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COMPLEXIFICATION AND HOMOTOPY

WOJCIECH KUCHARZ AND ŁUKASZ MACIEJEWSKI

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Abstract

Let Y be a real algebraic variety. We are interested in determining the supremum, $\beta(Y)$, of all nonnegative integers n with the following property: For every n-dimensional compact connected nonsingular real algebraic variety X, every continuous map from X into Y is homotopic to a regular map. We give an upper bound for $\beta(Y)$, based on a construction involving complexification of real algebraic varieties. In some cases, we obtain the exact value of $\beta(Y)$.

1. Introduction and main results

In the present paper we continue the line of research undertaken in [**2, 5**]. Our goal is to identify new obstructions to representing homotopy classes of continuous maps, between real algebraic varieties, by regular maps. We use the term *real algebraic variety* to mean a locally ringed space isomorphic to an algebraic subset of $\mathbb{R}^N$, for some N, endowed with the Zariski topology and the sheaf of real-valued regular functions (such an object is called an affine real algebraic variety in [**1**]). Morphisms between real algebraic varieties are called *regular maps*. Each real algebraic variety carries also the Euclidean topology, which is induced by the usual metric on $\mathbb{R}$. Unless explicitly stated otherwise, all topological notions relating to real algebraic varieties refer to the Euclidean topology.

In [**2**], a numerical invariant $\beta(Y)$ was defined for any real algebraic variety Y. Recall that $\beta(Y)$ is the supremum of all nonnegative integers n with the following property: For every n-dimensional compact connected nonsingular real algebraic variety X, every continuous map from X into Y is homotopic to a regular map. The exact value of $\beta(Y)$ is known only in some special cases. The main result of [**2**] is an upper bound for $\beta(Y)$. For any nonnegative integer k, let $H^k_{\mathrm{alg}}(Y;\mathbb{Z}/2)$ denote the subgroup consisting of all algebraic cohomology classes in the cohomology group $H^k(Y;\mathbb{Z}/2)$ (cf. [**1**] for Y compact and nonsingular, and [**2**] for Y arbitrary). According to [**2**, Theorem 2.9], $\beta(Y) \leqslant k$ if $H^k_{\mathrm{alg}}(Y,\mathbb{Z}/2) \neq 0$ for some $k \geqslant 1$.

In this paper we make use of a cohomology subgroup $H^k_{\mathbb{C}}(Y;\mathbb{Q})$ of $H^k(Y;\mathbb{Q})$, defined below, and prove that $\beta(Y) \leqslant k$ if $H^k_{\mathbb{C}}(Y;\mathbb{Q}) \neq 0$ for some $k \geqslant 1$ (cf. Theorem 1.2). Furthermore, $\beta(Y) = 0$ if $H^1_{\mathbb{C}}(Y;\mathbb{Q}) \neq 0$ (cf. Theorem 1.3), whereas $\beta(Y) =$

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$k-1$ if Y is $(k-1)$-connected and $H^k_{\mathbb{C}}(Y;\mathbb{Q}) \neq 0$ for some $k \geqslant 2$ (cf. Theorem 1.4).

Let V be a compact nonsingular real algebraic variety. A *nonsingular projective complexification* of V is a pair $(\mathbb{V}, e)$, where $\mathbb{V}$ is a nonsingular projective scheme over $\mathbb{R}$ and $e\colon V \to \mathbb{V}(\mathbb{C})$ is an injective map such that $\mathbb{V}(\mathbb{R})$ is Zariski dense in $\mathbb{V}$, $e(V) = \mathbb{V}(\mathbb{R})$, and e induces a biregular isomorphism between V and $\mathbb{V}(\mathbb{R})$. Here the set $\mathbb{V}(\mathbb{R})$ of real points of $\mathbb{V}$ is regarded as a subset of the set $\mathbb{V}(\mathbb{C})$ of complex points of $\mathbb{V}$. The existence of $(\mathbb{V}, e)$ follows from Hironaka's theorem on resolution of singularities [**6**] (cf. also [**7**] for a very readable exposition). If $\dim V \geqslant 2$, then V admits infinitely many pairwise nonisomorphic projective complexifications, for $\mathbb{V}$ can be blown up along a nonsingular center disjoined form $\mathbb{V}(\mathbb{R})$. In view of this nonuniqueness, it is remarkable that for any commutative ring R and any nonnegative integer k, the submodule

