{1+}_{\varphi_{G\mathbb{F}}}(h)(id_y\otimes id_y)=D^{1+}_{\varphi_{G\mathbb{F}}}(h)(id_y\otimes \sigma)=D^{1+}_{\varphi_{G\mathbb{F}}}(h)(\sigma\otimes id_x)=f_y+f_{\sigma^{-1}},$$
$$D^{1+}_{\varphi_{G\mathbb{F}}}(h)(\sigma\otimes\sigma^{-1})=adf_y+bcf_{\sigma^{-1}},\ D^{1+}_{\varphi_{G\mathbb{F}}}(h)(\sigma^{-1}\otimes\sigma)=bcf_x+adf_\sigma.$$

The neutral element of $Reg^+_{\varphi_{G\mathbb{F}}}(\mathbb{F}G^2,G\mathbb{F})$ is $u_2=\varphi_{G\mathbb{F}}\circ(\mathbb{F}G\otimes u_1)$. Then

$$u_2(id_x\otimes id_x)=u_2(id_x\otimes\sigma^{-1})=u_2(\sigma^{-1}\otimes id_y)=u_2(\sigma^{-1}\otimes\sigma)=f_x+f_\sigma,$$
$$u_2(id_y\otimes id_y)=u_2(id_y\otimes\sigma)=u_2(\sigma\otimes id_x)=u_2(\sigma\otimes\sigma^{-1})=f_y+f_{\sigma^{-1}}.$$

Thus, $h\in Ker(D^{1+}_{\varphi_{G\mathbb{F}}})$ if and only if $d=a^{-1}$, $c=b^{-1}$. Hence the groups $Ker(D^{1+}_{\varphi_{G\mathbb{F}}})$ and $Im(D^{1+}_{\varphi_{G\mathbb{F}}})$ are isomorphic to $\mathbb{F}^*\times\mathbb{F}^*$. Moreover, $\phi\in Reg^+_{\varphi_{G\mathbb{F}}}(\mathbb{F}G^2,G\mathbb{F})$ if and only if

$$\phi(id_x\otimes id_x)=\phi(id_x\otimes\sigma^{-1})=\phi(\sigma^{-1}\otimes id_y)=f_x+f_\sigma,$$
$$\phi(id_y\otimes id_y)=\phi(id_y\otimes\sigma)=\phi(\sigma\otimes id_x)=f_y+f_{\sigma^{-1}},$$
$$\phi(\sigma\otimes\sigma^{-1})=mf_y+nf_{\sigma^{-1}},\qquad \phi(\sigma^{-1}\otimes\sigma)=pf_x+qf_\sigma,$$

with $m,n,p,q\in\mathbb{F}^*$. The inverse of ϕ is the morphism ϕ^{-1} defined by

$$\phi^{-1}(id_x\otimes id_x)=\phi^{-1}(id_x\otimes\sigma^{-1})=\phi^{-1}(\sigma^{-1}\otimes id_y)=f_x+f_\sigma,$$
$$\phi^{-1}(id_y\otimes id_y)=\phi^{-1}(id_y\otimes\sigma)=\phi^{-1}(\sigma\otimes id_x)=f_y+f_{\sigma^{-1}},$$
$$\phi^{-1}(\sigma\otimes\sigma^{-1})=m^{-1}f_y+n^{-1}f_{\sigma^{-1}},\qquad \phi^{-1}(\sigma^{-1}\otimes\sigma)=p^{-1}f_x+q^{-1}f_\sigma.$$

Then $Reg^+_{\varphi_{G\mathbb{F}}}(\mathbb{F}G^2,G\mathbb{F})$ is isomorphic to the group $\mathbb{F}^*\times\mathbb{F}^*\times\mathbb{F}^*\times\mathbb{F}^*$ and $\phi\in Im(D^{1+}_{\varphi_{G\mathbb{F}}})$ if and only if $m=q$ and $n=p$. In this case $D^{1+}_{\varphi_{G\mathbb{F}}}(h)=\phi$ for h defined by

$$h(id_x)=f_x+f_\sigma,\quad h(id_y)=f_y+f_{\sigma^{-1}},\quad h(\sigma)=mf_y+f_{\sigma^{-1}},\quad h(\sigma^{-1})=f_x+nf_\sigma.$$

