Homology, Homotopy and Applications, vol. 15(2), 2013, pp.361–383

MANIFOLD CALCULUS AND HOMOTOPY SHEAVES

PEDRO BOAVIDA DE BRITO AND MICHAEL WEISS

(*communicated by J. Daniel Christensen*)

Abstract

Manifold calculus is a form of functor calculus concerned with contravariant functors from some category of manifolds to spaces. A weakness in the original formulation is that it is not continuous in the sense that it does not handle the natural enrichments well. In this paper, we correct this by defining an enriched version of manifold calculus that essentially extends the discrete setting. Along the way, we recast the Taylor tower as a tower of homotopy sheafifications. As a spin-off we obtain a natural connection to operads: the limit of the Taylor tower is a certain (derived) space of right module maps over the framed little discs operad.

1. Introduction

Let M be a smooth manifold without boundary and denote by $\mathcal{O}(M)$ the poset of open subsets of M, ordered by inclusion. Manifold calculus, as defined in [**Wei99**], is a way to study (say, the homotopy type of) contravariant functors F from $\mathcal{O}(M)$ to spaces that take isotopy equivalences to (weak) homotopy equivalences. In essence, it associates to such a functor a tower—called the *Taylor tower*—of polynomial approximations that in good cases converges to the original functor, very much like the approximation of a function by its Taylor series.

The remarkable fact, which is where the geometry of manifolds comes in, is that the Taylor tower can be explicitly constructed: the k^{th} Taylor polynomial of a functor F is a functor T_kF that is in some sense the universal approximation to F with respect to the subposet of $\mathcal{O}(M)$ consisting of open sets diffeomorphic to k or fewer open balls.

A weakness in the traditional discrete approach is that in cases where F has obvious continuity properties, T_kF does not obviously inherit them, where by continuous we mean enriched over spaces. For example, let $F(U)$ for $U \in \mathcal{O}(M)$ be the space of smooth embeddings from U to a fixed smooth manifold N. It is clear that the group of diffeomorphisms $M \to M$ acts in a continuous manner on $F(M)$. One would expect a similar continuous action of the same group on $T_kF(M)$, for all k. But with the standard description of T_kF we only get an action of the underlying discrete group.

Received February 20, 2013, revised July 13, 2013; published on December 2, 2013.
2010 Mathematics Subject Classification: 18F10, 18D50, 18G55, 18G55, 57R40.
Key words and phrases: manifold calculus, functor calculus, embedding, operad, homotopy sheaf.
Article available at `http://intlpress.com/HHA/v15/n2/a20` and `doi:10.4310/HHA.2013.v15.n2.a20`

As a solution to this problem in the particular case of the embedding functor, a continuous model was proposed in [**GKW03**].

In this paper, we correct this lack of continuity by defining an enriched (or ∞) version of manifold calculus. Along the way, we reapproach the foundations of the theory by focusing on the wider notion of homotopy sheaves rather than on polynomial functors, which had the central role in [**Wei99**].

We now give a brief overview of the paper. Let $\mathcal{S}$ be a category of *spaces*; i.e., compactly generated Hausdorff spaces or simplicial sets (more on this at the end of the introduction). To have an enriched setting we replace the category $\mathcal{O}(M)$ by the topological category Man of smooth manifolds of a fixed dimension d and (codimension zero) embeddings. We then want to consider contravariant functors that are enriched over $\mathcal{S}$, namely functors $F\colon \mathsf{Man}^{\mathrm{op}} \to \mathcal{S}$ inducing *continuous* (or simplicial) maps

$$\mathrm{emb}(M,N) \to \mathrm{Hom}_{\mathcal{S}}(F(N),F(M))$$

that preserve composition, associativity, and units.

Moreover, there is the usual Grothendieck topology $\mathcal{J}_1$ on Man given by open covers. For each positive k, we can define a multi-local version of $\mathcal{J}_1$ where we only admit covers that have the property that every set of k (or fewer) points is contained in some open set of the cover. These form the Grothendieck topologies $\mathcal{J}_k$. Equipping Man with one of these Grothendieck topologies, we view it as a site and thus refer to $\mathcal{S}$-functors on Man as $\mathcal{S}$-enriched presheaves, or simply *presheaves*.

Definition 1.1. The **Taylor tower** of F is the *tower of homotopy sheafifications* of F with respect to the Grothendieck topologies $\mathcal{J}_k$.

For the precise meaning of this statement, see section 3. The enriched analogue of T_kF is an $\mathcal{S}$-enriched presheaf denoted by $\mathcal{T}_kF$ (Definition 4.2). It is the best homotopical approximation to F with respect to the subcategories Disc_k of Man whose objects are disjoint unions of k or fewer balls, and it comes with a natural "evaluation" map

$$F \to \mathcal{T}_kF \tag{1}$$

One of the main results of this paper is

Theorem 1.2. *The map (1) is a homotopy $\mathcal{J}_k$-sheafification.*

As a byproduct, we obtain a very natural connection to operads,

$$\mathcal{T}_\infty F(M) \simeq \mathbb{R}\mathrm{Hom}_{\mathbb{E}^{fr}}(\mathrm{emb}_M, F)$$

where the right hand side is the derived space of right module maps over the framed little d-dim discs operad $\mathbb{E}^{fr} := \mathbb{E}^{fr}_d$, and emb_M and F are the right $\mathbb{E}^{fr}$-modules defined by $\{\mathrm{emb}(\amalg_n\mathbb{R}^d, M)\}_{n\geqslant 0}$ and $\{F(\amalg_n\mathbb{R}^d)\}_{n\geqslant 0}$, respectively. This answers a question posed by Greg Arone and Victor Turchin; see [**AT11**, Conjecture 4.14].

In the case where F is the embedding functor $\mathrm{emb}(-,N)$ and $\dim N - \dim M \geqslant 3$ we get, as an immediate corollary of Goodwillie-Klein excision estimates, that

$$\mathrm{emb}(M,N) \simeq \mathbb{R}\mathrm{Hom}_{\mathbb{E}^{fr}}(\mathrm{emb}_M, \mathrm{emb}_N)$$

A version of this connection to operads appears in the work of Turchin [**T13**], and Arone-Turchin [**AT11**]. In the latter, coupled with formality results, it is further used to obtain explicit descriptions of the rational homology and homotopy of certain spaces of embeddings.

Finally, we point out that the framework in this paper is rather general and can be applied to categories other than Man. Namely, for a topological (or ∞) category $\mathcal{C}$ equipped with a Grothendieck topology possessing *good covers* and a presheaf F on $\mathcal{C}$, one can construct the tower of homotopy sheafifications of F—its Taylor Tower—and give an explicit model for it as a tower of homotopical approximations with respect to certain subcategories of $\mathcal{C}$. Examples include the category of topological spaces, the category of d-dimensional manifolds with boundary, the category of all manifolds, and the analogous versions where instead of smooth manifolds one considers topological manifolds.

1.1. Outline of the paper

In section 2 we define homotopy sheaves. We relax the definition of a Grothendieck topology to that of a coverage and we introduce two coverages $\mathcal{J}_k^\circ$ and $\mathcal{J}_k^h$. We show in section 5 and 7, respectively, that $\mathcal{J}_k^\circ$ and $\mathcal{J}_k^h$ form a basis for the Grothendieck topology $\mathcal{J}_k$ by proving that the three coverages generate the same homotopy sheaves. To set the ground, we first introduce the local model structure on the category of presheaves in section 3 and, in section 4, we discuss enriched homotopical (or ∞) Kan extensions. Finally, in section 8 we show that $\mathcal{T}_k$ is really an "enrichment" of T_k. Specifically, we show that for functors F on $\mathcal{O}(M)$ that, like $\mathrm{emb}(-, N)$, factor through Man, we have a weak equivalence

$$T_k F(U) \simeq \mathcal{T}_k F(U)$$

for every open set U of M.

1.2. Spaces, enrichments and notation.

We do not want to be very imposing on which category of spaces we work with. However, we need it to be cartesian closed, considered as enriched over itself, and have small limits and colimits. The category of compactly generated weak Hausdorff spaces is a natural candidate and the one we opt for, but everything can easily be formulated simplicially (see Appendix). We denote this category of spaces by $\mathcal{S}$. To make $\mathcal{S}$ enriched over itself give the Hom-sets, $\mathrm{Hom}_{\mathcal{S}}(X, Y)$, the (Kelleyfication of the) weak (alias compact-open) topology.

Similarly, the category Man of d-dimensional smooth manifolds without boundary is enriched over $\mathcal{S}$: give the C^∞ weak topology to the space of smooth embeddings $\mathrm{emb}(M, N)$. One important property of the weak topology is that it is metrizable, hence compactly generated and Hausdorff. All manifolds in this paper are assumed to be paracompact and, except in section 9, without boundary.

For any $\mathcal{S}$-enriched category $\mathcal{C}$, the notation $\mathrm{Hom}_{\mathcal{C}}$ always refers to the mapping object in $\mathcal{S}$.

Acknowledgments

We would like to thank Victor Turchin for numerous helpful comments and corrections. We also thank Assaf Libman and Mike Shulman for interesting discussions on Kan extensions and higher categories, and Dimitri Zaganidis for a correction. Last, but not least, it is a pleasure to acknowledge the influence of the writings of Dan Dugger on this paper.

The first author was supported by FCT, Fundação para a Ciência e a Tecnologia, grant SFRH/BD/61499/2009. The second author was supported through a Humboldt foundation professorship.

2. Homotopy sheaves

Definition 2.1. Let $\mathcal{C}$ be a (small) category. A *coverage* τ is an assignment to each object $X \in \mathcal{C}$ of a set $Cov_\tau(X)$ of collections of objects in the overcategory $\mathcal{C} \downarrow X$ subject to the following condition:

> Given $\mathcal{U} := \{U_i \to X\}_{i \in I}$ in $Cov_\tau(X)$ and elements $i_0, \ldots, i_n$ of I, where $n \geqslant 0$, the iterated pullback $U_{i_0} \times_X \cdots \times_X U_{i_n}$ exists in $\mathcal{C}$.

