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KEI MODULES AND UNORIENTED LINK INVARIANTS

MICHAEL GRIER AND SAM NELSON

(*communicated by Ronald Brown*)

Abstract

We define invariants of unoriented knots and links by enhancing the integral kei counting invariant $\Phi_X^{\mathbb{Z}}(K)$ for a finite kei X using representations of the *kei algebra*, $\mathbb{Z}_K[X]$, a quotient of the quandle algebra $\mathbb{Z}[X]$ defined by Andruskiewitsch and Graña. We give an example that demonstrates that the enhanced invariant is stronger than the unenhanced kei counting invariant. As an application, we use a quandle module over the Takasaki kei on $\mathbb{Z}_3$ which is not a $\mathbb{Z}_K[X]$-module to detect the non-invertibility of a virtual knot.

1. Introduction

In [**10**], Mituhisa Takaski introduced an algebraic structure known as *kei* (or 圭 in the original kanji). In [**7**] this same structure was reintroduced under the name *involutory quandle*, a special case of a more general algebraic structure related to oriented knots known as *quandles*. These algebraic structures can be understood as arising from the unoriented and oriented Reidemeister moves respectively via a certain labeling scheme, encoding knot structures in algebra.

In [**1**], an associative algebra $\mathbb{Z}[X]$ was defined for every finite quandle X; in [**3**] a geometric interpretation of $\mathbb{Z}[X]$ was given, with generators representing coefficients of "beads" indexed by quandle labelings of arcs, with relations defined from the Reidemeister moves. Representations of $\mathbb{Z}[X]$, known as *quandle modules*, were used to define new invariants of oriented knots and links in [**3**]. In [**8**] a modification of $\mathbb{Z}[X]$ for finite racks (a generalization of quandles to the case of blackboard-framed isotopy) was used to define invariants of framed and unframed oriented knots and links.

In this paper we define a modification of the quandle algebra we call the *kei algebra* $\mathbb{Z}_K[X]$ and use it to extend the invariants defined in [**8**] to unoriented knots and links. The paper is organized as follows. In section 2 we review the basics of kei and the kei counting invariant. In section 3 we define the kei algebra and kei modules. In section 4 we define the kei module enhanced counting invariant. As an application, we use a module over $\mathbb{Z}[X]$ for a kei X which is not a $\mathbb{Z}_K[X]$-module to detect the

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non-invertibility of a virtual knot. In section 5 we collect a few questions for future research.

2. Kei

Kei or *involutory quandles* were introduced by Mituhisa Takasaki in 1945 [**10**] and later reintroduced independently by David Joyce and S.V. Matveev in the early 1980s [**7, 9**].

Definition 2.1. A *kei* or *involutory quandle* is a set X with a binary operation $\triangleright$ satisfying for all $x, y, z \in X$

(i) $x \triangleright x = x$,

(ii) $(x \triangleright y) \triangleright y = x$, and

(iii) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

Example 2.2. Let X be any abelian group regarded as a $\mathbb{Z}$-module. Then X is a kei under the operation

$$x \triangleright y = 2y - x.$$

Such a kei is known as a *Takasaki kei.* If $X \cong \mathbb{Z}_n$ then X is often denoted as R_n in the knot theory literature, known as the *dihedral quandle* on n elements. R_n can also be understood as the set of reflections of a regular n-gon.

Example 2.3. Let X be any module over $\mathbb{Z}[t]/(t^2 - 1)$. Then X is a kei known as an *Alexander kei* under the operation

$$x \triangleright y = tx + (1 - t)y.$$

A Takasaki kei is an Alexander kei with $t = -1$.

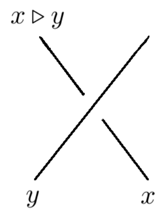
Example 2.4. Let L be an unoriented link diagram and let $A = \{a_1, \dots, a_n\}$ be a set of generators corresponding bijectively with the set of arcs of L. The *Fundamental Kei* of L, $FK(L)$, is defined in the following way. First, let $W(L)$ be the set of *kei words* in A, defined recursively by the rules

- $a \in A \Rightarrow a \in W(L)$ and
- $x, y \in W(L) \Rightarrow x \triangleright y \in W(L)$.

