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HOMOTOPY COLIMITS IN STABLE REPRESENTATION THEORY

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Abstract

We study the problem of existence and uniqueness of homotopy colimits in stable representation theory, where one typically does not have model category structures to guarantee that these homotopy colimits exist or have good properties. We get both negative results (homotopy cofibers fail to exist if there exist any objects of positive finite projective dimension!) and positive results (reasonable conditions under which homotopy colimits exist and are unique, even when model category structures fail to exist). We describe some applications to Waldhausen K-theory and to deformation-theoretic methods in stable representation theory.

Contents

1. Introduction.

Suppose $\mathcal{C}$ is an abelian category—for examples, the category of modules over a ring, or the category of abelian sheaves on a scheme. By *stable representation*

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theory one means the study of $\mathcal{C}$ under the equivalence relation in which one regards two maps f, g in $\mathcal{C}$ as being equivalent, or "homotopic," if $f - g$ factors through a projective object. One says that two objects in $\mathcal{C}$ are "stably equivalent" if they become isomorphic after imposing this equivalence relation on maps in $\mathcal{C}$. Since stably equivalent objects in $\mathcal{C}$ have the same $\mathrm{Ext}^n_{\mathcal{C}}$ groups for all $n > 0$, stable representation theory is a natural topic of study if one wants to compute the higher $\mathrm{Ext}_{\mathcal{C}}$ groups for a large family of objects (or perhaps *all* objects) in $\mathcal{C}$.

In this paper we consider the problem of the existence and uniqueness of *homotopy colimits* in stable representation theory. Specifically, if one has a diagram of objects in an abelian category $\mathcal{C}$, and all of the morphisms in the diagram are monomorphisms, one wants to know that replacing an object in the diagram with a stably equivalent object will not change the colimit of the diagram, up to stable equivalence. Here are some reasons why one wants to do this:

1. One wants to study and compute the stable algebraic G-theory associated to $\mathcal{C}$, that is, one wants to study *derived stable representation theory,* in the sense that $G_0(\mathcal{C})$ is the Grothendieck group completion of a monoid of stable equivalence classes of objects in $\mathcal{C}$, and the higher G-theory groups capture more subtle K-theoretic invariants of the stable representation theory of $\mathcal{C}$. We do some of this in our paper [**14**], using results from the present paper.

To construct the relevant G-theory, one needs the structure of a Waldhausen category on $\mathcal{C}$ in which the weak equivalences are the stable equivalences. But one of the axioms required of a Waldhausen category, Waldhausen's axiom **Weq 2** from [**15**], is that, given a commutative diagram in $\mathcal{C}$

$$\begin{tikzcd} X' \arrow[d] & Y' \arrow[l] \arrow[r] \arrow[d] & Z' \arrow[d] \\ X & Y \arrow[l] \arrow[r] & Z \end{tikzcd}$$

in which the horizontal maps are cofibrations and the vertical maps are weak equivalences, the induced map of pushouts $X' \amalg_{Y'} Z' \to X \amalg_Y Z$ is a weak equivalence. In other words, *homotopy pushouts are well-defined in $\mathcal{C}$.* So one must know something about well-definedness of homotopy pushouts in order to do any K-theory or G-theory.

2, One wants to be able to make constructions in stable representation theory which come from *geometric realization of simplicial objects* and *totalization of cosimplicial objects.* For example, topological Hochschild homology and topological Andre-Quillen homology occur as geometric realizations. Because of their applications in deformation theory and algebraic K-theory, one wants to be able to form the necessary geometric realizations to construct these objects in the context of stable representation theory.

Geometric realizations are particular kinds of homotopy colimits and we study their existence and uniqueness in this paper.

Existence and uniqueness of homotopy colimits is well-understood in the context of a model category, but *abelian categories frequently do not admit the structure of a model category* in which the weak equivalences are the stable equivalences and the cofibrations are the monomorphisms. So one cannot rely on general model-category-theoretic methods.

In fact, we get some negative results, which preclude the existence of such a model category structure (or even a Waldhausen category structure) on an abelian category under surprising conditions: a special case of our Cor. 3.11 is that *if an abelian category $\mathcal{C}$ with enough projectives has any objects of finite, positive projective dimension, then homotopy cofibres fail to be well-defined in $\mathcal{C}$.* As a consequence, if there exists a single object of projective dimension $\neq 0, \infty$, then $\mathcal{C}$ cannot have a model category structure or a Waldhausen category structure with the desired cofibrations and weak equivalences.

On the other hand, suppose that $\mathcal{C}$ has enough projectives and enough injectives, and suppose that every projective object is injective. Then homotopy cofibers (and homotopy pushouts in general) are unique up to stable equivalence; this is a special case of our Cor. 4.4. As a consequence, $\mathcal{C}$ then satisfies Waldhausen's axiom **Weq 2**. This is substantially weaker than the assumption that $\mathcal{C}$ is quasi-Frobenius (i.e., projective objects coincide with injective objects), which is the known condition under which $\mathcal{C}$ admits a model category structure with the desired properties, as in [**6**].

We also show that, when $\mathcal{C}$ has enough projectives and enough injectives, when every projective object is injective, and when every object can be embedded appropriately into a projective object, then geometric realization of simplicial objects is well-defined in $\mathcal{C}$; this is a special case of our Cor. 6.8.

Throughout this paper, we work in the context of *relative homological algebra*. A good treatment of the basics of this subject is in Mac Lane's book [**7**], but the appendix to this paper is a self-contained introduction to the subject, so that the reader will not have to look elsewhere for the basic definitions.

There are two reasons we work in the context of relative homological algebra:

1. In [**14**] we study the effect of localization, i.e., change of allowable class, on algebraic G-theory. Our results in [**14**] require the results on well-definedness of homotopy pushouts, in particular Cor. 4.5, from the present paper.

2. Our main area of applications for these results is in the stable representation theory of comodules over Hopf algebroids, especially those arising in stable homotopy theory. The Ext groups in the category of comodules over various Hopf algebroids are the E_2-terms of generalized Adams spectral sequences which are used to compute stable homotopy groups of various spaces and spectra, so the stable representation theory of these comodules is quite important for topology. If (A, Γ) is a Hopf algebroid, the relevant homological algebra is the one in which the relative projective objects are the comodules which are tensored up from A-modules; see Appendix 1 of [**12**] for these ideas. Since comodules over certain Hopf algebroids are equivalent to quasicoherent modules over certain Artin stacks, this direction is relevant to algebraic geometry as well.

We note that three essential technical tools in this paper are the relative-homological-algebraic versions of classical theorems in the theory of abelian categories: namely, our Lemmas 3.6 and 3.7 are the relative versions of the main results of the 1961 paper [**5**] of Hilton and Rees, and our Prop. 4.1 is a relative variant of the main result of the 1963 paper [**10**] of Oort. These results are, to our knowledge, new, but they are not difficult: one can simply mimic the proofs of Hilton-Rees and Oort, with appropriate adjustments for the more general setting.

Even if one has no interest in relative homological algebra or in abelian categories aside from categories of modules over a ring, our positive results still have some "teeth": there is an open conjecture in pure algebra that the category of finitely-generated modules over a ring R is quasi-Frobenius if and only if every finitely-generated R-module embeds in a projective R-module. See [**11**] for some discussion of this problem. This conjecture is the analogue for *finitely-generated* modules of the theorem of Faith and Walker (a good reference is [**3**]), which states that the category of *all* R-modules is quasi-Frobenius if and only if every R-module embeds in a projective R-module. The main point of the section on homotopy pushouts and homotopy cofibres in the present paper is that one weakening of the (relative) quasi-Frobenius condition—the condition that every relatively projective object be relatively injective—suffices to ensure that homotopy pushouts are well-defined. This result is Prop. 4.4. So if one wants to study the stable representation theory of *finitely-generated* R-modules (which is what one must do in order for K-theoretic approaches like stable G-theory to be applicable, to avoid an Eilenberg swindle forcing all K-groups to be trivial), then being able to embed such modules in projectives is *not* known to imply the quasi-Frobenius property and hence such module categories are *not* known to admit the structure of a model category—but one still has some good properties (e.g., homotopy cofibres, and Waldhausen's axiom **Weq 2**) in such categories of modules, by the results in the present paper.

Finally, we list the main results in this paper, for ease of reference:

- Cor. 3.11 states that, in a relative abelian category with enough relative projectives and in which there exists an object of positive, finite relative projective dimension, homotopy pushouts (including homotopy cofibers) fail to be unique up to homotopy equivalence.
- Cor. 3.12 states that, under the same hypotheses, such a relative abelian category does not satisfy Waldhausen's axiom **Weq 2**.
- Prop. 4.4 states that, under mild assumptions on a relative abelian category, if every relatively projective object is relatively injective, then homotopy pushouts (including homotopy cofibers) *are* unique up to homotopy equivalence.
- Cor. 4.5 states that, under the same hypotheses, the relative abelian category *does* satisfy Waldhausen's axiom **Weq 2**.
- Prop. 5.2 states that, under mild assumptions on a relative abelian category, if every relatively projective object is relatively injective, then sequential homotopy colimits are unique up to homotopy equivalence.
- Cor. 6.8 states that, under the same hypotheses, if the abelian category is AB3 and every object embeds appropriately in a relatively projective object, then geometric realizations of simplicial objects exist and are unique up to homotopy equivalence.

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2. Definitions.

Definition 2.1. By a *weak Waldhausen abelian category* we mean an abelian category $\mathcal{C}$ together with a pair of subcategories $cof(\mathcal{C})$ and $we(\mathcal{C})$ of it, satisfying the axioms:

- For each object X of $\mathcal{C}$, the identity map on X is in $we(\mathcal{C})$. (In other words, $we(\mathcal{C})$ is "lluf.")
- There exists an allowable class E in $\mathcal{C}$ with the property that $we(\mathcal{C})$ is equal to the class of E-stable equivalences, and $cof(\mathcal{C})$ is equal to the class of E-monomorphisms.

If E is such an allowable class for a given weak Waldhausen abelian category $\mathcal{C}$, we will say that E is *allowable for* $\mathcal{C}$.

The appendix to this paper provides some useful classical definitions for the reader unfamiliar with relative homological algebra. In particular, "allowable class" is Def. 7.1 and "E-stable equivalence" is Def. 7.5.

The following is Waldhausen's axiom **Weq 2**, which we will be concerned with:

Definition 2.2. Let $\mathcal{C}$ be a weak Waldhausen category. We say that $\mathcal{C}$ *satisfies Waldhausen's axiom* **Weq 2** if, for each commutative diagram

$$\begin{tikzcd} X' \arrow[d] & Y' \arrow[l] \arrow[r] \arrow[d] & Z' \arrow[d] \\ X & Y \arrow[l] \arrow[r] & Z \end{tikzcd}$$

in which the maps $Y' \to X'$ and $Y \to X$ are cofibrations and the vertical maps are all weak equivalences, then the map $X' \amalg_{Y'} Z' \to X \amalg_Y Z$ is a weak equivalence.

Definition 2.3. Let $\mathcal{D}$ be a small category and let $\mathcal{C}$ be a category with a distinguished class of morphisms $cof(\mathcal{C})$ (for example, $\mathcal{C}$ could be a weak Waldhausen abelian category). Suppose $\mathcal{C}$ has an initial object 0. By a *$\mathcal{D}$-indexed homotopy colimit diagram in* $\mathcal{C}$ we mean a functor $F\colon \mathcal{D} \to \mathcal{C}$ with the following properties:

- For each object X of $\mathcal{D}$, the map $0 \to F(X)$ is in $cof(\mathcal{C})$.
- For each map $f\colon X \to Y$ in $\mathcal{D}$, the map $F(f)\colon F(X) \to F(Y)$ is in $cof(\mathcal{C})$.