An element $\mathcal{U} \in Cov_\tau(X)$ is called a *covering* of X. If $\mathcal{U} = \{U_i \to X\}_{i \in I}$ and S is a finite nonempty subset of I, with distinct elements $i_0, \ldots, i_n$, we often write U_S instead of $U_{i_0} \times_X \cdots \times_X U_{i_n}$.

Let $(\mathcal{C}, \tau)$ be a simplicial or topological (i.e., $\mathcal{S}$-enriched) category equipped with a coverage τ. We denote by $\mathsf{PSh}(\mathcal{C})$ the category of simplicial or topological presheaves on $\mathcal{C}$ (i.e., $\mathcal{S}$-enriched functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}$). Since we will mostly be dealing with $\mathcal{S}$-enriched objects, we will often drop the adjective "enriched."

Definition 2.2. A presheaf $F \in \mathsf{PSh}(\mathcal{C})$ is said to satisfy *descent* for a covering $\{U_i \to X\}_{i \in I}$ if the natural map

$$F(X) \to \operatorname*{holim}_{S \subset I} F(U_S)$$

is a weak equivalence of spaces. The homotopy limit ranges over all nonempty finite subsets S of I.

A presheaf F is a *homotopy τ-sheaf* (or satisfies τ-descent) if it satisfies descent for every covering in τ.

Remark 2.3. A presheaf F is said to satisfy Čech descent for a covering $\mathcal{U}$ of X if the natural map

$$F(X) \to \operatorname*{holim}_{[n] \in \Delta} \prod_{i_0, \ldots, i_n} F(U_{\{i_0, \ldots, i_n\}})$$

is a weak equivalence. By cofinality [**BK87**, p. 317] of the functor that to a map $u : [n] \to I$ associates the image of u, this definition is equivalent to our definition 2.2 if every morphism $U_i \to X$ in $\mathcal{U}$ is a monomorphism. For the coverages τ that we are going to consider this always holds.

2.1. Coverages on the category of manifolds

Fix $d \geqslant 0$ once and for all. Let Man be the category of d-dimensional smooth manifolds and codimension zero embeddings. To ensure we have a small category, we consider its objects to be d-dimensional smooth submanifolds of $\mathbb{R}^\infty$.

Since Man has pullbacks (which are given by intersection), the condition defining a coverage on this category is vacuous. Manifold calculus provides us with two standard examples of coverages for a given non-negative integer k.

Definition 2.4 (Coverage $\mathcal{J}_k$)**.** The coverings of M in $\mathcal{J}_k$ are given by the collection of morphisms in Man of the form

$$\{f_i \colon U_i \to M\}_{i \in I}$$

where $I \neq \varnothing$, such that every set of k or fewer points is contained in $f_i(U_i)$ for some $i \in I$. These are called *k-covers*.

The condition $I \neq \varnothing$ is a little unusual. It amounts to saying that the set of coverings of the empty set, $Cov(\varnothing)$, consists of a single element $\{id \colon \varnothing \to \varnothing\}$. Notice that if we included the empty collection of morphisms in the set of coverings of the empty set, then our sheaves F would have the property that $F(\varnothing)$ is contractible.

Clearly, a 1-cover is the usual notion of an open cover of a manifold.

Definition 2.5 (Coverage $\mathcal{J}_k^h$)**.** The coverings of M in $\mathcal{J}_k^h$ are given by the collection of morphisms in Man of the form

$$\{f_i \colon M \backslash A_i \to M\}_{i \in \{0, \dots, k\}}$$

where $A_0, \dots, A_k$ are disjoint closed subsets of M.

Remark 2.6. A homotopy sheaf for $\mathcal{J}_k^h$ is usually called a *polynomial functor* of degree less than or equal to k.

2.2. Generalised good covers

Definition 2.7. Define the full subcategory Disc_k of Man by

$$Ob(\mathsf{Disc}_k) := \Big\{\text{manifolds diffeomorphic to } \textstyle\coprod_{i=1}^{j} \mathbb{R}^d \text{ for some } j \in \{0, 1, \dots, k\}\Big\}$$

Remark 2.8. The empty set $\varnothing$ is also an object of Disc_k (this is, by convention, the case $j = 0$).

It was realised long ago that every manifold M admits a covering $\{U_i \to M\}$ such that all finite intersections belong to Disc_1 (see for instance [**BT82**, Theorem 5.1]). In other words, every manifold can be covered by open balls $\{U_i\}$ such that every finite non-empty intersection $U_{i_0} \cap \dots \cap U_{i_n}$ is again diffeomorphic to an open ball. These are usually called *good covers*. Good covers define a coverage $\mathcal{J}_1^\circ$ on Man.

Definition 2.9. A cover $\{U_i \to M\}_{i \in I}$ of a manifold M, with $I \neq \varnothing$, is called a *good k-cover* if

1. every set of k or fewer points is contained in U_i for some i in I
2. every finite intersection $U_{i_0} \cap \dots \cap U_{i_n}$ belongs to Disc_k

A good 1-cover is simply a good cover. The multi-local analogue of the paragraph above is

Proposition 2.10. *Every manifold M admits a good k-cover.*

Proof. Equip M with a Riemannian metric, which we may take to be complete. Then there is, between any two points x and y of M, a (non-necessarily unique) geodesic from x to y of minimal length (corollary of Hopf-Rinow Theorem). Recall that a subset V of M is geodesically convex set if for distinct points x and y in V there exists a unique minimal geodesic segment connecting x and y, and that unique segment is contained in V. For every x in M and $\epsilon > 0$, there exists an open subset V of M that is geodesically convex, has diameter less than ϵ, and contains x (the diameter is the supremum of the lengths of any minimal geodesic segment in V).

Let U be an open subset of M. Let us say that U is *k-good* if it has not more than k path components, if there exists $\epsilon > 0$ such that each path component of U is geodesically convex and of diameter less than ϵ, and the (geodesic) distance between any two points in distinct path components is at least 100ϵ, say. The collection of all k-good subsets of M forms a good k-cover of M. This follows from the next lemma. ☐

Lemma 2.11. *Suppose open subsets $U, V \subset M$ are k-good. Then $U \cap V$ is also k-good.*

Proof. Choose ϵ_1 that works for U and ϵ_2 that works for V. Without loss of generality, ϵ_1 is less than or equal to ϵ_2. Since the intersection of two geodesically convex open subsets of M is a geodesically convex open subset of M, the components of $U \cap V$ are open, geodesically convex, and of diameter less than ϵ_1.

To see that there are at most k components, we show that the map from $\pi_0(U \cap V)$ to $\pi_0(U)$ induced by the inclusion is injective. Suppose not. Then there exist two distinct path components of V that make a nonempty intersection with a single path component of U. It follows that there are points x, y in those two distinct path components of V whose geodesic distance is less than ϵ_1, and therefore also less than ϵ_2. This contradicts our assumptions on V. The above argument also shows that the distance between any two points x, y in distinct components of $U \cap V$ is at least $100\epsilon_1$. Therefore ϵ_1 works for $U \cap V$. ☐

Definition 2.12 (Coverage $\mathcal{J}_k^\circ$)**.** The coverings in $\mathcal{J}_k^\circ$ are the good k-covers.

Remark 2.13. The coverage $\mathcal{J}_k$ satisfies the required axioms to be called a Grothendieck topology. The coverages $\mathcal{J}_k^h$ and $\mathcal{J}_k^\circ$ do not, and the Grothendieck topologies that they generate are too rigid to be interesting. Instead, we show in section 5 and 7 that, as *homotopy* (or ∞) Grothendieck topologies, these coverages do generate $\mathcal{J}_k$. We approach this by proving that the three coverages define the same homotopy sheaves. We shall not need any particular prerequisites on homotopy topos theory, but we suggest the reader interested in that connection consult the works of Toën and Vezzosi (in particular, [**TV05**]), Rezk, Simpson, and Lurie.

3. Homotopy sheafification

3.1. Projective model structure

The category $\mathsf{PSh}(\mathcal{C})$ of presheaves on $\mathcal{C}$ has a model structure, the so-called projective model structure, where weak equivalences and fibrations are determined objectwise[1] and cofibrations by a left lifting property with respect to acyclic fibrations. The simplicial or topological enrichment (as discussed in the appendix) makes $\mathsf{PSh}(\mathcal{C})$ into simplicial or topological model category.

---

[1]Meaning that a map of presheaves $F \to G$ is said to be an objectwise equivalence (resp. objectwise fibration) if the maps $F(M) \to G(M)$ are weak equivalences (resp. fibrations) in $\mathcal{S}$, for each $M \in \mathcal{C}$.

With this structure,

1. Every presheaf is fibrant (since every object in $\mathcal{S}$ is fibrant).
2. Every representable presheaf is cofibrant. This follows from the enriched Yoneda Lemma, which states that the natural map
$$\mathrm{Hom}_{\mathsf{PSh}(\mathcal{C})}(\mathrm{Hom}_{\mathcal{C}}(-, X), F) \xrightarrow{\cong} F(X)$$
is a homeomorphism.

Definition 3.1. The derived morphism space is the right derived functor of Hom,
$$\mathbb{R}\mathrm{Hom}_{\mathsf{PSh}(\mathcal{C})}(X, Y) = \mathrm{Hom}_{\mathsf{PSh}(\mathcal{C})}(QX, Y) \in \mathcal{S}$$
where Q denotes a cofibrant replacement functor on $\mathsf{PSh}(\mathcal{C})$ with the projective model structure.

Remark 3.2. The usual caveat applies here: if $\mathcal{S}$ is chosen to be the category of simplicial sets, then we do need to take an objectwise fibrant replacement of Y. Since we are working with compactly generated weak Hausdorff spaces in mind (and every space is fibrant), this is not needed here. See the appendix for further details.