Then the *free kei on A* is the set of equivalence classes of kei words in A under the equivalence relation generated by relations of the forms

- $x \triangleright x \sim x$,
- $(x \triangleright y) \triangleright y \sim x$, and
- $(x \triangleright y) \triangleright z \sim (x \triangleright z) \triangleright (y \triangleright z)$

for all $x, y, z \in W(L)$. The free kei is a kei under the operation $[x] \triangleright [y] = [x \triangleright y]$. Now, at each crossing in L, we have a *crossing relation* given by $z = x \triangleright y$ where y is the overcrossing arc and x and z are the undercrossing arcs. That is, we have



Then the *fundamental kei* of L, $FK(L)$, is the set of equivalence classes of free kei elements modulo the crossing relations of L, or equivalently $FK(L)$ is the set of equivalence classes of kei words in A modulo the equivalence relation determined by the crossing relations together with the free kei relations.

It is convenient to describe a finite kei $X = \{x_1, \dots, x_n\}$ with a matrix encoding the operation table of X, i.e. a matrix M_X whose (i, j) entry is k where $x_k = x_i \triangleright x_j$. For example, the Takasaki kei on $\mathbb{Z}_3$ has matrix

$$M_X = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

where we set $x_1 = 1, x_2 = 2$, and $x_3 = 3 = 0$.

As with groups and other algebraic structures, we have the following standard notions:

Definition 2.5. Let X and Y be kei.

- A map $f\colon X \to Y$ is a *kei homomorphism* if for all $x, x' \in X$ we have $f(x \triangleright x') = f(x) \triangleright f(x')$.
- A subset $Y \subset X$ which is itself a kei under the kei operation $\triangleright$ of X is a *subkei* of X. It is easy to check that $Y \subset X$ is a subkei if and only if Y is closed under $\triangleright$.

For defining invariants of unoriented links, we have the following well-known result:

Theorem 2.6. *If L and L' are ambient isotopic unoriented links, then there is an isomorphism of kei $\phi\colon FK(L) \to FK(L')$. For any finite kei X, the sets of homomorphisms $\mathrm{Hom}(FK(L), X)$ and $\mathrm{Hom}(FK(L'), X)$ are finite and there is an induced bijection $\phi_*\colon \mathrm{Hom}(FK(L), X) \to \mathrm{Hom}(FK(L'), X)$. In particular, the cardinality of the set of rack homomorphisms $\Phi_X^{\mathbb{Z}}(L) = |\mathrm{Hom}(FK(L), X)|$ is a non-negative integer-valued invariant of unoriented links known as the integral kei counting invariant.*

A kei homomorphism $f\colon FK(L) \to X$ can be represented as a labeling of the arcs of L with elements of X satisfying the crossing relations at every crossing—such a labeling defines a unique homomorphism, and every $f \in \mathrm{Hom}(FK(L), X)$ can be so represented.

Example 2.7. We can use the kei counting invariant to see that the trefoil knot 3_1 is nontrivially knotted. Let X be the Takasaki kei on $\mathbb{Z}_3$; we have $x \triangleright y = 2y - x =$

$2y+2x$. The crossing relations in 3_1 give us the system of linear equations

$$\begin{array}{rcl} z & = & 2x+2y \\ y & = & 2z+2x \\ x & = & 2x+2y \end{array} \rightarrow \left[\begin{array}{ccc} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

and the solution space is two-dimensional, giving us a total of $\Phi_X^{\mathbb{Z}}(3_1)=9$ solutions. Since $\Phi_X^{\mathbb{Z}}(\text{Unknot})=3$, the integral kei counting invariant detects the knottedness of the trefoil.

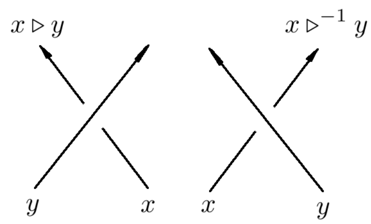
Remark 2.8. Replacing the second kei axiom with the alternative axiom

(ii′) There exists a second operation $\triangleright^{-1}$ satisfying

$$(x \triangleright y) \triangleright^{-1} y = x = (x \triangleright^{-1} y) \triangleright y$$

for all $x, y \in X$

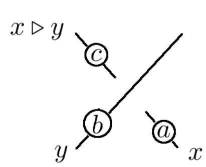
yields an algebraic object known as a *quandle*, which is the oriented analog of kei. Labeling oriented links according to the signed crossing conditions



defines homomorphisms from the fundamental quandle of the link L into X; the *integral quandle counting invariant* $\Phi_X^{\mathbb{Z}}(L)$ is then an invariant of oriented links.

