VERSAL DEFORMATION THEORY OF ALGEBRAS OVER A QUADRATIC OPERAD

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Abstract

We develop deformation theory of algebras over quadratic operads where the parameter space is a commutative local algebra. We also give a construction of a distinguised deformation of an algebra over a quadratic operad with a complete local algebra as its base—the so-called versal deformation—which induces all other deformations of the given algebra.

In memory of our good friend Jean-Louis Loday

1. Introduction

Formal one-parameter deformation theory for algebras was originally introduced for associative algebras by M. Gerstenhaber in the 1960s (see [**7**, **8**]). Since then it has been applied to many other algebraic categories. Most of these cases turned out to be algebras over a suitable quadratic operad. Gerstenhaber's theory was generalized to an algebra over a binary quadratic operad by D. Balavoine in 1997 (see [**1**]) and further developed by M. Kontsevich and Y. Soibelman (see [**12**]). In [**13**], J.L. Loday and B. Vallette introduced a deformation complex for any $\mathcal{P}$-algebra A over a quadratic operad $\mathcal{P}$. It turns out that this deformation complex is the resulting dg Lie algebra obtained from convolution dg Lie algebra $\mathfrak{g}_{\mathcal{P},A} = (\mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{\mathrm{i}}, \mathrm{End}_A)), [,])$, where $\mathcal{P}^{\mathrm{i}}$ is the Koszul dual co-operad of $\mathcal{P}$, twisted by the algebra structure.

Classical deformation theory is not general enough to describe all nonequivalent deformations of a given object. To take care of this, one needs to enlarge the base of deformations from a one parameter power series ring to a local commutative algebra, or more generally, to a complete local algebra. It is known that under certain cohomological restrictions there exists a "characteristic" versal deformation of a type of algebras, with complete local algebra base, that induces all nonequivalent deformations and is universal at the infinitesimal level (see, e.g., [**18**]). An explicit construction of versal deformation has been given for Lie [**3**, **4**], associative, infinity [**5**], and Leibniz algebras [**6**].

The aim of this paper is to show that the above method of construction of versal deformations of a particular type of algebra can be extended to construct versal deformations of an algebra over any quadratic operad. For this it is necessary to develop the relevant operadic tools. To this end we develop an obstruction theory for extending a given deformation, of an algebra over a quadratic operad with a finite-dimensional local commutative algebra base $\mathcal{R}$, to a deformation over a suitable extension of $\mathcal{R}$ using the operadic calculus as developed in [**13**, Chapter 12].

The paper is organized as follows: In Section 2 we briefly review the basic definitions and results involving algebras over a quadratic operad, and the associated deformation complex necessary for our purpose, and we fix the notation that we follow throughout the text. In Section 3 we develop the deformation theory of algebras over an operad with local commutative algebra base. In Section 4 we study infinitesimal deformations and their properties. Given a deformation λ of a $\mathcal{P}$-algebra A, with a finite-dimensional base $\mathcal{R}$ and a local abelian extension $0 \to M \to \mathcal{R}' \to \mathcal{R} \to 0$ of $\mathcal{R}$ by an $\mathcal{R}$-module M with $\dim_{\mathcal{R}} M < \infty$, we address in Section 5 the question of extending λ to a deformation of A with base $\mathcal{R}'$. It turns out that, in general, this extension is not always possible. The associated obstruction can be interpreted as a 2-cochain in the deformation complex associated to A. We prove that obstructions are 2-cocycles and that vanishing of the associated cohomology class is a necessary and sufficient condition for extending a given deformation to a deformation over a larger base. The results of Section 5 are crucial in order to construct a versal deformation of an algebra over a quadratic operad, which is the content of the next section. In Section 6 we introduce the notion of formal deformation of algebras over a quadratic operad with a complete local algebra base, and we define the notion of versal deformation. Finally, we give a construction of a versal deformation of an algebra over a quadratic operad, using the results developed in Sections 4 and 5.

2.   Preliminaries on operads and operadic cohomology

In this section we recall some basic definitions and results about (algebraic) operads and the deformation complex of an algebra over a quadratic operad as in [**13**]. General references for operads and related results are [**9, 10, 13, 14, 15, 16, 17**].

The symbol $\mathbb{N}$ denotes the set of non-negative integers. Throughout this paper, we work over a fixed field $\mathbf{k}$ of characteristic 0. We denote the category of vector spaces over $\mathbf{k}$ by **Vect**. The tensor product of vector spaces over $\mathbf{k}$ is denoted by $\otimes$. For any positive integer n, $\mathbb{S}_n$ denotes the group of permutations on n elements. For any map $f \colon \otimes_{i=1}^{n} E_i \to F$, $f(x_1, \cdots, x_n)$ will stand for $f(x_1 \otimes \cdots \otimes x_n)$.

2.1. $\mathbb{S}$-module
Definition 2.1. An $\mathbb{S}$-module over $\mathbf{k}$ is a family

$$M = \{M(0), M(1), \cdots, M(n), \cdots\}$$

of right $\mathbf{k}[\mathbb{S}_n]$ modules $M(n)$. An $\mathbb{S}$-module M is finite dimensional if $M(n)$ is finite dimensional for all n. A morphism $f \colon M \longrightarrow N$ between two $\mathbb{S}$-modules M and N is a family of maps $f(n) \colon M(n) \longrightarrow N(n)$ that are $\mathbb{S}_n$-equivariant for all n.

Note that to every $\mathbb{S}$-module M is associated a functor, called the *Schur functor*,

$\widetilde{M} \colon \mathbf{Vect} \longrightarrow \mathbf{Vect}$ such that $\widetilde{M}(V) = \oplus_{n \geqslant 0} M(n) \otimes_{\mathbb{S}_n} V^{\otimes n}$, where the left action of $\mathbb{S}_n$ on $V^{\otimes n}$ is given by $\sigma(v_1, v_2, \cdots, v_n) = (v_{\sigma^{-1}(1)}, v_{\sigma^{-1}(2)}, \cdots, v_{\sigma^{-1}(n)})$.

The *tensor product* of two $\mathbb{S}$-modules is the $\mathbb{S}$-module $M \otimes N$ defined by

$$(M \otimes N)(n) = \oplus_{i+j=n} Ind_{\mathbb{S}_i \times \mathbb{S}_j}^{\mathbb{S}_n} M(i) \otimes N(j).$$

The *composite* of two $\mathbb{S}$-modules M and N is the $\mathbb{S}$-module

$$M \circ N = \oplus_{n \geqslant 0} M(n) \otimes N^{\otimes n},$$

where $N^{\otimes n}$ is the tensor product of n copies of the $\mathbb{S}$-module N. The category $(\mathbb{S}\text{-Mod}, \circ, I)$ is a monoidal category with respect to this composite product, where I is the $\mathbb{S}$-module $(0, \mathbf{k}, 0, \cdots)$.

Recall that the composite of the two $\mathbb{S}$-modules M and N satisfies the formula

$$(\widetilde{M \circ N}) = \widetilde{M} \circ \widetilde{N},$$

where on the right-hand side the symbol $\circ$ stands for the composition of functors.

2.2.   Operads and cohomology

Definition 2.2. A *symmetric operad* $\mathcal{P} = (\mathcal{P}, \gamma, \eta)$ is a monoid in the monoidal category $(\mathbb{S}\text{-Mod}, \circ, I)$. Explicitly, a symmetric operad $\mathcal{P} = (\mathcal{P}, \gamma, \eta)$ is an $\mathbb{S}$-module $\mathcal{P} = \{\mathcal{P}(n)\}_{n \geqslant 0}$ endowed with morphisms of $\mathbb{S}$-modules

$$\gamma \colon \mathcal{P} \circ \mathcal{P} \longrightarrow \mathcal{P},$$

called the composition map, and

$$\eta \colon I \longrightarrow \mathcal{P},$$

called the unit map, which makes $\mathcal{P}$ into a monoid.

Throughout the paper, by an *operad* we shall mean a *symmetric operad*.

Definition 2.3. Let $\mathcal{P}, \mathcal{Q}$ be operads. A *morphism of operads* from $\mathcal{P}$ to $\mathcal{Q}$ is a morphism of $\mathbb{S}$-modules $\alpha \colon \mathcal{P} \longrightarrow \mathcal{Q}$ that is compatible with the composition maps.

Example 2.4. Let V be a vector space over $\mathbf{k}$, and for every $n \in \mathbb{N}$ let $\mathrm{End}_V(n) = \mathrm{Hom}_{\mathbf{k}}(V^{\otimes n}, V)$. Then $\mathrm{End}_V = \{\mathrm{End}_V(n), \ n \in \mathbb{N}\}$ is naturally a symmetric operad, called the *endomorphism operad of* V.

Definition 2.5. Let $\mathcal{P}$ be an operad. A $\mathcal{P}$-*algebra*, or an algebra over $\mathcal{P}$, denoted by (A, α), is a vector space A over $\mathbf{k}$, equipped with a morphism of operads $\alpha \colon \mathcal{P} \rightarrow \mathrm{End}_A$.

A *morphism of* $\mathcal{P}$-*algebras* $\phi \colon (A, \alpha) \longrightarrow (B, \beta)$ is a $\mathbf{k}$-linear map $\phi \colon A \longrightarrow B$ such that for any $a_1, \cdots, a_n \in A$ and $\mu \in \mathcal{P}(n)$,

$$\phi(\alpha(\mu)(a_1, \cdots, a_n)) = \beta(\mu)(\phi(a_1), \cdots, \phi(a_n)).$$

Definition 2.6. The *free operad* over the $\mathbb{S}$-module M is an operad $\mathcal{F}(M)$ equipped with an $\mathbb{S}$-module morphism $\eta(M) \colon M \longrightarrow \mathcal{F}(M)$ that satisfies the following universal condition: any $\mathbb{S}$-module morphism $f \colon M \longrightarrow \mathcal{P}$, where $\mathcal{P}$ is an operad, extends uniquely to an operad morphism $\widetilde{f} \colon \mathcal{F}(M) \longrightarrow \mathcal{P}$, that is, $\widetilde{f} \circ \eta(M) = f$.

