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THE HOMOLOGY GRAPH OF A PRECUBICAL SET

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Abstract

Precubical sets are used to model concurrent systems. We introduce the homology graph of a precubical set, which is a directed graph whose nodes are the homology classes of the precubical set. We show that the homology graph is invariant under weak morphisms that are homeomorphisms.

1. Introduction

Precubical sets (i.e., cubical sets without degeneracies) are the principal structural ingredient of higher-dimensional automata, which provide a powerful model for concurrent systems [**4, 7, 8**]. A higher-dimensional automaton (HDA) is a precubical set with initial and final states and labels on 1-cubes. The labeled edges of an HDA represent the actions of the system modeled by the HDA. Squares and higher-dimensional cubes indicate independence of actions: if two actions a and b are enabled in a state and are independent in the sense that they may be executed in any order or even simultaneously without any observable difference, then this is indicated, as in Figure 1, by a square linking the two execution sequences ab and ba. Similarly, the independence of n actions is represented by an n-cube.

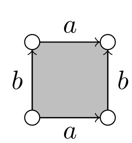

Figure 1: Cubes represent independence of actions

The 1-skeleton of a precubical set is a directed (multi)graph, and therefore a precubical set can be called a "directed topological object." Various notions of directed homology have been defined in the literature for precubical sets and other directed topological objects [**1, 5, 9, 10, 11**]. In this paper, we take a still different approach

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to directed homology and introduce the homology graph of a precubical set. This is a directed graph whose nodes are the homology classes of the precubical set. The definition and some basic properties of the homology graph of a precubical set are given in Section 4.

Our main result on the homology graph is that it is invariant under weak morphisms [**12**] that are homeomorphisms. Roughly speaking, a weak morphism of precubical sets is a continuous map between the geometric realizations that sends vertices to vertices and subdivided cubes to subdivided cubes. The precise definition is given in Section 3. A weak morphism that is a homeomorphism is a kind of T-homotopy equivalence in the sense of Gaucher and Goubault [**6**].

The concept of weak morphism can be extended to HDAs: a weak morphism of HDAs is a weak morphism of the underlying precubical sets that preserves initial and final states and labels of paths. Weak morphisms of HDAs have been used in [**12**] to define a preorder relation for HDAs, called homeomorphic abstraction. An HDA $\mathcal{A}$ is said to be a homeomorphic abstraction of an HDA $\mathcal{B}$ if there exists a weak morphism from $\mathcal{A}$ to $\mathcal{B}$ that is a homeomorphism and that is a bijection on initial and on final states. A homeomorphic abstraction of an HDA provides a smaller representation of the modeled system. By the main result of this paper, the homology graph of an HDA, i.e., the one of its underlying precubical set, is invariant under homeomorphic abstraction.

2. Preliminaries on precubical sets and HDAs

This section, which is taken from [**12**], contains some basic and well-known material on precubical sets and higher-dimensional automata.

2.1. Precubical sets

A *precubical set* is a graded set $P = (P_n)_{n \geqslant 0}$ with *boundary operators* $d_i^k : P_n \to P_{n-1}$ $(n > 0,\ k = 0,1,\ i = 1,\dots,n)$ satisfying the relations $d_i^k \circ d_j^l = d_{j-1}^l \circ d_i^k$ $(k,l = 0,1,\ i < j)$ [**2, 4, 6, 8**]. The least $n \geqslant 0$ such that $P_i = \emptyset$ for all $i > n$ is called the *dimension* of P. If no such n exists, then the dimension of P is ∞. If $x \in P_n$, we say that x is of *degree* n and write $\deg(x) = n$. The elements of degree n are called the *n-cubes* of P. The elements of degree 0 are also called the *vertices* or the *nodes* of P. A morphism of precubical sets is a morphism of graded sets that is compatible with the boundary operators.

The category of precubical sets can be seen as the presheaf category of functors $\square^{\mathrm{op}} \to \mathbf{Set}$ where $\square$ is the small subcategory of the category of topological spaces whose objects are the standard n-cubes $[0,1]^n$ $(n \geqslant 0)$ and whose non-identity morphisms are composites of the maps $\delta_i^k \colon [0,1]^n \to [0,1]^{n+1}$ $(k \in \{0,1\},\ n \geqslant 0,\ i \in \{1,\dots,n+1\})$ given by $\delta_i^k(u_1,\dots,u_n) = (u_1,\dots,u_{i-1},k,u_i\dots,u_n)$. Here, we use the convention that given a topological space X, X^0 denotes the one-point space $\{()\}$.

2.2. Precubical subsets

A *precubical subset* of a precubical set P is a graded subset of P that is stable under the boundary operators. It is clear that a precubical subset is itself a precubical set. Note that unions and intersections of precubical subsets are precubical subsets and that images and preimages of precubical subsets under a morphism of precubical sets are precubical subsets.

2.3. Intervals

Let k and l be two integers such that $k \leqslant l$. The *precubical interval* $[\![k,l]\!]$ is the at most 1-dimensional precubical set defined by $[\![k,l]\!]_0 = \{k,\dots,l\}$, $[\![k,l]\!]_1 = \{[k,k+1],\dots,[l-1,l]\}$, $d_1^0[j-1,j] = j-1$ and $d_1^1[j-1,j] = j$. We shall use the abbreviations $]\!]k,l[\![= [\![k,l]\!] \setminus \{k,l\}$, $[\![k,l[\![= [\![k,l]\!] \setminus \{l\}$, and $]\!]k,l]\!] = [\![k,l]\!] \setminus \{k\}$.

2.4. Tensor product

Given two graded sets P and Q, the *tensor product* $P \otimes Q$ is the graded set defined by $(P \otimes Q)_n = \coprod\limits_{p+q=n} P_p \times Q_q$. If P and Q are precubical sets, then $P \otimes Q$ is a precubical set with respect to the boundary operators given by

$$d_i^k(x,y) = \begin{cases} (d_i^k x, y), & 1 \leqslant i \leqslant \deg(x), \\ (x, d_{i-\deg(x)}^k y), & \deg(x)+1 \leqslant i \leqslant \deg(x)+\deg(y) \end{cases}$$

(cf. [**2**]). The tensor product turns the categories of graded and precubical sets into monoidal categories.

The n-fold tensor product of a graded or precubical set P is denoted by $P^{\otimes n}$. Here, we use the convention $P^{\otimes 0} = [\![0,0]\!] = \{0\}$. The *precubical n-cube* is the precubical set $[\![0,1]\!]^{\otimes n}$. The only element of degree n in $[\![0,1]\!]^{\otimes n}$ will be denoted by ι_n. We thus have $\iota_0 = 0$ and $\iota_n = (\underbrace{[0,1],\dots,[0,1]}_{n \text{ times}})$ for $n > 0$.

2.5. The morphism corresponding to an element

Let x be an element of degree n of a precubical set P. Then there exists a unique morphism of precubical sets $x_\sharp \colon [\![0,1]\!]^{\otimes n} \to P$ such that $x_\sharp(\iota_n) = x$. Indeed, by the Yoneda lemma, there exist unique morphisms of precubical sets $f \colon \square(-,[0,1]^n) \to P$ and $g \colon \square(-,[0,1]^n) \to [\![0,1]\!]^{\otimes n}$ such that $f(id_{[0,1]^n}) = x$ and $g(id_{[0,1]^n}) = \iota_n$. The map g is an isomorphism, and $x_\sharp = f \circ g^{-1}$.

2.6. Paths

A *path of length* k in a precubical set P is a morphism of precubical sets $\omega \colon [\![0,k]\!] \to P$. The set of paths in P is denoted by $P^{\mathbb{I}}$. If $\omega \in P^{\mathbb{I}}$ is a path of length k, we write $\mathrm{length}(\omega) = k$. The *concatenation* of two paths $\omega \colon [\![0,k]\!] \to P$ and $\nu \colon [\![0,l]\!] \to P$ with $\omega(k) = \nu(0)$ is the path $\omega \cdot \nu \colon [\![0,k+l]\!] \to P$ defined by

$$\omega \cdot \nu(j) = \begin{cases} \omega(j), & 0 \leqslant j \leqslant k, \\ \nu(j-k), & k \leqslant j \leqslant k+l \end{cases}$$

and

$$\omega \cdot \nu([j-1,j]) = \begin{cases} \omega([j-1,j]) & 0 < j \leqslant k, \\ \nu([j-k-1,j-k]) & k < j \leqslant k+l. \end{cases}$$

Clearly, concatenation is associative. Note that for any path $\omega \in P^{\mathbb{I}}$ of length $k \geqslant 1$ there exists a unique sequence $(x_1,\dots,x_k)$ of elements of P_1 such that $d_1^0 x_{j+1} = d_1^1 x_j$ for all $1 \leqslant j < k$ and $\omega = x_{1\sharp} \cdots x_{k\sharp}$.