3. Kei algebras and modules

Let X be a finite kei. We would like to define an associative algebra on X generated by "beads" such that secondary labelings of X-labeled link diagrams by beads are preserved by Reidemeister moves. Specifically, at a crossing in a link diagram with arcs labeled x, y and $x \triangleright y$, we define the following relationship between the beads a, b, and c:



$$c = t_{x,y}a + s_{x,y}b.$$

The *kei algebra* of X, $\mathbb{Z}_K[X]$, is the quotient of the polynomial algebra $\mathbb{Z}[t_{x,y}, s_{x,y}]$ by the ideal I required to obtain invariance under unoriented Reidemeister moves.

First, we note that the bead relationship above also requires that

$$a = t_{x\triangleright y,y}c + s_{x\triangleright y,y}b;$$

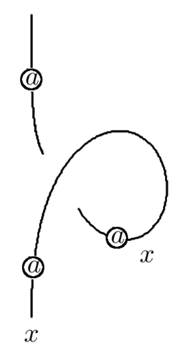
together these imply

$$a = t_{x,y}t_{x\triangleright y,y}a + (t_{x,y}s_{x\triangleright y,y} + s_{x,y})b,$$

which yields

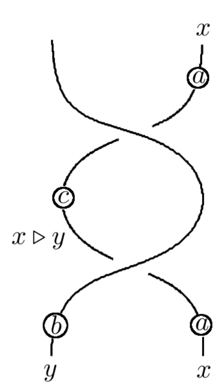
$$t_{x,y}t_{x\triangleright y,y} = 1 \quad \text{and} \quad t_{x,y}s_{x\triangleright y,y} + s_{x,y} = 0. \tag{1}$$

From the Reidemeister I move, we must have $t_{x,x} + s_{x,x} = 1$:



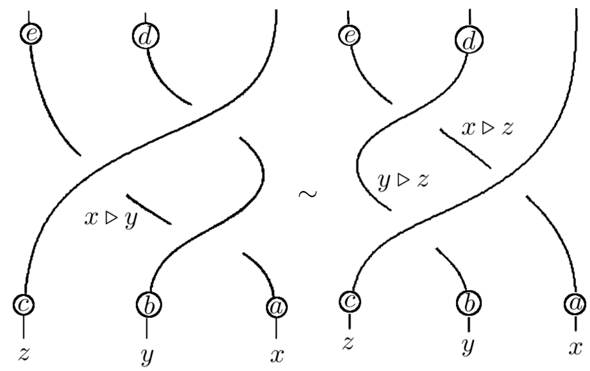
$$a = t_{x,x}a + s_{x,x}a \tag{2}$$

The Reidemeister II move yields conditions equivalent to equation (1):



$$\begin{array}{rcl} c & = & t_{x,y}a + s_{x,y}b \\ a & = & t_{x\triangleright y,y}c + s_{x\triangleright y,y}b \\ \Rightarrow a & = & t_{x\triangleright y,y}t_{x,y}a + (t_{x\triangleright y,y}s_{x,y} + s_{x\triangleright y,y})b \end{array}$$

The Reidemeister III move yields the defining equations for the original rack algebra $\mathbb{Z}[X]$ from [**1**], comparing coefficients on bead e:



$$t_{x\triangleright y,z}t_{x,y} = t_{x\triangleright z,y\triangleright z}t_{x,z}, \quad t_{x\triangleright y,z}s_{x,y} = s_{x\triangleright z,y\triangleright z}t_{y,z} \tag{3}$$
$$\text{and} \quad s_{x\triangleright y,z} = s_{x\triangleright z,y\triangleright z}s_{y,z} + t_{x\triangleright z,y\triangleright z}s_{x,z}.$$

Definition 3.1. Let X be a finite kei. The *kei algebra* $\mathbb{Z}_K[X]$ of X is the quotient of the polynomial algebra $\mathbb{Z}[t_{x,y}, s_{x,y}]$ for all $x, y \in X$ by the ideal I generated by all elements of the forms