$$H^k_{\mathbb{C}}(V;R) := e^*(H^k(\mathbb{V}(\mathbb{C});R))$$

of the cohomology R-module $H^k(V;R)$, where

$$e^*\colon H^*(\mathbb{V}(\mathbb{C});R) \to H^*(V;R)$$

denotes the homomorphism induced by e, does not depend on the choice of $(\mathbb{V}, e)$. This is proved in [**9**] for V orientable over R, and in [**4**] for arbitrary V. Note that in [**4, 9**] the authors use different notation for our $H^k_{\mathbb{C}}(-;R)$. As proved in [**4, 9**], the R-modules $H^k_{\mathbb{C}}(-;R)$ have the expected functorial property: If $h\colon V \to W$ is a regular map between compact nonsingular real algebraic varieties, then

$$h^*(H^k_{\mathbb{C}}(W;R)) \subseteq H^k_{\mathbb{C}}(V;R).$$

The reader who wishes to find results comparing $H^k_{\mathbb{C}}(-;R)$ and $H^k(-;R)$ may consult [**8**].

We extend the definition of $H^k_{\mathbb{C}}(-;R)$ as follows. For any real algebraic variety X, let $H^k_{\mathbb{C}}(X;R)$ denote the set of all cohomology classes u in $H^k(X;R)$ of the form $u = \varphi^*(v)$, where $\varphi\colon X \to V$ is a regular map into a compact nonsingular real algebraic variety V and v is a cohomology class in $H^k_{\mathbb{C}}(V;R)$.

Proposition 1.1. *For any real algebraic variety X and any nonnegative integer k, the set $H^k_{\mathbb{C}}(X;R)$ forms a submodule of the cohomology R-module $H^k(X;R)$. Furthermore, if $f\colon X \to Y$ is a regular map between real algebraic varieties, then*

$$f^*(H^k_{\mathbb{C}}(Y;R)) \subseteq H^k_{\mathbb{C}}(X;R).$$

Proof. Let $\varphi_i\colon X \to V_i$ be a regular map into a compact nonsingular real algebraic variety V_i for $i = 1, 2$. The regular map

$$(\varphi_1, \varphi_2)\colon X \to V_1 \times V_2$$

satisfies $\pi_i \circ (\varphi_1, \varphi_2) = \varphi_i$, where $\pi_i\colon V_1 \times V_2 \to V_i$ is the canonical projection. If v_i is a cohomology class in $H^k(V_i;R)$ for $i = 1, 2$, then

$$\varphi_1^*(v_1) + \varphi_2^*(v_2) = (\varphi_1, \varphi_2)^*(\pi_1^*(v_1) + \pi_2^*(v_2)).$$

If v_i is in $H^k_{\mathbb{C}}(V_i;R)$ for $i = 1, 2$, then $\pi_1^*(v_1) + \pi_2^*(v_2)$ is in $H^k_{\mathbb{C}}(V_1 \times V_2;R)$. It follows that $\varphi_1^*(v_1) + \varphi_2^*(v_2)$ belongs to $H^k_{\mathbb{C}}(X;R)$. Consequently, $H^k_{\mathbb{C}}(X;R)$ is a submodule of the R-module $H^k(X;R)$.

Let $\psi\colon Y \to W$ be a regular map into a compact nonsingular real algebraic variety W. For any cohomology class w in $H^k(W; R)$,

$$f^*(\psi^*(w)) = (\psi \circ f)^*(w).$$

Since $\psi \circ f$ is a regular map, it follows that $f^*(H^k_{\mathbb{C}}(Y; R)) \subseteq H^k_{\mathbb{C}}(X; R)$. □

We now announce three results whose proofs will be given in Section 2.

Theorem 1.2. *Let Y be a real algebraic variety. If $H^k_{\mathbb{C}}(Y;\mathbb{Q}) \neq 0$ for some positive integer k, then $\beta(Y) \leqslant k$.*

In some cases, we get stronger results.

Theorem 1.3. *Let Y be a real algebraic variety. If $H^1_{\mathbb{C}}(Y, \mathbb{Q}) \neq 0$, then $\beta(Y) = 0$.*

We also have a criterion for the equality $\beta(Y) = k - 1$, where $k \geqslant 2$.