2.7. Geometric realization

The *geometric realization* of a precubical set P is the quotient space

$$|P| = (\coprod_{n \geqslant 0} P_n \times [0,1]^n)/\sim,$$

where the sets P_n are given the discrete topology and the equivalence relation is given by

$$(d_i^k x, u) \sim (x, \delta_i^k(u)), \quad x \in P_{n+1},\ u \in [0,1]^n,\ i \in \{1, \dots, n+1\},\ k \in \{0,1\}$$

(see [**2, 4, 6, 8**]). The geometric realization of a morphism of precubical sets $f\colon P \to Q$ is the continuous map $|f|\colon |P| \to |Q|$ given by $|f|([x,u]) = [f(x), u]$. We remark that the geometric realization is a functor from the category of precubical sets to the category **Top** of topological spaces. The geometric realization is left adjoint to the *singular precubical set* functor S defined by $S(X)_n = \mathbf{Top}([0,1]^n, X)$, $d_i^k \sigma = \sigma \circ \delta_i^k$ and $S(f)(\sigma) = f \circ \sigma$.

Examples 2.1. (i) The geometric realization of the precubical n-cube can be identified with the standard n-cube by means of the homeomorphism $[0,1]^n \to |[\![0,1]\!]^{\otimes n}|$, $u \mapsto [\iota_n, u]$.

(ii) The geometric realization of the precubical interval $[\![k,l]\!]$ can be identified with the closed interval $[k,l]$ by means of the homeomorphism $|[\![k,l]\!]| \to [k,l]$ given by $[j,()] \mapsto j$ and $[[j-1,j],t] \mapsto j-1+t$. Using this correspondence, the geometric realization of a precubical path $[\![0,k]\!] \to P$ can be seen as a path $[0,k] \to |P|$, and under this identification we have that $|\omega \cdot \nu| = |\omega| \cdot |\nu|$.

We note that for every element $a \in |P|$ there exist a unique integer $n \geqslant 0$, a unique element $x \in P_n$, and a unique element $u \in]0,1[^n$ such that $a = [x,u]$.

The geometric realization of a precubical set P is a CW-complex [**6**]. The n-skeleton of $|P|$ is the geometric realization of the precubical subset $P_{\leqslant n}$ of P defined by $(P_{\leqslant n})_m = P_m$ $(m \leqslant n)$ and $(P_{\leqslant n})_m = \emptyset$ $(m > n)$. The closed n-cells of $|P|$ are the spaces $|x_\sharp([\![0,1]\!]^{\otimes n})|$, where $x \in P_n$. The characteristic map of the cell $|x_\sharp([\![0,1]\!]^{\otimes n})|$ is the map $[0,1]^n \stackrel{\approx}{\to} |[\![0,1]\!]^{\otimes n}| \stackrel{|x_\sharp|}{\to} |P|, u \mapsto [x,u]$. The geometric realization of a precubical subset Q of P is a subcomplex of $|P|$.

The geometric realization is a comonoidal functor with respect to the natural continuous map $\psi_{P,Q}\colon |P \otimes Q| \to |P| \times |Q|$ given by

$$\begin{aligned}&\psi_{P,Q}([(x,y),(u_1, \dots, u_{\deg(x)+\deg(y)})])\\ &= ([x,(u_1,\dots,u_{\deg(x)})],[y,(u_{\deg(x)+1}, \dots u_{\deg(x)+\deg(y)}]).\end{aligned}$$

If P and Q are finite, then $\psi_{P,Q}$ is a homeomorphism and permits us to identify $|P \otimes Q|$ with $|P| \times |Q|$. We may thus identify the geometric realization of a precubical set of the form $[\![k_1,l_1]\!] \otimes \cdots \otimes [\![k_n,l_n]\!]$ $(k_i < l_i)$ with the product $[k_1,l_1] \times \cdots \times [k_n,l_n]$ by means of the correspondence

$$[([i_1,i_1+1], \dots, [i_n,i_n+1]),(u_1,\dots,u_n)] \mapsto (i_1+u_1, \dots, i_n+u_n).$$

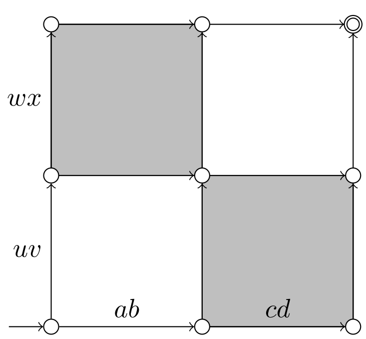
2.8. Higher-dimensional automata

Let M be a monoid. A *higher dimensional automaton over M* (which we shall abbreviate to M-HDA or simply HDA) is a tuple $\mathcal{A} = (P, I, F, \lambda)$, where P is a precubical set, $I \subseteq P_0$ is a set of *initial states*, $F \subseteq P_0$ is a set of *final states*, and $\lambda\colon P_1 \to M$ is a map, called the *labeling function*, such that $\lambda(d_i^0 x) = \lambda(d_i^1 x)$ for all $x \in P_2$ and $i \in \{1, 2\}$. A *morphism* from an M-HDA $\mathcal{A} = (P, I, F, \lambda)$ to an M-HDA $\mathcal{B} = (P', I', F', \lambda')$ is a morphism of precubical sets $f\colon P \to P'$ such that $f(I) \subseteq I'$, $f(F) \subseteq F'$, and $\lambda'(f(x)) = \lambda(x)$ for all $x \in P_1$. The *extended labeling function* of an M-HDA $\mathcal{A} = (P, I, F, \lambda)$ is the map $\overline{\lambda}\colon P^{\mathbb{I}} \to M$ defined as follows: If $\omega = x_{1\sharp} \cdots x_{k\sharp}$ for a sequence $(x_1, \dots, x_k)$ of elements of P_1 such that $d_1^0 x_{j+1} = d_1^1 x_j$ $(1 \leqslant j < k)$, then we set $\overline{\lambda}(\omega) = \lambda(x_1) \cdots \lambda(x_k)$; if ω is a path of length 0, then we set $\overline{\lambda}(\omega) = 1$.

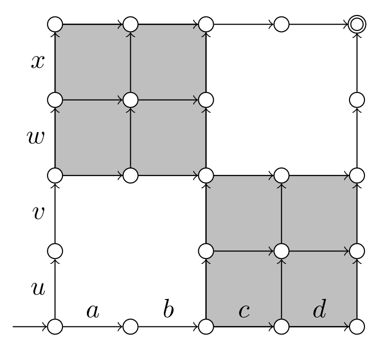
The definition of higher-dimensional automata given here is essentially the same as the one in [**7**]. Besides the fact that we consider a monoid and not just a set of labels, the only difference is that in [**7**] an HDA is supposed to have exactly one initial state. The use of a monoid of labels permits one to equip the edges of an HDA with decomposable labels and to model a system by HDAs of different size (see Figure 2 for an example). Note that 1-dimensional M-HDAs and morphisms of 1-dimensional M-HDAs are the same as automata over M and automata morphisms as defined in [**13**].

3. Weak morphisms

Morphisms of HDAs are often too rigid to be useful for the comparison of HDAs. For instance, the two HDAs $\mathcal{A}$ and $\mathcal{B}$ in Figure 2 model the same system, but there are no morphisms of HDAs between them. In order to overcome this problem, weak morphisms have been introduced in [**12**]. In Figure 2, there exists a weak morphism from $\mathcal{A}$ to $\mathcal{B}$ (but not from $\mathcal{B}$ to $\mathcal{A}$). This section contains the definition and the basic properties of weak morphisms.



(a) $\mathcal{A}$



(b) $\mathcal{B}$

Figure 2: Two $\{a, b, c, d, u, v, w, x\}^*$-HDAs modeling the same system. Parallel edges are meant to have the same label.

3.1. Weak morphisms of precubical sets

A *weak morphism* from a precubical set P to a precubical set Q is a continuous map $f\colon |P| \to |Q|$ such that the following two conditions hold:

1. For every vertex $v \in P_0$ there exists a (necessarily unique) vertex $f_0(v) \in Q_0$ such that $f([v,()]) = [f_0(v),()]$.
2. For all integers $n, k_1, \dots, k_n \geqslant 1$ and every morphism of precubical sets

$$\xi\colon [\![0,k_1]\!] \otimes \cdots \otimes [\![0,k_n]\!] \to P,$$

there exist integers $l_1, \dots, l_n \geqslant 1$, a morphism of precubical sets

$$\chi\colon [\![0,l_1]\!] \otimes \cdots \otimes [\![0,l_n]\!] \to Q,$$

and a homeomorphism

$$\begin{aligned} \phi\colon & |[\![0,k_1]\!] \otimes \cdots \otimes [\![0,k_n]\!]| = [0,k_1] \times \cdots \times [0,k_n] \\ & \to |[\![0,l_1]\!] \otimes \cdots \otimes [\![0,l_n]\!]| = [0,l_1] \times \cdots \times [0,l_n] \end{aligned}$$

such that $f \circ |\xi| = |\chi| \circ \phi$ and ϕ is a dihomeomorphism, i.e., ϕ and ϕ^{-1} preserve the natural partial order of $\mathbb{R}^n$.

It is clear that the geometric realization of a morphism of precubical sets is a weak morphism. We remark that weak morphisms are stable under composition. It is also important to note that the integers $l_1, \dots, l_n \geqslant 1$, the morphism of precubical sets χ, and the dihomeomorphism ϕ in condition 2 above are unique and that ϕ is itself a weak morphism [**12**, Proposition 2.3.5]. In the case of the morphism of precubical sets $x_\sharp\colon [\![0,1]\!]^{\otimes n} \to P$ corresponding to an element $x \in P_n$ $(n > 0)$, we shall use the notation $R_x = [\![0,l_1]\!] \otimes \cdots \otimes [\![0,l_n]\!]$, $\phi_x = \phi$, and $x_\flat = \chi$. We shall also write $\dot{R}_x =]\!]0,l_1[\![\otimes \cdots \otimes]\!]0,l_n[\![$ and $\partial R_x = R_x \setminus \dot{R}_x$.