We shall see, in Lemma 3.4, that if $\mathcal{C}$ is a weak Waldhausen abelian category with an allowable class that has sectile epics, then the first condition (that the map $0 \to F(X)$ be a cofibration) in Def. 2.3 is automatically satisfied. The second condition in Def. 2.3 is the significant one. See Def. 7.6 for the definition of "having sectile epics."

The essential property that one wants in a homotopy colimit is that it should be homotopy-invariant. In a model category, one always knows that this is so. But in our much, much more general situation, that of a weak Waldhausen abelian category, some homotopy colimit diagrams may fail to have homotopy-invariant colimits. When this is so, we say that the homotopy colimit in question *fails to be well-defined.* Precisely:

Definition 2.4. Let $\mathcal{D}$ be a small category and let $\mathcal{C}$ be a weak Waldhausen abelian category. Suppose $\mathcal{C}$ has all $\mathcal{D}$-indexed colimits. We say that *$\mathcal{D}$-indexed homotopy colimits are well-defined,* or *unique up to homotopy, in* $\mathcal{C}$ if, for any pair of $\mathcal{D}$-indexed homotopy colimit diagrams $F, G\colon \mathcal{D} \to \mathcal{C}$ and any map of diagrams $\phi\colon F \to G$ such that $\phi(X)\colon F(X) \to G(X)$ is in $we(\mathcal{C})$ for every object X of $\mathcal{D}$, the induced map $\operatorname{colim} F \to \operatorname{colim} G$ is in $we(\mathcal{C})$.

Finally, we include definitions related to lengths of E-projective resolutions, which we will use in the next section:

Definition 2.5. Suppose $\mathcal{C}$ is an abelian category, E an allowable class in $\mathcal{C}$. We say that a long exact sequence $\dots \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \dots \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \to 0$ is *E-long exact* if each short exact sequence $0 \to \ker f_{i+1} \to P_i \to \operatorname{coker} f_{i+1} \to 0$ is in E. If each P_i is an E-projective object, we say that the E-long exact sequence is an *E-projective resolution of X*.

Definition 2.6. Suppose $\mathcal{C}$ is an abelian category, E an allowable class in $\mathcal{C}$. Suppose n is a nonnegative integer. We say that an object X of $\mathcal{C}$ *has E-projective dimension* $\leqslant n$ if there exists an E-projective resolution of X

$$0 \to P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \dots \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \to 0.$$

We say that X *has E-projective dimension* n if it has E-projective dimension $\leqslant n$ but does not have E-projective dimension $\leqslant n-1$.

3. Negative results on all homotopy colimits.

In this section we prove that homotopy colimits in a weak Waldhausen abelian category fail to be unique up to homotopy unless colimits of appropriately-shaped relative projectives are themselves relatively projective. A precise statement is in Prop. 3.10. An important application of Prop. 3.10 is the case of homotopy pushouts in Cor. 3.11, and the question of whether Waldhausen's axiom **Weq 2** is satisfied, which we address in Cor. 3.12.

Lemma 3.1. *Let $\mathcal{C}$ be an abelian category and let E be an allowable class in $\mathcal{C}$. Suppose $\mathcal{C}$ has enough E-projectives. Then an object X of $\mathcal{C}$ is E-projective if and only if $\operatorname{Ext}^1_{\mathcal{C}/E}(X, Y) \cong 0$ for all objects Y of $\mathcal{C}$.*

Proof. Suppose X is E-projective. Then vanishing of $\operatorname{Ext}^1_{\mathcal{C}/E}(X, Y)$ is classical (and easy).

Now suppose X is not E-projective. Then there exists some E-epimorphism $f\colon A \to B$ and a map $g\colon X \to B$ which does not lift through f. In other words, the element $g \in \hom_{\mathcal{C}}(X, B)$ is not in the image of the map $\hom_{\mathcal{C}}(X, A) \to \hom_{\mathcal{C}}(X, B)$. But we have the exact sequence $\hom_{\mathcal{C}}(X, A) \to \hom_{\mathcal{C}}(X, B) \to \operatorname{Ext}^1_{\mathcal{C}/E}(X, \ker f)$ and so g must have nonzero image in $\operatorname{Ext}^1_{\mathcal{C}/E}(X, \ker f)$. So $\operatorname{Ext}^1_{\mathcal{C}/E}(X, \ker f)$ is nontrivial. So by contrapositive, vanishing of $\operatorname{Ext}^1_{\mathcal{C}/E}(X, Y)$ for all Y implies that X is E-projective. □

Lemma 3.2. *Suppose $\mathcal{C}$ is an abelian category, E an allowable class in $\mathcal{C}$ with sectile epics. Suppose $\mathcal{C}$ has enough E-projectives. Then any finite direct sum of members of E is in E.*

Proof. Let I be a finite set and $0 \to X_i \to Y_i \to Z_i \to 0$ be a member of E for every $i \in I$. Then, for any E-projective object P of $\mathcal{C}$, we have the commutative diagram

$$\begin{tikzcd} \hom_{\mathcal{C}}(P, \oplus_i Y_i) \arrow[r] \arrow[d, "\cong"] & \hom_{\mathcal{C}}(P, \oplus_i Z_i) \arrow[d, "\cong"] \\ \oplus_i \hom_{\mathcal{C}}(P, Y_i) \arrow[r] & \oplus_i \hom_{\mathcal{C}}(P, Z_i). \end{tikzcd}$$

The bottom horizontal map is a surjection of abelian groups, so the top horizontal map is as well. Now by Heller's theorem 7.7, the map $\oplus_i Y_i \to \oplus_i Z_i$ is an E-epimorphism. So the short exact sequence $0 \to \oplus_i X_i \to \oplus_i Y_i \to \oplus_i Z_i \to 0$ is in E. $\square$

Lemma 3.3. **(Shearing E-monics.)** *Let $\mathcal{C}$ be an abelian category and let E be an allowable class in $\mathcal{C}$. Suppose X, Y, Z are objects in $\mathcal{C}$ and suppose we have E-monomorphisms $e\colon X \to Y$ and $f\colon Z \to Y$. Let s be the morphism $s\colon X \oplus Z \to Y \oplus Z$ given by the matrix of maps $s = \begin{bmatrix} e & f \\ 0 & \mathrm{id}_Z \end{bmatrix}$. Then* $\operatorname{coker} s$ *is naturally isomorphic to* $\operatorname{coker} e$*. Furthermore, if $\mathcal{C}$ has enough E-injectives and E has retractile monics, then s is an E-monomorphism.*

Proof. We first show that $\operatorname{coker} e \cong \operatorname{coker} s$. But this follows immediately from the commutative diagram with exact rows and exact columns:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] \arrow[d] & X \arrow[r, "e"] \arrow[d, "i"] & Y \arrow[r] \arrow[d, "i"] & \operatorname{coker} e \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] \arrow[d] & X \oplus Z \arrow[r, "s"] \arrow[d, "\pi"] & Y \oplus Z \arrow[r] \arrow[d, "\pi"] & \operatorname{coker} s \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] \arrow[d] & Z \arrow[r, "\mathrm{id}"] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

in which the maps marked π are projections to the second summand, and the maps marked i are inclusions as the first summand.

Now assume that E has retractile monics, and let $t\colon Y \oplus Z \to Y \oplus Y$ be the map given by the matrix of maps $t = \begin{bmatrix} \mathrm{id}_Y & -f \\ 0 & f \end{bmatrix}$. Then a matrix multiplication reveals that the composite map $t \circ s\colon X \oplus Z \to Y \oplus Y$ is the direct sum map $e \oplus f$, a direct sum of E-monomorphisms, hence by Lemma 3.2, itself an E-monomorphism. (Note that, by taking the opposite category and noticing that the definition of an allowable class in an abelian category is self-dual, we get the conclusion of Lemma 3.2 if E has retractile monics and $\mathcal{C}$ has enough E-injectives.) Now since $t \circ s$ is an E-monomorphism and E is assumed to have retractile monics, s is an E-monomorphism. $\square$

Lemma 3.4. *Suppose $\mathcal{C}$ is an abelian category, E an allowable class in $\mathcal{C}$ which has sectile epics. Suppose $\mathcal{C}$ has enough E-projectives. Any split monomorphism in $\mathcal{C}$ is an E-monomorphism. Dually, any split epimorphism in $\mathcal{C}$ is an E-epimorphism.*

Proof. Any split monomorphism f fits into a short exact sequence

$$0 \to X \xrightarrow{f} Y \to \operatorname{coker} f \to 0 \tag{1}$$

in which Y decomposes as $X \oplus \operatorname{coker} f$, i.e., short exact sequence 1 is a direct sum of the short exact sequences $0 \to X \to X \to 0 \to 0$ and $0 \to 0 \to \operatorname{coker} f \to \operatorname{coker} f \to 0$, both of which are in E by the definition of an allowable class. Now by Lemma 3.2, short exact sequence 1 is in E. So f is an E-monomorphism. □

Lemma 3.5. *Suppose $\mathcal{C}$ is an abelian category, E an allowable class in $\mathcal{C}$ with sectile epics. Suppose $\mathcal{C}$ has enough E-projectives. A composite of E-epimorphisms is an E-epimorphism.*

Proof. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be E-epimorphisms. Let P be an E-projective object equipped with a map $P \to Z$. Then, since g is an E-epimorphism, $P \to Z$ lifts over g to a map $P \to Y$, which in turn lifts over f since f is an E-epimorphism. So every map from an E-projective to Z lifts over $g \circ f$. Now, by Heller's theorem 7.7, $g \circ f$ is an E-epimorphism. □

The following two lemmas are the relative-homological-algebraic generalizations of the main results of Hilton and Rees's paper [**5**]. We provide proofs of the lemmas, but they are fairly easy generalizations of those already in the literature. There are also some similar results already in the literature on Auslander-Reiten theory, e.g., those of section 9.2 of [**4**], but our results are more general than any already-existing results which we aware of. (We are grateful to C. Weibel and an anonymous referee for pointing out the relevance of that section in Gabriel and Roiter's book to us.)

Lemma 3.6. *Suppose $\mathcal{C}$ is an abelian category with an allowable class E. Suppose $\mathcal{C}$ has enough E-projectives. Then E-stable equivalence classes of morphisms $X \to Y$ in $\mathcal{C}$ are in bijection with natural transformations of functors $\operatorname{Ext}^1_{\mathcal{C}/E}(Y,-) \to \operatorname{Ext}^1_{\mathcal{C}/E}(X,-)$. This bijection is natural in X and Y.*

Proof. This proof is a straightforward generalization of Margolis' proof of the Hilton-Rees result, as in Prop. 9 of section 14.1 of [**9**]. Write $[X,Y]$ for $\hom_{\mathcal{C}}(X,Y)$ modulo E-stable equivalence. We have the morphism of abelian groups

$$\alpha\colon \hom_{\mathcal{C}}(X,Y) \to \operatorname{nat}(\operatorname{Ext}^1_{\mathcal{C}/E}(Y,-), \operatorname{Ext}^1_{\mathcal{C}/E}(X,-))$$

defined by the functoriality of $\operatorname{Ext}^1_{\mathcal{C}/E}$ in the first variable. If $f\colon X \to Y$ factors through a E-projective then clearly $\alpha(f) = 0$, so α factors as

$$\begin{tikzcd} \hom_{\mathcal{C}}(X,Y) \arrow[rr, "\alpha"] \arrow[dr] & & \operatorname{nat}(\operatorname{Ext}^1_{\mathcal{C}/E}(Y,-), \operatorname{Ext}^1_{\mathcal{C}/E}(X,-)) \\ & {[X,Y]} \arrow[ur, "\beta"'] & \end{tikzcd}$$

We now check that β is a bijection. Suppose $\beta(f) = 0$. Then choose E-projective covers $s_X\colon PX \to X$ and $s_Y\colon PY \to Y$. We have the commutative diagram with exact rows