- $t_{x,y}s_{x\triangleright y,y} + s_{x,y}$
- $t_{x,y}t_{x\triangleright y,y} - 1$
- $t_{x,x} + s_{x,x} - 1$
- $t_{x\triangleright y,z}t_{x,y} - t_{x\triangleright z,y\triangleright z}t_{x,z}$
- $t_{x\triangleright y,z}s_{x,y} - s_{x\triangleright z,y\triangleright z}t_{y,z}$
- $s_{x\triangleright y,z} - s_{x\triangleright z,y\triangleright z}s_{y,z} - t_{x\triangleright z,y\triangleright z}s_{x,z}$

for all $x, y, z \in X$. A $\mathbb{Z}_K[X]$-*module* or just an X-module is a representation of $\mathbb{Z}_K[X]$, i.e. an abelian group A with a family of automorphisms $t_{x,y}\colon A \to A$ and endomorphisms $s_{x,y}\colon A \to A$ such that each of the bulleted maps listed above are zero.

We can now define the kei algebra of a finite kei X.

Example 3.2. Let X be a kei. Any ring R becomes a $\mathbb{Z}_K[X]$-module by choosing invertible elements $t_{x,y}$ and elements $s_{x,y}$ for $x, y \in X$ satisfying the conditions (1), (2), and (3). In particular, if $X = \{x_1, x_2, \ldots, x_n\}$ is a finite kei, we can specify a $\mathbb{Z}_K[X]$-module structure on R with an $n \times 2n$ block matrix

$$M_R = \left[\begin{array}{cccc|cccc} t_{1,1} & t_{1,2} & \dots & t_{1,n} & s_{1,1} & s_{1,2} & \dots & s_{1,n} \\ t_{2,1} & t_{2,2} & \dots & t_{2,n} & s_{2,1} & s_{2,2} & \dots & s_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n,1} & t_{n,2} & \dots & t_{n,n} & s_{n,1} & s_{n,2} & \dots & s_{n,n} \end{array}\right]$$

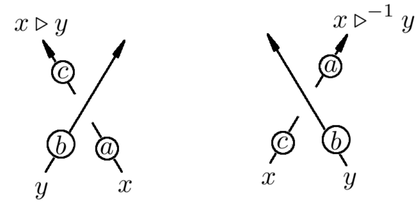
such that the entries satisfy the conditions (1), (2), and (3) above.

Remark 3.3. The *quandle algebra* defined in [**1**] is the quotient of the polynomial algebra $\mathbb{Z}[t^{\pm 1}_{x,y}, s_{x,y}]$ by the ideal generated by the relations coming from the Reidemeister I and III moves, i.e.,

- $t_{x,x} + s_{x,x} - 1$

- $t_{x\triangleright y,z}t_{x,y} - t_{x\triangleright z,y\triangleright z}t_{x,z}$,
- $t_{x\triangleright y,z}s_{x,y} - s_{x\triangleright z,y\triangleright z}t_{y,z}$,
- $s_{x\triangleright y,z} - s_{x\triangleright z,y\triangleright z}s_{y,z} - t_{x\triangleright z,y\triangleright z}s_{x,z}$,

with the type II move condition handled by the bead labeling rule below.



$$c = t_{x,y}a + s_{x,y}b$$

The kei algebra $\mathbb{Z}_K[X]$ is a quotient of the quandle algebra by the additional relations

$$t_{x,y}s_{x\triangleright y,y} + s_{x,y} \quad \text{and} \quad 1 - t_{x,y}t_{x\triangleright y,y}.$$

Example 3.4. For a specific instance of the type of kei module defined in example 3.2, let X be the 3-element Takasaki kei with kei matrix

$$M_X = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

and let the ring $R = \mathbb{Z}_5$. Our `python` computations indicate that there are 48 $\mathbb{Z}_K[X]$-module structures on R, including for instance

$$M_R = \left[\begin{array}{ccc|ccc} 4 & 1 & 3 & 2 & 4 & 1 \\ 3 & 4 & 2 & 3 & 2 & 3 \\ 2 & 1 & 4 & 4 & 1 & 2 \end{array}\right].$$