The functor $\mathcal{F}$ from the category of $\mathbb{S}$-modules to the category of **k**-operads is left adjoint to the forgetful functor from the category of **k**-operads to $\mathbb{S}$-modules. An explicit construction of the free operad can be found in [**2**].

Definition 2.7. Let $\mathcal{P}$ be an operad. An ideal of $\mathcal{P}$ is a sub-$\mathbb{S}$-module $\mathcal{I}$ of $\mathcal{P}$ such that the operad structure of $\mathcal{P}$ passes on to the quotient $\mathcal{P}/\mathcal{I}$.

Let E be an $\mathbb{S}$-module, and R be a sub-$\mathbb{S}$-module such that $R \subseteq \mathcal{F}(E)^{(2)}$, where $\mathcal{F}(E)^{(2)}$ is the graded sub-$\mathbb{S}$-module of the free operad $\mathcal{F}(E)$, which is spanned by the composites of two elements of E [**13**, Section 5.5.3]. Such a pair (E, R) is called a quadratic data.

Definition 2.8. Given a quadratic data (E, R), let

$$\mathcal{P}(E, R) = \mathcal{F}(E)/(R)$$

be the quotient of the free operad $\mathcal{F}(E)$ over E by the operadic ideal (R) generated by R. Then the operad $\mathcal{P}(E, R)$ is called the quadratic operad associated to the pair (E, R).

Definition 2.9. An $\mathbb{S}$-module $\mathcal{C}$ is a *cooperad* if it is a comonoid in the monoidal category $(\mathbb{S}\text{-mod}, \circ, I)$. A *cofree cooperad* on an $\mathbb{S}$-module M is the cooperad $\mathcal{F}^c(M)$, which is cofree in the category of conilpotent cooperads.

Recall that the Koszul dual cooperad of a quadratic operad is defined as follows:

Definition 2.10. The *quadratic cooperad* $\mathcal{C}(E, R)$ associated to the quadratic data (E, R) is the sub-cooperad of the cofree cooperad $\mathcal{F}^c(E)$ that is universal among the sub-cooperads of $\mathcal{F}^c(E)$ such that the following composite is 0,

$$\mathcal{C} \hookrightarrow \mathcal{F}^c(E) \twoheadrightarrow \mathcal{F}^c(E)^{(2)}/(R).$$

The *Koszul dual cooperad* of the quadratic operad $\mathcal{P}(E, R)$ is the quadratic cooperad

$$\mathcal{P}^{\mathsf{i}} := \mathcal{C}(sE, s^2 R),$$

where sE denotes the $\mathbb{S}$-module E whose degree is shifted by 1.

We recall the definition of the deformation complex of an algebra over a quadratic operad [**13**].

Let A be a $\mathcal{P}$-algebra, where $\mathcal{P} = \mathcal{P}(E, R)$ is a quadratic operad. Let

$$\mathfrak{g} = \mathfrak{g}_{\mathcal{P}, A} = (\mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{\mathsf{i}}, \mathrm{End}_A), [,], \partial)$$

be the *convolution dg Lie algebra* associated to A. Since A is concentrated in degree 0, the cohomological degree on the differential graded Lie algebra

$$\mathfrak{g} \colon \mathrm{Hom}(A, A) \xrightarrow{\partial} \mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{\mathsf{i}(1)}, \mathrm{End}_A) \xrightarrow{\partial} \mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{\mathsf{i}(2)}, \mathrm{End}_A) \cdots$$

is induced by the weight grading. Thus $\mathrm{Hom}(A, A)$ is the 0th cochain module and $\mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{\mathsf{i}(n)}, \mathrm{End}_A)$ is the nth cochain module. Moreover, since $\mathcal{P}$ is homogeneous quadratic, the coboundary map ∂ in the above complex is null. We note that the set of $\mathcal{P}$-algebra structures on a space A is in one-to-one correspondence with the set of Maurer–Cartan elements of degree 1 of $\mathfrak{g}$, i.e., all elements ϕ in $\mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{\mathsf{i}(1)}, \mathrm{End}_A)$

satisfying $[\phi, \phi] = 0$. Given such an element ϕ, one can define a differential ∂_ϕ on the Lie algebra $\mathfrak{g}$, which makes it into a differential graded Lie algebra. The differential ∂_ϕ is called the twisted differential, and $\mathfrak{g}^\phi = \mathfrak{g}^\phi_{\mathcal{P},A} = (\mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{\mathrm{i}}, \mathrm{End}_A)), [,], \partial_\phi)$ is called the twisted differential graded Lie algebra. The twisted differential ∂_ϕ is given by: $\partial_\phi(f) = [\phi, f]$. The underlying cochain complex of this twisted dg Lie algebra is the deformation complex that we intend to work with. We end this section with the following definition.

Definition 2.11. For any $\mathcal{P}$-algebra (A, π) where $\mathcal{P}$ is a quadratic operad, we define

$$H^*_{\mathcal{P}}(A) := H_*(\mathfrak{g}^\pi) = H_*(\mathfrak{g}, \partial_\pi).$$

3.   Deformations

Let $\mathcal{R}$ be a commutative local unital algebra with unit $1_{\mathcal{R}}$ over $\mathbf{k}$. Let $\epsilon \colon \mathcal{R} \to \mathbf{k}, \epsilon(1_{\mathcal{R}}) = 1$ be the canonical augmentation map, and let $\mathfrak{M} = ker(\epsilon)$ be the unique maximal ideal in $\mathcal{R}$. In this section, we study the notion of deformation of an algebra over a quadratic operad with base $\mathcal{R}$, and its properties.

Let $\mathcal{P} = \mathcal{P}(E, R)$ be a quadratic operad. We will denote by $\mathcal{P}_{\mathcal{R}}$ the operad that is obtained by extension of $\mathcal{P}$ to the category of modules over $\mathcal{R}$; in other words, $\mathcal{P}_{\mathcal{R}}(n) = \mathcal{R} \otimes \mathcal{P}(n)$ for all $n \in \mathbb{N}$. Let (A, π) be a $\mathcal{P}$-algebra. Let $A_{\mathcal{R}} = \mathcal{R} \otimes A$ denote the extension of A. Then $A_{\mathcal{R}}$ can be viewed as a $\mathcal{P}_{\mathcal{R}}$-algebra by extending $\pi \colon \mathcal{P} \longrightarrow \mathrm{End}_A$ to $\pi_{\mathcal{R}} \colon \mathcal{P}_{\mathcal{R}} \longrightarrow \mathrm{End}_{A_{\mathcal{R}}}$, since

$$\mathrm{Hom}_{\mathcal{R}}(A_{\mathcal{R}}^{\otimes_{\mathcal{R}} n}, A_{\mathcal{R}}) \cong \mathcal{R} \otimes \mathrm{Hom}_{\mathbf{k}}(A^{\otimes n}, A).$$

Moreover, $A = \mathbf{k} \otimes A$ can be viewed as a $\mathcal{P}_{\mathcal{R}}$-algebra by considering A as a module over $\mathcal{R}$ via ϵ; that is, the $\mathcal{R}$-module structure on A is given by $r \cdot a = \epsilon(r)a$, for $r \in \mathcal{R}$ and $a \in A$.

Definition 3.1. A deformation λ of a $\mathcal{P}$-algebra (A, π) with base $(\mathcal{R}, \mathfrak{M})$ is a morphism of operads $\lambda \colon \mathcal{P}_{\mathcal{R}} \longrightarrow \mathrm{End}_{A_{\mathcal{R}}}$ such that $(\epsilon \otimes \mathrm{Id}) \colon A_{\mathcal{R}} \longrightarrow \mathbf{k} \otimes A \cong A$ is a $\mathcal{P}_{\mathcal{R}}$-algebra morphism.

In other words, a deformation of (A, π) with base $(\mathcal{R}, \mathfrak{M})$ is a $\mathcal{P}_{\mathcal{R}}$-algebra structure on $A_{\mathcal{R}}$ that reduces to π, modulo $\mathfrak{M}$: $A \cong \mathbf{k} \otimes_{\mathcal{R}} (\mathcal{R} \otimes A)$.

We have the following result [**13**, Proposition 12.2.6].

Proposition 3.2. *For any $\mathcal{P}$-algebra (A, π), the set of all deformations of (A, π) with base $(\mathcal{R}, \mathfrak{M})$ is in one-to-one correspondence with the set of all Maurer–Cartan elements in the dg Lie algebra $\mathfrak{M} \otimes \mathfrak{g}^\pi$.*

Definition 3.3. Suppose λ_1 and λ_2 are two deformations of a $\mathcal{P}$-algebra (A, π) with the same base $(\mathcal{R}, \mathfrak{M})$. They are said to be equivalent if there exists a $\mathcal{P}_{\mathcal{R}}$-algebra isomorphism $\phi \colon (A_{\mathcal{R}}, \lambda_1) \longrightarrow (A_{\mathcal{R}}, \lambda_2)$ such that

$$
\begin{array}{ccc}
A_{\mathcal{R}} & \overset{\phi}{\longrightarrow} & A_{\mathcal{R}} \\
{\scriptstyle \epsilon \otimes \mathrm{Id}} \searrow & & \swarrow {\scriptstyle \epsilon \otimes \mathrm{Id}} \\
& \mathbf{k} \otimes A &
\end{array}
$$

commutes.