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d] & \ker s_x \arrow[r, "i_X"] \arrow[d, "\ker s_f"'] & PX \arrow[r, "s_X"] \arrow[d, "Pf"] \arrow[dl, dashed, "\ell"'] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[d] \\
0 \arrow[r] & \ker s_Y \arrow[r, "i_Y"'] & PY \arrow[r, "s_Y"'] & Y \arrow[r] & 0.
\end{tikzcd} \tag{2}
$$

(The map as in the dotted line has not yet been shown to exist.) After applying $\hom_{\mathcal{C}}(-, \ker s_Y)$, we have the commutative diagram with exact rows

$$
\begin{tikzcd}
0 & \mathrm{Ext}^1_{\mathcal{C}/E}(X, \ker s_Y) \arrow[l] & \hom_{\mathcal{C}}(\ker s_X, \ker s_Y) \arrow[l] & \hom_{\mathcal{C}}(PX, \ker s_Y) \arrow[l, "\phi"'] \\
0 \arrow[u] & \mathrm{Ext}^1_{\mathcal{C}/E}(Y, \ker s_Y) \arrow[l] \arrow[u, "0"] & \hom_{\mathcal{C}}(\ker s_Y, \ker s_Y) \arrow[l] \arrow[u, "\lambda"] & \hom_{\mathcal{C}}(PY, \ker s_Y) \arrow[l] \arrow[u]
\end{tikzcd}
$$

Commutativity of the diagram together with exactness of the rows and triviality of the far left-hand vertical map implies that the map λ factors through the image of ϕ, i.e., there exists a map ℓ as in the dotted line in diagram 2 making the triangle involving $\ker s_f, i_X$, and ℓ commute. We now replace Pf with $g = Pf - i_Y \circ \ell$ to get the commutative diagram with exact rows

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d] & \ker s_x \arrow[r, "i_X"] \arrow[d, "\ker s_f"] & PX \arrow[r, "s_X"] \arrow[d, "g"] & X \arrow[r] \arrow[d, "f"] \arrow[dl, dashed, "\mu"] & 0 \arrow[d] \\
0 \arrow[r] & \ker s_Y \arrow[r, "i_Y"'] & PY \arrow[r, "s_Y"'] & Y \arrow[r] & 0.
\end{tikzcd} \tag{3}
$$

Since $g \circ i_X = Pf \circ i_X - i_Y \circ \ell \circ i_X = 0$, there exists a map as in the dotted line in diagram 3 to make the triangle involving s_X, g, and μ commute. Now we have

$$f \circ s_X = s_Y \circ g = s_Y \circ \mu \circ s_X$$

and s_X is E-epic, hence epic, i.e., right-cancellable, so $f = s_Y \circ \mu$. So f factors through the E-projective PY, i.e., f is E-stably equivalent to zero. So β is one-to-one.

Now choose a natural transformation $\mathrm{Ext}^1_{\mathcal{C}/E}(Y, -) \to \mathrm{Ext}^1_{\mathcal{C}/E}(X, -)$. We choose E-projective covers $s_X \colon PX \to X$ and $s_Y \colon PY \to Y$ as above. Write $\chi \in \mathrm{Ext}^1_{\mathcal{C}/E}(Y, \ker s_Y)$ for the class of the extension $0 \to \ker s_Y \to PY \to Y \to 0$. Notice that the natural map $\hom_{\mathcal{C}}(\ker s_X, \ker s_Y) \to \mathrm{Ext}^1_{\mathcal{C}/E}(X, \ker s_Y)$ is surjective since PX is E-projective and hence $\mathrm{Ext}^1_{\mathcal{C}/E}(PX, \ker s_Y) \cong 0$. So we can choose an element $h \in \hom_{\mathcal{C}}(\ker s_X, \ker s_Y)$ whose image in $\mathrm{Ext}^1_{\mathcal{C}/E}(X, \ker s_Y)$ agrees with the image of χ under the given map $\mathrm{Ext}^1_{\mathcal{C}/E}(Y, \ker s_Y) \to \mathrm{Ext}^1_{\mathcal{C}/E}(X, \ker s_Y)$. The map $\mathrm{Ext}^1_{\mathcal{C}/E}(Y, \ker s_Y) \to \mathrm{Ext}^1_{\mathcal{C}/E}(Y, PY)$ is automatically zero, so from the commutative diagram

$$
\begin{tikzcd}
\mathrm{Ext}^1_{\mathcal{C}/E}(Y, \ker s_Y) \arrow[r, "0"] \arrow[d] & \mathrm{Ext}^1_{\mathcal{C}/E}(Y, PY) \arrow[d] \\
\mathrm{Ext}^1_{\mathcal{C}/E}(X, \ker s_Y) \arrow[r] & \mathrm{Ext}^1_{\mathcal{C}/E}(X, PY)
\end{tikzcd}
$$

we know that the image of χ in $\mathrm{Ext}^1_{\mathcal{C}/E}(X, PY)$ is zero. Hence also the image of h in $\mathrm{Ext}^1_{\mathcal{C}/E}(X, PY)$ is zero. Hence in the commutative diagram with exact rows

$$\begin{tikzcd} & \hom_{\mathcal{C}}(\ker s_X, \ker s_Y) \arrow[r] \arrow[d] & \mathrm{Ext}^1_{\mathcal{C}/E}(X, \ker s_Y) \arrow[d] \\ \hom_{\mathcal{C}}(PX, PY) \arrow[r] & \hom_{\mathcal{C}}(\ker s_X, PY) \arrow[r] & \mathrm{Ext}^1_{\mathcal{C}/E}(X, PY) \end{tikzcd}$$

the image of h in $\hom_{\mathcal{C}}(\ker s_X, PY)$ lifts to an element in $\hom_{\mathcal{C}}(PX, PY)$, i.e., we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \ker s_X \arrow[r] \arrow[d, "h"] & PX \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \ker s_Y \arrow[r] & PY \arrow[r] & Y \arrow[r] & 0 \end{tikzcd}$$

and the map $X \to Y$ is the desired map inducing the given natural transformation in $\mathrm{Ext}^1_{\mathcal{C}/E}$. Hence β is surjective, hence an isomorphism. □

Lemma 3.7. *Suppose $\mathcal{C}$ is an abelian category with an allowable class E with sectile epics. Suppose $\mathcal{C}$ has enough E-projectives. Then a map $f\colon X \to Y$ in $\mathcal{C}$ induces a natural isomorphism*

$$\mathrm{Ext}^1_{\mathcal{C}/E}(Y, -) \to \mathrm{Ext}^1_{\mathcal{C}/E}(X, -) \tag{4}$$

if and only if there exist E-projective objects P, Q and an isomorphism $g\colon X \oplus P \to Y \oplus Q$ such that the composite $X \xrightarrow{i} X \oplus P \xrightarrow{g} Y \oplus Q \xrightarrow{p} Y$ is equal to f. (Here we write i for inclusion of the first summand and p for projection to the first summand.)

Proof. If an isomorphism g exists as described then we have natural isomorphisms

$$\begin{aligned} \mathrm{Ext}^1_{\mathcal{C}/E}(Y, -) &\cong \mathrm{Ext}^1_{\mathcal{C}/E}(Y \oplus Q, -) \\ &\cong \mathrm{Ext}^1_{\mathcal{C}/E}(X \oplus P, -) \\ &\cong \mathrm{Ext}^1_{\mathcal{C}/E}(X, -) \end{aligned}$$

as desired.

For the converse: suppose $f\colon X \to Y$ induces the natural isomorphism 4. Then, by Lemma 3.6, f is an E-stable equivalence. So there exists a map $g\colon Y \to X$ such that $\mathrm{id}_X - g \circ f$ and $\mathrm{id}_Y - f \circ g$ each factors through an E-projective object. Suppose P is an E-projective object and $i\colon Y \to P$ and $s\colon P \to Y$ maps in $\mathcal{C}$ such that

$$s \circ i = \mathrm{id}_Y - f \circ g.$$

Since $\mathcal{C}$ has enough E-projectives, we can choose P so that s is epic. Then we have a short exact sequence in E

$$0 \to \ker m \to X \oplus P \xrightarrow{m} Y \to 0 \tag{5}$$

where m is the map given by the matrix of maps $m = \begin{bmatrix} f & s \end{bmatrix}$. That m is an E-epimorphism follows from the composite

$$P \longrightarrow X \oplus P \xrightarrow{m} Y \tag{6}$$

being s, which is epic, and hence m is epic since E has sectile epics; here the left-hand map in 6 is inclusion of the right-hand summand. (We are grateful to the anonymous referee for suggesting this argument for m being an E-epimorphism. The argument we had in the original draft of this paper was more complicated.)

Since m is an E-epimorphism, by definition the short exact sequence 5 is in E. So short exact sequence 5 induces a natural long exact sequence in $\mathrm{Ext}_{\mathcal{C}/E}$ for any object M of $\mathcal{C}$:

$$\begin{tikzcd} \mathrm{Ext}^i_{\mathcal{C}/E}(X \oplus P, M) & & \mathrm{Ext}^{i-1}_{\mathcal{C}/E}(Y, M) \arrow[d, "\cong"] \\ \mathrm{Ext}^i_{\mathcal{C}/E}(Y, M) \arrow[u, "\cong"] & \mathrm{Ext}^{i-1}_{\mathcal{C}/E}(\ker m, M) \arrow[l] & \mathrm{Ext}^{i-1}_{\mathcal{C}/E}(X \oplus P, M) \arrow[l] \end{tikzcd} \tag{7}$$

where the maps marked as isomorphisms are isomorphisms for $i \geqslant 2$ since $X \to Y$ is an E-stable equivalence. Exactness of long exact sequence 7 gives us that $\mathrm{Ext}^{i-1}_{\mathcal{C}/E}(\ker m, M) \cong 0$ for all M in $\mathcal{C}$ and all $i \geqslant 2$. So by Lemma 3.1, $\ker m$ is an E-projective. Part of the long exact sequence induced in $\mathrm{Ext}_{\mathcal{C}/E}$ by short exact sequence 5 reads:

$$\begin{tikzcd} \mathrm{Ext}^1_{\mathcal{C}/E}(X \oplus P, M) & \mathrm{Ext}^1_{\mathcal{C}/E}(Y, M) \arrow[l, "\cong"'] & & \\ \hom_{\mathcal{C}}(\ker m, M) \arrow[ur] & \hom_{\mathcal{C}}(X \oplus P, M) \arrow[l] & \hom_{\mathcal{C}}(Y, M) \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

i.e., $0 \to \hom_{\mathcal{C}}(Y, M) \to \hom_{\mathcal{C}}(X \oplus P, M) \to \hom_{\mathcal{C}}(\ker m, M) \to 0$ is exact for every object M in $\mathcal{C}$. Hence short exact sequence 5 is in fact *split*, and $X \oplus P \cong Y \oplus \ker m$, proving the lemma. ☐

Lemma 3.8. (Shearing isomorphism.) *Suppose X, Y are objects in an abelian category $\mathcal{C}$ and $f\colon X \to Y$ is a monomorphism. Then the pushout $Y \amalg_X Y$ is naturally isomorphic to $Y \oplus \operatorname{coker} f$.*

Proof. Let $g\colon Y \oplus Y \to Y \oplus Y$ be the map given by the (invertible, hence an isomorphism) matrix of maps $g = \begin{bmatrix} \mathrm{id} & 0 \\ -\mathrm{id} & \mathrm{id} \end{bmatrix}$. Then we have the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X \arrow[r, "{[f\ f]^{\perp}}"] \arrow[d, "\mathrm{id}"] & Y \oplus Y \arrow[r] \arrow[d, "g"] & Y \amalg_X Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & X \arrow[r, "{[f\ 0]^{\perp}}"] & Y \oplus Y \arrow[r] & \operatorname{coker} f \oplus Y \arrow[r] & 0 \end{tikzcd}$$

and hence the isomorphism $Y \amalg_X Y \xrightarrow{\cong} \operatorname{coker} f \oplus Y$. (We have written $\perp$ above as notation for the transpose of a matrix of maps.) ☐

Lemma 3.9. *Suppose $\mathcal{C}$ is an abelian category, E an allowable class in $\mathcal{C}$. Suppose $\mathcal{C}$ has enough E-projectives. If there exists an object of finite E-projective dimension $n \geqslant 2$ in $\mathcal{C}$, then there exists an object of E-projective dimension 1 in $\mathcal{C}$.*

Proof. Let X have E-projective dimension $n \geqslant 2$. Choose an E-projective resolution of X

$$0 \to P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \longrightarrow X \to 0.$$

Then

$$0 \to P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} P_2 \longrightarrow \operatorname{coker} f_2 \to 0 \tag{8}$$

is an E-projective resolution of $\operatorname{coker} f_2$. If $\operatorname{coker} f_2$ is E-projective, then

$$0 \to \operatorname{coker} f_2 \to P_0 \to X \to 0$$

is a length 1 E-projective resolution of X, and we are done. So suppose $\operatorname{coker} f_2$ is not E-projective. So it is not of E-projective dimension zero. Then diagram 8 expresses $\operatorname{coker} f_2$ as having E-projective dimension at most $n-1$. Now we continue by induction: either $\operatorname{coker} f_3$ is E-projective or has E-projective dimension at most $n-2$, etc. After at most n steps this process terminates with an object of E-projective dimension 1. □

The preceding lemmas suffice for us to prove the following proposition, which is really a negative result: it shows that, if $\mathcal{D}$-indexed colimits of E-projectives are not always E-projective, then $\mathcal{D}$-indexed homotopy colimits fail to be unique up to homotopy.