Theorem 1.4. *Let Y be a real algebraic variety. Assume that Y is $(k-1)$-connected for some integer $k \geqslant 2$. If $H^k_{\mathbb{C}}(Y;\mathbb{Q}) \neq 0$, then $\beta(Y) = k - 1$.*

In some cases, our results are stronger than those that can be deduced from [**2, 5**].

Example 1.5. For any positive integer k, the real algebraic variety

$$\Sigma^k = \{(x_0, \dots, x_k) \in \mathbb{R}^{k+1} \mid x_0^4 + \cdots + x_k^4 = 1\}$$

is nonsingular and diffeomorphic to the unit k-sphere. By [**8**, Example 2.3], we have

$$H^k_{\mathbb{C}}(\Sigma^k; \mathbb{Q}) = H^k(\Sigma; \mathbb{Q})$$

and hence Theorems 1.3 and 1.4 imply the equality $\beta(\Sigma^k) = k - 1$. Since the real curve Σ^1 is not rational, one easily obtains $\beta(\Sigma^1) = 0$ without referring to Theorem 1.3; cf. [**2**, Example 1.7(v)]. On the other hand, for $k \geqslant 2$, the methods developed in [**2, 5**] give only the inequalities $k - 1 \leqslant \beta(\Sigma^k) \leqslant k$.

2. Proofs

For any k-dimensional compact oriented smooth (of class $\mathcal{C}^\infty$) manifold K, let $[K]$ denote its fundamental class in $H_k(K;\mathbb{Z})$. If K is a subspace of a topological space P, we denote by $[K]_P$ the homology class in $H_k(P;\mathbb{Z})$ represented by K, that is, $[K]_P = i_*([K])$, where $i\colon K \hookrightarrow P$ is the inclusion map.

As usual, for any nonnegative integer d, we denote by $\mathbb{S}^d$ the unit d-sphere,

$$\mathbb{S}^d = \{(u_0, \dots, u_d) \in \mathbb{R}^{d+1} \mid u_0^2 + \cdots + u_d^2 = 1\}.$$

The following refinement of Thom's representability theorem [**11**, Théorème III.4] will play a key role.

Theorem 2.1. *Let Y be a topological space that is homotopically equivalent to a CW-complex, k a positive integer, and α a homology class in $H_k(Y;\mathbb{Z})$. Then there exist a k-dimensional compact oriented stably parallelizable smooth manifold K, a continuous map $f\colon K \to Y$, and a positive integer c such that $f_*([K]) = c\,\alpha$. Furthermore, if α is represented by a singular cycle with support contained in a connected component of Y, then the manifold K can be chosen connected.*

Proof. We may assume without loss of generality that Y is a compact and connected CW-complex that is embedded in $\mathbb{R}^p$ for some $p \geqslant 2k+2$. The argument used in [**11**, pp. 57, 58] implies the existence of a retraction $Q \to Y$, where $Q \subseteq \mathbb{R}^p$ is a p-dimensional compact connected smooth submanifold with boundary, containing Y in its interior. Let P be the double of Q. By construction, P is a compact connected parallelizable smooth manifold and there exists a retraction $r\colon P \to Y$. Choose an orientation of P. Let $i\colon Y \hookrightarrow P$ be the inclusion map. Let $d = p - k$ and let u be the cohomology class in $H^d(P;\mathbb{Z})$ that corresponds via the Poincaré duality to the homology class $i_*(\alpha)$ in $H_k(P;\mathbb{Z})$. Since $p \leqslant 2d-2$, according to Serre's result [**10**, p. 289, Proposition 2′] we can find a continuous map $\varphi\colon P \to \mathbb{S}^d$ and a positive integer c such that

$$\varphi^*(s_d) = c\,u,$$

where s_d is a generator of the cohomology group $H^d(\mathbb{S}^d;\mathbb{Z}) \cong \mathbb{Z}$. We can assume that the map φ is smooth. By Sard's theorem, there exists a regular value y in $\mathbb{S}^d$ of the map φ. If the smooth submanifold $L := \varphi^{-1}(y)$ of P is suitably oriented, then