Remark 3.5. For a given kei X, the set of $\mathbb{Z}_K[X]$-modules over a given ring R is a subset of the set of $\mathbb{Z}[X]$-modules, and can be a proper subset depending on R, since a $\mathbb{Z}[X]$-module satisfies the conditions in equations (1) and (3) but not necessarily those of equation (2). For instance, our `python` computations reveal a total of 32 $\mathbb{Z}[X]$-modules on the kei X and ring R in example 3.4 which are not $\mathbb{Z}_K[X]$-modules, including for instance

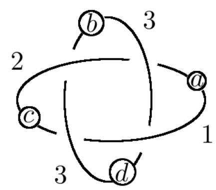
$$M_R = \left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 4 & 2 & 3 \\ 1 & 2 & 2 & 2 & 4 & 3 \\ 4 & 4 & 2 & 4 & 4 & 4 \end{array}\right].$$

The invariants defined in the next section associated with such modules are invariants of oriented links but not invariants of unoriented links.

Example 3.6. Another important example of a $\mathbb{Z}_K[X]$ module is the *fundamental $\mathbb{Z}_K[X]$-module of an X-labeled link.* Let L be an unoriented link with a labeling $f\colon FK(L) \to X$ by a kei X. On each arc of L, we place a bead; the set of crossing relations then determines a presentation for a $\mathbb{Z}_K[X]$-module, denoted $\mathbb{Z}_f[X]$, which we can represent concretely with a coefficient matrix of the resulting homogeneous system of linear equations. For instance, let X be the kei with matrix

$$M_X = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \\ 3 & 3 & 3 \end{bmatrix};$$

then the (4,2)-torus link with the X-labeling below has fundamental $\mathbb{Z}_k[X]$-module presented by the matrix $M_{\mathbb{Z}_f[X]}$:



$$M_{\mathbb{Z}_f[X]} = \begin{bmatrix} t_{13} & s_{13} & -1 & 0 \\ 0 & t_{32} & s_{32} & -1 \\ -1 & 0 & t_{23} & s_{23} \\ s_{31} & -1 & 0 & t_{31} \end{bmatrix}.$$

4. Kei module enhancements of the counting invariant

We can now define invariants of unoriented knots and links using kei modules. The idea is to use the set of homomorphisms $g\colon \mathbb{Z}_f[X] \to R$ from the fundamental kei module of an X-labeled diagram L to the kei module R as a signature for each kei homomorphism $f\colon FK(L) \to X$.

Definition 4.1. Let L be an unoriented knot or link, X a finite kei, and R a finite $\mathbb{Z}_K[X]$-module. The *kei module enhanced multiset* invariant of L associated to X and R is the multiset of cardinalities of the sets of $\mathbb{Z}_k[X]$-module homomorphisms, i.e.,

$$\Phi_{X,R}^{K,M}(L) = \left\{ |\mathrm{Hom}_{\mathbb{Z}_K[X]}(\mathbb{Z}_f[X], R)| \ : \ f \in \mathrm{Hom}(FK(L), X) \right\}.$$

Taking the generating function of this multiset gives us a polynomial-form invariant for easy comparison: the *kei module enhanced invariant* of L with respect to X and R is

$$\Phi_{X,R}^{K}(L) = \sum_{f \in \mathrm{Hom}(FK(L),X)} u^{|\mathrm{Hom}_{\mathbb{Z}_K[X]}(\mathbb{Z}_f[X],R)|}.$$

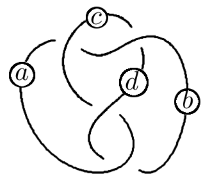
By construction, we have the following:

Theorem 4.2. *If L and L' are ambient isotopic unoriented links, X is a finite kei, and R is a $\mathbb{Z}_K[X]$-module, then $\Phi_{X,R}^{K,M}(L) = \Phi_{X,R}^{K,M}(L')$ and $\Phi_{X,R}^{K}(L) = \Phi_{X,R}^{K}(L')$.*

Remark 4.3. If R is not finite, we can replace the cardinality $|\mathrm{Hom}_{\mathbb{Z}_K[X]}(\mathbb{Z}_f[X], R)|$ with the rank of the $\mathbb{Z}_K[X]$-module $\mathrm{Hom}_{\mathbb{Z}_K[X]}(\mathbb{Z}_f[X], R)$.