3.2. Local model structure

Homotopy τ-sheaves are the "local" objects with respect to the maps of presheaves
$$\operatorname*{hocolim}_{S\subset I} \mathrm{Hom}_{\mathcal{C}}(-, U_S) \to \mathrm{Hom}_{\mathcal{C}}(-, M) \tag{2}$$
for each covering $\mathcal{U} := \{U_i \to M\}_{i\in I}$ in τ. More precisely,

Proposition 3.3. *Homotopy τ-sheaves are the presheaves F for which the map*
$$\mathbb{R}\mathrm{Hom}_{\mathsf{PSh}(\mathcal{C})}(\mathrm{Hom}_{\mathcal{C}}(-, M), F) \to \mathbb{R}\mathrm{Hom}_{\mathsf{PSh}(\mathcal{C})}(\operatorname*{hocolim}_{S\subset I} \mathrm{Hom}_{\mathcal{C}}(-, U_S), F)$$
is a weak equivalence for each covering $\mathcal{U} := \{U_i \to M\}_{i\in I}$ in τ.

Proof. The homotopy colimit on the right hand side is cofibrant (by [**Hir03**, Theorem 18.5.2]) so we can consider the honest (i.e., non-derived) space of morphisms functor $\mathrm{Hom}_{\mathsf{PSh}(\mathcal{C})}$ instead. Moreover,
$$\mathrm{Hom}_{\mathsf{PSh}(\mathcal{C})}(\operatorname*{hocolim}_{S\subset I} \mathrm{Hom}_{\mathcal{C}}(-, U_S), F) \simeq \operatorname*{holim}_{S\subset I} \mathrm{Hom}_{\mathsf{PSh}(\mathcal{C})}(\mathrm{Hom}_{\mathcal{C}}(-, U_S), F)$$
which one can check by using the usual formulas computing hocolim/holim and cartesian closedness. The assertion now follows by applying the enriched Yoneda Lemma to both sides. □

Definition 3.4. A morphism $F \to G$ in $\mathsf{PSh}(\mathcal{C})$ is a *τ-local equivalence* if
$$\mathbb{R}\mathrm{Hom}_{\mathsf{PSh}(\mathcal{C})}(G, Z) \xrightarrow{\simeq} \mathbb{R}\mathrm{Hom}_{\mathsf{PSh}(\mathcal{C})}(F, Z)$$
is a weak equivalence for every homotopy τ-sheaf Z.

Note that if $F \to G$ is an objectwise equivalence, then it is a τ-local equivalence.

Remark 3.5. One can say that τ-local equivalences are the maps that are seen as a weak equivalence by every homotopy τ-sheaf. In this terminology, homotopy τ-sheaves are precisely the presheaves that see all maps (2) as weak equivalences.

Theorem 3.6. *There is a model structure on the underlying category* $\mathsf{PSh}(\mathcal{C})$, *called the* τ*-local model structure and denoted by* $\mathsf{PSh}^{\tau}(\mathcal{C})$, *in which*

- *the weak equivalences are the* τ*-local equivalences;*
- *the cofibrations are the same as in the projective model structure on* $\mathsf{PSh}(\mathcal{C})$*;*
- *the fibrant objects are the homotopy* τ*-sheaves.*

Moreover, the identity maps

$$id\colon \mathsf{PSh}(\mathcal{C}) \leftrightarrows \mathsf{PSh}^{\tau}(\mathcal{C}) : id \tag{3}$$

form a simplicial/topological Quillen adjunction.

Proof. This model structure is the (left) Bousfield localisation of the projective model structure on $\mathsf{PSh}(\mathcal{C})$ at the set of all maps of the form (2). The statement is then a consequence of the general theory of Bousfield localisations (for more details, see [**Hir03**]). □

Remark 3.7. In classical topos theory, further conditions are usually imposed on the allowable coverages in order to guarantee that the forgetful functor from sheaves to presheaves has a left adjoint (called sheafification) which preserves finite limits. These conditions are guaranteed by the structure of a Grothendieck topology on τ. This point of view extends naturally to the simplicial (or ∞) setting. If $(\mathcal{C}, \tau)$ is a site (or more generally, a homotopy site), then the homotopy left adjoint in (3) commutes with finite homotopy limits; i.e., it is homotopy left exact (for details, consult [**TV05**]).

By definition, homotopy sheafification is a fibrant replacement in $\mathsf{PSh}^{\tau}(\mathcal{C})$; i.e., it consists of a homotopy τ-sheaf F^{sh} together with a τ-local equivalence $F \to F^{sh}$. Two homotopy sheafifications are necessarily weakly equivalent by uniqueness (up to weak equivalence) of fibrant replacements.

3.3. Taylor tower

We return to the category of d-manifolds Man. Recall from section 2 that Man has topologies $\mathcal{J}_k$, one for each non-negative k. By definition, every covering in $\mathcal{J}_{k+1}$ is a covering in $\mathcal{J}_k$, so, given a presheaf F in $\mathsf{PSh}(\mathsf{Man})$, we obtain a tower of sheafifications

$$\begin{tikzcd} F \arrow[d] \arrow[dr] \arrow[drr] \arrow[drrr] \arrow[drrrr] & & & & \\ F^{(0)} & F^{(1)} \arrow[l] & F^{(2)} \arrow[l] & F^{(3)} \arrow[l] & \dots \arrow[l] \end{tikzcd}$$

More precisely,

1. the map $F \to F^{(k)}$ is a homotopy $\mathcal{J}_k$-sheafification of F;
2. the map $F^{(k+1)} \to F^{(k)}$ is a homotopy $\mathcal{J}_k$-sheafification of $F^{(k+1)}$.

This tower is called the *Taylor tower*. The existence of such a tower is guaranteed by the existence of Bousfield localisations in our setting. In section 5 we shall give an explicit model for the Taylor tower. It is clear that any two models are weakly equivalent by uniqueness of fibrant replacements.

4. Enriched Kan extensions

4.1. Non-derived version

Recall the category Disc_k whose objects are given by k or fewer open balls (definition 2.7) and let i be the inclusion $\mathsf{Disc}_k \hookrightarrow \mathsf{Man}$. The restriction map $i^*\colon F \mapsto F \circ i$ fits into an $\mathcal{S}$-enriched adjunction

$$i^*\colon \mathsf{PSh}(\mathsf{Man}) \leftrightarrows \mathsf{PSh}(\mathsf{Disc}_k) : \mathrm{Ran}_i \tag{4}$$

where the right adjoint is given by Ran_i, the terminal or right Kan extension along i, as we will see below. It can be calculated [**Dub70**, Theorem I.4.2] as the equaliser, taken in spaces, of

$$\prod_{U \in \mathsf{Disc}_k} \mathrm{Hom}_{\mathcal{S}}(\mathrm{emb}(U, M), G(U)) \rightrightarrows \prod_{U,V \in \mathsf{Disc}_k} \mathrm{Hom}_{\mathcal{S}}(\mathrm{emb}(U, V) \times \mathrm{emb}(V, M), G(U))$$

when evaluated at $M \in \mathsf{Man}$.

Proposition 4.1. *The enriched terminal Kan extension* $\mathrm{Ran}_i G$ *of* G *along* i *is* $\mathcal{S}$*-naturally isomorphic to the presheaf which assigns to a manifold* M *the space of natural transformations from* $\mathrm{emb}(-, M)$ *to* G*; i.e.,* $\mathrm{Hom}_{\mathsf{PSh}(\mathsf{Disc}_k)}(\mathrm{emb}(-, M), G)$.

Proof. By direct checking, using the fact that $\mathcal{S}$ is cartesian closed. □

The Yoneda lemma provides a natural transformation

$$\epsilon\colon i^*\mathrm{Ran}_i G \to G$$

and hence a map of spaces

$$\mathrm{Hom}_{\mathsf{PSh}(\mathsf{Man})}(Z, \mathrm{Ran}_i G) \to \mathrm{Hom}_{\mathsf{PSh}(\mathsf{Disc}_k)}(i^* Z, G) \tag{5}$$

natural in $G \in \mathsf{PSh}(\mathsf{Man})$ and $F \in \mathsf{PSh}(\mathsf{Disc}_k)$, obtained by applying i^* and then post-composing with ϵ. Saying that this map is a natural homeomorphism is equivalent to saying that Ran_i is the right adjoint to i^*. One can then check this by reduction to the case of representables. Indeed, for $Z = \mathrm{emb}(-, M)$, the map (5) is a homeomorphism by the Yoneda lemma. Given an arbitrary presheaf Z, write it as a colimit of representables and then use the fact that $\mathrm{Hom}(\mathrm{colim}\, Z_i, G) \simeq \lim \mathrm{Hom}(Z_i, G)$.

Since i is a full embedding, ϵ is a natural homeomorphism; i.e., $\epsilon_V\colon \mathrm{Ran}_i G(V) \cong G(V)$ for every $V \in \mathsf{Disc}_k$. To sum up, given a presheaf F in $\mathsf{PSh}(\mathsf{Man})$, $\mathrm{Ran}_i(F \circ i)$ is the best terminal approximation to F by a presheaf that agrees with F on Disc_k.

4.2. Homotopical version

For homotopy-theoretic purposes Ran_i is not appropriate, however. We need to consider the homotopical counterpart of the adjunction (4), a simplicial or topological Quillen adjunction

$$i^*\colon \mathsf{PSh}(\mathsf{Man}) \leftrightarrows \mathsf{PSh}(\mathsf{Disc}_k) : \mathcal{T}_k \tag{6}$$

where $\mathcal{T}_k$ denotes the homotopy right adjoint to i^*. We proceed like in the non-homotopical case by defining a candidate for $\mathcal{T}_k$ and showing it is indeed a homotopy right adjoint.