Definition 3.4. Let $\mathcal{R}$ be a complete local algebra, $\mathcal{R} = \overleftarrow{\lim} \, (\mathcal{R}/\mathfrak{M}^n)$, $\mathfrak{M}$ denoting the maximal ideal in $\mathcal{R}$. A formal deformation of a $\mathcal{P}$-algebra (A, π) with base $(\mathcal{R}, \mathfrak{M})$ is a $\mathcal{P}_{\mathcal{R}}$-algebra structure on the completed tensor product $\mathcal{R} \widehat{\otimes} A = \overleftarrow{\lim} \, ((\mathcal{R}/\mathfrak{M}^n) \otimes A)$, such that $\epsilon \widehat{\otimes} \, \mathrm{Id} \colon \mathcal{R} \widehat{\otimes} A \longrightarrow \mathbf{k} \otimes A \cong A$ is a $\mathcal{P}_{\mathcal{R}}$-algebra morphism.

Example 3.5. Let $\mathcal{R} = \mathbf{k}[[t]]$ be the ring of formal power series with coefficients in $\mathbf{k}$. Then a formal deformation of a $\mathcal{P}$-algebra (A, π) over $\mathcal{R}$ is precisely the formal "1-parameter" deformation of (A, π).

Definition 3.6. Let λ be a deformation of the $\mathcal{P}$-algebra (A, π) with base $(\mathcal{R}, \mathfrak{M})$ and augmentation $\epsilon \colon \mathcal{R} \longrightarrow \mathbf{k}$. Let $\mathcal{R}'$ be another commutative local algebra with unit, and augmentation $\epsilon' \colon \mathcal{R}' \longrightarrow \mathbf{k}$ with $\mathrm{Ker}(\epsilon') = \mathfrak{M}'$. Let $f \colon \mathcal{R} \longrightarrow \mathcal{R}'$ be an algebra homomorphism with $f(1_{\mathcal{R}}) = 1_{\mathcal{R}'}$. Then $\epsilon' \circ f = \epsilon$.

Consider $\mathcal{R}'$ as an $\mathcal{R}$-module via the map $f \colon r' \cdot r = r' f(r)$ so that

$$\mathcal{R}' \otimes A = (\mathcal{R}' \otimes_{\mathcal{R}} \mathcal{R}) \otimes A = \mathcal{R}' \otimes_{\mathcal{R}} (\mathcal{R} \otimes A).$$

Then the *push-out of the deformation* λ is the deformation $f_* \lambda$ of (A, π) with base $(\mathcal{R}', \mathfrak{M}')$, defined by

$$
\begin{aligned}
f_* \lambda(1_{\mathcal{R}'} \otimes \mu)\{r_1' \otimes_{\mathcal{R}} (r_1 \otimes a_1), r_2' \otimes_{\mathcal{R}} (r_2 \otimes a_2), \cdots, r_n' \otimes_{\mathcal{R}} (r_n \otimes a_n)\} \\
= r_1' r_2' \cdots r_n' \otimes_{\mathcal{R}} \lambda(1_{\mathcal{R}} \otimes \mu)(r_1 \otimes a_1, r_2 \otimes a_2, \cdots, r_n \otimes a_n), \quad \mu \in \mathcal{P}^{\mathrm{i}(1)}(n).
\end{aligned}
$$

It is straightforward to see that $f_* \lambda$ is a deformation of (A, π) with base $(\mathcal{R}', \mathfrak{M}')$.

Remark 3.7. Note that by the proof of Proposition 3.2, as in [**13**], if λ is a deformation of the $\mathcal{P}$-algebra (A, π) with a base $(\mathcal{R}, \mathfrak{M})$, then λ can be expressed as

$$\lambda = \pi + \sum_{i=1}^{s} m_i \otimes \phi_i, \tag{3.1}$$

where π is a Maurer–Cartan element of $\mathfrak{g}^{\pi}$ and $\sum_{i=1}^{s} m_i \otimes \phi_i$ is a Maurer–Cartan element of $\mathfrak{M} \otimes \mathfrak{g}^{\pi}$. Also, the push-out $f_* \lambda$ is given by

$$f_* \lambda = \pi + \sum_{i=1}^{s} f(m_i) \otimes \phi_i, \tag{3.2}$$

where $\sum_{i=1}^{s} f(m_i) \otimes \phi_i$ is a Maurer–Cartan element of $\mathfrak{M}' \otimes \mathfrak{g}^{\pi}$.

Definition 3.8. A deformation λ of (A, π) with base $(\mathcal{R}, \mathfrak{M})$ is called *infinitesimal* if, in addition, $\mathfrak{M}^2 = 0$.

To consider the equivalence of infinitesimal deformations, cohomology comes into play naturally.

Let λ be an infinitesimal deformation of (A, π), with base $(\mathcal{R}, \mathfrak{M})$ such that

$$\lambda = \pi + \sum_{i=1}^{s} m_i \otimes \phi_i.$$

Let $\xi \in \mathfrak{M}' = \mathrm{Hom}_{\mathbf{k}}(\mathfrak{M}, \mathbf{k})$. Clearly, ξ can be viewed as an element of $\mathrm{Hom}_{\mathbf{k}}(\mathcal{R}, \mathbf{k})$ with $\xi(1_{\mathcal{R}}) = 0$.

Define a 1-cochain $\alpha_{\lambda, \xi}$ of $\mathfrak{g}^{\pi}$ by

$$\alpha_{\lambda, \xi} = (\xi \otimes \mathrm{Id})\lambda.$$

Theorem 3.9. *For any infinitesimal deformation λ of a $\mathcal{P}$-algebra (A, π), $\alpha_{\lambda,\xi}$ is a 1-cocycle.*

Proof. As in Remark 3.7, we write $\lambda = \pi + \sum_{i=1}^{s} m_i \otimes \phi_i$. Then, using the fact that $\xi(1_{\mathcal{R}}) = 0$, the 1-cochain $\alpha_{\lambda,\xi}$ can be expressed as

$$\alpha_{\lambda,\xi} = (\xi \otimes \mathrm{Id})\lambda = (\xi \otimes \mathrm{Id})(\pi + \sum_{i=1}^{s} m_i \otimes \phi_i) = \sum_{i=1}^{s} \xi(m_i)\phi_i.$$

The fact that $\sum_{i=1}^{s} m_i \otimes \phi_i$ ia a Maurer–Cartan element in $\mathfrak{M} \otimes \mathfrak{g}^{\pi}$, and the fact that $\mathfrak{M}^2 = 0$, implies that

$$\sum_{i=1}^{s} m_i \otimes [\phi_i, \pi] = 0.$$

Therefore,

$$\sum_{i=1}^{s} \xi(m_i)[\pi, \phi_i] = (\xi \otimes \mathrm{Id})(\sum_{i=1}^{s} m_i \otimes [\pi, \phi_i]) = 0.$$

Using the above observations, we deduce that

$$\partial_{\pi}(\alpha_{\lambda,\xi}) = [\pi, \alpha_{\lambda,\xi}] = [\pi, \sum_{i=1}^{s} \xi(m_i)\phi_i] = \sum_{i=1}^{s} \xi(m_i)[\pi, \phi_i] = 0. \qquad \square$$

Let us define for $\xi \in \mathfrak{M}'$ the cohomology class of the cocycle $\alpha_{\lambda,\xi}$ by $a_{\lambda,\xi}$. The correspondence $\xi \longmapsto a_{\lambda,\xi}$ defines a map

$$a_{\lambda} \colon \mathfrak{M}' \longrightarrow H^1_{\mathcal{P}}(A).$$

Theorem 3.10. *Let λ_1 and λ_2 be two infinitesimal deformations of (A, π) with base $(\mathcal{R}, \mathfrak{M})$. Assume that $\mathcal{R}$ is of finite dimension. Then the deformations λ_1 and λ_2 are equivalent iff $\alpha_{\lambda_1,\xi}$ and $\alpha_{\lambda_2,\xi}$ represent the same cohomology class; that is, $a_{\lambda_1,\xi} = a_{\lambda_2,\xi}$ for $\xi \in \mathfrak{M}'$.*

Proof. This result is proved in [**13**, Theorem 12.2.7] for $R = \mathbf{k}[t]/(t^2)$. The present theorem is proved by following the same idea. We only give the essential steps.

Let $\{m_i\}_{1 \leqslant i \leqslant r}$ be a basis of $\mathfrak{M}$, and let $\{\xi_i\}_{1 \leqslant i \leqslant r}$ be the dual basis of $\mathfrak{M}'$. By definition, λ_1 and λ_2 are equivalent if, and only if, there exists a $\mathcal{P}_{\mathcal{R}}$-algebra isomorphism

$$\rho \colon A_{\mathcal{R}} \longrightarrow A_{\mathcal{R}}, \text{ such that } (\epsilon \otimes \mathrm{Id}) \circ \rho = \epsilon \otimes \mathrm{Id}. \tag{3.3}$$

Since $A_{\mathcal{R}} = \mathcal{R} \otimes A = (\mathbf{k} \oplus \mathfrak{M}) \otimes A \cong A \oplus (\mathfrak{M} \otimes A)$, the isomorphism ρ can be written as $\rho = \rho_1 + \rho_2$, where $\rho_1 \colon A \longrightarrow A$ and $\rho_2 \colon A \longrightarrow \mathfrak{M} \otimes A$.