3.2. Weak morphisms and paths

Let $f\colon |P| \to |Q|$ be a weak morphism of precubical sets, and let $\omega\colon [\![0,k]\!] \to P$ $(k \geqslant 0)$ be a path. If $k > 0$, we denote by $f^{\mathbb{I}}(\omega)$ the unique path $\nu\colon [\![0,l]\!] \to Q$ for which there exists a dihomeomorphism $\phi\colon |[\![0,k]\!]| = [0,k] \to |[\![0,l]\!]| = [0,l]$ such that

$$f \circ |\omega| = |\nu| \circ \phi.$$

If $k = 0$, $f^{\mathbb{I}}(\omega)$ is defined to be the path in Q of length 0 given by $f^{\mathbb{I}}(\omega)(0) = f_0(\omega(0))$. Note that if $g\colon P \to Q$ is a morphism of precubical sets such that $f = |g|$, then $f^{\mathbb{I}}(\omega) = g \circ \omega$. Note also that the path $f^{\mathbb{I}}(\omega)$ leads from $f_0(\omega(0))$ to $f_0(\omega(k))$ and that the map $f^{\mathbb{I}}\colon P^{\mathbb{I}} \to Q^{\mathbb{I}}$ is compatible with composition and concatenation [**12**, 2.5].

3.3. Weak morphisms of HDAs

A *weak morphism* from an M-HDA $\mathcal{A} = (P,I,F,\lambda)$ to an M-HDA $\mathcal{B} = (Q,J,G,\mu)$ is a weak morphism $f\colon |P| \to |Q|$ such that $f_0(I) \subseteq J$, $f_0(F) \subseteq G$ and $\overline{\mu} \circ f^{\mathbb{I}} = \overline{\lambda}$.

3.4. Weak morphisms and precubical subsets

Weak morphisms are more flexible than morphisms of precubical sets, but they are much more rigid than arbitrary continuous maps. Here, we show that, like morphisms of precubical sets, they send precubical subsets to precubical subsets:

Proposition 3.1. *Let $f\colon |P| \to |Q|$ be a weak morphism of precubical sets, and let X be a precubical subset of P. Then there exists a unique precubical subset A of Q such that $f(|X|) = |A|$. It satisfies $\dim(A) = \dim(X)$. If X is finite, so is A.*

Proof. Set $A = f_0(X_0) \cup \bigcup_{x \in X_{>0}} x_\flat(R_x)$. Then A is a precubical subset of Q such $\dim(A) = \dim(X)$. If X is finite, so is A. We have $X = X_0 \cup \bigcup_{x \in X_{>0}} x_\sharp([\![0,1]\!]^{\otimes \deg(x)})$ and

$$|X| = |X_0| \cup \bigcup_{x \in X_{>0}} |x_\sharp([\![0,1]\!]^{\otimes \deg(x)})|.$$

Hence

$$f(|X|) = f(|X_0|) \cup \bigcup_{x \in X_{>0}} f(|x_\sharp([\![0,1]\!]^{\otimes \deg(x)})|) = |f_0(X_0)| \cup \bigcup_{x \in X_{>0}} |x_\flat(R_x)| = |A|.$$

Suppose that A' is another precubical subset of Q such that $f(|X|) = |A'|$. Then $|A'| = |A|$. Consider an element $a \in A$. Then $[a, (\frac{1}{2}, \dots, \frac{1}{2})] \in |A'|$. It follows that there exist elements $a' \in A'$ and $u \in]0,1[^{\deg(a')}$ such that $[a, (\frac{1}{2}, \dots, \frac{1}{2})] = [a', u]$. This implies that $a = a'$. Thus, $A \subseteq A'$. Similarly, $A' \subseteq A$. □

Definition 3.2. Let $f\colon |P| \to |Q|$ be a weak morphism of precubical sets, and let X be a precubical subset of P. The unique precubical subset A of Q such that $f(|X|) = |A|$ is called the *image of* X *under* f and will be denoted by $f(X)$.

Remarks 3.3. (i) Let $g\colon P \to Q$ be a morphism of precubical sets, and let X be a precubical subset of P. Then $|g|(X) = g(X)$.

(ii) Let $f\colon |P| \to |Q|$ be a weak morphism of precubical sets, and let $\{X_i\}_{i\in I}$ be a family of precubical subsets of P. Then $f(\bigcup_{i\in I} X_i) = \bigcup_{i \in I} f(X_i)$ and $f(\bigcap_{i\in I} X_i) \subseteq \bigcap_{i\in I} f(X_i)$. Given two precubical subsets $X, Y \subseteq P$ such that $X \subseteq Y$, one has $f(X) \subseteq f(Y)$.

(iii) Consider a weak morphism of precubical sets $f\colon |P| \to |Q|$ and an element $x \in P_n$. If $n > 0$, then $f(x_\sharp([\![0,1]\!]^{\otimes n})) = x_\flat(R_x)$. Indeed, $f(|x_\sharp([\![0,1]\!]^{\otimes n})|) = |x_\flat(R_x)|$. If $n = 0$, then $f(x_\sharp([\![0,1]\!]^{\otimes n})) = \{f_0(x)\}$. Indeed, $f(|\{x\}|) = |\{f_0(x)\}|$.

4. The homology graph

In this section, we define the homology graph of a precubical set and establish its basic properties. We consider singular homology with coefficients in an arbitrary commutative unital ring R.

4.1. The pointing relation and the homology graph

Let P be a precubical set. We say that a homology class $\alpha \in H_*(|P|)$ *points* to a homology class (of a possibly different degree) $\beta \in H_*(|P|)$ and write $\alpha \nearrow \beta$ if there exist precubical subsets $X, Y \subseteq P$ such that $\alpha \in \operatorname{im} H_*(|X| \hookrightarrow |P|)$, $\beta \in \operatorname{im} H_*(|Y| \hookrightarrow |P|)$ and for all $x \in X_0$ and $y \in Y_0$ there exists a path in P from x to y. The *homology graph* of P is the directed graph whose vertices are the homology classes of $|P|$ and whose edges are given by the relation $\nearrow$. The *homology graph* of an M-HDA $\mathcal{A} = (P, I, F, \lambda)$ is defined to be the homology graph of P.

Examples 4.1. The directed circle is the precubical set with exactly one vertex and exactly one edge. Every homology class of the directed circle points to every homology class.

The precubical set with two vertices v and w and two edges from v to w has the same homology as the directed circle but not the same homology graph. No non-trivial 1-dimensional class points to a non-trivial 1-dimensional class.

Another such example is depicted in Figure 3. The precubical sets P and Q have the same homology but distinct homology graphs. In the case of P, the homology class representing the lower hole points to the homology class representing the upper hole. In the homology graph of Q, there are no edges between non-trivial homology classes of degree 1.

(a) P (b) Q

Figure 3: Precubical sets with the same homology but different homology graphs

4.2. Some basic properties

Let P be a precubical set.

Proposition 4.2. *Let $\alpha, \beta \in H_*(|P|)$ be homology classes such that $\alpha \nearrow \beta$. Then for all $r, s \in R$, $r\alpha \nearrow s\beta$.*

Proof. This is obvious. □

Proposition 4.3. *Consider a homology class $\alpha \in H_*(|P|)$ and the class 0 of an arbitrary degree. Then $0 \nearrow \alpha$ and $\alpha \nearrow 0$.*

Proof. This follows from the fact that $0 \in \operatorname{im} H_*(|\emptyset| \hookrightarrow |P|)$. □

As a consequence, we obtain that the pointing relation is in general very far from being a partial order:

Corollary 4.4. *The relation $\nearrow$ is*

(i) anti-symmetric if and only if $P = \emptyset$;

(ii) transitive if and only if $\alpha \nearrow \beta$ for all homology classes $\alpha, \beta \in H_*(|P|)$.

Examples 4.5. (i) An example of a homology class that is not related to any non-trivial class is given in Figure 4.

(ii) In a precubical set with exactly one vertex, every homology class points to every homology class.

4.3. Compatibility with weak morphisms

Consider a weak morphism of precubical sets $f\colon |P| \to |Q|$.

Theorem 4.6. *Let $\alpha, \beta \in H_*(|P|)$ be two homology classes such that $\alpha \nearrow \beta$. Then $f_*(\alpha) \nearrow f_*(\beta)$.*

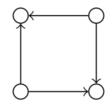


Figure 4: The homology class representing the hole is not incident with any edge from or to a non-trivial class

Proof. Let $X, Y \subseteq P$ be precubical subsets such that $\alpha \in \operatorname{im} H_*(|X| \hookrightarrow |P|)$, $\beta \in \operatorname{im} H_*(|Y| \hookrightarrow |P|)$, and for all vertices $x \in X_0$ and $y \in Y_0$ there exists a path from x to y. Then $f_*(\alpha) \in \operatorname{im} H_*(|f(X)| \hookrightarrow |Q|)$ and $f_*(\beta) \in \operatorname{im} H_*(|f(Y)| \hookrightarrow |Q|)$.