Definition 4.2. Let $F \in \mathsf{PSh}(\mathsf{Disc}_k)$. Define the presheaf $\mathcal{T}_k F$ in $\mathsf{PSh}(\mathsf{Man})$ as

$$(\mathcal{T}_k F)(M) = \mathbb{R}\mathrm{Hom}_{\mathsf{PSh}(\mathsf{Disc}_k)}(\mathrm{emb}(-, M), F)$$

Remark 4.3. Note that we are restricting $\mathrm{emb}(-, M)$ to the full subcategory Disc_k. So, strictly speaking, $\mathcal{T}_k F(M) := \mathbb{R}\mathrm{Hom}_{\mathsf{PSh}(\mathsf{Disc}_k)}(i^*\mathrm{emb}(-, M), F)$, although we suppress this redundant information in the notation.

A few comments are in order:

1. If M is in Disc_k, then $\mathcal{T}_k F(M) \simeq \mathrm{Hom}_{\mathsf{PSh}(\mathsf{Disc}_k)}(\mathrm{emb}(-, M), F)$ since representables are cofibrant[2] in the projective model structure on $\mathsf{PSh}(\mathsf{Disc}_k)$. The enriched Yoneda lemma then gives a natural weak equivalence
$$\epsilon \colon i^*(\mathcal{T}_k F) \to F \tag{7}$$
Hence $\mathcal{T}_k F$ agrees with (meaning, is objectwise equivalent to) F on Disc_k.
2. For $F \in \mathsf{PSh}(\mathsf{Man})$, the adjoint of the evaluation map
$$\begin{array}{rccc} ev\colon & F(M) \times \mathrm{emb}(U, M) & \longrightarrow & F(U) \\ & (x, f) & \longmapsto & F(f)(x) \end{array}$$
gives rise to a morphism $F \to \mathrm{Hom}_{\mathsf{PSh}(\mathsf{Disc}_k)}(\mathrm{emb}(-, M), F)$ and, composing with the cofibrant replacement functor Q, to a morphism
$$\eta \colon F \to \mathcal{T}_k(i^* F) \tag{8}$$
called the k^{th} **Taylor approximation** to F.
3. The value of $\mathcal{T}_k F$ (or of $\mathcal{T}_k(i^*F)$ in case $F \in \mathsf{PSh}(\mathsf{Man})$) at a manifold M can be presented as the totalisation of the cosimplicial object given by
$$[j] \mapsto \prod_{U_0, \ldots, U_j \in \mathsf{Disc}_k} \mathrm{Hom}_{\mathcal{S}}\Big(\Big(\prod_{i=0}^{j-1} \mathrm{emb}(U_i, U_{i+1})\Big) \times \mathrm{emb}(U_j, M), F(U_0)\Big)$$
We defer the proof of this fact to the appendix.

Notice (7) and (8) are the counit and unit, respectively, of the hypothetical adjunction, which we now show exists.

Proposition 4.4. *Let $F \in \mathsf{PSh}(\mathsf{Disc}_k)$. Then*

$$\mathbb{R}\mathrm{Hom}_{\mathsf{PSh}(\mathsf{Man})}(G, \mathcal{T}_k F) \xrightarrow{\simeq} \mathbb{R}\mathrm{Hom}_{\mathsf{PSh}(\mathsf{Disc}_k)}(i^* G, F) \tag{9}$$

natural in $G \in \mathsf{PSh}(\mathsf{Man})$.

Proof. The case of a representable functor $G = \mathrm{emb}(-, M)$ is straightforward from the Yoneda lemma. Given a presheaf G, we can resolve it by representables as in the appendix and remark 3 above. Namely, $G \simeq |\mathcal{L}(G)_\bullet|$, where $\mathcal{L}(G)_i$ is essentially a coproduct of representables. By bringing geometric realisation outside the Hom as a totalisation on both sides, we reduce the problem to the case of representables, thus proving the claim. ∎

[2] Note, however, we are not claiming that there is a *functorial* cofibrant replacement Q that is the identity on representables.

Remark 4.5. Moreover, $\mathcal{T}_k F$ is an ∞-full embedding in the sense that

$$\mathbb{R}\mathrm{Hom}_{\mathsf{PSh}(\mathsf{Disc}_k)}(Z, F) \to \mathbb{R}\mathrm{Hom}_{\mathsf{PSh}(\mathsf{Man})}(\mathcal{T}_k Z, \mathcal{T}_k F)$$

is a natural weak equivalence. This follows automatically from (7) above; i.e., that the counit is a weak equivalence.

The proposition and remark above justify describing $\mathcal{T}_k F$ as the best homotopical terminal approximation of F by a functor on Man that agrees with F on Disc_k. In what follows, we will mostly consider presheaves F in $\mathsf{PSh}(\mathsf{Man})$, so we often write $\mathcal{T}_k F$ to mean $\mathcal{T}_k(i^* F) = \mathcal{T}_k(F \circ i)$ when there is no danger of confusion.

5. A model for the Taylor tower

This section is the heart of the paper. We show that the Taylor approximation $\mathcal{T}_k F$ is a model for the homotopy sheafification of F with respect to $\mathcal{J}_k$. This identifies the Taylor tower with the tower of homotopical approximations with respect to the subcategories Disc_k of Man.

Theorem 5.1. *The presheaf $\mathcal{T}_k F$ is a homotopy $\mathcal{J}_k$-sheaf.*

Proof. Let $\{U_i \to M\}_{i \in I}$ be a $\mathcal{J}_k$-cover of M. For V in Disc_k, the spaces $\mathrm{emb}(V, M)$ and $\mathrm{emb}(V, U_S)$ are homotopy equivalent to the spaces of (ordered) framed configurations of j points in M and U_S respectively, where j is the number of components of V. The homotopy equivalence is obtained by taking the value and first derivative of the embedding at the origin of each component. Hence, the canonical map of presheaves on Disc_k

$$\operatorname*{hocolim}_{S \subset I} \mathrm{emb}(-, U_S) \to \mathrm{emb}(-, M) \tag{10}$$

is an objectwise equivalence. It follows that

$$\begin{aligned} \mathcal{T}_k F(M) &\stackrel{\mathrm{def}}{=} \mathbb{R}\mathrm{Hom}_{\mathsf{PSh}(\mathsf{Disc}_k)}(\mathrm{emb}(-, M), F) \\ &\simeq \mathbb{R}\mathrm{Hom}_{\mathsf{PSh}(\mathsf{Disc}_k)}(\operatorname*{hocolim}_{S \subset I} \mathrm{emb}(-, U_S), F) \\ &\simeq \operatorname*{holim}_{S \subset I} \mathbb{R}\mathrm{Hom}_{\mathsf{PSh}(\mathsf{Disc}_k)}(\mathrm{emb}(-, U_S), F) \\ &= \operatorname*{holim}_{S \subset I} \mathcal{T}_k F(U_S) \end{aligned}$$

The first equivalence holds since the derived Hom preserves weak equivalences by definition. The second equivalence follows from [**Hir03**, Theorem 19.4.4]. $\square$

Theorem 5.2. *The following are equivalent for a presheaf $F \in \mathsf{PSh}(\mathsf{Man})$.*

(1) *F is a homotopy $\mathcal{J}_k$-sheaf.*
(2) *F is a homotopy $\mathcal{J}_k^\circ$-sheaf.*
(3) *The k^{th} Taylor approximation of F*

$$\eta_M \colon F(M) \to \mathcal{T}_k F(M)$$

is a weak equivalence for each $M \in \mathsf{Man}$.

Proof. (1) $\Rightarrow$ (2) is clear since a good k-cover is a k-cover. For (2) $\Rightarrow$ (3) take a good k-cover $\{U_i \to M\}_{i\in I}$ of M and let F be a homotopy $\mathcal{J}_k^\circ$-sheaf. We have the following commutative diagram

$$\begin{tikzcd} F(M) \arrow[r, "\simeq"] \arrow[d] & \operatorname{holim}_{S\subset I} F(U_S) \arrow[d, "\simeq"] \\ \mathcal{T}_k F(M) \arrow[r, "\simeq"] & \operatorname{holim}_{S\subset I} \mathcal{T}_k F(U_S) \end{tikzcd}$$

where the bottom arrow is a weak equivalence by Theorem 5.1 and, by hypothesis, so is the top arrow. The right hand arrow is an equivalence since F and $\mathcal{T}_k F$ agree on Disc_k and $U_S \in \mathsf{Disc}_k$ by definition of a good k-cover.

Finally, (3) $\Rightarrow$ (1) is immediate from Theorem 5.1. $\square$

Theorem 5.3. *The k^{th} Taylor approximation of a presheaf F*

$$\eta\colon F \to \mathcal{T}_k F$$

is a homotopy $\mathcal{J}_k$-sheafification.

Proof. In Theorem 5.1 we established that $\mathcal{T}_k F$ is a homotopy $\mathcal{J}_k$-sheaf. We now show that the Taylor approximation is a $\mathcal{J}_k$-local equivalence.

Let Z be a homotopy $\mathcal{J}_k$-sheaf. By Theorem 5.2, the Taylor approximation of Z is an objectwise equivalence, so we are required to show

$$\mathbb{R}\mathrm{Hom}_{\mathsf{PSh}(\mathsf{Man})}(\mathcal{T}_k F, \mathcal{T}_k Z) \to \mathbb{R}\mathrm{Hom}_{\mathsf{PSh}(\mathsf{Man})}(F, \mathcal{T}_k Z)$$

is a weak equivalence. By Proposition 4.4, the source and target of this map are weakly equivalent to $\mathbb{R}\mathrm{Hom}_{\mathsf{PSh}(\mathsf{Disc}_k)}(i^*F, i^*Z)$. $\square$

Corollary 5.4. *Let $\phi\colon F \to G$ be a map of homotopy $\mathcal{J}_k$-sheaves such that $i^*\phi$ is an objectwise equivalence. Then ϕ is an objectwise equivalence in* $\mathsf{PSh}(\mathsf{Man})$.