By compatibility (3.3), we get $\rho_1 = \mathrm{Id}$. Using the adjunction property of tensor products, we have

$$\mathrm{Hom}_{\mathbf{k}}(A; \mathfrak{M} \otimes A) \cong \mathfrak{M} \otimes \mathrm{Hom}_{\mathbf{k}}(A, A) \cong \mathrm{Hom}_{\mathbf{k}}(\mathfrak{M}'; \mathrm{Hom}_{\mathbf{k}}(A, A)),$$

where the isomorphisms are given by

$$\rho_2 \longmapsto \sum_{1}^{r} m_i \otimes \phi_i \longmapsto \sum_{i}^{r} \chi_i. \tag{3.4}$$

Here $\phi_i = (\xi_i \otimes \mathrm{id}) \circ \rho_2$, $\chi_i(\xi_j) = \delta_{i,j}\phi_i$.

Thus we can write

$$\rho = \mathrm{Id} + \sum_1^r m_i \otimes \phi_i.$$

Recall that ρ is a $\mathcal{P}_\mathcal{R}$-algebra morphism iff

$$\rho(\lambda_1(1_\mathcal{R} \otimes \mu)) = \lambda_2(1_\mathcal{R} \otimes \mu)(\rho, \rho, \cdots, \rho), \qquad (3.5)$$

$\mu \in \mathcal{P}^{\mathrm{i}(1)}(n)$, where on the right-hand side we have n many copies of ρ.

Let us set $\psi_i^k = \alpha_{\lambda_k, \xi_i}$, $i = 1, 2, \ldots, r$, and $k = 1, 2$. Then we have

$$\lambda_k(1_\mathcal{R} \otimes \mu) = \pi(\mu) + \sum_1^r m_i \otimes \psi_i^k(\mu) \qquad (3.6)$$

for all $\mu \in \mathcal{P}^{\mathrm{i}(1)}$. We explicitly write both sides of the equation (3.5), using the fact that the deformations involved are infinitesimal:

$$\lambda_2(1_\mathcal{R} \otimes \mu)(\rho, \rho, \cdots, \rho)$$
$$= \pi(\mu) + \sum_{i=1}^r m_i \otimes \psi_i^2(\mu) + \sum_{k=1}^n \sum_{i=1}^r m_i \otimes (\pi(\mu)(\mathrm{Id}, \mathrm{Id}, \cdots, \phi_i, \mathrm{Id}, \cdots)),$$

where in the last summand ϕ_i is in the kth slot.

On the other hand,

$$\rho(\lambda_1(1_\mathcal{R} \otimes \mu)) = \rho(\pi(\mu) + \sum_1^r m_i \otimes \psi_i^1(\mu))$$
$$= \pi(\mu) + \sum_1^r m_i \otimes \phi_i \circ \pi(\mu) + \sum_1^r m_i \otimes \psi_i^1(\mu).$$

It follows from (3.5) that

$$\sum_1^r m_i \otimes (\psi_i^2 - \psi_i^1) + \sum_1^r m_i \otimes \partial_\pi \phi_i = 0.$$

Hence,

$$\partial_\pi \phi_i = \psi_i^2 - \psi_i^1 = \alpha_{\lambda_2, \xi_i} - \alpha_{\lambda_1, \xi_i} \qquad \text{for all } i = 1, \ldots, r.$$

So, $a_{\lambda_1, \xi} = a_{\lambda_2, \xi}$, for all $\xi \in \mathfrak{M}'$. $\qquad\qquad\square$

Let $\mathcal{R}$ be a local algebra with maximal ideal $\mathfrak{M}$. Then $\mathcal{R}/\mathfrak{M}^2$ is local with maximal ideal $\mathfrak{M}/\mathfrak{M}^2$ and $(\mathfrak{M}/\mathfrak{M}^2)^2 = 0$. Let $q \colon \mathcal{R} \longrightarrow \mathcal{R}/\mathfrak{M}^2$ be the projection map. Let λ be a deformation of (A, π) with base $(\mathcal{R}, \mathfrak{M})$. Then the deformation $q_*\lambda$ is infinitesimal and we have a map

$$a_{q_*\lambda} \colon (\mathfrak{M}/\mathfrak{M}^2)' \longrightarrow H_\mathcal{P}^1(A).$$

Definition 3.11. The dual space $(\mathfrak{M}/\mathfrak{M}^2)'$ is called the *tangent space* of $\mathcal{R}$ and is denoted by $T\mathcal{R}$. The mapping

$$a_{q_*\lambda} \colon (\mathfrak{M}/\mathfrak{M}^2)' \longrightarrow H_\mathcal{P}^1(A)$$

is called the *differential* of λ and is denoted by $d\lambda$. In particular, if $\mathfrak{M}^2 = 0$, then the differential $d\lambda$ of the infinitesimal deformation λ is the map a_λ.

Corollary 3.12. *If two deformations λ_1 and λ_2 of a $\mathcal{P}_{\mathcal{R}}$-algebra (A, π) are equivalent, then their* differentials *are equal.*

4. Universal Infinitesimal deformation

In this section we construct a specific example of an infinitesimal deformation of a $\mathcal{P}$-algebra satisfying finite dimensionality of the first cohomology module. We shall also prove a fundamental property of this deformation. In the last section we will see that this infinitesimal deformation is the first step of an inductive construction of a versal deformation.

Let (A, π) be a given $\mathcal{P}$-algebra satisfying the condition $\dim H^1_{\mathcal{P}}(A) < \infty$. Let us denote $H^1_{\mathcal{P}}(A)$ by $\mathbb{H}$. Consider the $\mathbf{k}$-algebra $C_1 = \mathbf{k} \oplus \mathbb{H}'$ with the following structure:

$$(k_1, h_1) \cdot (k_2, h_2) = (k_1 k_2, k_1 h_2 + k_2 h_1).$$

Clearly, C_1 is local with maximal ideal $\mathfrak{M} = \mathbb{H}'$ and $\mathfrak{M}^2 = 0$.

Fix a homomorphism

$$\sigma \colon \mathbb{H} \longrightarrow \mathfrak{g}^1_{\mathcal{P},A} = \operatorname{Hom}_{\mathbb{S}}(\mathcal{P}^{\mathrm{i}(1)}, \operatorname{End}_A)$$

that takes a cohomology class into a representative cocycle. We note that

$$\mathbb{H}' \otimes A = \operatorname{Hom}_{\mathbf{k}}(H^1_{\mathcal{P}}(A), \mathbf{k}) \otimes A \cong \operatorname{Hom}_{\mathbf{k}}(H^1_{\mathcal{P}}(A), A)$$

and

$$A_{\mathcal{C}_1} = C_1 \otimes A \cong (\mathbf{k} \oplus \mathbb{H}') \otimes A \cong A \oplus \operatorname{Hom}_{\mathbf{k}}(\mathbb{H}, A).$$

Define a $\mathcal{P}_{C_1}$-algebra structure η_1 on A_{C_1} by

$$\eta_1 = \pi + \sum_{i=1}^{n} h_i^{\,*} \otimes \sigma(h_i) \tag{4.1}$$

as a map $C_1 \otimes \mathcal{P}^{\mathrm{i}(1)} \longrightarrow (\mathbf{k} \oplus \mathbb{H}') \otimes \operatorname{End}_A = C_1 \otimes \operatorname{End}_A$, where $\{h_i\}_{1 \leqslant i \leqslant n}$ denotes a finite basis of $\mathbb{H}$ and $\{h_i^{\,*}\}_{1 \leqslant i \leqslant n}$ denotes the dual basis.

Proposition 4.1. *For any homomorphism* $\sigma \colon \mathbb{H} \longrightarrow \mathfrak{g}^1_{\mathcal{P},A} = Hom_{\mathbb{S}}(\mathcal{P}^{\mathrm{i}(1)}, End_A)$, (A_{C_1}, η_1) *is a $\mathcal{P}_{C_1}$-algebra.*

Proof. We need to show that η_1 is a Maurer–Cartan element in $C_1 \otimes \mathfrak{g}^{\pi}$. The coboundary of the complex $C_1 \otimes \mathfrak{g}^{\pi}$ is given by

$$\bar{\partial}_{\pi} := \operatorname{Id}_{C_1} \otimes \partial_{\pi}.$$

From (4.1),

$$\eta_1 = \pi + \sum_i h_i^{\,*} \otimes \sigma(h_i).$$

So

$$\bar{\partial}_{\pi}(\eta_1) = \operatorname{Id}_{C_1} \otimes [\pi, \eta_1] = \operatorname{Id}_{C_1} \otimes [\pi, \pi] + \sum_i h_i^{*} \otimes [\pi, \sigma(h_i)]) = 0,$$

using the fact that $\sigma(h_i)$ is a 1-cocycle in $\mathfrak{g}^{\pi}$. $\qquad \square$

Proposition 4.2. *Up to isomorphism, the $\mathcal{P}_{C_1}$-algebra structures of A_{C_1} does not depend on the choice of σ.*