$$[L]_P = c\,i_*(\alpha).$$

Obviously, the normal bundle of L in P is trivial. Since $\dim L = k \geqslant 1$, we can perform the connected sum operation on the connected components of L. This can be done inside P since $k \leqslant p-2$ and hence $P \setminus L$ is connected. In other words, we join, in a suitable way, the connected components of L with k-dimensional tubes in P. Thus we obtain a compact connected oriented smooth submanifold K of P, which is homologous to L and whose normal bundle in P is trivial. Note that

$$[K]_P = c\,i_*(\alpha).$$

If $j\colon K \hookrightarrow P$ is the inclusion map and $f := r \circ j\colon K \to Y$, then $j_*([K]) = [K]_P = c\,i_*(\alpha)$ and

$$f_*([K]) = r_*(j_*([K])) = c\,r_*(i_*(\alpha)) = c\,(r \circ i)_*(\alpha) = c\,\alpha.$$

It remains to show that the smooth manifold K is stably parallelizable. For any smooth manifold M, let τ_M denote its tangent bundle. We have $\tau_K \oplus \nu = \tau_P|_K$, where ν is the normal bundle of K in P. Hence K is stably parallelizable, the vector bundles ν and τ_P being trivial. □

Let S be a topological space. For any cohomology class u in $H^k(S;\mathbb{Q})$ and any homology class α in $H_k(S;\mathbb{Z})$, we denote by $\langle u, \alpha\rangle$ their Kronecker index. If $u \neq 0$, then we can choose α so that $\langle u, \alpha\rangle \neq 0$.

Recall that any real algebraic variety is homotopically equivalent to a compact polyhedron (thus, homotopically equivalent to a compact CW-complex); cf. [**1**, Theorem 9.2.1, Corollary 9.3.7].

Proof of Theorem 1.2. Assume that $H^k_{\mathbb{C}}(Y;\mathbb{Q}) \neq 0$, where $k \geqslant 1$. Let u be a nonzero cohomology class in $H^k_{\mathbb{C}}(Y;\mathbb{Q})$. Choose a homology class α in $H_k(Y;\mathbb{Z})$ satisfying

$$\langle u, \alpha\rangle \neq 0$$

and such that it is represented by a singular cycle with support contained in a connected component of Y. According to Theorem 2.1, there exist a k-dimensional compact connected oriented smooth manifold K, a continuous map $f\colon K \to Y$, and a positive integer c such that

$$f_*([K]) = c\,\alpha$$

and K is stably parallelizable. By [**8**, Corollary 2.9], there exists a nonsingular real algebraic variety X diffeomorphic to $K \times \mathbb{S}^1$ and satisfying

$$H^k_{\mathbb{C}}(X; \mathbb{Q}) = 0.$$

Let $\varphi\colon X \to K \times \mathbb{S}^1$ be a smooth diffeomorphism and let $\pi\colon K \times \mathbb{S}^1 \to K$ be the canonical projection. It suffices to prove that the continuous map

$$g := f \circ \pi \circ \varphi\colon X \to Y$$

is not homotopic to a regular map. This can be done as follows. Let z_0 be a point in $\mathbb{S}^1$ and $K_0 := \varphi^{-1}(K \times \{z_0\})$. Then

$$g_*([K_0]_X) = f_*(\pi_*([K \times \{z_0\}]_{K\times\mathbb{S}^1})) = f_*([K]) = c\,\alpha.$$

Consequently,

$$\langle g^*(u), [K_0]_X\rangle = \langle u, g_*([K_0]_X)\rangle = c\,\langle u, \alpha\rangle \neq 0,$$

which implies $g^*(u) \neq 0$. In view of Proposition 1.1 and the equality $H^k_{\mathbb{C}}(X; \mathbb{Q}) = 0$, we would have $g^*(u) = 0$ if g were homotopic to a regular map. The proof is complete. □

The following fact will be useful.

Example 2.2. If $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ is the n-fold product, then

$$H^k_{\mathbb{C}}(\mathbb{T}^n; \mathbb{Q}) = 0 \quad \text{for every } k \geqslant 1.$$