Proof. The statement follows from the commutative diagram below.

$$\begin{tikzcd} F \arrow[r, "\phi"] \arrow[d, "\simeq"'] & G \arrow[d, "\simeq"] \\ \mathcal{T}_k F \arrow[r, "\simeq"] & \mathcal{T}_k G \end{tikzcd}$$

The vertical arrows are weak equivalences by Theorem 5.2. The bottom arrow is a weak equivalence by the universal property of Kan extensions (or by direct checking using the formula defining $\mathcal{T}_k$). $\square$

5.1. $\mathcal{T}_k$-local structure

The homotopy idempotent functor $\mathcal{T}_k\colon \mathsf{PSh}(\mathsf{Man}) \to \mathsf{PSh}(\mathsf{Man})$ defines yet another model structure on $\mathsf{PSh}(\mathsf{Man})$ by the Bousfield-Friedlander localisation of the projective model structure (see Section 9 in [**Bou01**], in particular Theorem 9.3). For this new model structure, which we refer to as the $\mathcal{T}_k$-local model structure, a morphism $Q\colon F \to G$ is

- a weak equivalence if the map
$$\mathcal{T}_k Q\colon \mathcal{T}_k F \to \mathcal{T}_k G$$
is an objectwise equivalence in $\mathsf{PSh}(\mathsf{Man})$.
- a fibration if it is an objectwise fibration in $\mathsf{PSh}(\mathsf{Man})$ and the diagram
$$\begin{tikzcd} F \arrow[r, "Q"] \arrow[d, "\eta"'] & G \arrow[d, "\eta"] \\ \mathcal{T}_k F \arrow[r, "\mathcal{T}_k Q"] & \mathcal{T}_k G \end{tikzcd}$$
is a homotopy pullback square.

Lemma 5.5. *Suppose $Q\colon F \to G$ is a morphism in $\mathsf{PSh}(\mathsf{Man})$. Then Q is a $\mathcal{T}_k$-local equivalence if and only if it is a $\mathcal{J}_k$-local equivalence.*

Proof. We show that both statements are equivalent to the assertion

$$\text{the restriction } i^*Q \text{ is an objectwise weak equivalence on } \mathsf{PSh}(\mathsf{Disc}_k) \qquad (*)$$

If Q is a $\mathcal{T}_k$-local equivalence, this is immediate from the definition of $\mathcal{T}_k$. Suppose now that Q is a $\mathcal{J}_k$-local equivalence. Using Theorem 5.2, which identifies a homotopy sheaf Z with $\mathcal{T}_k Z$, we have that the induced map

$$\mathbb{R}\mathrm{Hom}_{\mathsf{PSh}(\mathsf{Man})}(G, \mathcal{T}_k Z) \to \mathbb{R}\mathrm{Hom}_{\mathsf{PSh}(\mathsf{Man})}(F, \mathcal{T}_k Z)$$

is a weak equivalence for every homotopy $\mathcal{J}_k$-sheaf Z in $\mathsf{PSh}(\mathsf{Man})$. By adjunction, this is equivalent to

$$\mathbb{R}\mathrm{Hom}_{\mathsf{PSh}(\mathsf{Disc}_k)}(i^*G, i^*Z) \to \mathbb{R}\mathrm{Hom}_{\mathsf{PSh}(\mathsf{Disc}_k)}(i^*F, i^*Z) \qquad (11)$$

being a weak equivalence for every homotopy sheaf Z. Since homotopy $\mathcal{J}_k$-sheaves are determined by their value on Disc_k, we now see that this is equivalent to $(*)$ rephrased as saying that the natural map

$$\mathbb{R}\mathrm{Hom}_{\mathsf{PSh}(\mathsf{Disc}_k)}(i^*G, W) \to \mathbb{R}\mathrm{Hom}_{\mathsf{PSh}(\mathsf{Disc}_k)}(i^*F, W) \qquad (12)$$

is a weak equivalence for every $W \in \mathsf{PSh}(\mathsf{Disc}_k)$. ☐

The two model structures have the same cofibrations by definition and the same weak equivalences by the preceding lemma, so the fibrations coincide.

Corollary 5.6. *The $\mathcal{T}_k$-local and $\mathcal{J}_k$-local model structures on $\mathsf{PSh}(\mathsf{Man})$ coincide. In particular, the identity functors yield a Quillen equivalence.*

It is worth emphasising that via the $\mathcal{T}_k$-local structure we have completely described the fibrations in the $\mathcal{J}_k$-local structure, something which a priori was not known.

6. Connection to operads

Let $\mathsf{Disc}_\infty \subset \mathsf{Man}$ be the colimit or union of the full subcategories Disc_k for $k \geqslant 0$. For each positive k, fix an embedding e_k of the disjoint union of k copies of $\mathbb{R}^d$

in $\mathbb{R}^\infty$. By taking the images of the embeddings e_k we obtain a category that is a topological skeleton of Disc_∞. This category (which we will still refer to as Disc_∞) is a topological PROP; i.e., its objects are identified with the non-negative integers, and it has a symmetric monoidal structure (here given by disjoint union) that corresponds to the addition of integers. Disc_∞ is called *framed little d-discs* PROP. The framed little d-discs operad [**MSS02**, p. 203] is the part $\mathsf{Disc}_\infty(m,1) := \mathrm{emb}(\amalg_m \mathbb{R}^d, \mathbb{R}^d)$ of the PROP. Since

$$\mathrm{emb}(m,n) \cong \coprod_{f\colon \underline{m}\to\underline{n}} \mathrm{emb}(m_1,1)\times\cdots\times\mathrm{emb}(m_n,1)$$

where m_i denotes the preimage of $i \in \underline{m} = \{1,\dots,m\}$ by f, we can reconstruct the PROP from the operad and vice-versa and use the two words interchangeably.

For $F \in \mathsf{PSh}(\mathsf{Disc}_\infty)$ we can define $\mathcal{T}_\infty F$ either as the homotopy inverse limit, over the positive integers k, of the $\mathcal{T}_k F$, or directly as a homotopy right Kan extension:

$$\mathcal{T}_\infty F(M) = \mathbb{R}\mathrm{Hom}_{\mathsf{PSh}(\mathsf{Disc}_\infty)}(\mathrm{emb}(-,M), F).$$

It follows from Theorem A.5 that these two definitions are equivalent.

Moreover, the category $\mathsf{PSh}(\mathsf{Disc}_\infty)$ is $\mathcal{S}$-isomorphic to the category of right modules over the framed little discs operad $\mathbb{E}^{fr}$, denoted $\mathbf{Mod}_{\mathbb{E}^{fr}}$. Therefore, for a given $F \in \mathsf{PSh}(\mathsf{Man})$, we obtain a description of $\mathcal{T}_\infty F$ as a derived space of right module maps over the framed little discs operad,

$$\mathcal{T}_\infty F(M) \simeq \mathbb{R}\mathrm{Hom}_{\mathbb{E}^{fr}}(\mathrm{emb}_M, F) \tag{13}$$

where the right $\mathbb{E}^{fr}$-modules are given by $\mathrm{emb}_M(n) := \mathrm{emb}(\amalg_n \mathbb{R}^d, M)$ and $F(n) := F(\amalg_n \mathbb{R}^d)$.

Combining (13) with the analyticity results of Goodwillie-Klein for the embedding functor, one has the following immediate consequence.

Proposition 6.1. *Suppose* $\dim N - \dim M \geqslant 3$. *Then*

$$\mathrm{emb}(M,N) \simeq \mathbb{R}\mathrm{Hom}_{\mathbb{E}^{fr}}(\mathrm{emb}_M, \mathrm{emb}_N)$$

Remark 6.2. For finite k, we obtain "truncated" versions of (13). Indeed, $\mathsf{PSh}(\mathsf{Disc}_k)$ is $\mathcal{S}$-isomorphic to the category of k-truncated right modules over the k-truncated framed little discs operad. The composition product on the the category of k-truncated (symmetric) sequences is the obvious one,

$$M(n)\times M(m_1)\times\cdots\times M(m_n) \to M(m_1+\cdots+m_n)$$

only defined when $m_1+\cdots+m_n \leqslant k$.

Specialising to $n \leqslant k$, we view emb_M and F as k-truncated sequences of spaces. In particular, emb_M and F are k-truncated modules over the k-truncated framed little discs operad $(\mathbb{E}^{fr})_{\leqslant k} := \{\mathbb{E}^{fr}(n)\}_{n\leqslant k}$, and we see that

$$\mathcal{T}_k F(M) \simeq \mathbb{R}\mathrm{Hom}_{(\mathbb{E}^{fr})_{\leqslant k}}(\mathrm{emb}_M, F) \tag{14}$$

Another example of interest is $S_*\mathrm{emb}(-,N)$, the singular chains of the embedding functor. We briefly sketch how to obtain a chain complex version of $\mathcal{T}_k$. Write S_* for the normalised singular chains functor $\mathsf{Top} \to \mathsf{Ch}_{\geqslant 0}$. Since it is a lax monoidal functor, we can use it to enrich Man over chain complexes.

Rename, for the rest of this section, $\mathsf{PSh}(\mathsf{Man})$ (resp. $\mathsf{PSh}(\mathsf{Disc}_k)$) as the category of $\mathsf{Ch}_{\geqslant 0}$-enriched presheaves from Man (resp. Disc_k) to $\mathsf{Ch}_{\geqslant 0}$ and define

$$\mathcal{T}_k^{\mathsf{Ch}} F(M) := \mathbb{R}\mathrm{Hom}_{\mathsf{PSh}(\mathsf{Disc}_k)}(S_*\mathrm{emb}(-,M), F) \in \mathsf{Ch}_{\geqslant 0}$$

The arguments of the previous sections show that $F \to \mathcal{T}_k^{\mathsf{Ch}} F$ is a homotopy $\mathcal{J}_k$-sheafification. It is also not hard to show that $\mathsf{PSh}(\mathsf{Disc}_k)$ is $\mathsf{Ch}_{\geqslant 0}$-equivalent to the category of right modules over $S_*(\mathbb{E}^{fr})_{\leqslant k}$, the chains of the framed little d-discs operad. Hence,

$$\mathcal{T}_\infty^{\mathsf{Ch}} F(M) \simeq \mathbb{R}\mathrm{Hom}_{S_*\mathbb{E}^{fr}}(S_*\mathrm{emb}_M, F)$$

In the particular case when F is the (normalised) singular chains of $\mathrm{emb}(-,N)$ we obtain the following result, by the analyticity results in [**Wei04**].