Consider vertices $a \in f(X)_0$ and $b \in f(Y)_0$. We have to show that there exists a path in Q from a to b. Since $|f(X)| = f(|X|)$, there exist an integer n and elements $x \in X_n$ and $u \in]0,1[^n$ such that $[a, ()] = f([x, u])$. We choose a path σ in Q beginning in a as follows: If $n = 0$, then σ is the constant path from a to a. Suppose $n > 0$. Let $R_x = [\![0, k_1]\!] \otimes \cdots \otimes [\![0, k_n]\!]$. Consider elements $\tilde{a} \in R_x$ and $w \in]0,1[^{\deg(\tilde{a})}$ such that $\phi_x(u) = [\tilde{a}, w]$. Then $[a, ()] = f \circ |x_\sharp|(u) = |x_\flat| \circ \phi_x(u) = [x_\flat(\tilde{a}), w]$. It follows that $\deg(\tilde{a}) = 0$ and $a = x_\flat(\tilde{a})$. Let ρ be a path in R_x from $\tilde{a}$ to $(k_1, \ldots, k_n)$, and set $\sigma = x_\flat \circ \rho$. Then σ is a path in Q from a to $x_\flat(k_1, \ldots, k_n)$.

Since $|f(Y)| = f(|Y|)$, there exist an integer m and elements $y \in Y_m$ and $u' \in]0,1[^m$ such that $[b, ()] = f([y, u'])$. In a similar fashion as above, we choose a path τ in Q ending in b: If $m = 0$, then τ is the constant path from b to b. Suppose $m > 0$. Then there exists a vertex $\tilde{b} \in R_y$ such that $y_\flat(\tilde{b}) = b$. Let θ be a path in R_y from $(0, \ldots, 0)$ to $\tilde{b}$, and set $\tau = y_\flat \circ \theta$. Then τ is a path in Q from $y_\flat(0, \ldots, 0)$ to b.

Consider the vertices $v \in X$ and $w \in Y$ defined by

$$v = \begin{cases} x, & n = 0, \\ x_\sharp(1, \ldots, 1), & n > 0 \end{cases} \quad \text{and} \quad w = \begin{cases} y, & m = 0, \\ y_\sharp(0, \ldots, 0), & m > 0. \end{cases}$$

Let ω be a path from v to w. If $n = 0$, then $f^{\mathbb{I}}(\omega)(0) = f_0(\omega(0)) = f_0(v) = f_0(x) = a$. If $n > 0$, then $f^{\mathbb{I}}(\omega)(0) = f_0(v) = f_0(x_\sharp(1, \ldots, 1)) = x_\flat(k_1, \ldots, k_n)$. The last equality holds because

$$\begin{aligned} [f_0(x_\sharp(1, \ldots, 1)), ()] &= f([x_\sharp(1, \ldots, 1), ()]) \\ &= f \circ |x_\sharp|(1, \ldots, 1) \\ &= |x_\flat| \circ \phi_x(1, \ldots, 1) \\ &= |x_\flat|(k_1, \ldots, k_n) \\ &= [x_\flat(k_1, \ldots, k_n), ()]. \end{aligned}$$

A similar argument shows that the end point of $f^{\mathbb{I}}(\omega)$ is b if $m = 0$ and $y_\flat(0, \ldots, 0)$ else. It follows that $\sigma \cdot f^{\mathbb{I}}(\omega) \cdot \tau$ is a path in Q from a to b. □

4.4. A remark on d-spaces

The geometric realization of a precubical set is a d-space in the sense of Grandis [**10**]. A d-space is a topological space together with a set of paths, called d-paths, which is subject to certain conditions. It is straightforward to adapt the definition of the homology graph of a precubical set to define the homology graph of a d-space:

one basically substitutes precubical subsets by subsets (or d-subspaces) and paths by d-paths. It is then natural to ask whether the homology graph of a precubical set P coincides with the homology graph of the d-space $|P|$. Unfortunately, although this seems reasonable to conjecture, it appears that such a result would not be very easy to establish.

Indeed, consider two homology classes $\alpha, \beta \in H_*(|P|)$ and two subspaces $A, B \subseteq |P|$ such that $\alpha \in \operatorname{im} H_*(A \hookrightarrow |P|)$, $\beta \in \operatorname{im} H_*(B \hookrightarrow |P|)$, and for all $a \in A$ and $b \in B$ there exists a d-path from a to b. If one wanted to show that the two concepts of homology graph coincide, one would have to construct, from these data, two precubical subsets $X, Y \subseteq P$ such that $\alpha \in \operatorname{im} H_*(|X| \hookrightarrow |P|)$, $\beta \in \operatorname{im} H_*(|Y| \hookrightarrow |P|)$, and for all vertices $x \in X$ and $y \in Y$ there exists a precubical path from x to y. In order to replace the d-paths by combinatorial paths, one could try to generalize the constructions of [**3**], which have been developed for certain particular precubical sets, called non-self-intersecting $\square$-sets. One could restrict oneself to non-self-intersecting $\square$-sets, but this would exclude important examples of precubical sets such as the directed circle. One would also have to construct the precubical subsets X and Y from the subspaces A and B. In the particular case where A and B are the geometric realizations of two precubical subsets of some subdivision of P, this is done in the next section. In the general case, more work would be necessary.

Not surprisingly, there exist d-space versions of the results of this section and in particular of Theorem 4.6. The proof of this result is much easier in the setting of d-spaces than in the setting of precubical sets. However, in order to deduce the results of this or the next section from results on the homology graph of d-spaces, one would need the theorem that the two concepts of homology graph coincide for precubical sets.

In this paper, we are interested in the homology graph of precubical sets rather than in the homology graph of d-spaces. The reason for this is that the combinatorial homology graph is, at least in principle, computable. It should also be noted that it is not always true that results on the homology graph are easier to prove for d-spaces than for precubical sets. Consider, for example, the situation where one has a precubical set P and a precubical subset Q and one wants to show that there is an edge in the homology graph of Q between two given homology classes if there is an edge between them in the homology graph of P. In the proof of such a result, which could be useful for the reduction of the complexity to compute the homology graph, one would have to construct new paths connecting the homology classes from those existing in P and one would have to guarantee that the new paths stay in Q. If Q has the same 1-skeleton as P, then this is automatically true for precubical paths but not for d-paths.

5. Weak morphisms that are homeomorphisms

Our aim in this section is to establish that the homology graph is invariant under weak morphisms that are homeomorphisms. We fix a weak morphism of precubical sets $f\colon |P| \to |Q|$ that is a homeomorphism.

5.1. Carriers

The *carrier* of an element $a \in Q$ with respect to f is the unique element $c_f(a) \in P$ for which there exists an element $u \in]0,1[^{\deg(c_f(a))}$ such that

$$f([c_f(a), u]) = [a, (\tfrac{1}{2}, \dots, \tfrac{1}{2})].$$

The *carrier* of a precubical subset $A \subseteq Q$ with respect to f is the precubical subset $c_f(A)$ of P defined by

$$c_f(A) = \bigcup_{a \in A} c_f(a)_\sharp(\llbracket 0,1 \rrbracket^{\otimes \deg(c_f(a))}).$$

We shall normally suppress the subscript f and simply write $c(a)$ and $c(A)$ to denote carriers of elements and precubical subsets.

Remarks 5.1. (i) If $f = |g|$ for a morphism of precubical sets $g\colon P \to Q$, then g is an isomorphism and g^{-1} is given by $g^{-1}(a) = c(a)$.

(ii) Let $\{A_i\}_{i \in I}$ be a family of precubical subsets of Q. Then

$$c(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} c(A_i)$$

and $c(\bigcap_{i \in I} A_i) \subseteq \bigcap_{i \in I} c(A_i)$. Given two precubical subsets $A, B \subseteq Q$ such that $A \subseteq B$, one has $c(A) \subseteq c(B)$.

(iii) For any element $a \in Q$, $\deg(c(a)) \geqslant \deg(a)$. If $\deg(c(a)) = 0$, then

$$[a, (\tfrac{1}{2}, \dots, \tfrac{1}{2})] = f([c(a), ()]) = [f_0(c(a)), ()]$$

and hence $\deg(a) = 0$. Suppose that $\deg(c(a)) = n > 0$. Let $u \in]0,1[^n$ be the unique element such that $|c(a)_\flat|(\phi_{c(a)}(u)) = f([c(a), u]) = [a, (\tfrac{1}{2}, \dots, \tfrac{1}{2})]$. Consider elements $z \in R_{c(a)}$ and $v \in]0,1[^{\deg(z)}$ such that $\phi_{c(a)}(u) = [z, v]$. Then

$$[c(a)_\flat(z), v] = [a, (\tfrac{1}{2}, \dots, \tfrac{1}{2})].$$

Hence $a = c(a)_\flat(z)$ and therefore $\deg(a) \leqslant n$.

The easy proof of the following proposition can be found in [**12**]:

Proposition 5.2. *Consider a second weak morphism of precubical sets $g\colon |P'| \to |Q'|$ that is a homeomorphism and two morphisms of precubical sets $\xi\colon P' \to P$ and $\chi\colon Q' \to Q$ such that $f \circ |\xi| = |\chi| \circ g$. Then for all $a \in Q'$, $c_f(\chi(a)) = \xi(c_g(a))$.*

Images and carriers are related as follows:

Proposition 5.3. *Consider precubical subsets of $X \subseteq P$ and $A \subseteq Q$ and an element $a \in Q$. Then*

(i) $a \in f(X)$ if and only if $c(a) \in X$;

(ii) $A \subseteq f(c(A))$;

(iii) $c(f(X)) = X$.