To compute $\Phi_{X,R}^{K}$, for each kei labeling $f\colon FK(L) \to X$ of L by X, we first obtain the matrix for $\mathbb{Z}_f[X]$, replace each $t_{x,y}$ and $s_{x,y}$ with its value in R, and solve the resulting system of equations to obtain the contributions to $\Phi_{X,R}^{K}$ for f.

Example 4.4. Let L be the figure eight knot 4_1 and let X and R be the kei and kei module on $\mathbb{Z}_5$ from example 3.4. The set of X-labelings of L includes only constant labelings, i.e. every arc is labeled with a 1, 2, or 3. For example, the constant labeling with every arc labeled 1 yields the listed $\mathbb{Z}_f[X]$-presentation matrix:



$$M_{\mathbb{Z}_f[X]} = \begin{bmatrix} t_{11} & -1 & s_{11} & 0 \\ 0 & s_{11} & t_{11} & -1 \\ -1 & 0 & t_{11} & s_{11} \\ s_{11} & t_{11} & 0 & -1 \end{bmatrix}$$

Replacing the t_{xy} and $s_{x,y}$ with their values in R and row-reducing over $\mathbb{Z}_5$, we obtain

$$\begin{bmatrix} 4 & 4 & 2 & 0 \\ 0 & 2 & 4 & 4 \\ 4 & 0 & 4 & 2 \\ 2 & 4 & 0 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and this X-labeling contributes a u^{25} to the invariant $\Phi^{k}_{X,R}(4_1)$. Summing these contributions over the complete set of X-labelings gives us $\Phi^{K}_{X,R}(4_1) = 3u^{25}$. Comparing this to the unknot, which has $\Phi^{k}_{X,R}(\text{Unknot}) = 3u^5$, we see that $\Phi^{K}_{X,R}$ distinguishes the unoriented figure eight from the unoriented unknot despite the two having equal kei counting invariant values. In particular, since $\Phi^{\mathbb{Z}}_{X}(k)$ is obtained from $\Phi^{K}_{X,R}$ by evaluating at $u = 1$, $\Phi^{K}_{X,R}$ is a strictly stronger invariant than $\Phi^{\mathbb{Z}}_{X}(k)$.

Example 4.5. Our `python` computations[1] yield the listed values for $\Phi^{K}_{X,R}$ with X the 3-element Takasaki kei and the randomly selected $\mathbb{Z}_K[X]$-module over $\mathbb{Z}_7$ below for the prime knots with up to eight crossings and prime links with up to seven crossings as listed in the `knot atlas` [**2**]:

$$M_R = \left[\begin{array}{ccc|ccc} 6 & 3 & 5 & 2 & 5 & 3 \\ 5 & 6 & 3 & 3 & 2 & 5 \\ 3 & 5 & 6 & 5 & 3 & 2 \end{array}\right]$$

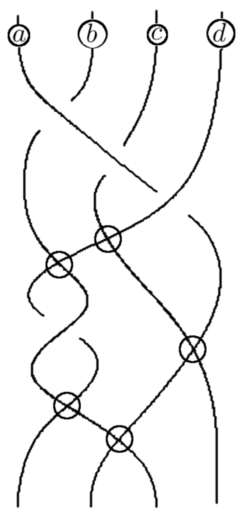
$\Phi^{K}_{X,M}(L)$	L
$3u^7$	unknot, $4_1, 5_1, 6_2, 6_3, 7_2, 7_3, 7_5, 7_6, 8_1, 8_2, 8_3, 8_4, 8_6, 8_7, 8_8, 8_9, 8_{12}, 8_{13}, 8_{14}, 8_{17}, L2a1, L4a1, L6a2, L6a4, L6n1, L7a2, L7a3, L7a4, L7a7, L7n1, L7n2$
$3u^7 + 6u^{49}$	$3_1, 6_1, 7_4, 8_{10}, 8_{11}, 8_{15}, 8_{19}, 8_{20}, 8_{21}, L6a1, L6a3, L6a5, L7a1, L7a5$
$3u^7 + 24u^{49}$	8_{18}
$3u^{49}$	$5_2, 7_1, 8_{16}, L7a6$
$9u^{49}$	$7_7, 8_5$

Remark 4.6. As with most enhancements of quandle-related counting invariants, $\Phi^{K}_{X,M}$ is well-defined for unoriented virtual links as well as classical links.