Proposition 3.10. *Let $\mathcal{D}$ be a small category and let $\mathcal{C}$ be a weak Waldhausen abelian category. Choose a class E allowable for $\mathcal{C}$. Suppose $\mathcal{D}$-indexed homotopy colimits are well-defined in $\mathcal{C}$. Then every $\mathcal{D}$-indexed colimit of E-projective objects of $\mathcal{C}$ is E-projective.*

Proof. We work by contrapositive. Suppose there exists a homotopy colimit diagram $F\colon \mathcal{D} \to \mathcal{C}$ with the property that $F(X)$ is E-projective for every object X of $\mathcal{D}$, and $\operatorname{colim} F$ is not E-projective. Let $G\colon \mathcal{D} \to \mathcal{C}$ be the zero diagram, i.e., $G(X) = 0$ for all objects X of $\mathcal{D}$. Then the unique map $\phi\colon F \to G$ has the property that $\phi(X)$ is an E-stable equivalence for every object X of $\mathcal{D}$, since any map with E-projective domain and E-projective codomain is trivially an E-stable equivalence. But $\operatorname{colim} F$ is nontrivial, so $\operatorname{Ext}^1_{\mathcal{C}/E}(\operatorname{colim} F, Y)$ is nontrivial for some object Y of $\mathcal{C}$, by Lemma 3.1. So the natural transformation $0 \cong \operatorname{Ext}^1_{\mathcal{C}/E}(\operatorname{colim} G, Y) \to \operatorname{Ext}^1_{\mathcal{C}/E}(\operatorname{colim} F, Y)$ is not an isomorphism. So by Lemma 3.6, the map $\operatorname{colim} F \to \operatorname{colim} G \cong 0$ is not an E-stable equivalence. □

Corollary 3.11. *Let $\mathcal{C}$ be a weak Waldhausen abelian category. Let E be a class allowable for $\mathcal{C}$. Suppose $\mathcal{C}$ has enough E-projectives. Then homotopy pushouts, and in particular homotopy cofibers, are well-defined in $\mathcal{C}$ only if every object in $\mathcal{C}$ has E-projective dimension 0 or ∞.*

Proof. Suppose an object in $\mathcal{C}$ has finite E-projective dimension $n > 0$. Then, by Lemma 3.9, there exists some object X in $\mathcal{C}$ of E-projective dimension 1. Choose an E-projective resolution $0 \to P_1 \xrightarrow{s} P_0 \to X \to 0$ of X. Then we have the commutative diagram

$$\begin{tikzcd} P_1 & P_1 \arrow[l, "\mathrm{id}"'] \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & P_1 \arrow[d, "s"] \\ P_0 \arrow[from=u, "s"] & P_1 \arrow[l, "s"'] \arrow[r, "s"] & P_0 \end{tikzcd}$$

in which all vertical arrows are E-stable equivalences (since any map between two E-projective objects is an E-stable equivalence) and all horizontal arrows are E-monomorphisms. We compute the induced map on pushouts by the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & P_1 \arrow[r, "\Delta"] \arrow[d, "\mathrm{id}"] & P_1 \oplus P_1 \arrow[r] \arrow[d, "s\oplus s"] & P_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & P_1 \arrow[r] & P_0 \oplus P_0 \arrow[r] & P_0 \amalg_{P_1} P_0 \arrow[r] & 0. \end{tikzcd}$$

Exactness of the top row, as well as an isomorphism $P_0 \amalg_{P_1} P_0 \cong X \oplus P_0$, both follow from Lemma 3.8. So the pushout map $P_1 \amalg_{P_1} P_1 \to P_0 \amalg_{P_1} P_0$ is, up to isomorphism, the map $P_1 \to X \oplus P_0$. Applying $\mathrm{Ext}^1_{\mathcal{C}/E}(-, M)$ to this map, we get

$$\begin{tikzcd} \mathrm{Ext}^1_{\mathcal{C}/E}(P_0 \amalg_{P_1} P_0, M) \arrow[r, "\cong"] \arrow[d] & \mathrm{Ext}^1_{\mathcal{C}/E}(X \oplus P_0, M) \arrow[r, "\cong"] \arrow[d] & \mathrm{Ext}^1_{\mathcal{C}/E}(X, M) \arrow[d] \\ \mathrm{Ext}^1_{\mathcal{C}/E}(P_1 \amalg_{P_1} P_1, M) \arrow[r, "\cong"] & \mathrm{Ext}^1_{\mathcal{C}/E}(P_1, M) \arrow[r, "\cong"] & 0 \end{tikzcd}$$

for all objects M of $\mathcal{C}$. Since X is assumed to be of E-projective dimension 1, it is not E-projective, so by Lemma 3.1, $\mathrm{Ext}^1_{\mathcal{C}/E}(X, M)$ is nonzero for some object M in $\mathcal{C}$. So by Lemma 3.6, the pushout map $P_1 \amalg_{P_1} P_1 \to P_0 \amalg_{P_1} P_0$ is not an E-stable equivalence. □

Corollary 3.12. *Let $\mathcal{C}$ be a weak Waldhausen abelian category. Let E be a class allowable for $\mathcal{C}$. Suppose $\mathcal{C}$ has enough E-projectives. Then $\mathcal{C}$ is not a Waldhausen category unless each object in $\mathcal{C}$ has E-projective dimension either 0 or ∞.*

Proof. Well-definedness of homotopy pushouts is implied by Waldhausen's axiom **Weq 2** in the definition of a Waldhausen category, from [**15**]. □

4. Positive results on homotopy pushouts and cofibers.

In this section we prove that homotopy pushouts in a weak Waldhausen abelian category $\mathcal{C}$ are well-defined if one makes some mild assumptions on $\mathcal{C}$, as well as one quite significant assumption on $\mathcal{C}$: that every relatively projective object is relatively injective. In the absolute case, i.e., the case where E is the class of all short exact sequences in $\mathcal{C}$, this condition is somewhat weaker than the assumption that $\mathcal{C}$ be quasi-Frobenius, which holds when $\mathcal{C}$ is the category of modules over any quasi-Frobenius ring (e.g., connected co-commutative finite-dimensional Hopf algebras over fields, such as finite-dimensional sub-Hopf-algebras of the Steenrod algebra). Recall that an abelian category $\mathcal{C}$ is said to be "quasi-Frobenius" if projective and injective objects coincide in $\mathcal{C}$.

The following proposition is a relative-homological-algebraic variant of the main result of Oort's paper [**10**]. The proof follows Oort's but with some adaptations to the relative situation. We also make the assumption, here, that relative projectives are relatively injective, which simplifies the proof.

Proposition 4.1. *Let $\mathcal{C}$ be an abelian category, E an allowable class in $\mathcal{C}$ with sectile epics and retractile monics. Suppose $\mathcal{C}$ has enough E-projectives, suppose every E-projective in $\mathcal{C}$ is E-injective, and suppose $i > 1$. Then a map $f\colon X \to Y$ in $\mathcal{C}$ induces a natural isomorphism $\mathrm{Ext}^i_{\mathcal{C}/E}(Y,-) \to \mathrm{Ext}^i_{\mathcal{C}/E}(X,-)$ if and only if there exists a short exact sequence in E*

$$0 \to Q \to P \oplus X \xrightarrow{p} Y \to 0 \tag{9}$$

with P an E-projective and Q of E-projective dimension $< i-1$, such that the composite $X \xrightarrow{i} P \oplus X \xrightarrow{p} Y$ is equal to f. (Here i is inclusion as the second summand.)

Proof. If short exact sequence 9 exists with the described properties, then for any object M of $\mathcal{C}$, the induced long exact sequence in $\mathrm{Ext}_{\mathcal{C}/E}$ reads

$$0 \longleftarrow \mathrm{Ext}^i_{\mathcal{C}/E}(P \oplus X, M) \longleftarrow \mathrm{Ext}^i_{\mathcal{C}/E}(Y, M) \longleftarrow 0$$

and since P is projective we now have $\mathrm{Ext}^i_{\mathcal{C}/E}(Y,M) \cong \mathrm{Ext}^i_{\mathcal{C}/E}(X,M)$.

Now instead assume that f induces an isomorphism in $\mathrm{Ext}^i_{\mathcal{C}/E}$. We want to construct a short exact sequence 9 with the described properties. Choose exact sequences for X, Y

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & N \arrow[r, "d^Q_{i-1}"] \arrow[d, "\theta_{i-1}"] & Q_{i-2} \arrow[r, "d^Q_{i-2}"] \arrow[d, "\theta_{i-2}"] & \cdots \arrow[r, "d^Q_2"] & Q_1 \arrow[r, "d^Q_1"] \arrow[d, "\theta_1"] & Q_0 \arrow[r, "d^Q_0"] \arrow[d, "\theta_0"] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[d] \\
0 \arrow[r] & M \arrow[r, "d^P_{i-1}"] & P_{i-2} \arrow[r, "d^P_{i-2}"] & \cdots \arrow[r, "d^P_2"] & P_1 \arrow[r, "d^P_1"] & P_0 \arrow[r, "d^P_0"] & Y \arrow[r] & 0
\end{tikzcd} \tag{10}$$

with each Q_j and P_j an E-projective, with each short exact sequence

$$0 \to \mathrm{im}\ d^Q_{j+1} \to Q_j \to \mathrm{im}\ d^Q_j \to 0$$

and

$$0 \to \mathrm{im}\ d^P_{j+1} \to P_j \to \mathrm{im}\ d^P_j \to 0$$

both in E, and such that each θ_j factors through the kernel of the map $Q_{j-1} \to Q_{j-2}$ for $j \geqslant 1$, and θ_0 factors through the kernel of the map $Q_0 \to X$. Then θ_{i-1} induces a natural isomorphism

$$\mathrm{Ext}^1_{\mathcal{C}/E}(N,-) \cong \mathrm{Ext}^i_{\mathcal{C}/E}(X,-) \cong \mathrm{Ext}^i_{\mathcal{C}/E}(Y,-) \cong \mathrm{Ext}^1_{\mathcal{C}/E}(M,-)$$

and hence, by Lemma 3.7, there exist E-projectives Q, P and an isomorphism $g\colon N \oplus Q \xrightarrow{\cong} M \oplus P$ extending θ_{n-1}. We choose component maps for g, so that we can write g as a matrix of maps