Proposition 6.3. *Suppose* $2\dim M + 1 < \dim N$. *The Taylor approximation gives a chain homotopy equivalence*

$$S_*\mathrm{emb}(M,N) \simeq \mathbb{R}\mathrm{Hom}_{S_*\mathbb{E}^{fr}}(S_*\mathrm{emb}_M, S_*\mathrm{emb}_N)$$

natural in M *and* N.

7. Homotopy $\mathcal{J}_k$-sheaf = Polynomial functor

Recall from section 2 that a polynomial functor of degree $\leqslant k$ is a homotopy sheaf for the coverage $\mathcal{J}_k^h$.

Definition 7.1. A presheaf $F \in \mathsf{PSh}(\mathsf{Man})$ is **good** if, for any monotone sequence $U_0 \subset U_1 \subset \cdots$ in Man whose union is M, the natural map

$$F(M) \to \operatorname*{holim}_i F(U_i)$$

is a weak equivalence of spaces.

Theorem 7.2. *The following are equivalent.*

(1) F *is a homotopy* $\mathcal{J}_k$*-sheaf.*

(2) F *is good and polynomial of degree* $\leqslant k$.

Proof. A covering in $\mathcal{J}_k^h$ is a covering in $\mathcal{J}_k$, so in order to show (1) $\Rightarrow$ (2) we need only prove goodness. Observe that a covering $\{U_i \to M\}_{i\in\mathbb{N}}$ of M with $U_i \subset U_{i+1}$ is a k-cover and

$$\operatorname*{holim}_{\varnothing \neq S \subset I} F(U_S) \simeq \operatorname*{holim}_i F(U_i)$$

so the homotopy $\mathcal{J}_k$-sheaf property for these coverings is precisely the condition of goodness.

Now, suppose $F \in \mathsf{PSh}(\mathsf{Man})$ is good and polynomial of degree $\leqslant k$. By Theorem 5.2, we are required to show that

$$F(M) \to \mathcal{T}_k F(M) \tag{15}$$

is a weak equivalence for every $M \in \mathsf{Man}$. The proof is now essentially the same as [**Wei99**, Theorem 5.1].

Any M in Man admits a proper smooth function $f\colon M \to [0,\infty)$. Any choice of an increasing and unbounded sequence of regular values for f exhibits M as the monotone union of a sequence of open submanifolds U_i, where the closure of each U_i in M is a compact smooth manifold (possibly with boundary). Therefore, due to the goodness of F, we can assume that M is the interior of a compact handlebody L. Take a handle decomposition of L with top-dimensional handles of index s.

The first case is $s = 0$; i.e., $M \cong \mathbb{R}^d \times \{1, 2, \dots, i\}$ for some integer $i \geqslant 0$. If $i \leqslant k$, then $F(M) \simeq \mathcal{T}_k F(M)$ since F and $\mathcal{T}_k F$ agree on Disc_k by construction. For $i > k$ we proceed inductively. Choose $k+1$ distinct components $A_0, \dots, A_k$ of M and consider the commutative diagram

$$\begin{tikzcd} F(M) \arrow[rr, "\eta"] \arrow[d, "\simeq"'] & & \mathcal{T}_k F(M) \arrow[d, "\simeq"] \\ \operatorname*{holim}_{\varnothing \neq S \subset \{0,\dots,k\}} F(M \backslash A_S) \arrow[rr, "\simeq"] & & \operatorname*{holim}_{\varnothing \neq S \subset \{0,\dots,k\}} \mathcal{T}_k F(M \backslash A_S) \end{tikzcd}$$

The vertical arrows are weak equivalences since F is polynomial of degree $\leqslant k$ by hypothesis, and the lower horizontal arrow is a weak equivalence by induction.

Now, suppose $s > 0$. Pick one of the s-handles

$$e\colon D^{d-s} \times D^s \to L$$

where $e^{-1}(\partial L) = D^{d-s} \times S^{s-1}$. Take pairwise disjoint closed discs $C_0, \dots, C_k$ in the interior of D^s and define

$$A_i := e(D^{d-s} \times C_i) \cap M$$

Then,

1. A_i is closed in M and $M \backslash A_i$ is the interior of a smooth handlebody with a handle decomposition with fewer s-handles, and so is any intersection of these, $\cap_{i \in S} M \backslash A_i$, for S a non-empty subset of $\{0, \dots, k\}$.
2. The family $\{M \backslash A_i \to M\}_{i \in \{0,\dots,k\}}$ is a covering of M for the coverage $\mathcal{J}_k^h$.

By induction, as in the case $s = 0$, the statement is easily verified. □

Remark 7.3. A way to paraphrase the Theorem above is as follows. Define a coverage by declaring $Cov(M)$ to consist of coverings in $\mathcal{J}_k^h$ and collections of morphisms of the form $\{f_i\colon U_i \to M\}_{i \in \mathbb{N}}$ with $f_i(U_i) \subset f_{i+1}(U_{i+1})$ and $\cup_i f_i(U_i) = M$. Then Theorem 7.2 says that this coverage and $\mathcal{J}_k$ define the same homotopy sheaves.

Polynomial functors are important in manifold calculus (and in functor calculus in general) as they can be given rather explicit descriptions in terms of cubical diagrams. The coverings in $\mathcal{J}_k^h$ are in practice much smaller than arbitrary or good k-covers so they are easier to handle.

Example 7.4. A polynomial functor of degree $\leqslant 1$ is a functor F which sends union-intersection squares to homotopy pullback squares.

7.1. Classification of linear functors

Following Goodwillie, we call a presheaf F *reduced* if $F(\varnothing) \simeq *$. Most examples of interest are reduced, but not every homotopy $\mathcal{J}_1$-sheaf is reduced: for instance, any constant presheaf is a homotopy $\mathcal{J}_1$-sheaf. One can reduce a presheaf F by setting

$$F^{red} := \text{hofiber}(F \to F(\varnothing))$$

Definition 7.5. A reduced polynomial functor of degree $\leqslant 1$ is called a *linear* functor.

For M in Man let frame(M) be the total space of the tangent frame bundle. If F is reduced, then

$$\begin{array}{rcl} \mathcal{T}_1 F(M) & \simeq & \mathrm{Hom}^{hO(d)}(\mathrm{emb}(\mathbb{R}^d, M), F(\mathbb{R}^d)) \\ & \simeq & \mathrm{Hom}^{O(d)}(\mathrm{frame}(M), F(\mathbb{R}^d)) \\ & \simeq & \Gamma(\mathrm{frame}(M) \times_{O(d)} F(\mathbb{R}^d) \to M) \end{array}$$

where $\mathrm{Hom}^{O(d)}(-,-)$ is the space of $O(d)$-maps and $\mathrm{Hom}^{hO(d)}(-,-)$ its derived functor. The first equivalence above uses $\mathrm{Hom}_{\mathsf{Disc}_1}(\mathbb{R}^d, \mathbb{R}^d) := \mathrm{emb}(\mathbb{R}^d, \mathbb{R}^d) \simeq O(d)$ and $F(\varnothing) \simeq *$.

Combining Theorems 5.2 and 7.2 with the above paragraph we obtain

Proposition 7.6. *The following are equivalent for a presheaf $F \in \mathsf{PSh}(\mathsf{Man})$.*

(1) *F is linear and good.*

(2) *The "scanning" map*

$$F(M) \longrightarrow \Gamma(\mathrm{frame}(M) \times_{O(d)} F(\mathbb{R}^d) \to M)$$

is a natural weak equivalence.

The scanning map can be made explicit by choosing a (smooth) exponential map $TM \to M$ which embeds every tangent space T_xM in M.

8. Relation to the unenriched model

Let $\mathcal{O}(M)$ be the poset of open sets of a manifold M; i.e., the discrete and relative version of Man. Clearly there is an "inclusion" functor

$$\mathcal{O}(M) \to \mathsf{Man}$$

given by inclusion $Ob(\mathcal{O}(M)) \hookrightarrow Ob(\mathsf{Man})$ on object-sets and sending a morphism $U \subset V$ in $\mathcal{O}(M)$ to the inclusion $i \in \mathrm{emb}(U, V)$.

Definition 8.1. A functor $f\colon \mathcal{O}(M)^{\mathrm{op}} \to \mathcal{S}$ is called *context-free*[3] if it factors through Man by an $\mathcal{S}$-functor $F\colon \mathsf{Man}^{\mathrm{op}} \to \mathcal{S}$.

Remark 8.2. If f is context-free, then it is necessarily isotopy invariant.

The discrete analogues of Disc_k and $\mathcal{T}_k$ are denoted $\mathcal{O}_k$ and T_k respectively. We refer the reader to [**Wei99**] for details on the unenriched setting. The following proposition says that $\mathcal{T}_k$ is really an enrichment of T_k.

---

[3]This terminology is due to G. Arone and V. Turchin

Proposition 8.3. *Let f be a context-free functor on $\mathcal{O}(M)$. Then*

$$T_k f(U) \simeq \mathcal{T}_k F(U)$$

for every $U \in \mathcal{O}(M)$.