Proof. Let $\sigma' \colon \mathbb{H} \longrightarrow \mathfrak{g}^1_{\mathcal{P},A}$ be another choice of σ, and denote the corresponding $\mathcal{P}_{C_1}$-algebra structure on $C_1 \otimes A$ by η'. Then for $h \in \mathbb{H}$, $\sigma(h)$ and $\sigma'(h)$ are two 1-cocycles of A, representing the same cohomology class; that is, $\sigma(h) - \sigma'(h)$ is a 1-coboundary. Let $\sigma(h) - \sigma'(h) = \partial_\pi(\gamma(h))$, where $\gamma \colon \mathbb{H} \longrightarrow \text{Hom}_{\mathbf{k}}(A, A)$. Here $\text{Hom}_{\mathbf{k}}(A, A)$ is the 0 cochains of $\mathfrak{g}^\pi$. Using the identification

$$C_1 \otimes A \cong A \oplus \text{Hom}_{\mathbf{k}}(\mathbb{H}, A),$$

define a C_1-linear automorphism $\rho \colon C_1 \otimes A \longrightarrow C_1 \otimes A$ by $\rho(a, \phi) = (a, \bar{\phi})$, where $\bar{\phi}(h) = \phi(h) + \gamma(h)(a)$. We need to show that ρ preserves the $\mathcal{P}_{C_1}$-algebra structure, that is,

$$\rho(\eta_1(1_{C_1} \otimes \mu)) = \eta'(1_{C_1} \otimes \mu) \circ (\rho, \rho, \cdots, \rho), \qquad (4.2)$$

where the number of copies of ρ on the right-hand side is the same as the (homogeneous) degree of μ in $\mathcal{P}^{\text{i}(1)}$. Now observe that, using the isomorphism

$$\text{Hom}_{\mathbf{k}}(\mathbb{H}, A) \cong \text{Hom}_{\mathbf{k}}(\mathbb{H}, \mathbf{k}) \otimes A,$$

we can rewrite the expression (4.1) of η_1 as

$$\eta_1(1_{C_1} \otimes \mu)((a_1, \phi_1), \cdots, (a_n, \phi_n)) = (\pi(\mu)(a_1, \cdots, a_n), \psi_\mu),$$

where $\psi_\mu(h) = \sigma(h)(\mu)(a_1, \cdots, a_n) + \sum_i \pi(\mu)(a_1, \cdots, \phi_i(h), \cdots, a_n)$, where in the summation, ϕ_i is at the ith slot. Similarly,

$$\eta'(1_{C_1} \otimes \mu)((a_1, \phi_1), \cdots, (a_n, \phi_n)) = (\pi(\mu)(a_1, \cdots, a_n), \psi'_\mu),$$

where

$$\begin{aligned}
\psi'_\mu(h) &= \sigma'(h)(\mu)(a_1, \cdots, a_n) + \sum_i \pi(\mu)(a_1, \cdots, \phi_i(h), \cdots, a_n) \\
&= (\sigma(h) - \partial_\pi(\gamma(h)))(\mu)(a_1, \cdots, a_n) + \sum_i \pi(\mu)(a_1, \cdots, \phi_i(h), \cdots, a_n).
\end{aligned}$$

We evaluate both sides of (4.2) using the expressions of η_1, η' as given above and the definitions of ρ and the coboundary ∂_π, to get $\psi'_\mu(h) - \psi_\mu(h) = 0$, for all $h \in \mathbb{H}$. Then equation (4.2) holds. $\square$

The main property of the infinitesimal deformation η_1 is its universality in the class of infinitesimal deformations with finite-dimensional base.

Theorem 4.3. *For any infinitesimal deformation λ of the $\mathcal{P}$-algebra A with a finite-dimensional local base $(\mathcal{R}, \mathfrak{M})$, there exists a unique homomorphism $\phi \colon C_1 \to \mathcal{R}$ such that λ is equivalent to the push-out $\phi_* \eta_1$.*

Proof. Let $a_\lambda \colon \mathfrak{M}' \to H^1_{\mathcal{P}}(A) = \mathbb{H}$ denote the differential of λ,

$$a_\lambda \colon \xi \mapsto a_{\lambda, \xi} = [\alpha_{\lambda, \xi}], \qquad \xi \in \mathfrak{M}'.$$

Consider the map $\phi = \text{Id} \oplus a'_\lambda \colon \mathbf{k} \oplus \mathbb{H}' \to \mathbf{k} \oplus \mathfrak{M} = \mathcal{R}$. Let $\{m_i\}_{i=1}^r$ be a basis of $\mathfrak{M}$, and let $\{\xi_i\}$ be the dual basis.

It is enough to show $\alpha_{\phi_* \eta_1} = \sigma \circ a_\lambda$ (Theorem 3.10). Let $\{h_1, \cdots, h_n\}$ be a basis of $\mathbb{H}$, and let $\{h_1{}^*, \cdots, h_n{}^*\}$ be the corresponding dual basis of $\mathbb{H}'$.

Then by Remark 3.7 and equation (3.2) we have

$$\phi_* \eta_1 = \pi + \sum_{j=1}^{n} \phi(h_j{}^*) \otimes \sigma(h_j).$$

Now,

$$a'_\lambda(h_j{}^*) = \sum_{i=1}^{r} \xi_i(a'_\lambda(h_j{}^*))m_i \qquad \text{and} \qquad a_\lambda(\xi_i) = \sum_{j=1}^{n} h_j{}^*(a_\lambda(\xi_i))h_j.$$

Thus

$$
\begin{aligned}
\alpha_{\phi_* \eta_1}&(\xi_i)(\mu; a_1, \cdots, a_n) \\
&= (\xi_i \otimes \mathrm{Id})\phi_* \eta_1(\mu)(1_{\mathcal{R}} \otimes a_1, \cdots, 1_{\mathcal{R}} \otimes a_n) \\
&= (\xi_i \otimes \mathrm{Id}) \left(\pi(\mu)(a_1, \cdots, a_n) + \sum_{j=1}^{n} \phi(h_j{}^*) \otimes \sigma(h_j)(\mu; a_1, \cdots, a_n) \right) \\
&= (\xi_i \otimes \mathrm{Id}) \left(\sum_{j=1}^{n} a'_\lambda(h_j{}^*) \otimes \sigma(h_j)(\mu; a_1, \cdots, a_n) \right) \\
&= \sum_{j=1}^{n} \xi_i(a'_\lambda(h_j{}^*)) \otimes \sigma(h_j)(\mu; a_1, \cdots, a_n) \\
&= \sum_{j=1}^{k} h_j{}^*(a_\lambda(\xi_i)) \otimes \sigma(h_j)(\mu; a_1, \cdots, a_n) \\
&= \sigma \left(\sum_{j=1}^{n} h_j{}^*(a_\lambda(\xi_i))h_j \right) (\mu; a_1, \cdots, a_n) \\
&= \sigma \circ a_\lambda(\xi_i)(\mu; a_1, \cdots, a_n).
\end{aligned}
$$

Therefore, $\alpha_{\phi_* \eta_1} = \sigma \circ a_\lambda$. The uniqueness of the homomorphism $\phi \colon C_1 \longrightarrow \mathcal{R}$ follows from the definition of ϕ and the fact that two infinitesimal deformations θ and θ' are equivalent if, and only if, the corresponding maps a_θ and $a_{\theta'}$ are equal, (cf. Theorem 3.10). $\qquad \square$

5. Deformation Extensions

Let us recall some definitions and results from [**11**]. Let $\mathcal{R}$ be a commutative algebra over $\mathbf{k}$. Let $(C_q(\mathcal{R}), \delta)$ denote the standard Hochschild complex, where $C_q(\mathcal{R})$ is the $\mathcal{R}$-module $\mathcal{R}^{\otimes(q+1)}$ with $\mathcal{R}$ acting on the first factor by multiplication of $\mathcal{R}$. Let $Sh_q(\mathcal{R})$ be the $\mathcal{R}$-submodule of $C_q(\mathcal{R})$ generated by chains

$$s_p(r_0, r_1, r_2, \cdots, r_q) = \sum_{\sigma \in Sh(p, q-p)} sgn(\sigma)(r_0, r_{\sigma^{-1}(1)}, r_{\sigma^{-1}(2)}, \cdots, r_{\sigma^{-1}(q)}) \in C_q(\mathcal{R})$$

for $r_1, r_2, \cdots, r_q \in \mathcal{R}$; $0 < p < q$.

Then Sh_* is a subcomplex of $C_*(\mathcal{R})$, and hence we have a complex called the

Harrison complex

$$Ch_*(\mathcal{R}) = \{Ch_q(\mathcal{R}), \delta\}; \qquad Ch_q(\mathcal{R}) = C_q(\mathcal{R})/Sh_q(\mathcal{R}).$$

For an $\mathcal{R}$-module M, the Harrison cochain complex defining the Harrison cohomology with coefficients in M is given by $Ch^*(\mathcal{R}; M) = Hom_\mathcal{R}(Ch_*(\mathcal{R}), M)$.

Definition 5.1. For an $\mathcal{R}$-module M, we define

$$H^q_{Harr}(\mathcal{R}; M) = Z^q(\mathcal{R}, M)/B^q(\mathcal{R}, M) = H^q(Ch^*(\mathcal{R}, M)),$$

where $Z^q(\mathcal{R}, M)$ and $B^q(\mathcal{R}, M)$ are the spaces of q-cocycles and q-coboundaries, respectively.

Proposition 5.2. *Let $\mathcal{R}$ be a commutative local algebra with maximal ideal $\mathfrak{M}$, and let M be an $\mathcal{R}$-module with $\mathfrak{M}M = 0$. Then we have the canonical isomorphism*

$$H^q_{Harr}(\mathcal{R};\ M) \cong H^q_{Harr}(\mathcal{R}; \mathbf{k}) \otimes M.$$

Definition 5.3. A *(split) abelian extension* or *square-zero extension* $\mathcal{R}'$ of $\mathcal{R}$ by an $\mathcal{R}$-module M is a $\mathbf{k}$-algebra $\mathcal{R}'$ together with an exact sequence of $\mathbf{k}$-modules

$$0 \longrightarrow M \xrightarrow{\ i\ } \mathcal{R}' \xrightarrow{\ p\ } \mathcal{R} \longrightarrow 0,$$

where p is an algebra homomorphism so that $N = i(M)$ is an $\mathcal{R}'$-module and this $\mathcal{R}'$-module structure is induced by the $\mathcal{R}$-module structure on M as $r'i(m) = i(p(r')m)$. In particular, N is an ideal in $\mathcal{R}'$ satisfying $N^2 = 0$.