$$g = \begin{bmatrix} \theta_{i-1} & \beta \\ \gamma & \delta \end{bmatrix}.$$

We take a direct sum of diagram 10 with the E-projectives P and Q to get a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & N \oplus Q \arrow[r, "d^Q_{i-1} \oplus \mathrm{id}_Q"] \arrow[d, "g"] & Q_{i-2} \oplus Q \arrow[r, "d^Q_{i-2} \circ \pi_1"] \arrow[d, "f"] & Q_{i-3} \arrow[r, "d^Q_{i-3}"] \arrow[d, "\theta_{i-3}"] & Q_{i-4} \arrow[r, "d^Q_{i-4}"] \arrow[d, "\theta_{i-4}"] & \cdots \\ 0 \arrow[r] & M \oplus P \arrow[r, "d^P_{i-1} \oplus \mathrm{id}_P"] & P_{i-2} \oplus P \arrow[r, "d^P_{i-2} \circ \pi_1"] & P_{i-3} \arrow[r, "d^P_{i-3}"] & P_{i-4} \arrow[r, "d^P_{i-4}"] & \cdots \end{tikzcd} \tag{11}$$

defined as follows: first, we are writing π_1 for projection to the first summand. The map f is given by the matrix of maps

$$f = \begin{bmatrix} \theta_{i-2} & d^P_{i-1} \circ \beta \\ f_{01} & \delta \end{bmatrix}$$

where $f_{01}\colon Q_{i-2} \to P$ is any map making the diagram

$$\begin{tikzcd} N \arrow[r, "d^Q_{i-1}"] \arrow[d, "\beta"'] & Q_{i-2} \arrow[dl, "f_{01}"] \\ P & \end{tikzcd}$$

commute. The existence of such a map f_{01} is guaranteed by d^Q_{i-1} being an E-monomorphism, by P being an E-projective, and by our assumption that E-projectives are E-injective; now existence of f_{01} follows from the universal property of an E-injective object. (This is the only place in this proof where we use the assumption that E-projectives are E-injective.) One can easily check (by matrix multiplication of the matrices of maps) that every square in diagram 11 commutes.

Now, regarding diagram 11 as a double complex and totalizing, we get an exact sequence

$$0 \to Q_{i-2} \oplus Q \to Q_{i-3} \oplus P_{i-2} \oplus P \to Q_{i-4} \oplus P_{i-3} \to \cdots \to X \oplus P_0 \xrightarrow{p} Y \to 0$$

which expresses that we have a short exact sequence

$$0 \to \ker p \to X \oplus P_0 \xrightarrow{p} Y \to 0 \tag{12}$$

with P_0 E-projective and $\ker p$ of E-projective dimension $\leqslant i-2$, and with p extending f as desired. All that remains is to check that short exact sequence 12 is in E. This follows from p being a difference of the map $f\colon X \to Y$ and the E-projective cover $d^P_0\colon P_0 \to Y$, as follows: by the dual of Lemma 3.3, the map $X \oplus P_0 \to X \oplus Y$ given by the matrix of maps

$$\begin{bmatrix} \mathrm{id}_X & 0 \\ f & p \end{bmatrix}$$

is an E-epimorphism. By Lemma 3.4, the projection $X \oplus Y \to Y$ is an E-epimorphism as well. So the composite map $X \oplus P_0 \to X \oplus Y \to Y$, which is equal to p, is an E-epimorphism by Lemma 3.5. $\square$

Lemma 4.2. *Let $\mathcal{C}$ be an abelian category and let E be an allowable class in $\mathcal{C}$ with sectile epics. Suppose $\mathcal{C}$ has enough E-projectives, and suppose that every E-projective object is E-injective. If $X \to Y$ is a map in $\mathcal{C}$ which induces a natural isomorphism of functors*

$$\mathrm{Ext}^2_{\mathcal{C}/E}(Y,-) \xrightarrow{\cong} \mathrm{Ext}^2_{\mathcal{C}/E}(X,-), \tag{13}$$

then $X \to Y$ is an E-stable equivalence.

Proof. Let $f\colon X \to Y$ be a map which induces a natural isomorphism of functors 13. Then, by Prop. 4.1, there exists a short exact sequence in E

$$0 \to Q \to P \oplus X \to Y \to 0 \tag{14}$$

where $P \oplus X \to Y$ extends f and both P and Q are E-projective. Now we use the assumption that every E-projective in $\mathcal{C}$ is E-injective: since Q is E-injective, short exact sequence 14 splits, and we get an isomorphism $P \oplus X \xrightarrow{\cong} Q \oplus Y$. Now Lemma 3.7 implies that f induces a natural isomorphism in $\mathrm{Ext}^1_{\mathcal{C}/E}$ and hence, by Lemma 3.6, f is an E-stable equivalence. □

Lemma 4.3. *Let $\mathcal{C}$ be a weak Waldhausen abelian category. Let E be a class allowable for $\mathcal{C}$. Suppose $\mathcal{C}$ has enough E-projectives and enough E-injectives, suppose E has sectile epics and retractile monics, and suppose every E-projective object is E-injective. Then, for any cofibration $f\colon X \to Y$ and any weak equivalence $g\colon X \to Z$ in $\mathcal{C}$, the pushout map $Y \to Y \amalg_X Z$ is also a weak equivalence.*

Proof. The given maps fit into a commutative diagram with exact rows in E

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[r] \arrow[d] & \operatorname{coker} f \arrow[r] \arrow[d, "\cong"] & 0 \arrow[d] \\
0 \arrow[r] & Z \arrow[r] & Y \amalg_X Z \arrow[r] & \operatorname{coker} f \arrow[r] & 0
\end{tikzcd}$$

and since g is an E-stable equivalence, Lemma 3.6 gives us, for any object M of $\mathcal{C}$, the marked isomorphisms in the commutative diagram with exact columns

$$\begin{tikzcd}
\mathrm{Ext}^1_{\mathcal{C}/E}(Z,M) \arrow[r, "\cong"] \arrow[d] & \mathrm{Ext}^1_{\mathcal{C}/E}(X,M) \arrow[d] \\
\mathrm{Ext}^2_{\mathcal{C}/E}(\operatorname{coker} f,M) \arrow[r, "\cong"] \arrow[d] & \mathrm{Ext}^2_{\mathcal{C}/E}(\operatorname{coker} f,M) \arrow[d] \\
\mathrm{Ext}^2_{\mathcal{C}/E}(Y \amalg_X Z,M) \arrow[r] \arrow[d] & \mathrm{Ext}^2_{\mathcal{C}/E}(Y,M) \arrow[d] \\
\mathrm{Ext}^2_{\mathcal{C}/E}(Z,M) \arrow[r, "\cong"] \arrow[d] & \mathrm{Ext}^2_{\mathcal{C}/E}(X,M) \arrow[d] \\
\mathrm{Ext}^3_{\mathcal{C}/E}(\operatorname{coker} f,M) \arrow[r, "\cong"] & \mathrm{Ext}^3_{\mathcal{C}/E}(\operatorname{coker} f,M).
\end{tikzcd}$$

So by the Five Lemma, $Y \to Y \amalg_X Z$ induces a natural isomorphism of functors

$$\mathrm{Ext}^2_{\mathcal{C}/E}(Y \amalg_X Z, -) \xrightarrow{\cong} \mathrm{Ext}^2_{\mathcal{C}/E}(Y, -)$$

and hence, by Lemma 4.2, $Y \to Y \amalg_X Z$ is an E-stable equivalence, hence a weak equivalence. □

Proposition 4.4. *Let $\mathcal{C}$ be a weak Waldhausen abelian category. Let E be a class allowable for $\mathcal{C}$. Suppose $\mathcal{C}$ has enough E-projectives and enough E-injectives, suppose E has sectile epics and retractile monics, and suppose that every E-projective object is E-injective. Then homotopy pushouts are well-defined in $\mathcal{C}$.*

Proof. Let $\mathcal{D}$ be the small category indexing pushout diagrams, i.e., $\mathcal{D}$ has three objects A_0, A_1, A_2, maps $A_0 \to A_1$ and $A_0 \to A_2$, and no other non-identity maps. Suppose $F, G\colon \mathcal{D} \to \mathcal{C}$ are homotopy colimit diagrams and $\phi\colon F \to G$ is a natural transformation such that $\phi(X)\colon F(X) \to G(X)$ is an E-stable equivalence for every object X of $\mathcal{D}$. Then we have the commutative diagram with rows short exact sequences in E:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & F(A_0) \arrow[r] \arrow[d] & F(A_1) \oplus F(A_2) \arrow[r] \arrow[d] & \operatorname{colim} F \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & G(A_0) \arrow[r] & G(A_1) \oplus G(A_2) \arrow[r] & \operatorname{colim} G \arrow[r] & 0.
\end{tikzcd}$$

That the maps $F(A_0) \to F(A_1) \oplus F(A_2)$ and $G(A_0) \to G(A_1) \oplus G(A_2)$ are E-monomorphisms, and hence that the rows are exact and in E, follows from Lemma 3.3. Now, for any object M of $\mathcal{C}$, we get the commutative diagram with exact columns

$$\begin{tikzcd}
\mathrm{Ext}^1_{\mathcal{C}/E}(G(A_1) \oplus G(A_2), M) \arrow[r, "\cong"] \arrow[d] & \mathrm{Ext}^1_{\mathcal{C}/E}(F(A_1) \oplus F(A_2), M) \arrow[d] \\
\mathrm{Ext}^1_{\mathcal{C}/E}(G(A_0), M) \arrow[r, "\cong"] \arrow[d] & \mathrm{Ext}^1_{\mathcal{C}/E}(F(A_0), M) \arrow[d] \\
\mathrm{Ext}^2_{\mathcal{C}/E}(\operatorname{coker} G, M) \arrow[r] \arrow[d] & \mathrm{Ext}^2_{\mathcal{C}/E}(\operatorname{coker} F, M) \arrow[d] \\
\mathrm{Ext}^2_{\mathcal{C}/E}(G(A_1) \oplus G(A_2), M) \arrow[r, "\cong"] \arrow[d] & \mathrm{Ext}^2_{\mathcal{C}/E}(F(A_1) \oplus F(A_2), M) \arrow[d] \\
\mathrm{Ext}^2_{\mathcal{C}/E}(G(A_0), M) \arrow[r, "\cong"] & \mathrm{Ext}^2_{\mathcal{C}/E}(F(A_0), M)
\end{tikzcd}$$

where the horizontal maps marked as isomorphisms are isomorphisms by Lemma 3.6. By the Five Lemma, the remaining horizontal map is an isomorphism. So we have a natural isomorphism of functors $\mathrm{Ext}^2_{\mathcal{C}/E}(\operatorname{colim} G, -) \xrightarrow{\cong} \mathrm{Ext}^2_{\mathcal{C}/E}(\operatorname{colim} F, -)$ and now, by Lemma 4.2, the map $\operatorname{colim} F \to \operatorname{colim} G$ is an E-stable equivalence. Hence homotopy pushouts are well-defined in $\mathcal{C}$. □

Corollary 4.5. *Let $\mathcal{C}$ be a weak Waldhausen abelian category. Let E be a class allowable for $\mathcal{C}$. Suppose $\mathcal{C}$ has enough E-projectives and enough E-injectives, suppose E has sectile epics and retractile monics, and suppose that every E-projective object is E-injective. Then $\mathcal{C}$ satisfies Waldhausen's axiom* **Weq 2**.

Proof. Follows immediately from Lemma 4.3 and Prop. 4.4. □

5. Positive results on sequential homotopy colimits.

In this section we show that, under the same conditions from the previous section (that relatively projective objects are relatively injective), sequential homotopy colimits are well-defined (Prop. 5.2). As a corollary, in the next section we will be able to show that geometric realization of simplicial objects in $\mathcal{C}$ is well-defined (Cor. 6.8).