Indeed, the real projective line $\mathbb{P}^1$ (regarded as a scheme over $\mathbb{R}$) is a nonsingular projective complexification of $\mathbb{S}^1$, and hence the n-fold product $\mathbb{V} = \mathbb{P}^1 \times_{\mathbb{R}} \cdots \times_{\mathbb{R}} \mathbb{P}^1$ is a nonsingular projective complexification of $\mathbb{T}^n$. Let

$$e\colon\ \mathbb{V}(\mathbb{R}) = \mathbb{P}^1(\mathbb{R}) \times \cdots \times \mathbb{P}^1(\mathbb{R}) \hookrightarrow \mathbb{V}(\mathbb{C}) = \mathbb{P}^1(\mathbb{C}) \times \cdots \times \mathbb{P}^1(\mathbb{C})$$

be the inclusion map. It suffices to note that $e^*(H^k(\mathbb{V}(\mathbb{C}); \mathbb{Q})) = 0$ for every $k \geqslant 1$. This follows from the Künneth formula in cohomology since $\mathbb{P}^1(\mathbb{R})$ is homeomorphic to $\mathbb{S}^1$ while $\mathbb{P}^1(\mathbb{C})$ is homeomorphic to $\mathbb{S}^2$.

Henceforth, for each nonnegative integer d, we choose an orientation of $\mathbb{S}^d$ and regard $\mathbb{S}^d$ as an oriented manifold.

Lemma 2.3. *Let Y be a real algebraic variety, k a positive integer, and u a cohomology class in $H^k_{\mathbb{C}}(Y; \mathbb{Q})$. Assume that there exists a continuous map $f\colon \mathbb{S}^k \to Y$ such that $\langle u, f_*([\mathbb{S}^k])\rangle \neq 0$. Then $\beta(Y) \leqslant k - 1$.*

Proof. Let $\mathbb{T}^k = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ be the k-fold product. We endow $\mathbb{T}^k$ with an orientation and choose a continuous map $\varphi\colon\ \mathbb{T}^k \to \mathbb{S}^k$ satisfying

$$\varphi_*([\mathbb{T}^k]) = [\mathbb{S}^k].$$

It suffices to prove that the continuous map

$$g := f \circ \varphi\colon\ \mathbb{T}^k \to Y$$

is not homotopic to a regular map. In view of Proposition 1.1 and Example 2.2, we would have $g^*(u) = 0$ if g were homotopic to a regular map. However

$$\langle g^*(u), [\mathbb{T}^k] \rangle = \langle u, g_*([\mathbb{T}^k]) \rangle = \langle u, f_*([\mathbb{S}^k]) \rangle \neq 0,$$

which implies $g^*(u) \neq 0$. □

Proof of Theorem 1.3. Assume that $H^1_{\mathbb{C}}(Y; \mathbb{Q}) \neq 0$, and let u be a nonzero cohomology class in $H^1_{\mathbb{C}}(Y; \mathbb{Q})$. We can find a continuous map $f\colon \mathbb{S}^1 \to Y$ for which $\langle u, f_*([\mathbb{S}^1]) \rangle \neq 0$. This assertion holds since the homology classes of the form $f_*([\mathbb{S}^1])$ generate the group $H_1(Y; \mathbb{Z})$. In order to complete the proof it suffices to apply Lemma 2.3 with $k = 1$. □

Proof of Theorem 1.4. Assume that $H^k_{\mathbb{C}}(Y; \mathbb{Q}) \neq 0$, and let u be a nonzero cohomology class in $H^k_{\mathbb{C}}(Y; \mathbb{Q})$. Since Y is $(k-1)$-connected, according to the Hurewicz theorem, each homology class in $H_k(Y; \mathbb{Z})$ is of the form $h_*([\mathbb{S}^k])$ for some continuous map $h\colon \mathbb{S}^k \to Y$. Hence there exists a continuous map $f\colon \mathbb{S}^k \to Y$ such that $\langle u, f_*([\mathbb{S}^k]) \rangle \neq 0$. It follows from Lemma 2.3 that $\beta(Y) \leqslant k-1$. We get the equality $\beta(Y) = k-1$, since, for every compact polyhedron X of dimension at most $k-1$, every continuous map from X into Y is null homotopic; cf. [**3**, p. 509, Corollary 13.14]. □

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Wojciech Kucharz Wojciech.Kucharz@im.uj.edu.pl

Department of Mathematics, Faculty of Mathematics and Computer Science, Jagiellonian University, Łojasiewicza 6, 30-348 Kraków, Poland

Łukasz Maciejewski Lukasz.Maciejewski@im.uj.edu.pl

Department of Mathematics, Faculty of Mathematics and Computer Science, Jagiellonian University, Łojasiewicza 6, 30-348 Kraków, Poland