Definition 5.4. An abelian extension $\mathcal{R}'$ of $\mathcal{R}$ by $\mathcal{R}$-module M is called a *local abelian extension* if, in addition, $\mathfrak{M}M = 0$, where $\mathfrak{M}$ is the maximal ideal of $\mathcal{R}$.

Henceforth, by an extension we shall always mean a local abelian extension.

Remark 5.5. Note that, as $\mathcal{R}$ is local, $\mathcal{R}'$ is also local with $\mathfrak{M}_{\mathcal{R}'} = p^{-1}(\mathfrak{M})$ as its maximal ideal. Moreover, the condition $\mathfrak{M}M = 0$ clearly implies that, for any $x \in \mathfrak{M}_{\mathcal{R}'}$ and $n \in N$, $xn = 0$.

We will use the following results relating Harrison cohomology and extensions of the algebra $\mathcal{R}$ by means of M, [**11**].

Proposition 5.6.   (i) *The space $H^1_{Harr}(\mathcal{R}; M)$ is isomorphic to the space of derivations $\mathcal{R} \longrightarrow M$.*

(ii) *Elements of $H^2_{Harr}(\mathcal{R}; M)$ correspond bijectively to isomorphism classes of extensions*

$$0 \longrightarrow M \longrightarrow \mathcal{R}' \longrightarrow \mathcal{R} \longrightarrow 0$$

of the algebra $\mathcal{R}$ by means of M.

(iii) *The space $H^1_{Harr}(\mathcal{R}; M)$ can also be interpreted as the group of automorphisms of any given extension of $\mathcal{R}$ by M.*

Corollary 5.7. *If $\mathcal{R}$ is a local algebra with the maximal ideal $\mathfrak{M}$, then*

$$H^1_{Harr}(\mathcal{R}; \mathbf{k}) \cong \left(\frac{\mathfrak{M}}{\mathfrak{M}^2} \right)' = T\mathcal{R}.$$

Let $\mathcal{R}$ be a finite-dimensional commutative, unital, local algebra with augmentation ϵ and maximal ideal $\mathfrak{M}$. Let λ be a deformation of a $\mathcal{P} = \mathcal{P}(\mathbf{k}, E, R)$-algebra (A, π) with the base $(\mathcal{R}, \mathfrak{M})$. Let $(\mathcal{R}', \mathfrak{M}_{\mathcal{R}'})$ be an extension of $(\mathcal{R}, \mathfrak{M})$ by an $\mathcal{R}$-module M, where $\dim_{\mathcal{R}} M < \infty$. In this section we consider the problem of extending the given deformation λ to a deformation with base $(\mathcal{R}', \mathfrak{M}_{\mathcal{R}'})$.

First, let us consider the case of a one-dimensional extension. Let

$$0 \longrightarrow \mathbf{k} \xrightarrow{i} \mathcal{R}' \xrightarrow{p} \mathcal{R} \longrightarrow 0$$

be any one-dimensional extension of $\mathcal{R}$. By the above proposition the isomorphism classes of one-dimensional extensions of $\mathcal{R}$ are in one-to-one correspondence with the Harrison cohomology $H^2_{Harr}(\mathcal{R}; \mathbf{k})$ of $\mathcal{R}$ with coefficients in $\mathbf{k}$ where the $\mathcal{R}$ module structure on $\mathbf{k}$ is given by $rk = \epsilon(r)k$.

Let $[f] \in H^2_{Harr}(\mathcal{R}; \mathbf{k})$. Suppose $0 \longrightarrow \mathbf{k} \xrightarrow{i} \mathcal{R}' \xrightarrow{p} \mathcal{R} \longrightarrow 0$ is a representative of the class of one-dimensional extensions of $\mathcal{R}$, corresponding to the cohomology class $[f]$.

Let us recall how the algebra structure on $\mathcal{R}'$ is related to f. Fix a splitting $q \colon \mathcal{R} \to \mathcal{R}'$. Let $\hat{\epsilon} = \epsilon \circ p \colon \mathcal{R}' \to \mathbf{k}$ denote the augmentation of $\mathcal{R}'$. Then the map $b \mapsto (p(b), i^{-1}(b - q \circ p(b)))$ is a $\mathbf{k}$-module isomorphism $\mathcal{R}' \cong \mathcal{R} \oplus \mathbf{k}$. Let $(r, k)_q \in \mathcal{R}'$ denote the inverse of $(r, k) \in \mathcal{R} \oplus \mathbf{k}$ under the above isomorphism. The cocycle f representing the extension is determined by $f(r_1, r_2) = i^{-1}((r_1, 0)_q (r_2, 0)_q - (r_1 r_2, 0)_q)$. On the other hand, f determines the algebra structure on $\mathcal{R}'$ by

$$(r_1, k_1)_q \cdot (r_2, k_2)_q := (r_1 r_2, r_1 \cdot k_2 + r_2 \cdot k_1 + f(r_1, r_2))_q.$$

As in Section 3, let $\{m_i\}_{i=1}^r$ be a fixed basis of the maximal ideal $\mathfrak{M}$ of $\mathcal{R}$ with the dual basis $\{\xi_i\}_{i=1}^r$. Let $\psi_i = \alpha_{\lambda, \xi_i}, 1 \leqslant i \leqslant r$ be the 1-cochain introduced as in Section 3. Then by Proposition 3.2, the deformation λ can be written as

$$\lambda = \pi + \sum_{i=1}^r m_i \otimes \psi_i,$$

where $\sum_{i=1}^r m_i \otimes \psi_i$ is a Maurer–Cartan element in $\mathfrak{M} \otimes \mathfrak{g}^\pi$.

Let $\{n_j\}_{1 \leqslant j \leqslant r+1}$ be defined by $n_j = (m_j, 0)_q$ for $1 \leqslant j \leqslant r$ and $n_{r+1} = (0, 1)_q$. Then $\{n_j\}_{1 \leqslant j \leqslant r+1}$ is a basis of the maximal ideal $\mathfrak{M}_{\mathcal{R}'}$ of $\mathcal{R}'$.

A deformation Γ with base $\mathcal{R}'$ extending λ is entirely determined by the following two facts:

- Γ defined on $\mathcal{P}^{\mathrm{i}(1)}$ can be extended to the category of $\mathcal{R}'$-modules, and
- if $\hat{\Gamma}$ is the unique extension of Γ, then $\hat{\Gamma}(r) = 0$ for every $r \in R$.

Let $\psi \in \mathfrak{g}^1$ be any cochain. Define

$$\Gamma(1_{\mathcal{R}'} \otimes \mu) = \pi(\mu) + \sum_{j=1}^r n_j \otimes \psi_j(\mu) + n_{r+1} \otimes \psi(\mu), \tag{5.1}$$

where $\mu \in \mathcal{P}^{\mathrm{i}(1)}$.

Extending to the category of $\mathcal{R}'$-modules, this defines a $\mathcal{R}'$-linear map

$$\Gamma \colon \mathcal{R}' \otimes \mathcal{P}^{\mathrm{i}(1)} \longrightarrow \mathcal{R}' \otimes \mathrm{End}_A.$$

By the universal property of free operads, we extend Γ to a morphism of operads

$\widetilde{\Gamma} \colon \mathcal{F}(E_{\mathcal{R}'}) \longrightarrow \mathrm{End}(A_{\mathcal{R}'})$. Now, Γ induces a $\mathcal{P}_{\mathcal{R}'}$-algebra structure on $A_{\mathcal{R}'}$ if, and only if, $\widetilde{\Gamma}((R)) = 0$, and it is clear from our construction of Γ that it extends the given deformation λ.

As λ defines an algebra structure on $\mathcal{R} \otimes A$, $[\lambda, \lambda] = 0$. This implies [**13**, Proposition 12.2.6])

$$\sum_{i=1}^{r} m_i \otimes \partial_\pi(\psi_i) + 1/2 \sum_{i,j=1}^{r} m_i m_j \otimes [\psi_i, \psi_j] = 0. \tag{5.2}$$

Note that saying that Γ induces an algebra structure on $\mathcal{R}' \otimes A$ is equivalent to saying that, as an element of the Lie algebra $\mathfrak{g}_{\mathcal{R}' \otimes \mathcal{P}, \mathcal{R}' \otimes A} \cong \mathcal{R}' \otimes \mathfrak{g}_{\mathcal{P}, A}$, $[\Gamma, \Gamma]$ vanishes. From the expression of Γ, we get

$$[\Gamma, \Gamma] = \left[\pi + \sum_{i=1}^{r} n_i \otimes \psi_i + n_{r+1} \otimes \psi, \pi + \sum_{i=1}^{r} n_i \otimes \psi_i + n_{r+1} \otimes \psi \right]$$

$$= [\pi, \pi] + 2 \sum_{i=1}^{r} n_i \otimes \partial_\pi \psi_i + 2 n_{r+1} \otimes \partial_\pi \psi + \sum_{i,j=1}^{r} n_i n_j \otimes [\psi_i, \psi_j]$$

$$+ 2 \sum_{i=1}^{r} n_i n_{r+1} \otimes [\psi_i, \psi] + n_{r+1}^2 \otimes [\psi, \psi],$$

using the fact that $\partial_\pi(f) = [\pi, f]$.