Proof. (i) Let $u \in]0,1[^{\deg(c(a))}$ be the unique element such that $f([c(a), u]) = [a, (\tfrac{1}{2}, \dots, \tfrac{1}{2})]$. Then $a \in f(X) \iff [a, (\tfrac{1}{2}, \dots, \tfrac{1}{2})] \in |f(X)| \iff f([c(a), u]) \in f(|X|) \iff [c(a), u] \in |X| \iff c(a) \in X$.

(ii) Let $b \in A$. Since $c(b) \in c(A)$, by (i), $b \in f(c(A))$.

(iii) For $b \in f(X)$, $c(b) \in X$ and hence $c(b)_\sharp(\llbracket 0,1\rrbracket^{\otimes \deg(c(b))}) \subseteq X$. It follows that $c(f(X)) = \bigcup\limits_{b\in f(X)} c(b)_\sharp(\llbracket 0,1\rrbracket^{\otimes \deg(c(b))}) \subseteq X$. For the reverse inclusion, consider an element $x \in X$. Suppose first that x is a vertex. Then $x = c(f_0(x)) \in c(f(X))$. Suppose now that $\deg(x) = n > 0$. Consider an element $z \in R_x$ such that $\deg(z) = n$. Then $c(z) = \iota_n$. Hence $x = x_\sharp(\iota_n) = x_\sharp(c(z)) = c(x_\flat(z))$. Since

$$x_\flat(R_x) = f(x_\sharp(\llbracket 0,1\rrbracket^{\otimes n}) \subseteq f(X),$$

we have $x_\flat(z) \in f(X)$ and therefore $x = c(x_\flat(z)) \in c(f(X))$. □

Proposition 5.4. [**12**] *The map $f^{\mathbb{I}}\colon P^{\mathbb{I}} \to Q^{\mathbb{I}}$ admits a right inverse $\rho\colon Q^{\mathbb{I}} \to P^{\mathbb{I}}$. If ω is a path in Q from a to b, then $\rho(\omega)$ is a path in P from $c(a)_\sharp(0,\dots,0)$ to $c(b)_\sharp(0,\dots,0)$.*

Lemma 5.5. *Consider a precubical subset B of Q and an element $b \in B$. Then $c(b_\sharp(0,\dots,0)) \in c(b)_\sharp(\llbracket 0,1\llbracket^{\otimes \deg(c(b))})$.*

Proof. Suppose first that $\deg(b) = 0$. Then $b_\sharp(0,\dots,0) = b$ and therefore

$$c(b_\sharp(0,\dots,0)) = c(b) = c(b)_\sharp(\iota_{\deg(c(b))}) \in c(b)_\sharp(\llbracket 0,1\llbracket^{\otimes \deg(c(b))}).$$

Suppose that $\deg(b) > 0$ and set $n = \deg(c(b))$. Then $b_\sharp(0,\dots,0) = d_1^0\cdots d_1^0 b$ and $n > 0$. Let $u \in]0,1[^n$ be the unique element such that

$$f([c(b),u]) = [b,(\tfrac12,\dots,\tfrac12)].$$

Consider elements $z \in R_{c(b)}$ and $v \in]0,1[^{\deg(z)}$ such that $\phi_{c(b)}(u) = [z,v]$. Then

$$[c(b)_\flat(z),v] = |c(b)_\flat| \circ \phi_{c(b)}(u) = f \circ |c(b)_\sharp|(u) = f([c(b),u]) = [b,(\tfrac12,\dots,\tfrac12)]$$

and hence $\deg(z) = \deg(b)$, $c(b)_\flat(z) = b$ and $v = (\frac12,\dots,\frac12)$. Write

$$\begin{aligned} z = (&i_1,\dots,i_{s_1-1},[i_{s_1},i_{s_1}+1],i_{s_1+1},\dots,i_{s_2-1},[i_{s_2},i_{s_2}+1],\dots,\\ &[i_{s_r},i_{s_r}+1],i_{s_r+1},\dots,i_n). \end{aligned}$$

Then

$$\begin{aligned} \phi_{c(b)}(u) &= [z,(\tfrac12,\dots,\tfrac12)]\\ &= (i_1,\dots,i_{s_1-1},i_{s_1}+\tfrac12,i_{s_1+1},\dots,i_{s_2-1},i_{s_2}+\tfrac12,\dots,i_{s_r}+\tfrac12,i_{s_r+1},\dots,i_n). \end{aligned}$$

Write $m = \deg(c(d_1^0\cdots d_1^0 z))$, and let $w \in]0,1[^m$ be the unique element such that $\phi_{c(b)}([c(d_1^0\cdots d_1^0 z),w]) = [d_1^0\cdots d_1^0 z,()] = [z,(0,\dots,0)] = (i_1,\dots,i_n)$. Since $\phi_{c(b)}$ is a dihomeomorphism, we have $[c(d_1^0\cdots d_1^0 z),w] \leqslant u$ and hence $[c(d_1^0\cdots d_1^0 z),w] \in [0,1[^n$. By Lemma 5.6, below, this implies that $c(d_1^0\cdots d_1^0 z) \in \llbracket 0,1\llbracket^{\otimes n}$. By Proposition 5.2, $c(d_1^0\cdots d_1^0 b) = c(c(b)_\flat(d_1^0\cdots d_1^0 z)) = c(b)_\sharp(c(d_1^0\cdots d_1^0 z))$. The result follows. □

Lemma 5.6. *Consider elements $b \in \llbracket 0,k_1\rrbracket \otimes\cdots\otimes \llbracket 0,k_n\rrbracket$ $(n,k_1,\dots,k_n \geqslant 1)$ and $u \in]0,1[^{\deg(b)}$. Then*

(i) $b \in \rrbracket 0,k_1\llbracket \otimes\cdots\otimes \rrbracket 0,k_n\llbracket \Leftrightarrow [b,u] \in]0,k_1[\times\cdots\times]0,k_n[$;

(ii) $b \in \rrbracket 0,k_1\rrbracket \otimes\cdots\otimes \rrbracket 0,k_n\rrbracket \Leftrightarrow [b,u] \in]0,k_1]\times\cdots\times]0,k_n]$;

(iii) $b \in \llbracket 0,k_1\llbracket \otimes\cdots\otimes \llbracket 0,k_n\llbracket \Leftrightarrow [b,u] \in [0,k_1[\times\cdots\times[0,k_n[$.

Proof. The last statement is [**12**, Lemma 4.7.1]. The proofs of the other statements are analogous. ☐

5.2. Top cubes

We say that an element $z \in P$ is a *top cube* of a precubical subset $X \subseteq P$ if $z \in X$ and there does not exist any element $x \in X$ such that $\deg(z) < \deg(x)$ and $z \in x_\sharp(\llbracket 0,1\rrbracket^{\otimes \deg(x)})$.

Lemma 5.7. *Let A be a precubical subset of Q, and let $z \in P$ be a top cube of $c(A)$. Then there exists an element $a \in A$ such that $z = c(a)$.*

Proof. Since $z \in c(A) = \bigcup_{a\in A} c(a)_\sharp(\llbracket 0,1\rrbracket^{\otimes \deg(c(a))})$, there exists an element $a \in A$ such that $z \in c(a)_\sharp(\llbracket 0,1\rrbracket^{\otimes \deg(c(a))})$. Since z is a top cube of $c(A)$, $\deg(z) \geqslant \deg(c(a))$. Thus, $z = c(a)$. ☐

Definition 5.8. Let A be a precubical subset of Q, and let z be a top cube of $c(A)$. We set $A^-_z = \{a \in A : c(a) \neq z\}$.

Proposition 5.9. *Let A be a precubical subset of Q, and let z be a top cube of $c(A)$. Then A^-_z is a precubical subset of A and $z \notin c(A^-_z)$.*

Proof. Consider an element $a \in A^-_z$ such that $\deg(a) > 0$. Suppose that $d^k_i a \notin A^-_z$. Then $c(d^k_i a) = z$. Since $c(a) \in c(a)_\sharp(\llbracket 0,1\rrbracket^{\otimes \deg(c(a))})$, $a \in f(c(a)_\sharp(\llbracket 0,1\rrbracket^{\otimes \deg(c(a))})) = c(a)_\flat(R_{c(a)})$. Write $a = c(a)_\flat(y)$. Then

$$z = c(d^k_i a) = c(d^k_i c(a)_\flat(y)) = c(c(a)_\flat(d^k_i y)) = c(a)_\sharp(c(d^k_i y)) \in c(a)_\sharp(\llbracket 0,1\rrbracket^{\otimes \deg(c(a))}).$$

Since z is a top cube of $c(A)$ and $c(a) \in c(A)$, we have $\deg(z) \geqslant \deg(c(a))$ and hence $z = c(a)$. This contradicts the fact that $a \in A^-_z$. It follows that A^-_z is a precubical subset of A.