Proof. By definition, $T_k f(U) := \operatorname*{holim}_{V \in \mathcal{O}_k(U)} f(V)$. Then,

$$\begin{aligned} T_k f(M) &= \operatorname*{holim}_{V \in \mathcal{O}_k(M)} F(V) \\ &\cong \operatorname*{holim}_{V \in \mathcal{O}_k(M)} \mathrm{Hom}_{\mathsf{PSh}(\mathsf{Disc}_k)}(\mathrm{emb}(-,V), F) \\ &\simeq \mathrm{Hom}_{\mathsf{PSh}(\mathsf{Disc}_k)}(\operatorname*{hocolim}_{V \in \mathcal{O}_k(M)} \mathrm{emb}(-,V), F) \\ &\simeq \mathbb{R}\mathrm{Hom}_{\mathsf{PSh}(\mathsf{Disc}_k)}(\mathrm{emb}(-,M), F) \end{aligned}$$

The first weak equivalence holds because f is context-free. The second is the enriched Yoneda lemma. The last equivalence follows from the fact that the map of presheaves in $\mathsf{PSh}(\mathsf{Disc}_k)$,

$$\operatorname*{hocolim}_{V \in \mathcal{O}_k(M)} \mathrm{emb}(-,V) \to \mathrm{emb}(-,M) \tag{16}$$

is an objectwise equivalence since, again, $\mathrm{emb}(-,V)$ and $\mathrm{emb}(-,M)$ in $\mathsf{PSh}(\mathsf{Disc}_k)$ are weakly equivalent to framed configuration spaces. Moreover, the left-hand side is cofibrant in the projective model structure since representables $\mathrm{emb}(-,V)$ are cofibrant and the homotopy colimit of an objectwise cofibrant diagram in a simplicial model category is cofibrant by [**Hir03**, Theorem 18.5.2]. □

9. Boundary case

Fix a $(d-1)$-manifold Z, and let Man^∂ denote the category of d-manifolds M with boundary ∂M, with a chosen diffeomorphism to Z. For simplicity, we assume Z is connected. Morphisms are (neat) embeddings respecting the identification of boundaries with Z. The replacement for Disc_k is Disc_k^∂, the full subcategory of Man^∂ whose objects are identified with non-negative integers (i.e., an object is the disjoint union of $Z \times [0,1)$ with n copies of the disc $\mathbb{R}^d$). Notice that a morphism in Disc_k^∂ may take some of the discs in the source to $Z \times [0,1)$.

In parallel with the non-boundary case, define $\mathcal{J}_k$ as the Grothendieck topology on Man^∂ with coverings given by collections $\{U_i \to M\}$ subject to the condition that every finite subset of cardinality k in the interior of M is contained in (the image of) U_j for some j.

A good k-cover $\{U_i \to M\}_{i \in I}$ is then a $\mathcal{J}_k$-cover with the property that every intersection U_S is in Disc_k^∂, for finite nonempty $S \subset I$.

The boundary versions of the statements in sections 5, 6, 7, and 8 follow from the propositions below.

Proposition 9.1. *Every manifold M with boundary admits a good k-cover.*

Proposition 9.2. *For every $\mathcal{J}_k$-covering $\{U_i \to M\}_{i \in I}$ and every V in Disc_k^∂, the map*
$$\operatorname*{hocolim}_{\varnothing \neq S \subset I} \mathrm{emb}^\partial(V, U_S) \to \mathrm{emb}^\partial(V, M)$$
is a weak equivalence.

Here $\mathrm{emb}^\partial(V, Y)$ denotes the space of embeddings of V into M fixing the boundary. Proposition 9.2 can be proved by analogy with the non-boundary case statement appearing in the proof of Theorem 5.1.

A variation of the argument in the proof of Proposition 2.10 proves Proposition 9.1, as we now briefly describe. Equip M with a complete Riemannian metric that restricts to a product metric on a fixed collar C of the boundary. We now say that an open subset U is k-good if it is the disjoint union of a sub-collar C' of C (on which the metric also restricts to a product) and m components $U_1, \ldots, U_m$ diffeomorphic to $\mathbb{R}^d$, with $m \leqslant k$, subject to the following conditions. Each path component of U is geodesically convex, there exists $\epsilon > 0$ such that the diameter of the U_i is less than ϵ, and the distance between any two points in distinct components (including the collar) of U is at least 100ϵ, say. The collection of all k-good subsets of M then forms a good k-cover.

We thus obtain a Taylor tower for a presheaf $F \in \mathsf{PSh}(\mathsf{Man}^\partial)$ where
$$\mathcal{T}_k F(M) = \mathbb{R}\mathrm{Hom}_{\mathsf{PSh}(\mathsf{Disc}_k^\partial)}(\mathrm{emb}^\partial(-, M), F)$$
is a model for the k^{th}-approximation of F.

As an application, let N be a smooth manifold with boundary, $\dim N = n \geqslant d$. Fix a smooth embedding $Z \to \partial N$. For M in Man let $\mathrm{emb}^\partial(M, N)$ be the space of smooth neat embeddings $M \to N$ that extend the specified embedding of $Z = \partial M$ in the boundary of N.

Corollary 9.3. *Suppose $n - d \geqslant 3$. Then*
$$\mathrm{emb}^\partial(M, N) \simeq \mathbb{R}\mathrm{Hom}_{\mathsf{PSh}(\mathsf{Disc}^\partial)}(\mathrm{emb}^\partial(-, M), \mathrm{emb}^\partial(-, N))$$

Remark 9.4. In the case where $M = \mathbb{D}^d$ and $N = \mathbb{D}^n$, Arone and Turchin [**AT11**] combine this with operad theoretic arguments to show that
$$\mathrm{hofiber}[\mathrm{emb}^\partial(M, N) \to \mathrm{imm}^\partial(M, N)]$$
is weakly eqivalent to the space of (right derived) *weak* $\mathbb{E}_d$ *bimodule* maps from $\mathbb{E}_d$ to $\mathbb{E}_n$. Here $\mathrm{imm}^\partial(M, N)$ is the space of smooth neat immersions $M \to N$ prescribed on ∂M, while $\mathbb{E}_d = \mathbb{E}$ is the unframed little d-discs operad and $\mathbb{E}_n$ is the unframed little n-discs operad. Note that $\mathrm{imm}^\partial(M, N) \simeq \mathcal{T}_1 \mathrm{emb}^\partial(M, N)$. In the meantime, it seems that the *weak bimodule* terminology has been dropped in favour of *infinitesimal bimodule.*

Appendix A. Derived mapping spaces and resolutions

References for this appendix are [**GJ09**, VIII], [**Hir03**], and [**MMSS01**]. Throughout, $\mathcal{S}$ denotes either the category of simplicial sets or compactly generated Hausdorff spaces (CGHS). The simplicial case is better documented in the literature so we will concentrate on the topological case.

Let $\mathcal{C}$ be a small $\mathcal{S}$-enriched category. Recall that an $\mathcal{S}$-enriched functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ consists of a collection of spaces $\{F(X)\colon X \in Ob\,\mathcal{C}\}$ together with maps in $\mathcal{S}$

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{S}}(F(Y),F(X))$$

which respect composition, associativity, and units.

Definition A.1. Given two enriched functors F and G, define $\mathrm{Hom}_{\mathsf{PSh}(\mathcal{C})}(F,G)$ as the limit (alias equaliser), taken in $\mathcal{S}$, of the diagram

$$\prod_{X\in\mathcal{C}} \mathrm{Hom}_{\mathcal{S}}(F(X),G(X)) \rightrightarrows \prod_{X,Y\in\mathcal{C}} \mathrm{Hom}_{\mathcal{S}}(\mathrm{Hom}_{\mathcal{C}}(X,Y), \mathrm{Hom}_{\mathcal{S}}(F(Y),G(X)))$$

where one of the morphisms in the diagram sends $\{\eta_X\colon F(X)\to G(X)\}_{X\in\mathcal{C}}$ to the map $(f\colon X\to Y)\mapsto \eta_X\circ F(f)$ and the other morphism sends it to $f\mapsto G(f)\circ\eta_Y$.

If $\mathcal{S}$ = simplicial sets, then $\mathrm{Hom}_{\mathsf{PSh}(\mathcal{C})}(F,G)$ is isomorphic to the simplicial set whose set of n-simplices is the set of natural transformations $F\otimes\Delta[n]\to G$, where $(F\otimes\Delta[n])(U) := F(U)\times\Delta[n]$.

If $\mathcal{S} = \mathrm{CGHS}$, then $\mathrm{Hom}_{\mathsf{PSh}(\mathcal{C})}(F,G)$ is equivalent to the space we obtain by topologising the set of natural transformations $F\to G$ by the (kelleyfication of the) subspace topology of the product

$$\prod_{U\in\mathcal{C}} \mathrm{Hom}_{\mathcal{S}}(F(U),G(U))$$

equipped with the product topology.

These definitions make the category of enriched functors into an $\mathcal{S}$-enriched category that we denote by $\mathsf{PSh}(\mathcal{C})$.

We call a map $\eta\colon F\to G$ in $\mathsf{PSh}(\mathcal{C})$ an *objectwise weak equivalence*, resp. *objectwise fibration,* if, for each $M\in\mathcal{C}$, the map

$$\eta_M\colon F(M)\to G(M)$$

is a weak equivalence, resp. fibration, in $\mathcal{S}$.

Theorem A.2. *There is a cofibrantly generated topological/simplicial model structure on* $\mathsf{PSh}(\mathcal{C})$*, the so-called projective model structure, in which a map* $\eta\colon F\to G$ *is a weak equivalence (resp. fibration) if it is an objectwise weak equivalence (resp. fibration). Cofibrations are retracts of transfinite compositions of pushouts along maps of the form*

$$\mathrm{Hom}_{\mathcal{C}}(-,X)\times S^{n-1} \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(-,X)\times D^n$$

or, in the simplicial case,

$$\mathrm{Hom}_{\mathcal{C}}(-,X)\times \partial\Delta^{n} \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(-,X)\times \Delta^n$$

for some $X\in\mathcal{C}$ *and* $n\geqslant 0$*. Here* S^{-1} *and* $\partial\Delta^0$ *are defined as* $\varnothing$*.*

Proof. One reference for this is [**MMSS01**, Theorem 6.5]. We sketch another proof that highlights a few important aspects that will be needed when we discuss resolutions. It is a variation of the proof of [**Hir03**, Theorem 11.6.1].