Let ϕ be in $Reg^+_{\varphi_{G\mathbb{F}}}(\mathbb{F}G^2,G\mathbb{F})$. As in the previous case, to compute $D^{2+}_{\varphi_{G\mathbb{F}}}(\phi)$ we need only to obtain the images of $\omega\otimes\tau\otimes\pi$ satisfying $s(\omega)=t(\tau)$ and $s(\tau)=t(\pi)$. Then,

$$\begin{aligned}
D^{2+}_{\varphi_{G\mathbb{F}}}(\phi)(id_x\otimes id_x\otimes id_x)&=D^{2+}_{\varphi_{G\mathbb{F}}}(\phi)(id_x\otimes id_x\otimes\sigma^{-1})=D^{2+}_{\varphi_{G\mathbb{F}}}(\phi)(id_x\otimes\sigma^{-1}\otimes id_y)\\
&=D^{2+}_{\varphi_{G\mathbb{F}}}(\phi)(id_x\otimes\sigma^{-1}\otimes\sigma)=D^{2+}_{\varphi_{G\mathbb{F}}}(\phi)(\sigma^{-1}\otimes id_y\otimes id_y)\\
&=D^{2+}_{\varphi_{G\mathbb{F}}}(\phi)(\sigma^{-1}\otimes id_y\otimes\sigma)=D^{2+}_{\varphi_{G\mathbb{F}}}(\phi)(\sigma^{-1}\otimes\sigma\otimes id_x)\\
&=f_x+f_\sigma,\\
D^{2+}_{\varphi_{G\mathbb{F}}}(\phi)(id_y\otimes id_y\otimes id_y)&=D^{2+}_{\varphi_{G\mathbb{F}}}(\phi)(id_y\otimes id_y\otimes\sigma)=D^{2+}_{\varphi_{G\mathbb{F}}}(\phi)(id_y\otimes\sigma\otimes id_x)\\
&=D^{2+}_{\varphi_{G\mathbb{F}}}(\phi)(id_y\otimes\sigma\otimes\sigma^{-1})=D^{2+}_{\varphi_{G\mathbb{F}}}(\phi)(\sigma\otimes id_x\otimes id_x)\\
&=D^{2+}_{\varphi_{G\mathbb{F}}}(\phi)(\sigma\otimes id_x\otimes\sigma^{-1})=D^{2+}_{\varphi_{G\mathbb{F}}}(\phi)(\sigma\otimes\sigma^{-1}\otimes id_y)\\
&=f_y+f_{\sigma^{-1}},\\
D^{2+}_{\varphi_{G\mathbb{F}}}(\phi)(\sigma\otimes\sigma^{-1}\otimes\sigma)&=qm^{-1}f_y+pn^{-1}f_{\sigma^{-1}},\\
D^{2+}_{\varphi_{G\mathbb{F}}}(\phi)(\sigma^{-1}\otimes\sigma\otimes\sigma^{-1})&=np^{-1}f_x+mq^{-1}f_\sigma.
\end{aligned}$$

Therefore, $\phi\in Ker(D^{2+}_{\varphi_{G\mathbb{F}}})$ if and only if $m=q$ and $n=f$ and then $Ker(D^{2+}_{\varphi_{G\mathbb{F}}})=$

$Im(D^{1+}_{\varphi_{G\mathbb{F}}})$. This implies that $H^2_{\varphi_{G\mathbb{F}}}(\mathbb{F}G, G\mathbb{F}) = \{1\}$. As a consequence, all weak crossed products $G\mathbb{F} \otimes_\phi \mathbb{F}G$ are equivalent.

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