Suppose that $z \in c(A^-_z) = \bigcup_{a\in A^-_z} c(a)_\sharp(\llbracket 0,1\rrbracket^{\otimes \deg(c(a))})$. Then there exists an element $a \in A^-_z$ such that $z \in c(a)_\sharp(\llbracket 0,1\rrbracket^{\otimes \deg(c(a))})$. Since $a \in A^-_z$, $c(a) \neq z$ and therefore $\deg(z) < \deg(c(a))$. This is impossible because z is a top cube of $c(A)$. ☐

Lemma 5.10. *Suppose that $\phi\colon |\llbracket 0,1\rrbracket^{\otimes n}| \to |\llbracket 0,k_1\rrbracket \otimes \cdots \otimes \llbracket 0,k_n\rrbracket|$ $(n > 0)$ is a weak morphism that is a homeomorphism, and let b be an element in $\llbracket 0,k_1\rrbracket \otimes \cdots \otimes \llbracket 0,k_n\rrbracket$. Then $b \in \rrbracket 0,k_1\llbracket \otimes \cdots \otimes \rrbracket 0,k_n\llbracket$ if and only if $c(b) = \iota_n$.*

Proof. Let $u \in \,]0,1[^{\deg(c(b))}$ be the element such that $\phi([c(b),u]) = [b,(\frac{1}{2},\dots,\frac{1}{2})]$. Since ϕ is a homeomorphism, $[b,(\frac{1}{2},\dots,\frac{1}{2})] \in \,]0,k_1[\, \times \cdots \times \,]0,k_m[$ if and only if $[c(b),u] \in]0,1[^n$. Lemma 5.6 implies the result. ☐

Lemma 5.11. *For any element $x \in P$ of degree $\geqslant 1$, the restriction of $x_\flat$ to $\dot{R}_x$ is injective.*

Proof. Let $\deg(x) = n$, and let $R_x = \llbracket 0,k_1\rrbracket \otimes \cdots \otimes \llbracket 0,k_n\rrbracket$. Let $a, b \in \dot{R}_x$ be distinct elements of the same degree. Then $[a,(\frac{1}{2},\dots,\frac{1}{2})] \neq [b,(\frac{1}{2},\dots,\frac{1}{2})] \in]0,k_1[\times\cdots\times]0,k_n[$ and therefore $\phi_x^{-1}([a,(\frac{1}{2},\dots,\frac{1}{2})]) \neq \phi_x^{-1}([b,(\frac{1}{2},\dots,\frac{1}{2})]) \in]0,1[^n$. Let $u \in]0,1[^{\deg(c(a))}$ and $v \in]0,1[^{\deg(c(b))}$ be the uniquely determined elements such that $\phi_x([c(a),u]) =$

$[a,(\frac{1}{2},\dots,\frac{1}{2})]$ and $\phi_x([c(b),v]) = [b,(\frac{1}{2},\dots,\frac{1}{2})]$. From Lemma 5.10, $c(a) = c(b) = \iota_n$. Thus $u \neq v$, and hence $|x_\sharp|([c(a),u]) \neq |x_\sharp|([c(b),v])$. Since f is injective, it follows that

$$|x_\flat|([a,(\tfrac{1}{2},\dots,\tfrac{1}{2})]) \neq |x_\flat|([b,(\tfrac{1}{2},\dots,\tfrac{1}{2})])$$

and hence that $x_\flat(a) \neq x_\flat(b)$. □

Proposition 5.12. *Let A be a precubical subset of Q, and let z be a top cube of $c(A)$ of degree $n > 0$. Then the diagram*

$$\begin{tikzcd} z_\flat^{-1}(A_z^-) \arrow[r, "z_\flat"] \arrow[d, hook] & A_z^- \arrow[d, hook] \\ z_\flat^{-1}(A) \arrow[r, "z_\flat"] & A \end{tikzcd}$$

is a pushout of precubical sets.

Proof. Let $R_z = [\![0,k_1]\!] \otimes \cdots \otimes [\![0,k_n]\!]$. Since A is the disjoint union of A_z^- and the graded set $B = \{a \in A : c(a) = z\}$, $z_\flat^{-1}(A)$ is the disjoint union of $z_\flat^{-1}(A_z^-)$ and $z_\flat^{-1}(B)$. We have $z_\flat^{-1}(B) \subseteq \dot{R}_z$. Indeed, if $y \in z_\flat^{-1}(B)$, then $z_\sharp(c(y)) = c(z_\flat(y)) = z$. This implies that $c(y) = \iota_n$ and hence, by Lemma 5.10, that $y \in]\!]0,k_1[\![\otimes \cdots \otimes]\!]0,k_n[\![$. By Lemma 5.11, it follows that $z_\flat \colon z_\flat^{-1}(B) \to B$ is injective. This map is also surjective because for $b \in B$, $c(b) = z \in z_\sharp([\![0,1]\!]^{\otimes n})$ and hence $b \in f(z_\sharp([\![0,1]\!]^{\otimes n})) = z_\flat(R_z)$. It follows that the diagram of the statement is a pushout of graded sets. This implies that it is a pushout of precubical sets. □

5.3. Broken cubes

Let A be a precubical subset of Q. An n-cube $x \in P$ is said to be *broken in A* if $x \in c(A)$ and $f(x_\sharp([\![0,1]\!]^{\otimes n})) \not\subseteq A$.

Proposition 5.13. *Let z be a top cube of $c(A)$ and $x \in P$ be an n-cube that is broken in A_z^-. Then x is broken in A.*

Proof. Suppose that x is not broken in A. Since $x \in c(A_z^-) \subseteq c(A)$, we have

$$f(x_\sharp([\![0,1]\!]^{\otimes n})) \subseteq A.$$

It follows that there exists an element $b \in f(x_\sharp([\![0,1]\!]^{\otimes n}))$ such that $b \notin A_z^-$ but $b \in A$. Since $b \in f(x_\sharp([\![0,1]\!]^{\otimes n}))$, $c(b) \in x_\sharp([\![0,1]\!]^{\otimes n})$. Since $b \notin A_z^-$, $c(b) = z$. It follows that $z \in x_\sharp([\![0,1]\!]^{\otimes n}) \subseteq c(A_z^-)$. By Proposition 5.9, this is impossible. □

Proposition 5.14. *Suppose that A is finite and that some element of P is broken in A. Then there exists top cube of $c(A)$ that is broken in A.*

Proof. Let $x \in P$ be broken in A. Since A is finite, so is

$$c(A) = \bigcup_{a \in A} c(a)_\sharp([\![0,1]\!]^{\otimes \deg(c(a))}).$$

It follows that there exists a top cube z of $c(A)$ such that $x \in z_\sharp([\![0,1]\!]^{\deg(z)})$. Thus, $x_\sharp([\![0,1]\!]^{\deg(x)}) \subseteq z_\sharp([\![0,1]\!]^{\deg(z)})$ and therefore $f(x_\sharp([\![0,1]\!]^{\deg(x)})) \subseteq f(z_\sharp([\![0,1]\!]^{\deg(z)}))$.

Since x is broken in A, we have $f(x_\sharp(\llbracket 0,1\rrbracket^{\deg(x)})) \nsubseteq A$ and hence also

$$f(z_\sharp(\llbracket 0,1\rrbracket^{\deg(z)})) \nsubseteq A.$$

Thus, z is broken in A. □

Proposition 5.15. *Let z be a top cube of $c(A)$ that is broken in A. Then* $\deg(z) > 0$.

Proof. Suppose that $\deg(z) = 0$. By Lemma 5.7, there exists an element $a \in A$ such that $z = c(a)$. Since $\deg(c(a)) = 0$, also $\deg(a) = 0$. We have

$$[f_0(c(a)), ()] = f([c(a), ()]) = [a, ()]$$

and hence $f_0(z) = f_0(c(a)) = a$. Thus, $f(z_\sharp(\llbracket 0,1\rrbracket^{\deg(z)})) = \{f_0(z)\} \subseteq A$. This contradicts the assumption that z is broken. □

Proposition 5.16. *Suppose that no element of $c(A)$ is broken in A. Then $A = f(c(A))$.*

Proof. By 5.3, $A \subseteq f(c(A))$. Since no element of $c(A)$ is broken in A, $f(c(A)) = f(\bigcup_{x\in c(A)} x_\sharp(\llbracket 0,1\rrbracket^{\otimes \deg(x)})) = \bigcup_{x\in c(A)} f(x_\sharp(\llbracket 0,1\rrbracket^{\otimes \deg(x)})) \subseteq A$. □

5.4. Past-complete cubes

Let A be a precubical subset of Q. Consider an n-cube $x \in c(A)$. We say that x is *past-complete in A* if either $n = 0$ or for all vertices $(l_1, \dots, l_n) \in R_x$,

$$x_\flat(l_1, \dots, l_n) \in A \Rightarrow x_\flat(\llbracket 0, l_1\rrbracket \otimes \cdots \otimes \llbracket 0, l_n\rrbracket) \subseteq A.$$

Proposition 5.17. *Let z be a top cube of $c(A)$. If all elements of $c(A)$ are past-complete in A, then all elements of $c(A_z^-)$ are past-complete in A_z^-.*

Proof. Suppose that there exists an n-cube $x \in c(A_z^-)$ that is not past-complete in A_z^-. Consider a vertex $(l_1, \dots, l_n) \in R_x$ such that $x_\flat(l_1, \dots, l_n) \in A_z^-$ but

$$x_\flat(\llbracket 0, l_1\rrbracket \otimes \cdots \otimes \llbracket 0, l_n\rrbracket) \nsubseteq A_z^-.$$

Consider an element $y \in \llbracket 0, l_1\rrbracket \otimes \cdots \otimes \llbracket 0, l_n\rrbracket$ such that $x_\flat(y) \notin A_z^-$. Since x is past-complete in A, $x_\flat(y) \in A$. Hence $c(x_\flat(y)) = z$. Since $x_\flat(y) \in f(x_\sharp(\llbracket 0,1\rrbracket^{\otimes n}))$, we have $c(x_\flat(y)) \in x_\sharp(\llbracket 0,1\rrbracket^{\otimes n})$. It follows that $z \in x_\sharp(\llbracket 0,1\rrbracket^{\otimes n}) \subseteq c(A_z^-)$. By Proposition 5.9, this is impossible. □

5.5. Order-convex sets

A subset B of a partially ordered set $(S, \leqslant)$ is called *order-convex* if for all $x, y \in B$, $\{z \in S : y \leqslant z \leqslant x\} \subseteq B$. A subset B of S with a minimal element m is order-convex if and only if for each element $x \in B$, $\{z \in S : m \leqslant z \leqslant x\} \subseteq B$.