We note that $[\pi, \pi] = 0$. As $n_i n_{r+1} = 0$ for $1 \leqslant i \leqslant r$ ($\mathfrak{M}\mathbf{k} = 0$) and $n_{r+1}^2 = 0$, the above expression is equal to

$$2 \sum_{i=1}^{r} n_i \otimes \partial_\pi \psi_i + 2 n_{r+1} \otimes \partial_\pi \psi + \sum_{i,j=1}^{r} n_i n_j \otimes [\psi_i, \psi_j].$$

Thus Γ defines a $\mathcal{R}' \otimes \mathcal{P}$-algebra structure on $\mathcal{R}' \otimes A$ extending λ, the $\mathcal{R} \otimes \mathcal{P}$-algebra structure on $\mathcal{R} \otimes A$, if, and only if,

$$[\Gamma, \Gamma] = 0$$

$$\iff \quad n_{r+1} \otimes \partial_\pi \psi = -\sum_{i=1}^{r} n_i \otimes \partial_\pi \psi_i - \frac{1}{2} \sum_{i,j=1}^{r} n_i n_j \otimes [\psi_i, \psi_j]$$

$$\iff (0,1)_q \otimes \partial_\pi(\psi) = -\sum_{i=1}^{r} (m_i, 0)_q \otimes \partial_\pi(\psi_i) - \frac{1}{2} \sum_{i,j=1}^{r} (m_i m_j, f(m_i, m_j))_q \otimes [\psi_i, \psi_j]$$

(using the isomorphism between $\mathcal{R}' \cong \mathcal{R} \oplus \mathbf{k}$)

$$\iff \quad \partial_\pi(\psi) = -\frac{1}{2} \sum_{i,j=1}^{r} f(m_i, m_j)[\psi_i, \psi_j].$$

Let us define a 2-cochain Φ on $\mathfrak{g}_{\mathcal{P}, A}$ as follows:

$$\Phi = \sum_{i,j=1}^{r} f(m_i, m_j)[\psi_i, \psi_j].$$

This cochain is called the *obstruction cochain*.

Proposition 5.8. *The obstruction cochain Φ is a 2-cocycle in $\mathfrak{g}_{\mathcal{P}, A}$.*

Proof. From equation 5.2,

$$\sum_{i=1}^{r}(m_i,0)_q \otimes \partial_\pi \psi_i + 1/2 \sum_{i,j=1}^{r} (m_i m_j,0)_q \otimes [\psi_i,\psi_j] = 0 \qquad (5.3)$$

as an element of $\mathcal{R}' \otimes \mathfrak{g}^\pi$.

Let ∂_π' be the differential of $\mathcal{R}' \otimes \mathfrak{g}$ induced by ∂_π in $\mathfrak{g}^\pi$. Then

$$\partial_\pi' \left(\sum_{i,j=1}^{r} n_i n_j \otimes [\psi_i,\psi_j] \right)$$

$$= \sum_{i,j=1}^{r} n_i n_j \otimes \partial_\pi [\psi_i,\psi_j]$$

$$= \left(\sum_{i,j=1}^{r} n_i n_j \otimes \left([\partial_\pi \psi_i, \psi_j] - [\psi_i, \partial_\pi \psi_j] \right) \right)$$

$$= \left(\left[\sum_{i=1}^{r} n_i \otimes \partial_\pi \psi_i, \sum_{j=1}^{r} n_j \otimes \psi_j \right] - \left[\sum_{i=1}^{r} n_i \otimes \psi_i, \sum_{j=1}^{r} n_j \otimes \partial_\pi \psi_j \right] \right)$$

$$= \left[\sum_{i=1}^{r} (m_i,0)_q \otimes \partial_\pi \psi_i, \sum_{j=1}^{r} (m_j,0)_q \otimes \psi_j \right]$$

$$\qquad - \left[\sum_{i=1}^{r} (m_i,0)_q \otimes \psi_i, \sum_{j=1}^{r} (m_j,0)_q \otimes \partial_\pi \psi_j \right]$$

$$= -\frac{1}{2} \left[\sum_{k,l=1}^{r} (m_k m_l,0)_q \otimes [\psi_k,\psi_l], \sum_{j=1}^{r} (m_j,0)_q \otimes \psi_j \right]$$

$$\qquad + \frac{1}{2} \left[\sum_{i=1}^{r} (m_i,0)_q \otimes \psi_i, \sum_{k,l=1}^{r} (m_k m_l,0)_q \otimes [\psi_k,\psi_l] \right], \text{by (5.3)}$$

$$= -\frac{1}{2} \sum_{i,j,k=1}^{r} (m_i m_j m_k, f(m_i m_j, m_k))_q \otimes [[\psi_i,\psi_j],\psi_k]$$

$$\qquad + \frac{1}{2} \sum_{i,j,k=1}^{r} (m_i m_j m_k, f(m_i, m_j m_k))_q \otimes [\psi_i,[\psi_j,\psi_k]]$$

$$= \frac{1}{2} \sum_{i,j,k=1}^{r} n_i n_j n_k \otimes \left([\psi_i,[\psi_j,\psi_k]] - [[\psi_i,\psi_j],\psi_k] \right)$$

as $f(m_i m_j, m_k) = f(m_i, m_j m_k)$, f being a 2-cocycle and $\mathfrak{M}\mathbf{k} = 0$

$$= \frac{1}{2} \sum_{i,j,k=1}^{r} n_i n_j n_k \otimes [[\psi_i,\psi_k],\psi_j]$$

$$= 0, \quad \text{by the Jacobi identity.}$$

On the other hand,

$$\partial_\pi'(\sum_{i,j=1}^{r} n_i n_j \otimes [\psi_i,\psi_j]) = \sum_{i,j=1}^{r} (m_i m_j, f(m_i, m_j))_q \otimes \partial_\pi [\psi_i,\psi_j].$$

Hence, by the previous argument,

$$\sum_{i,j=1}^{r} (m_i m_j, f(m_i, m_j))_q \otimes \partial_\pi [\psi_i, \psi_j] = 0.$$

By equation (5.2), $\sum_{i,j=1}^{r} m_i m_j \otimes \partial_\pi [\psi_i, \psi_j] = 0$. Hence,

$$\partial_\pi \Phi = \sum_{i,j=1}^{r} f(m_i, m_j) \partial_\pi [\psi_i, \psi_j] = 0. \qquad \square$$

The above proposition enables us to define a map from the set of 2-cocycles $Z^2(\mathcal{R}, \mathbf{k})$ to $H^2_\mathcal{P}(A)$. We shall see that this map passes to cohomology.

Let f and f_1 be two 2-cocycles in the same cohomology class in $H^2_{Harr}(\mathcal{R}; \mathbf{k})$; that is, $f - f_1 = \delta h$ for some 1-cochain $h \in Ch^1(\mathcal{R}, \mathbf{k})$. The obstruction cocycle for extending λ to an extension corresponding to f is given by $\sum_{i,j} f(m_i, m_j)[\psi_i, \psi_j]$. The obstruction cocycle corresponding to the cocycle f_1 is given by $\sum_{i,j} f_1(m_i, m_j)[\psi_i, \psi_j]$. Now,

$$\begin{aligned}
\sum_{i,j} f(m_i, m_j)[\psi_i, \psi_j] - \sum_{i,j} f_1(m_i, m_j)[\psi_i, \psi_j] &= \sum_{i,j} \delta h(m_i, m_j)[\psi_i, \psi_j] \\
&= \sum_{i,j} (m_i h(m_j) - h(m_i m_j) + h(m_i)m_j)[\psi_i, \psi_j] \\
&= -\sum_{i,j} h(m_i, m_j)[\psi_i, \psi_j].
\end{aligned}$$

The terms $m_i h(m_j)$ and $h(m_i)m_j$ vanish as $\epsilon(\mathfrak{M}) = 0$. On the other hand, from equation (5.2),

$$\sum_{i=1}^{r} m_i \otimes \partial_\pi(\psi_i) = -1/2 \sum_{i,j=1}^{r} m_i m_j \otimes [\psi_i, \psi_j].$$

Hence

$$(h \otimes \mathrm{id}) \sum_{i=1}^{r} m_i \otimes \partial_\pi \psi_i = -1/2 \sum_{i,j=1}^{r} h(m_i m_j)[\psi_i, \psi_j]$$

$$\Rightarrow \quad \partial_\pi \left(2 \sum_{i=1}^{r} h(m_i)\psi_i \right) = \sum_{i=1}^{r} h(m_i) \partial_\pi(\psi_i) = -\sum_{i,j=1}^{r} h(m_i m_j)[\psi_i, \psi_j].$$

The above consideration defines a map

$$\theta_\lambda \colon H^2_{Harr}(\mathcal{R}; \mathbf{k}) \longrightarrow H^2_\mathcal{P}(A), \quad \text{by } \theta_\lambda([f]) = [\Phi],$$

where $[\Phi]$ is the cohomology class of Φ. The map θ_λ is called the *obstruction map*. The proof of the following proposition is straightforward.

Proposition 5.9. *Let λ be a deformation of a $\mathcal{P}$-algebra A with base $\mathcal{R}$, and let $\mathcal{R}'$ be a one-dimensional extension of $\mathcal{R}$ corresponding to the cohomology class $[f] \in H^2_{Harr}(\mathcal{R}; \mathbf{k})$. Then λ can be extended to a deformation of A with base $\mathcal{R}'$ if, and only if, the obstruction $\theta_\lambda([f]) = 0$.* $\qquad \square$

We state the following proposition, the proof of which is similar to the proof of Corollary 5.8 in [6].