Let $\mathcal{C}^\delta$ denote the (topological) category with the same objects as $\mathcal{C}$ but only identity morphisms. The forgetful functor $U\colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{C}^\delta)$ fits into an enriched adjunction

$$L\colon \mathsf{PSh}(\mathcal{C}^\delta) \leftrightarrows \mathsf{PSh}(\mathcal{C}) : U \tag{17}$$

where L, the *free* functor, is given by

$$L(G) = \coprod_{V\in\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(-,V) \times G(V)$$

for $G \in \mathsf{PSh}(\mathcal{C}^\delta)$. It is rather straightforward to see that $\mathsf{PSh}(\mathcal{C}^\delta)$ is a topological model category since it is essentially an $Ob\,\mathcal{C}$-fold product of $\mathcal{S}$. We then use the adjunction to transfer the topological model structure on $\mathsf{PSh}(\mathcal{C}^\delta)$ to a topological model structure on $\mathsf{PSh}(\mathcal{C})$. The underlying 0-categorical statement follows from [**Hir03**, Theorem 11.3.2]. Quillen's axiom SM7 (or its topological incarnation) is easily verified. □

A.1. Resolutions

In this section we discuss the construction of a resolution of a presheaf $F \in \mathsf{PSh}(\mathcal{C})$ (c.f. 2.6 in [**Dug01**] for a discrete analogue). More precisely, we wish to find a cofibrant presheaf $F^\natural$ and a weak equivalence $F^\natural \to F$, where everything in sight should be enriched as always.

Associated to the free-forgetful adjunction (17) we construct a simplicial object in $\mathsf{PSh}(\mathcal{C})$, sometimes called the cotriple resolution. For G in $\mathsf{PSh}(\mathcal{C})$, let $\mathcal{L}(G)_\bullet$ be the simplicial object with n-simplices given by

$$\mathcal{L}(G)_n := (LU)^{n+1}(G) \in \mathsf{PSh}(\mathcal{C})$$

i.e.,

$$\coprod_{V_0,\dots,V_n\in\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(-,V_0) \times \cdots \times \mathrm{Hom}_{\mathcal{C}}(V_{n-1},V_n) \times G(V_n)$$

Note that $\mathcal{L}(G)_\bullet$ is naturally augmented via the counit map $LU(G) \to G$ given by the composition (remember G is contravariant)

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(W,V) \times G(V) & \longrightarrow & G(W) \\ (f,x) & \longmapsto & G(f)(x) \end{array}$$

Finally, define $|G| \in \mathsf{PSh}(\mathcal{C})$ as the geometric realisation (i.e., the coend over Δ) of $\mathcal{L}(G)_\bullet$,

$$|G| := |\mathcal{L}(G)_\bullet| = \Delta^\bullet \otimes_\Delta \mathcal{L}(G)_\bullet$$

The tensoring of $\mathsf{PSh}(\mathcal{C})$ over $\mathcal{S}$ is defined objectwise, thus $|G|(V) = |\mathcal{L}(G)_\bullet(V)|$.

Proposition A.3. *Suppose that, for each $V \in \mathcal{C}$,*

(1) *$G(V)$ is cofibrant;*
(2) *the map $* \to \mathrm{Hom}_{\mathcal{C}}(V,V)$ selecting the identity morphism is a cofibration.*

Then the presheaf $|G|$ is a cofibrant replacement of G in $\mathsf{PSh}(\mathcal{C})$.

Proof. The natural map $|G| \to G$ is a weak equivalence by general considerations of cotriple resolutions. The lemma below implies $|G|$ is cofibrant since geometric realisation preserves cofibrations ([**GJ09**, Proposition 3.6]). □

Lemma A.4. *Under the hypothesis of Proposition A.3, $\mathcal{L}(G)_\bullet$ is Reedy cofibrant.*

Proof. Recall that a simplicial object $W_\bullet$ in a model category $\mathcal{D}$ is Reedy cofibrant if, for each $n \geqslant 0$, the latching map

$$L_n W \hookrightarrow W_n$$

is a cofibration in $\mathcal{D}$. Here $L_n W$ can be described as $\cup_{i=0}^{n-1} s_i(W_{n-1})$, a subspace of W_n , if $\mathcal{D} = \mathcal{S}$. For general $\mathcal{D}$ it is defined as

$$\operatorname*{colim}_{[n]\twoheadrightarrow[k]} W_k$$

where the colimit is taken over the following category. The objects are monotone surjections $[n] \to [k]$ in Δ, with n fixed and $k < n$. A morphism from $f\colon [n] \to [k]$ to $g\colon [n] \to [\ell]$ is a surjective monotone $h\colon [k] \to [\ell]$ that satisfies $hf = g$.

The condition *Reedy cofibrant* uses only the degeneracy operators and can therefore also be formulated for any functor $W_\bullet\colon \Delta_s^{\mathrm{op}} \to \mathcal{D}$, where Δ_s is the subcategory of Δ with objects $[n]$ for $n \geqslant 0$ and monotone surjections only as morphisms.

Since cofibrations in $\mathsf{PSh}(\mathcal{C})$ are generated via the free functor (which is a left adjoint, so commutes with colimits), it suffices to prove that

$$[k] \mapsto W_k := U(LU)^k(G)$$

is Reedy cofibrant as a functor from Δ_s^{op} to $\mathsf{PSh}(\mathcal{C}^\delta)$. The first hypothesis guarantees that the 0^{th} latching map is a cofibration. The second hypothesis establishes the result for the higher latching maps. ☐

When $\mathcal{C}$ is the category Disc_k , where k is a positive integer or $k = \infty$, we obtain

Theorem A.5. *Let $F \in \mathsf{PSh}(\mathsf{Man})$. Then $\mathcal{T}_k F(M)$ is weakly equivalent to the totalisation of the cosimplicial object whose space of 0-simplices is*

$$\prod_{V\in\mathsf{Disc}_k} \mathrm{Hom}_{\mathcal{S}}(\mathrm{emb}(V,M), F(V))$$

and, for $n > 0$, whose space of n-simplices is

$$\prod_{V\in\mathsf{Disc}_k} \mathrm{Hom}_{\mathcal{S}}(\mathcal{L}(M)_{n-1}(V), F(V))$$

where $\mathcal{L}(M)_\bullet := \mathcal{L}(\mathrm{emb}(-,M))_\bullet$.

Proof. Apply the previous proposition to $\mathrm{emb}(-,M)$. For each $V \in \mathsf{Disc}_k$, $\mathrm{emb}(V,M)$ is a manifold, hence cofibrant. The morphism spaces in Disc_k are manifolds, so the inclusions $* \to \mathrm{emb}(V,V)$ selecting the identity morphism are cofibrations. Therefore,

$$\mathcal{T}_k F(M) \simeq \mathrm{Hom}_{\mathsf{PSh}(\mathsf{Disc}_k)}(|\mathcal{L}(M)_\bullet|, F) \simeq \mathrm{Tot}\,\mathrm{Hom}_{\mathsf{PSh}(\mathsf{Disc}_k)}(\mathcal{L}(M)_\bullet, F)$$

The result follows by applying the adjunction (17). ☐

References

[AT11] Gregory Arone and Victor Turchin, *On the rational homology of high dimensional analogues of spaces of long knots*, `http://arxiv.org/abs/1105.1576v4` (2013).

[BK87] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, vol. 304, Springer, 1987.

[Bou01] A. K. Bousfield, *On the telescopic homotopy theory of spaces*, Trans. Amer. Math. Soc. **353** (2001), no. 6, 2391–2426.

[BT82] Raoul Bott and Loring W. Tu, *Differential forms in algebraic topology*, Graduate Texts in Mathematics, vol. 82, Springer-Verlag, New York, 1982.

[DK80] W. G. Dwyer and D. M. Kan, *Function complexes in homotopical algebra*, Topology **19** (1980), no. 4, 427–440.

[Dub70] Eduardo J. Dubuc, *Kan extensions in enriched category theory*, Lecture Notes in Mathematics, Vol. 145, Springer-Verlag, Berlin, 1970.

[Dug01] Daniel Dugger, *Universal homotopy theories*, Advances in Mathematics **164** (2001), no. 1, 144–176.

[GJ09] Paul G. Goerss and John F. Jardine, *Simplicial homotopy theory*, vol. 174, Springer, 2009.

[GKW03] Thomas G. Goodwillie, John R. Klein, and Michael S. Weiss, *A Haefliger style description of the embedding calculus tower*, Topology **42** (2003), no. 3, 509–524.

[Hir03] Philip S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003.

[MMSS01] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley, *Model categories of diagram spectra*, Proceedings of the London Mathematical Society **82** (2001), no. 2, 441–512.

[MSS02] Martin Markl, Steve Shnider, and Jim Stasheff, *Operads in algebra, topology and physics*, Mathematical Surveys and Monographs, vol. 96, American Mathematical Society, 2002.

[TV05] Bertrand Toën and Gabriele Vezzosi, *Homotopical algebraic geometry I: topos theory*, Advances in Mathematics **193** (2005), no. 2, 257–372.

[T13] Victor Turchin, *Context-free manifold calculus and the Fulton-MacPherson operad*, Algebraic and Geometric Topology **13** (2013), 1243–1271.

[Wei99] Michael Weiss, *Embeddings from the point of view of immersion theory I*, Geom. Topol. **3** (1999), 67–101.

[Wei04] Michael S. Weiss, *Homology of spaces of smooth embeddings*, Q. J. Math. **55** (2004), no. 4, 499–504.

Pedro Boavida de Brito `p.boavida@uni-muenster.de`
Michael Weiss `m.weiss@uni-muenster.de`

Mathematisches Institut, Universität Münster, Einsteinstr. 62, D-48149 Münster, Germany