Lemma 5.1. *Suppose $\mathcal{C}$ is an abelian category and E is an allowable class in $\mathcal{C}$ with retractile monics. Suppose $\mathcal{C}$ has enough E-projectives and enough E-injectives. Suppose we have objects P, X, Y in $\mathcal{C}$ with P an E-projective, and suppose we have a map $f\colon X \to Y$ and an E-epimorphism $p\colon P \to X$. Then there exists an E-projective object Q, a split monomorphism $g\colon P \to Q$ with E-projective cokernel, and an E-epimorphism $q\colon Q \to Y$ making the diagram*

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Q \arrow[r, "q"] & Y \end{tikzcd} \tag{15}$$

commute.

Proof. Choose an E-projective P_0 and an E-epimorphism $s\colon P_0 \to Y$. Since P is E-projective, the composite map $f \circ p\colon P \to Y$ lifts over s to give a map $\ell\colon P \to P_0$, i.e., $s \circ \ell = f \circ p$. Let $g\colon P \to P \oplus P_0$ be the map given by the matrix of maps $g = \begin{bmatrix} \mathrm{id}_P \\ \ell \end{bmatrix}$ and let $q\colon P \oplus P_0 \to Y$ be the map given by the matrix of maps $q = \begin{bmatrix} 0 & s \end{bmatrix}$. It is trivial to check that the diagram

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "g"] & X \arrow[d, "f"] \\ P \oplus P_0 \arrow[r, "q"] & Y \end{tikzcd} \tag{16}$$

commutes. The map q is an E-epimorphism since s is, by the dual of Lemma 3.4 combined with Lemma 3.2. We also have that $\pi \circ g = \mathrm{id}_P$, where $\pi\colon P \oplus P_0 \to P$ is projection to the first summand. So g is a split monomorphism. That its cokernel is E-projective follows from an easy application of the Nine Lemma to get the commutative diagram with exact rows and exact columns:

$$
\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & P_0 \arrow[r, "\mathrm{id}"] \arrow[d, "i"] & P_0 \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & P \arrow[r, "g"] \arrow[d, "\mathrm{id}"] & P \oplus P_0 \arrow[r, "\pi'"] \arrow[d, "\pi"] & P_0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & P \arrow[r, "\mathrm{id}"] \arrow[d] & P \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & 0 \arrow[r] & 0 &
\end{tikzcd}
$$

where i is inclusion as the second summand and π' is projection to the second summand. So P_0, an E-projective, is the cokernel of g.

So diagram 16 is the desired diagram 15. $\square$

For the next proposition, we use the phrase "sequential colimit" to describe any colimit indexed by the partially-ordered set of the natural numbers regarded as a category, i.e., a colimit with shape $\bullet \to \bullet \to \bullet \to \dots$. We also include, in the next proposition, a requirement that sequential colimits of E-long exact sequences be E-long exact. This is a mild assumption; in the absolute case, when E is the class of all short exact sequences in $\mathcal{C}$, this assumption is equivalent to Grothendieck's axiom AB5. We remind the reader that axiom AB5 on an abelian category $\mathcal{C}$ stipulates that small colimits exist in $\mathcal{C}$ and that a sequential colimit of exact sequences in $\mathcal{C}$ remains exact. This axiom is satisfied, for example, by the category of R-modules, for any ring R.

Proposition 5.2. *Let $\mathcal{C}$ be a weak Waldhausen abelian category. Let E be a class allowable for $\mathcal{C}$, and suppose that a sequential colimit of E-long exact sequences in $\mathcal{C}$ is E-long exact. Suppose $\mathcal{C}$ has enough E-projectives and enough E-injectives, suppose E has retractile monics and sectile epics, and suppose that every E-projective object is E-injective. Then sequential homotopy colimits are well-defined in $\mathcal{C}$.*

Proof. Let $F, G\colon \mathbb{N} \to \mathcal{C}$ be homotopy colimit diagrams and let $\phi\colon F \to G$ be a natural transformation (i.e., map of diagrams) such that $\phi(n)\colon F(n) \to G(n)$ is an E-stable equivalence for every $n \in \mathbb{N}$. In other words, we have a commutative diagram

$$
\begin{tikzcd}
F(0) \arrow[r] \arrow[d] & F(1) \arrow[r] \arrow[d] & F(2) \arrow[r] \arrow[d] & \cdots \\
G(0) \arrow[r] & G(1) \arrow[r] & G(2) \arrow[r] & \cdots
\end{tikzcd}
\tag{17}
$$

in which all horizontal maps are cofibrations (in particular, E-monomorphisms) and all vertical maps are E-stable equivalences. Then, by Lemma 3.6, the vertical maps in the diagram

$$\begin{tikzcd}
\mathrm{Ext}^n_{\mathcal{C}/E}(F(0),M) & \mathrm{Ext}^n_{\mathcal{C}/E}(F(1),M) \arrow[l] & \mathrm{Ext}^n_{\mathcal{C}/E}(F(2),M) \arrow[l] & \cdots \arrow[l] \\
\mathrm{Ext}^n_{\mathcal{C}/E}(G(0),M) \arrow[u] & \mathrm{Ext}^n_{\mathcal{C}/E}(G(1),M) \arrow[l] \arrow[u] & \mathrm{Ext}^n_{\mathcal{C}/E}(G(2),M) \arrow[l] \arrow[u] & \cdots \arrow[l]
\end{tikzcd}$$

are isomorphisms for all $n \geqslant 1$ and for any object M of $\mathcal{C}$, and these isomorphisms are natural in M. Consequently, we have natural isomorphisms

$$\lim{}_i \mathrm{Ext}^n_{\mathcal{C}/E}(G(i),M) \xrightarrow{\cong} \lim{}_i \mathrm{Ext}^n_{\mathcal{C}/E}(F(i),M) \tag{18}$$

and

$$\lim{}^1_i \mathrm{Ext}^n_{\mathcal{C}/E}(G(i),M) \xrightarrow{\cong} \lim{}^1_i \mathrm{Ext}^n_{\mathcal{C}/E}(F(i),M) \tag{19}$$

for all $n \geqslant 1$, where we are writing $\lim^1$ for the first right-derived functor of lim.

Now we can use Lemma 5.1 to choose E-projective resolutions for each $F(i)$ to get a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P^F_{2,0} \arrow[r] \arrow[d] & P^F_{1,0} \arrow[r] \arrow[d] & P^F_{0,0} \arrow[r] \arrow[d] & F(0) \arrow[r] \arrow[d] & 0 \arrow[d] \\
\cdots \arrow[r] & P^F_{2,1} \arrow[r] \arrow[d] & P^F_{1,1} \arrow[r] \arrow[d] & P^F_{0,1} \arrow[r] \arrow[d] & F(1) \arrow[r] \arrow[d] & 0 \arrow[d] \\
\cdots \arrow[r] & P^F_{2,2} \arrow[r] \arrow[d] & P^F_{1,2} \arrow[r] \arrow[d] & P^F_{0,2} \arrow[r] \arrow[d] & F(2) \arrow[r] \arrow[d] & 0 \arrow[d] \\
 & \vdots & \vdots & \vdots & \vdots & \vdots
\end{tikzcd}$$

in which each row is an E-projective resolution and each vertical map $P^F_{i,j} \to P^F_{i,j+1}$ is a split monomorphism.

We claim that, for every $i \in \mathbb{N}$, the colimit $\mathrm{colim}_j P^F_{i,j}$ is E-projective. Let $f\colon X \to Y$ be an E-epimorphism and $g\colon \mathrm{colim}_j P^F_{i,j} \to Y$ be a map. We write $g_j\colon P^F_{i,j} \to Y$ for the jth component map of g. Since $P^F_{i,0}$ is E-projective, there exists a lift $\ell_0\colon P^F_{i,0} \to X$ of g_0 over f, which provides the first step of an induction. Suppose we have a map $\ell_j\colon P^F_{i,j} \to X$ such that $f \circ \ell_j = g_j$. Then, since the map $P^F_{i,j} \to P^F_{i,j+1}$ is a split monomorphism with E-projective cokernel, there exists a map $\ell_{j+1}\colon P^F_{i,j+1} \to X$ making the diagram

$$\begin{tikzcd}
F(i) \arrow[r] \arrow[d, "\ell_i"] & F(i+1) \arrow[d, "g_{i+1}"] \arrow[dl, dashed, "\ell_{i+1}"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

commute. So we can assemble the maps $\{\ell_j\}_{j\in\mathbb{N}}$ into a map $\ell\colon \mathrm{colim}_j P^F_{i,j} \to X$ such that $f \circ \ell = g$. So $\mathrm{colim}_j P^F_{i,j}$ has the universal property defining an E-projective object, so $\mathrm{colim}_j\, P^F_{i,j}$ is E-projective. We make the same constructions for G as well as F, writing $P^G_{i,j}$ rather than $P^F_{i,j}$ for the E-projectives constructed in this way.

Now the chain complex

$$\cdots \to \mathrm{colim}_j\, P^F_{2,j} \to \mathrm{colim}_j\, P^F_{1,j} \to \mathrm{colim}_j\, P^F_{0,j} \to \mathrm{colim}\, F \to 0 \tag{20}$$

is E-long exact, due to our assumption that sequential colimits of E-long exact sequences in $\mathcal{C}$ are E-long exact. Furthermore, we have shown that each $\mathrm{colim}_j\, P^F_{i,j}$ is E-projective. So long exact sequence 20 is an E-projective resolution of $\mathrm{colim}\, F$. So, for any object M of $\mathcal{C}$, the cohomology of the cochain complex

$$\begin{tikzcd}
\cdots & \hom_{\mathcal{C}}(\mathrm{colim}_j\, P^F_{1,j}, M) \arrow[l] \arrow[d, "\cong"] & \hom_{\mathcal{C}}(\mathrm{colim}_j\, P^F_{0,j}, M) \arrow[l] \arrow[d, "\cong"] & 0 \arrow[l] \arrow[d] \\
\cdots & \lim_j \hom_{\mathcal{C}}(P^F_{1,j}, M) \arrow[l] & \lim_j \hom_{\mathcal{C}}(P^F_{0,j}, M) \arrow[l] & 0 \arrow[l]
\end{tikzcd} \tag{21}$$

is $\mathrm{Ext}^*_{\mathcal{C}/E}(\mathrm{colim}\, F, M)$, and this isomorphism is natural in M.

Now we have the usual short exact sequence relating the cohomology of a sequential limit of cochain complexes of abelian groups to the sequential limit of their cohomologies (we have left off the zeroes to fit within the margins):

$$\lim{}^1_j H^{n-1} \hom_{\mathcal{C}}(P^F_{\bullet,j}, M) \to H^n \lim_j \hom_{\mathcal{C}}(P^F_{\bullet,j}, M) \to \lim_j H^n \hom_{\mathcal{C}}(P^F_{\bullet,j}, M).$$

Due to isomorphisms 18 and 19, we now have the commutative diagram with rows short exact sequences:

$$\begin{tikzcd}
\lim^1_j \mathrm{Ext}^{n-1}_{\mathcal{C}/E}(G(j), M) \arrow[r] \arrow[d, "\cong"] & \mathrm{Ext}^n_{\mathcal{C}/E}(\mathrm{colim}\, G, M) \arrow[r] \arrow[d, "\cong"] & \lim_j \mathrm{Ext}^n_{\mathcal{C}/E}(G(j), M) \arrow[d, "\cong"] \\
\lim^1_j H^{n-1} \hom_{\mathcal{C}}(P^G_{\bullet,j}, M) \arrow[r] \arrow[d, "b"] & H^n \lim_j \hom_{\mathcal{C}}(P^G_{\bullet,j}, M) \arrow[r] \arrow[d, "a"] & \lim_j H^n \hom_{\mathcal{C}}(P^G_{\bullet,j}, M) \arrow[d, "c"] \\
\lim^1_j H^{n-1} \hom_{\mathcal{C}}(P^F_{\bullet,j}, M) \arrow[r] \arrow[d, "\cong"] & H^n \lim_j \hom_{\mathcal{C}}(P^F_{\bullet,j}, M) \arrow[r] \arrow[d, "\cong"] & \lim_j H^n \hom_{\mathcal{C}}(P^F_{\bullet,j}, M) \arrow[d, "\cong"] \\
\lim^1_j \mathrm{Ext}^{n-1}_{\mathcal{C}/E}(F(j), M) \arrow[r] & \mathrm{Ext}^n_{\mathcal{C}/E}(\mathrm{colim}\, F, M) \arrow[r] & \lim_j \mathrm{Ext}^n_{\mathcal{C}/E}(F(j), M)
\end{tikzcd}$$

and the vertical map marked b is an isomorphism if $n \geqslant 2$, and the vertical map marked c is an isomorphism if $n \geqslant 1$. Hence, by the Five Lemma, the vertical map marked a is an isomorphism if $n \geqslant 2$.