Proposition 5.10. *Suppose that for a deformation λ of a $\mathcal{P}$-algebra A with base $\mathcal{R}$, the differential $d\lambda \colon T\mathcal{R} \longrightarrow \mathbb{H}$ is onto. Then the group of automorphisms $\mathcal{A}$ of the extension*

$$0 \longrightarrow \mathbf{k} \xrightarrow{\ i\ } \mathcal{R}' \xrightarrow{\ p\ } \mathcal{R} \longrightarrow 0 \tag{5.4}$$

operates transitively on the set of equivalence classes of deformations Γ of A with base $\mathcal{R}'$ such that $p_*\Gamma = \lambda$. In other words, if Γ exists, it is unique up to an isomorphism and an automorphism of this extension.

More generally, for an extension

$$0 \longrightarrow M \xrightarrow{\ i\ } \mathcal{R}' \xrightarrow{\ p\ } \mathcal{R} \longrightarrow 0$$

with $\dim_{\mathcal{R}} M < \infty$, the above arguments can be generalized.

The obstruction map in this more general situation is defined by

$$\theta_\lambda \colon H^2_{Harr}(\mathcal{R}; M) \longrightarrow M \otimes H^2_{\mathcal{P}}(A)$$
$$[f] \ \mapsto \ [\Phi].$$

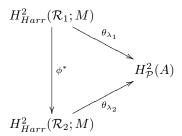Then, as in the case of one-dimensional extension, we have the following.

Proposition 5.11. *Let λ be a deformation of a $\mathcal{P}$-algebra A with base $(\mathcal{R}, \mathfrak{M})$, and let M be a finite-dimensional $\mathcal{R}$-module with $\mathfrak{M}M = 0$. Consider an extension $\mathcal{R}'$ of $\mathcal{R}$*

$$0 \longrightarrow M \xrightarrow{\ i\ } \mathcal{R}' \xrightarrow{\ p\ } \mathcal{R} \longrightarrow 0$$

corresponding to some $[f] \in H^2_{Harr}(\mathcal{R}; M)$. A deformation Γ of A with base $\mathcal{R}'$ such that $p_*\Gamma = \lambda$ exists if, and only if, the obstruction $\theta_\lambda([f]) = 0$. If $d\lambda : T\mathcal{R} \longrightarrow \mathbb{H}$ is onto, then the deformation Γ, if it exists, is unique up to an isomorphism and an automorphism of the above extension.

We end this section with the following naturality property of the obstruction map, the proof of which is similar to Proposition 5.10 in [**6**].

Proposition 5.12. *Suppose $\mathcal{R}_1$ and $\mathcal{R}_2$ are finite-dimensional unital local algebras with augmentations ε_1 and ε_2, respectively. Let $\phi \colon \mathcal{R}_2 \longrightarrow \mathcal{R}_1$ be an algebra homomorphism with $\phi(1) = 1$ and $\varepsilon_1 \circ \phi = \varepsilon_2$. Suppose λ_2 is a deformation of a $\mathcal{P}$-algebra A with base $\mathcal{R}_2$ and $\lambda_1 = \phi_*\lambda_2$ is the push-out via ϕ. Then the following diagram commutes:*

6. Construction of a Versal Deformation

We begin the last section with the definition of the notion of a versal deformation of a $\mathcal{P}$-algebra. The importance of versal deformation lies in the fact that it includes

information of all other non-equivalent deformations of a given object. We use the results developed in the last two sections to give a constructive proof of the existence of versal deformations of a given $\mathcal{P}$-algebra. In fact, starting with the infinitesimal deformation as introduced in Section 4, we give an explicit construction of a versal deformation of a $\mathcal{P}$-algebra by an inductive argument, using the obstruction theory developed in Section 5.

Definition 6.1. A formal deformation η of a $\mathcal{P}$-algebra A with base $\mathcal{R}'$ is called *versal* if it satisfies the conditions

(i) for any formal deformation λ of A with base $\mathcal{R}$ there exists a homomorphism $f\colon \mathcal{R}' \longrightarrow \mathcal{R}$ such that the deformation λ is equivalent to $f_*\eta$;

(ii) if $\mathcal{R}$ satisfies the condition $\mathfrak{M}^2 = 0$, then f is unique.

We proceed to construct a versal deformation of a $\mathcal{P}$-algebra A. As before, let $\mathbb{H} = H^1_{\mathcal{P}}(A)$, and assume that $dim(\mathbb{H}) < \infty$. Let η_1 be the universal infinitesimal deformation with base $\mathcal{C}_1$ as constructed in Section 4. Suppose for some $k \geqslant 1$ we have constructed a finite-dimensional local algebra $\mathcal{C}_k$ and a deformation η_k of A with base $\mathcal{C}_k$. Let

$$\mu\colon H^2_{Harr}(\mathcal{C}_k; \mathbf{k}) \longrightarrow (\mathcal{C}h_2(\mathcal{C}_k))'$$

be a homomorphism sending a cohomology class to a cocycle representing the class. Let

$$f_{\mathcal{C}_k}\colon Ch_2(\mathcal{C}_k) \longrightarrow H^2_{Harr}(\mathcal{C}_k; \mathbf{k})'$$

be the dual of μ. By Proposition 5.6 (ii), we have the following extension of $\mathcal{C}_k$:

$$0 \longrightarrow H^2_{Harr}(\mathcal{C}_k; \mathbf{k})' \overset{\bar{i}_{k+1}}{\longrightarrow} \bar{\mathcal{C}}_{k+1} \overset{\bar{p}_{k+1}}{\longrightarrow} \mathcal{C}_k \longrightarrow 0. \tag{6.1}$$

The corresponding obstruction $\theta_{\eta_k}([f_{\mathcal{C}_k}]) \in H^2_{Harr}(\mathcal{C}_k; \mathbf{k})' \otimes H^2_{\mathcal{P}}(A)$ gives a linear map $\omega_k\colon H^2_{Harr}(\mathcal{C}_k; \mathbf{k}) \longrightarrow H^2_{\mathcal{P}}(A)$ with the dual map

$$\omega_k'\colon H^2_{\mathcal{P}}(A)' \longrightarrow H^2_{Harr}(\mathcal{C}_k; \mathbf{k})'.$$

We have an induced extension

$$0 \longrightarrow coker(\omega_k') \longrightarrow \bar{\mathcal{C}}_{k+1}/\big(\bar{i}_{k+1} \circ \omega_k'(H^2_{\mathcal{P}}(A)')\big) \longrightarrow \mathcal{C}_k \longrightarrow 0.$$

Since $coker(\omega_k') \cong (ker(\omega_k))'$, it yields an extension

$$0 \longrightarrow (ker(\omega_k))' \overset{i_{k+1}}{\longrightarrow} \mathcal{C}_{k+1} \overset{p_{k+1}}{\longrightarrow} \mathcal{C}_k \longrightarrow 0, \tag{6.2}$$

where $\mathcal{C}_{k+1} = \bar{\mathcal{C}}_{k+1}/\big(\bar{i}_{k+1} \circ \omega_k'(H^2_{\mathcal{P}}(A)')\big)$ and i_{k+1}, p_{k+1} are the mappings induced by $\bar{i}_{k+1}$ and $\bar{p}_{k+1}$, respectively. Observe that the algebra $\mathcal{C}_{k+1}$ is also local. Since $\mathcal{C}_k$ is finite dimensional, the cohomology group $H^2_{Harr}(\mathcal{C}_k; \mathbf{k})$ is also finite dimensional, and hence $\mathcal{C}_{k+1}$ is finite dimensional as well.

Remark 6.2. It follows from Proposition 5.2 that the specific extension (6.1) has the following "universality property." For any $\mathcal{C}_k$-module M with $\mathfrak{M}M = 0$, (6.1) admits a unique morphism into an arbitrary extension of $\mathcal{C}_k$:

$$0 \longrightarrow M \longrightarrow \mathcal{R}' \longrightarrow \mathcal{C}_k \longrightarrow 0.$$

Proposition 6.3. *The deformation η_k with base $\mathcal{C}_k$ of a $\mathcal{P}$-algebra A admits an extension to a deformation with base $\mathcal{C}_{k+1}$, that is unique up to an isomorphism and an automorphism of the extension*

$$0 \longrightarrow (ker(\omega_k))' \xrightarrow{i_{k+1}} \mathcal{C}_{k+1} \xrightarrow{p_{k+1}} \mathcal{C}_k \longrightarrow 0.$$

Proof. From the above construction of the extension in 6.2, it is clear that $\theta_{\eta_k}([f_{\mathcal{C}_k}]) = \omega_k|_{ker(\omega_k)} = 0$. Therefore, the proof is complete by Proposition 5.11. $\square$

By induction, the above process gives a sequence of finite-dimensional local algebras C_k and the deformation η_k of the $\mathcal{P}$-algebra A with base C_k:

$$\mathbf{k} \xleftarrow{p_1} \mathcal{C}_1 \xleftarrow{p_2} \cdots \xleftarrow{p_k} \mathcal{C}_k \xleftarrow{p_{k+1}} \mathcal{C}_{k+1}$$

such that $(p_{k+1})_*\eta_{k+1} = \eta_k$.

As a consequence, we obtain a formal deformation η of the $\mathcal{P}$-algebra A with base $\mathcal{C} = \varprojlim_{k \to \infty} \mathcal{C}_k$.

The proof of the following theorem is analogous to the proof of Theorem 6.8 of [**6**].

Theorem 6.4. *Let A be a $\mathcal{P}$-algebra with $dim(\mathbb{H}) < \infty$. Then the formal deformation η with base $\mathcal{C}$ constructed above is a versal deformation of A.*

Acknowledgments

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