Let B be an order-convex subset of $[0, k_1] \times \cdots \times [0, k_n] \setminus \{(k_1, \dots, k_n)\}$ such that $(0, \dots, 0) \in B$. Then B is contractible. A contraction $G\colon B \times [0,1] \to B$ is given by $G(x, t) = (1 - t)x$.

Write $\partial([0, k_1] \times \cdots \times [0, k_n]) = ([0, k_1] \times \cdots \times [0, k_n]) \setminus (]0, k_1[\times \cdots \times]0, k_n[)$. Let B be an order-convex subset of $\partial([0, k_1] \times \cdots \times [0, k_n]) \setminus \{(k_1, \dots, k_n)\}$ such that

$(0,\dots,0) \in B$. Then B is contractible. A contraction $\Phi\colon B\times[0,1]\to B$ is given by

$$\Phi(x,t)=\begin{cases} x-\dfrac{2t\cdot\min\limits_{1\leqslant i\leqslant n}\frac{x_i}{k_i}}{1-\min\limits_{1\leqslant i\leqslant n}\frac{x_i}{k_i}}((k_1,\dots,k_n)-x), & t\leqslant\frac{1}{2},\\[2em] (2-2t)(x-\dfrac{\min\limits_{1\leqslant i\leqslant n}\frac{x_i}{k_i}}{1-\min\limits_{1\leqslant i\leqslant n}\frac{x_i}{k_i}}((k_1,\dots,k_n)-x)), & t\geqslant\frac{1}{2}.\end{cases}$$

Let B be an order-convex subset of $[0,k_1]\times\cdots\times[0,k_n]\setminus\{(k_1,\dots,k_n)\}$ such that $(0,\dots 0)\in B$. Then $B\cap\partial([0,k_1]\times\cdots\times[0,k_n])$ is an order-convex subset of

$$\partial([0,k_1]\times\cdots\times[0,k_n])\setminus\{(k_1,\dots,k_n)\}$$

and $(0,\dots,0)\in B\cap\partial([0,k_1]\times\cdots\times[0,k_n])$. Therefore, $B\cap\partial([0,k_1]\times\cdots\times[0,k_n])$ is contractible.

5.6. A homotopy equivalence

Let A be a precubical subset of Q, and let z be a top cube of $c(A)$ that is broken but past-complete in A. We shall show that the inclusion $|A_z^-|\hookrightarrow|A|$ is a homotopy equivalence. Set $n=\deg(z)$. By 5.15, $n>0$. Let $R_z=[\![0,k_1]\!]\otimes\cdots\otimes[\![0,k_n]\!]$.

Lemma 5.18. $(0,\dots,0)\in z_\flat^{-1}(A)$.

Proof. By Lemma 5.7, there exists an element $a\in A$ such that $z=c(a)$. We have $a\in f(c(a)_\sharp([\![0,1]\!]^{\otimes n}))=z_\flat(R_z)$. It follows that there exists an element $y\in R_z$ such that $z_\flat(y)=a$. Since $a\in A$, $y\in z_\flat^{-1}(A)$ and therefore $z_\flat^{-1}(A)\neq\emptyset$. Let $(l_1,\dots,l_n)$ be a vertex in $z_\flat^{-1}(A)$. Since z is past-complete in A, $z_\flat([\![0,l_1]\!]\otimes\cdots\otimes[\![0,l_n]\!])\subseteq A$. This implies that $(0,\dots,0)\in z_\flat^{-1}(A)$. □

Lemma 5.19. $(k_1,\dots,k_n)\notin z_\flat^{-1}(A)$.

Proof. Suppose that $(k_1,\dots,k_n)\in z_\flat^{-1}(A)$. Since z is past-complete in A,

$$f(z_\sharp([\![0,1]\!]^{\otimes n}))=z_\flat([\![0,k_1]\!]\otimes\cdots\otimes[\![0,k_n]\!])\subseteq A.$$

This contradicts the fact that z is broken in A. □

Lemma 5.20. *$|z_\flat^{-1}(A)|$ is an order-convex subset of*

$$[0,k_1]\times\cdots\times[0,k_n]\setminus\{(k_1,\dots,k_n)\}$$

and $(0,\dots,0)\in|z_\flat^{-1}(A)|$.

Proof. By Lemmas 5.18 and 5.19, $|z_\flat^{-1}(A)|$ is a subset of $[0,k_1]\times\cdots\times[0,k_n]\setminus\{(k_1,\dots,k_n)\}$ that contains $(0,\dots,0)$. It remains to show that $|z_\flat^{-1}(A)|$ is order-convex. Consider an element $[y,u]\in|z_\flat^{-1}(A)|$, where $y\in z_\flat^{-1}(A)$ and $u\in]0,1[^{\deg(y)}$. Write $y=(y_1,\dots,y_n)$ and suppose $\deg(y)=p$. Then there exist indices $1\leqslant i_1<\cdots<i_p\leqslant n$ such that $\deg(y_{i_q})=1$ for $q\in\{1,\dots,p\}$ and $y_i\in\{0,\dots,k_i\}$ for $i\notin\{i_1,\dots,i_p\}$. For each $q\in\{1,\dots,p\}$, there exists an element $j_q\in\{0,\dots,k_q-1\}$ such that $y_{i_q}=[j_q,j_q+1]$. We have

$$[y,u]=[y,(u_1,\dots,u_p)]=(t_1,\dots,t_n)\in[0,k_1]\times\cdots\times[0,k_n],$$

where $t_i=y_i$ for $i\notin\{i_1,\dots,i_p\}$ and $t_{i_q}=j_q+u_q$ for $q\in\{1,\dots,p\}$. Set $l_i=y_i$ for $i\notin\{i_1,\dots,i_p\}$ and $l_{i_q}=j_q+1$ for $q\in\{1,\dots,p\}$. Then $(l_1,\dots,l_n)=[y,(1,\dots,1)]\in$

$y_\sharp([\![0,1]\!]^{\otimes p}) \subseteq z_\flat^{-1}(A)$ and $[y,u] \leqslant (l_1,\dots,l_n)$. Since z is past-complete in A,

$$[\![0,l_1]\!] \otimes \cdots \otimes [\![0,l_n]\!] \subseteq z_\flat^{-1}(A)$$

and hence $[0,l_1] \times \cdots \times [0,l_n] \subseteq |z_\flat^{-1}(A)|$. Consider an element

$$x \in [0,k_1] \times \cdots \times [0,k_n] \setminus \{(k_1,\dots,k_n)\}$$

such that $x \leqslant [y,u]$. Then $x \leqslant (l_1,\dots,l_n)$ and therefore $x \in [0,l_1] \times \cdots \times [0,l_n] \subseteq |z_\flat^{-1}(A)|$. □

Lemma 5.21. $|z_\flat^{-1}(A_z^-)| = |z_\flat^{-1}(A)| \cap \partial([0,k_1] \times \cdots \times [0,k_n])$.

Proof. We have $|\partial R_z| = \partial([0,k_1] \times \cdots \times [0,k_n])$. It is therefore enough to show that $z_\flat^{-1}(A_z^-) = z_\flat^{-1}(A) \cap \partial R_z$. Consider an element $y \in z_\flat^{-1}(A)$. We have $y \in z_\flat^{-1}(A_z^-) \Leftrightarrow z_\flat(y) \in A_z^- \Leftrightarrow c(z_\flat(y)) \neq z \Leftrightarrow z_\sharp(c(y)) \neq z \Leftrightarrow c(y) \neq \iota_n \Leftrightarrow y \in \partial R_z$. □

Proposition 5.22. *The inclusion $|A_z^-| \hookrightarrow |A|$ is a homotopy equivalence.*

Proof. As a left adjoint, the geometric realization preserves pushouts. Hence, by Proposition 5.12, the diagram

$$\begin{tikzcd} {|z_\flat^{-1}(A_z^-)|} \arrow[r, "|z_\flat|"] \arrow[d, hook] & {|A_z^-|} \arrow[d, hook] \\ {|z_\flat^{-1}(A)|} \arrow[r, "|z_\flat|"] & {|A|} \end{tikzcd}$$

is a pushout of topological spaces. By Lemmas 5.5, 5.20, and 5.21, $|z_\flat^{-1}(A)|$ and $|z_\flat^{-1}(A_z^-)|$ are contractible. Therefore the inclusion $|z_\flat^{-1}(A_z^-)| \hookrightarrow |z_\flat^{-1}(A)|$ is a homotopy equivalence. Since it is the inclusion of a sub CW-complex, it is a closed cofibration. The result follows. □

5.7. Invariance of the homology graph

After the next two lemmas, we will finally be ready to show that P and Q have isomorphic homology graphs.