We conclude that the $\mathrm{Ext}^n_{\mathcal{C}/E}(\mathrm{colim}\, G, M) \to \mathrm{Ext}^n_{\mathcal{C}/E}(\mathrm{colim}\, F, M)$ is an isomorphism if $n \geqslant 2$. Hence, by Lemma 4.2, the assumption that every E-projective is E-injective implies that the map $\mathrm{colim}\, F \to \mathrm{colim}\, G$ is an E-stable equivalence. So sequential homotopy colimits in $\mathcal{C}$ are unique up to homotopy. $\square$

6. Positive results on geometric realization.

In this section we prove that, under the same assumptions made in the previous section plus the assumption that our abelian category satisfies Grothendieck's axiom AB3, geometric realization of simplicial objects is well-defined (Cor. 6.8).

One can approach geometric realization as a particular kind of colimit called a "co-end"; this approach is taken in e.g. [**8**]. We take a different approach to geometric realization in this section, by regarding geometric realization as the sequential colimit of a certain sequence of homotopy cofibers. We give an abbreviated description of this approach in Def. 6.5, but it is well-known in the special case of a pointed (e.g., stable) model category, and, for example, it appears in the context of a triangulated category in [**2**].

Lemma 6.1. *A pullback of a surjective map of abelian groups is surjective.*

Proof. The forgetful functor from abelian groups to sets is a right adjoint, hence preserves limits. It also clearly preserves surjections. So the lemma is true if a pullback of a surjective maps of sets is surjective, which is an elementary exercise. □

Lemma 6.2. *Let $\mathcal{C}$ be an abelian category and let E be an allowable class with retractile monics. Suppose $\mathcal{C}$ has enough E-injectives. Then E-monics are closed under pushout in $\mathcal{C}$. That is, if $X \to Z$ is an E-monic and $X \to Y$ is any morphism in $\mathcal{C}$, then the canonical map $Y \to Y \amalg_X Z$ is an E-monic.*

Proof. Suppose $f\colon X \to Z$ is an E-monic and $X \to Y$ any morphism. We have the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d] & Z \arrow[r] \arrow[d] & \operatorname{coker} f \arrow[r] \arrow[d] & 0 \arrow[d] \\ & Y \arrow[r] & Y \amalg_X Z \arrow[r] & \operatorname{coker} f \arrow[r] & 0 \end{tikzcd}$$

and hence, for every E-injective I, the induced commutative diagram of abelian groups

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \hom_{\mathcal{C}}(\operatorname{coker} f, I) \arrow[r] & \hom_{\mathcal{C}}(Z, I) \arrow[r] & \hom_{\mathcal{C}}(X, I) \arrow[r] & 0 \\ 0 \arrow[r] & \hom_{\mathcal{C}}(\operatorname{coker} f, I) \arrow[r] \arrow[u, "\cong"] & \hom_{\mathcal{C}}(Y \amalg_X Z, I) \arrow[r] \arrow[u] & \hom_{\mathcal{C}}(Y, I). \arrow[u] & \end{tikzcd}$$

Exactness of the top row follows from f being an E-monic together with E having retractile monics, hence E is its own retractile closure, hence E-monics are precisely the maps which induce a surjection after applying $\hom_{\mathcal{C}}(-, I)$ for every E-injective I. Now in particular we have a commutative square in the above commutative diagram:

$$\begin{tikzcd} \hom_{\mathcal{C}}(Z, I) \arrow[r] & \hom_{\mathcal{C}}(X, I) \\ \hom_{\mathcal{C}}(Y \amalg_X Z, I) \arrow[r] \arrow[u] & \hom_{\mathcal{C}}(Y, I), \arrow[u] \end{tikzcd}$$

which is a pullback square of abelian groups, by the universal property of the pushout.

The top map in the square is a surjection, hence so is the bottom map, Lemma 6.1. So $\hom_{\mathcal{C}}(Y \amalg_X Z, I) \to \hom_{\mathcal{C}}(Y, I)$ is a surjection for every E-injective I. Again since E is its own retractile closure, this implies that $Y \to Y \amalg_X Z$ is an E-monic. $\square$

We now define a weak form of the quasi-Frobenius condition that will allow us to factor maps into cofibrations followed by weak equivalences:

Definition 6.3. Suppose $\mathcal{C}$ is a weak Waldhausen abelian category and E a class allowable for $\mathcal{C}$. We say that $\mathcal{C}$ is *quasi-cone-Frobenius* if there exists a functor $J\colon \mathcal{C} \to \mathcal{C}$ and a natural transformation $\eta\colon \mathrm{id}_{\mathcal{C}} \to J$ such that:

1. $J(X)$ is E-projective for every object X of $\mathcal{C}$,
2. $\eta(X)\colon X \to J(X)$ is an E-monomorphism for every object X of $\mathcal{C}$, and
3. if $f\colon X \to Y$ is an E-monomorphism then so is $J(f)\colon J(X) \to J(Y)$.

We sometimes call the pair J, η a *quasi-cone functor on $\mathcal{C}$.*

Here is an example of a quasi-cone functor: suppose R is a Noetherian ring, and let U be the injective envelope of the direct sum $\bigoplus R/I$, where I ranges across all right ideals of R. For each right R-module M, let

$$J(M) = \prod_{\hom_R(M,U)} U,$$

and let $\eta(M)\colon M \to J(M)$ send m to the map whose component in the factor corresponding to $f \in \hom_R(M, U)$ is $f(m)$. This gives a functorial embedding of every R-module into an injective R-module, and (due to the characterizing property of an injective envelope) J sends monomorphisms to monomorphisms. This construction appears in Bass's paper [**1**], and Bass writes there that he was told of it by C. Watts. We do not know if there is an earlier reference. We have a paper in preparation, [**13**], which provides a more general version of this construction and develops its basic properties. In any case, if every injective R-module is projective, then this functor J is a quasi-cone functor. So the category of modules over any Noetherian quasi-Frobenius ring is quasi-cone-Frobenius, for example. Similarly, the category of finitely-generated modules over any finite-dimensional quasi-Frobenius algebra over a finite field is also quasi-cone-Frobenius.

Now we will begin assuming our weak Waldhausen abelian category $\mathcal{C}$ is quasi-cone-Frobenius. The notation we will usually use is this: for any object X of $\mathcal{C}$, we will write $i_X\colon X \to P_X$ for the chosen E-monomorphism from X to an E-projective P_X.

The following lemma gives us conditions under which a homotopy pushout can be computed as the pushout of a diagram in which only *one* map is a cofibration, rather than both maps.

Lemma 6.4. *Let $\mathcal{C}$ be a quasi-cone-Frobenius weak Waldhausen abelian category. Let E be a class allowable for $\mathcal{C}$. Suppose $\mathcal{C}$ has enough E-projectives and enough*

E-injectives, suppose E has sectile epics and retractile monics, suppose every E-projective object is E-injective. Then any map $f\colon X \to Y$ can be factored as a composite

$$X \xrightarrow{f_0} \tilde{Y} \xrightarrow{f_1} Y, \tag{22}$$

where f_0 is a cofibration and f_1 is a weak equivalence, and furthermore, the pushout of the diagram

$$\begin{tikzcd} X \arrow[r, "f_0"] \arrow[d, "i_X"] & \tilde{Y} \\ P_X & \end{tikzcd} \tag{23}$$

is weakly equivalent to the pushout of the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_X"] & Y \\ P_X. & \end{tikzcd} \tag{24}$$

Proof. We let $\tilde{Y}$ be $P_X \oplus Y$, we let f_0 be given by the matrix of maps $f_0 = \begin{bmatrix} i_X \\ f \end{bmatrix}$, and we let f_1 be the projection to the second summand. It is trivial to check that $f = f_1 \circ f_0$. That f_1 is a weak equivalence follows from its being a split epimorphism with E-projective kernel. That f_0 is a cofibration follows from its being the composite of the inclusion in the first summand $X \to X \oplus Y$ (which is an E-monomorphism, hence cofibration, by Lemma 3.4) followed by the map $X \oplus Y \to P_X \oplus Y$ given by the matrix of maps $\begin{bmatrix} i_x & 0 \\ f & \mathrm{id}_Y \end{bmatrix}$, which is an E-monomorphism, hence cofibration, by Lemma 3.3. So f_0 is a composite of two cofibrations, hence itself a cofibration. So we have the desired factorization 22.

Now we have the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f_0"] \arrow[d, "i_X"] & P_X \oplus Y \arrow[r, "f_1"] \arrow[d] & Y \arrow[d] \\ P_X \arrow[r] & P_X \amalg_X (P_X \oplus Y) \arrow[r] & (P_X \amalg_X (P_X \oplus Y)) \amalg_{P_X \oplus Y} Y \end{tikzcd} \tag{25}$$

in which the two squares are pushout squares, hence the outer rectangle is a pushout diagram, i.e., we have a natural isomorphism

$$(P_X \amalg_X (P_X \oplus Y)) \amalg_{P_X \oplus Y} Y \cong P_X \amalg_X Y.$$

In diagram 25, the maps i_X and f_0 are cofibrations, hence the central vertical map is as well, by Lemma 6.2. Hence the bottom map $P_X \amalg_X (P_X \oplus Y) \to P_X \amalg_X Y$ is a pushout of a weak equivalence (f_1) along a cofibration (the central vertical map), hence itself a weak equivalence by Lemma 4.3. So we have a weak equivalence, as desired, between the pushout of the diagram 23 and the pushout of diagram 24. $\square$

Recall that an abelian category satisfies Grothendieck's axiom AB3 if it has arbitrary (small) colimits. Since any abelian category has coequalizers, axiom AB3 is equivalent to having arbitrary (small) coproducts.