[1] Code available at http://www.esotericka.org.

In our final example, we use a quandle module which is not a kei module to detect the non-invertibility of a virtual knot.

Example 4.7. Let X be the kei from example 3.4 and M the quandle module from remark 3.5. Since M is not a kei module, Φ_X^M is an invariant of oriented knots and links, but not unoriented knots and links. Thus, we can potentially use Φ_X^M to compare the two orientations of a non-invertible knot. In particular, consider the virtual knot numbered 4.97 in the Knot Atlas [**2**]; it is the closure of the virtual braid below. Let us denote 4.97 with the upward orientation by $4.97_\uparrow$ and 4.97 with the downward orientation as $4.97_\downarrow$. The only labelings of 4.97 by X are constant labelings, of which there are three for both orientations, the unenhanced integral kei counting invariant $\Phi_X^{\mathbb{Z}}(4.97_\uparrow) = 3 = \Phi_X^{\mathbb{Z}}(4.97_\downarrow)$, and $\Phi_X^{\mathbb{Z}}$ does not distinguish $4.97_\uparrow$ from $4.97_\downarrow$. However, the constant labeling with every arc labeled with a $1 \in X$ yields the listed fundamental kei module presentation matrices. Replacing $t_{1,1}$ and $s_{1,1}$ with their values from M yields the listed matrices, which we row-reduce over $\mathbb{Z}_5$ to obtain the cardinalities of the solution spaces which form the signature of the constant labeling by the element $1 \in X$.



$M_{\mathbb{Z}[f]}(4.97_\downarrow)$:

$$\begin{bmatrix} s_{11} & -1 & t_{1,1} & 0 \\ s_{11} & 0 & -1 & t_{1,1} \\ t_{11} & -1 & 0 & s_{1,1} \\ -1 & 0 & s_{11} & t_{1,1} \end{bmatrix} \to \begin{bmatrix} 4 & 4 & 2 & 0 \\ 4 & 0 & 4 & 2 \\ 2 & 4 & 0 & 4 \\ 4 & 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$M_{\mathbb{Z}[f]}(4.97_\uparrow)$:

$$\begin{bmatrix} s_{1,1} & t_{1,1} & -1 & 0 \\ s_{1,1} & 0 & t_{1,1} & -1 \\ -1 & t_{1,1} & 0 & s_{1,1} \\ t_{1,1} & 0 & s_{1,1} & -1 \end{bmatrix} \to \begin{bmatrix} 4 & 2 & 4 & 0 \\ 4 & 0 & 2 & 4 \\ 4 & 2 & 0 & 4 \\ 2 & 0 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $t_{1,1} = t_{2,2} = t_{3,3} = 2$ and $s_{1,1} = s_{2,2} = s_{3,3} = 4$, we get the same signatures for all three labelings for each knot, respectively u^{25} and u^5, and thus we have

$$\Phi_X^M(4.97^\downarrow) = 3u^{25} \neq 3u^5 = \Phi_X^M(4.97_\uparrow)$$

and for non-kei module quandle modules M over a finite kei, X, the quandle module enhanced counting invariant Φ_X^M is capable of detecting invertibility of virtual (and hence classical) knots.

5. Questions

In this section we collect a few open questions for future research.

In our computations we have only considered the simplest type of $\mathbb{Z}_K[X]$ modules, namely $\mathbb{Z}_k[X]$-module structures on $\mathbb{Z}_n$ with the action of $t_{x,y}$ and $s_{x,y}$ given by multiplication by fixed elements of $\mathbb{Z}_K[X]$. Expanding to other abelian groups with other automorphisms $t_{x,y}\colon X \to X$ and endomorphisms $s_{x,y}\colon X \to X$ should give interesting results. We are particularly interested in the case of non-commuting $t_{x,y}$ and $s_{x,y}$ values.

We have generalized the rack module bead counting invariant from [**8**], but several other oriented link invariants using the quandle algebra were defined in [**3**]; these invariants should have generalizations to the unoriented case using the kei algebra.

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Michael Grier `mhg02007@mymail.pomona.edu`
Sam Nelson `knots@esotericka.org`

Department of Mathematical Sciences, Claremont McKenna College, 850 Columbia Ave., Claremont, CA 91711, USA