Lemma 5.23. *Let A be a finite precubical subset of Q such that all elements of $c(A)$ are past complete in A. Then there exists a precubical subset $\tilde{A}$ of A such that the inclusion $|\tilde{A}| \hookrightarrow |A|$ is a homotopy equivalence and no element of P is broken in $\tilde{A}$.*

Proof. Suppose that the result is not true. Let B be a precubical subset of A such that all elements of $c(B)$ are past-complete in B, the inclusion $|B| \hookrightarrow |A|$ is a homotopy equivalence, and the number q of elements of P that are broken in B is minimal. By the hypothesis, $q > 0$. By Proposition 5.14, there exists a top cube $z \in c(B)$ that is broken in B. By Proposition 5.17, all elements of $c(B_z^-)$ are past-complete in B_z^-. By Proposition 5.22, the inclusion $|B_z^-| \hookrightarrow |B|$ is a homotopy equivalence. It follows that the inclusion $|B_z^-| \hookrightarrow |A|$ is a homotopy equivalence. Since A is finite, so are B, B_z^-, $c(B)$, and $c(B_z^-)$. Hence q and the number r of elements of P that are broken in B_z^- are finite. By Proposition 5.13, any element of P that is broken in B_z^- is also broken in B. By Proposition 5.9, $z \notin c(B_z^-)$. Hence z is broken in B but not in B_z^-. It follows that $r < q$. This contradicts the minimality of q. □

Lemma 5.24. *Consider homology classes $\alpha, \beta \in H_*(|P|)$. Let $X \subseteq P$ and $B \subseteq Q$ be precubical subsets such that $\alpha \in \operatorname{im} H_*(|X| \hookrightarrow |P|)$, $f_*(\beta) \in \operatorname{im} H_*(|B| \hookrightarrow |Q|)$, and for all vertices $a \in f(X)_0$ and $b \in B_0$ there exists a path in Q from a to b. Then $\alpha \nearrow \beta$.*

Proof. Since $B \subseteq f(c(B))$, we have $f_*(\beta) \in \operatorname{im} H_*(|f(c(B))| \hookrightarrow |Q|)$. Therefore $\beta \in \operatorname{im} H_*(|c(B)| \hookrightarrow |P|)$. Consider vertices $x \in X_0$ and $y \in c(B)_0$. Let $b \in B$ be such that $y \in c(b)_\sharp(\llbracket 0,1\rrbracket^{\otimes \deg(c(b))})$, and let ω be a path in Q from $f_0(x)$ to $b_\sharp(0,\dots,0)$. Then, by Proposition 5.4, $\rho(\omega)$ is a path in P from $c(f_0(x)) = x$ to $c(b_\sharp(0,\dots,0))_\sharp(0,\dots,0)$. By Lemma 5.5, we have that $c(b_\sharp(0,\dots,0)) \in c(b)_\sharp(\llbracket 0,1\llbracket^{\otimes \deg(c(b))})$. Since $\llbracket 0,1\llbracket^{\otimes \deg(c(b))}$ is closed under the operators d_i^0, so is $c(b)_\sharp(\llbracket 0,1\llbracket^{\otimes \deg(c(b))})$. It follows that $c(b_\sharp(0,\dots,0))_\sharp(0,\dots,0) \in c(b)_\sharp(\llbracket 0,1\llbracket^{\otimes \deg(c(b))})$ and hence that

$$c(b_\sharp(0,\dots,0))_\sharp(0,\dots,0) = c(b)_\sharp(0,\dots,0).$$

Let ν be a path in P from $c(b)_\sharp(0,\dots,0)$ to y. Then $\rho(\omega)\cdot\nu$ is a path in P from x to y. It follows that $\alpha \nearrow \beta$. $\square$

Theorem 5.25. *For all homology classes $\alpha, \beta \in H_*(|P|)$, we have $\alpha \nearrow \beta$ if and only if $f_*(\alpha) \nearrow f_*(\beta)$.*

Proof. We only have to show the if part of the statement. Consider homology classes $\alpha, \beta \in H_*(|P|)$ such that $f_*(\alpha) \nearrow f_*(\beta)$. Let A and B be precubical subsets of Q such that $f_*(\alpha) \in \operatorname{im} H_*(|A| \hookrightarrow |Q|)$, $f_*(\beta) \in \operatorname{im} H_*(|B| \hookrightarrow |Q|)$, and for all vertices $a \in A_0$ and $b \in B_0$ there exists a path in Q from a to b. We may suppose that A is finite. Then also $f(c(A))$ is finite. Let A' be the largest precubical subset of $f(c(A))$ such that $A \subseteq A'$ and for all vertices $a \in A'_0$ and $b \in B_0$ there exists a path in Q from a to b. Then A' is finite and $f_*(\alpha) \in \operatorname{im} H_*(|A'| \hookrightarrow |Q|)$. We have $c(A) \subseteq c(A') \subseteq c(f(c(A))) = c(A)$ and hence $c(A') = c(A)$.

We show that all elements of $c(A')$ are past-complete in A'. Suppose that there exists an element $x \in c(A')$ that is not past-complete in A'. Then $\deg(x) = n > 0$ and there exists a vertex $(l_1,\dots,l_n) \in R_x$ such that $x_\flat(l_1,\dots,l_n) \in A'$ but

$$x_\flat(\llbracket 0,l_1\rrbracket \otimes \dots \otimes \llbracket 0,l_n\rrbracket) \not\subseteq A'.$$

Since $x_\flat(l_1,\dots,l_n) \in A'$, for all vertices $a \in x_\flat(\llbracket 0,l_1\rrbracket \otimes \dots \otimes \llbracket 0,l_n\rrbracket)$ and $b \in B$, there exists a path in Q from a to b. Since

$$x_\flat(\llbracket 0,l_1\rrbracket \otimes \dots \otimes \llbracket 0,l_n\rrbracket) \subseteq x_\flat(R_x) = f(x_\sharp(\llbracket 0,1\rrbracket^{\otimes n})) \subseteq f(c(A')) = f(c(A)),$$

the maximality of A' implies that $A' \cup x_\flat(\llbracket 0,l_1\rrbracket \otimes \dots \otimes \llbracket 0,l_n\rrbracket) \subseteq A'$ and hence that $x_\flat(\llbracket 0,l_1\rrbracket \otimes \dots \otimes \llbracket 0,l_n\rrbracket) \subseteq A'$, a contradiction. It follows that all elements of $c(A')$ are past-complete in A'.

By Lemma 5.23, there exists a precubical subset $\tilde{A}$ of A' such that the inclusion $|\tilde{A}| \hookrightarrow |A'|$ is a homotopy equivalence and no element of P is broken in $\tilde{A}$. Since $\tilde{A} \subseteq A'$, for all vertices $a \in \tilde{A}_0$ and $b \in B_0$ there exists a path in Q from a to b. Since the inclusion $|\tilde{A}| \hookrightarrow |A'|$ is a homotopy equivalence, $f_*(\alpha) \in \operatorname{im} H_*(|\tilde{A}| \hookrightarrow |Q|)$. Since no element of P is broken in $\tilde{A}$, by Proposition 5.16, $f(c(\tilde{A})) = \tilde{A}$. It follows that $\alpha \in \operatorname{im} H_*(|c(\tilde{A})| \hookrightarrow |P|)$ and consequently, by Lemma 5.24, that $\alpha \nearrow \beta$. $\square$

Remark 5.26. It is not true that the homology graph of d-spaces is invariant under morphisms that are homeomorphisms. If one wanted to derive thereom 5.25 from results on the homology graph of d-spaces, then one would have to use the invariance of the homology graph of d-spaces under isomorphisms. Of course, one would also need to show that the homology graph of a precubical set and the homology graph of its geometric realization coincide.

5.8. Homeomorphic abstraction

We say that an M-HDA $\mathcal{A} = (P, I, F, \lambda)$ is a *homeomorphic abstraction* of an M-HDA $\mathcal{B} = (Q, J, G, \mu)$, or that $\mathcal{B}$ is a *homeomorphic refinement* of $\mathcal{A}$, if there exists a weak morphism f from $\mathcal{A}$ to $\mathcal{B}$ that is a homeomorphism and satisfies $f_0(I) = J$ and $f_0(F) = G$ [**12**]. For instance, in Figure 2, $\mathcal{A}$ is a homeomorphic abstraction of $\mathcal{B}$. It follows from Theorem 5.25 that if an M-HDA $\mathcal{A}$ is a homeomorphic abstraction of an M-HDA $\mathcal{B}$, then $\mathcal{A}$ and $\mathcal{B}$ have isomorphic homology graphs.

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