Definition 6.5. Suppose $\mathcal{C}$ is a quasi-cone-Frobenius weak Waldhausen abelian category satisfying Grothendieck's axiom AB3 as well as the assumptions of Lemma 6.4. Let $F\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ be a simplicial object in $\mathcal{C}$. We will write F_n for the nth object of F and $d_i\colon F_n \to F_{n-1}$, $i = 0, \ldots, n$, for the face maps of F. We write ΣX for the pushout of the diagram

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[d, "i_x"] & P_X \\ P_X. & \end{tikzcd}$$

Then by a *geometric realization tower of* F we mean the diagram $GR_F\colon \mathbb{N} \to \mathcal{C}$ defined inductively as follows: $GR_F(0) = F_0 \oplus \coprod_{n>0} P_{F_n}$, and if $GR_F(i)$ has already been defined for $i = 0, \ldots, n-1$, we define $GR_F(n)$ as the pushout in the diagram

$$\begin{tikzcd} \Sigma^{n-1}(F_n \oplus \coprod_{i>n} P_{F_i}) \arrow[r, "f_n"] \arrow[d, "\Sigma^{n-1}(i_{F_n} \oplus \mathrm{id})"] & GR_F(n-1) \arrow[d, "g_n"] \\ \Sigma^{n-1}\left(\coprod_{i\geqslant n} P_{F_i}\right) \arrow[r] & GR_F(n) \end{tikzcd} \tag{26}$$

where $f_n\colon \Sigma^{n-1}(F_n \oplus \coprod_{i>n} P_{F_i}) \to GR_F(n-1)$ is the map obtained from the two nulhomotopies (i.e., factorizations through an E-projective) of the composite map

$$\begin{tikzcd} \Sigma^{n-2}(F_n \oplus \coprod_{i>n} P_{F_i}) \arrow[r, "\Sigma^{n-2}d"] & \Sigma^{n-2}(F_{n-1} \oplus \coprod_{i>n-1} P_{F_i}) \arrow[dl, "f_{n-1}"'] \\ GR_F(n-2) \arrow[r, "g_{n-1}"'] & GR_F(n-1), \end{tikzcd}$$

where we write

$$d\colon F_n \oplus \coprod_{i>n} P_{F_i} \to F_{n-1} \oplus \coprod_{i>n-1} P_{F_i}$$

for the map given on the summands $F_n, P_{F_{n+1}}, P_{F_{n+2}}, \ldots$ of its domain as follows:

- on the summand F_n of its domain, it is the alternating sum

$$d_0 - d_1 + d_2 - \cdots + (-1)^{n-1} d_{n-1}\colon F_n \to F_{n-1}$$

  plus the map $i_{F_n}\colon F_n \to P_{F_n}$, and
- on each summand P_{F_i} for $i > n$, it is simply the inclusion of the summand P_{F_i} into the codomain.

Finally, by the *geometric realization of* F we mean the colimit $\operatorname{colim} GR_F$ of the geometric realization tower of F.

Here is a simple way of describing the maps d appearing in Def. 6.5: if one applies the factorization 22 from Lemma 6.4 to the alternating sum map $d_0 - d_1 + d_2 - \cdots + (-1)^{n-1} d_{n-1} \colon F_n \to F_{n-1}$, one gets a cofibration $f_0 \colon F_n \to F_{n-1} \oplus P_{F_n}$. The map d is simply the direct sum of f_0 with the identity map on all the E-projectives P_{F_i} that will appear later on in the geometric realization tower. Note that, if P is E-projective, then so is ΣP, since the assumptions made in Def. 6.5 imply that homotopy pushouts are well-defined in $\mathcal{C}$ due to Prop. 4.4, which in turn implies that a homotopy pushout of E-projective objects is E-projective, by Prop. 3.10. Furthermore, the quasi-cone-Frobenius assumption implies that i_{F_n} being a cofibration forces Σi_{F_n} to be a cofibration. So the left-hand vertical map in square 26 really is an E-monomorphism into an E-projective object. So by Lemma 6.4, square 26 is computing a homotopy pushout.

Lemma 6.6. *Under the assumptions made in Def. 6.5, making different choices of the quasi-cone functor, in particular the E-projective objects P_X and cofibrations i_X, does not change the E-stable equivalence type of each object $GR_F(n)$ in the geometric realization tower.*

Proof. Suppose we have two choices P_X, P'_X of E-projective object and two choices of cofibration, $i_X \colon X \to P_X$ and $i'_X \colon X \to P'_X$. For any cofibration $f \colon X \to Y$, we have the two pushout diagrams

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[d, "f"] & P_X \\ Y & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "i'_X"] \arrow[d, "f"] & P'_X \\ Y & \end{tikzcd} .$$

By Lemma 6.4, the pushout of each diagram is E-stably equivalent to the pushout of the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & 0 \\ Y. & \end{tikzcd}$$

So the E-stable equivalence type of the homotopy cofiber of a cofibration doesn't depend on the choice of i_x, P_X.

Now we handle the dependence of the factorization 22 on the choices of i_X, P_X. We have the two pushout diagrams

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[d, "f_0"] & P_X \\ P_X \oplus Y & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "i_X"] \arrow[d, "f'_0"] & P_X \\ P'_X \oplus Y & \end{tikzcd} \tag{27}$$

arising from factorization 22, and we want to know that the pushouts of these two diagrams are E-stably equivalent. We accomplish this with the pushout diagram

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[d, "m"] & P_X \\ P_X \oplus P'_X \oplus Y & \end{tikzcd} \tag{28}$$

where m is given the matrix of maps

$$m = \begin{bmatrix} i_X \\ i'_X \\ f \end{bmatrix},$$

and is a cofibration by Lemma 3.3. We note that diagram 28 maps to each of the two diagrams 27; we handle the map to the left-hand diagram 27, and the map to the right-hand diagram is handled similarly. Since i_X, f_0, and m are all cofibrations, Lemma 6.2 implies that, in the pushout square

$$\begin{tikzcd} P_X \oplus P'_X \oplus Y \arrow[r] \arrow[d] & P_X \amalg_X (P_X \oplus P'_X \oplus Y) \arrow[d] \\ P_X \oplus Y \arrow[r] & P_X \amalg_X (P_X \oplus Y), \end{tikzcd}$$

the top horizontal map is a cofibration. The left-hand vertical map is a weak equivalence, since it is a split epimorphism with E-projective kernel; so by Lemma 4.3, the right-hand vertical map is also a weak equivalence. A similar argument holds for the right-hand pushout diagram 27, so since the pushouts of the two pushout diagrams in 27 are each E-stably equivalent to the pushout of diagram 28, the two pushout diagrams in 27 are E-stably equivalent to one another. So the choice of i_X, P_X used in the construction of the factorization 22 doesn't affect the E-stable equivalence type of the resulting homotopy pushouts.

Finally, since GR_F is constructed entirely from these two operations (factorizations as in 22 and homotopy cofibers of cofibrations), up to levelwise E-stable equivalence, GR_F does not depend on the choices of i_X, P_X. □

Proposition 6.7. *Let $\mathcal{C}$ be a quasi-cone-Frobenius weak Waldhausen abelian category. Let E be a class allowable for $\mathcal{C}$. Suppose $\mathcal{C}$ has enough E-projectives and enough E-injectives, suppose E has sectile epics and retractile monics, and suppose every E-projective object is E-injective. Then, for any simplicial object $F\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ in $\mathcal{C}$, the geometric realization tower $GR_F\colon \mathbb{N} \to \mathcal{C}$ is a homotopy colimit diagram. Furthermore, if $F, G\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ are two simplicial objects and $\phi\colon F \to G$ a levelwise weak equivalence, then the induced natural transformation $GR_\phi\colon GR_F \to GR_G$ has the property that, for every natural number n, the map $GR_\phi(n)\colon GR_F(n) \to GR_G(n)$ is a weak equivalence.*

Proof. First we check that GR_F is a homotopy pushout diagram. All we need to check is that, for every natural number n, the map $GR_F(n) \to GR_F(n+1)$ is a cofibration. But this map is the pushout of diagram 26, which is the pushout of a cofibration along a cofibration, hence itself a cofibration by Lemma 6.2.

Now suppose F, G, ϕ are as in the statement of the proposition. The maps $GR_\phi(n)\colon GR_F(n) \to GR_G(n)$ are, by construction, pushouts of weak equivalences along cofibrations, hence by Lemma 4.3, they are themselves weak equivalences. □

Corollary 6.8. *Let $\mathcal{C}, E$ be as in Def. 6.5, and suppose that a sequential colimit of E-long exact sequences in $\mathcal{C}$ is E-long exact. Then geometric realization of simplicial objects in $\mathcal{C}$ is well defined. That is, if we have two simplicial objects $F, G\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ in $\mathcal{C}$ and a natural transformation $\phi\colon F \to G$ such that $\phi(n)\colon F(n) \to G(n)$ is a weak equivalence for every natural number n, then the induced map of geometric realizations* $\operatorname{colim} GR_F \to \operatorname{colim} GR_G$ *is a weak equivalence.*

Proof. By Prop. 6.7, ϕ induces a natural transformation $GR_\phi\colon GR_F \to GR_G$ of homotopy colimit diagrams which is a levelwise weak equivalence. Then by Prop. 5.2, GR_ϕ induces a weak equivalence of colimits $\operatorname{colim} GR_F \to \operatorname{colim} GR_G$. □

7. Appendix on basic notions of relative homological algebra.

Here is an appendix on some ideas in relative homological algebra. The definitions are all classical, except for Def. 7.6.

Definition 7.1. An *allowable class in $\mathcal{C}$* consists of a collection E of short exact sequences in $\mathcal{C}$ which is closed under isomorphism of short exact sequences and which contains every short exact sequence in which at least one object is the zero object of $\mathcal{C}$. (See section IX.4 of [**7**] for this definition and basic properties.)

The usual "absolute" homological algebra in an abelian category $\mathcal{C}$ is recovered by letting the allowable class E consist of *all* short exact sequences in $\mathcal{C}$.

Definition 7.2. Let E be an allowable class in $\mathcal{C}$. A monomorphism $f\colon M \to N$ in $\mathcal{C}$ is called an *E-monomorphism* or an *E-monic* if the short exact sequence $0 \to M \xrightarrow{f} N \to \operatorname{coker} f \to 0$ is in E.

Dually, an epimorphism $g\colon M \to N$ is called an *E-epimorphism* or an *E-epic* if the short exact sequence $0 \to \ker f \to M \xrightarrow{f} N \to 0$ is in E.

In the absolute case, the case that E is all short exact sequences in $\mathcal{C}$, the E-monomorphisms are simply the monomorphisms, and the E-epimorphisms are simply the epimorphisms.

Definition 7.3. Let E be an allowable class in $\mathcal{C}$. An object X of $\mathcal{C}$ is said to be an *E-projective* if, for every diagram

$$\begin{tikzcd} & X \arrow[d] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

in which f is an E-epic, there exists a morphism $X \to M$ making the above diagram commute.

Dually, an object X of $\mathcal{C}$ is said to be an *E-injective* if, for every diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & N \\ X & \end{tikzcd}$$

in which f is an E-monic, there exists a morphism $N \to X$ making the above diagram commute.

When the allowable class E is clear from context we sometimes refer to E-projectives and E-injectives as *relative projectives* and *relative injectives,* respectively.

In the absolute case, the case that E is all short exact sequences in $\mathcal{C}$, the E-projectives are simply the projectives, and the E-injectives are simply the injectives.

Definition 7.4. Let E be an allowable class in $\mathcal{C}$. Let $f, g\colon M \to N$ be morphisms in $\mathcal{C}$. We say that f and g are *E-stably equivalent* and we write $f \simeq g$ if $f - g$ factors through an E-projective object of $\mathcal{C}$.

Definition 7.5. We say that a map $f\colon M \to N$ is a *E-stable equivalence* if there exists a map $h\colon N \to M$ such that $f \circ h \simeq \mathrm{id}_N$ and $h \circ f \simeq \mathrm{id}_M$.

In the absolute case where E consists of all short exact sequences in $\mathcal{C}$, this is the usual notion of stable equivalence of modules over a ring.

Here is a new definition which makes many arguments substantially smoother:

Definition 7.6. An allowable class E is said to *have retractile monics* if, whenever $g \circ f$ is an E-monic, f is also an E-monic.

Dually, an allowable class E is said to *have sectile epics* if, whenever $g \circ f$ is an E-epic, g is also an E-epic.

Here is a fundamental theorem of relative homological algebra, due to Heller (see [**7**]), whose statement is slightly cleaner is one is willing to use the phrase "having sectile epics." The consequence of Heller's theorem is that, in order to specify a "reasonable" allowable class in an abelian category, it suffices to specify its associated relative projective objects.

Theorem 7.7. (Heller.) *If $\mathcal{C}$ is an abelian category and E is an allowable class in $\mathcal{C}$ with sectile epics and enough E-projectives, then an epimorphism $M \to N$ in $\mathcal{C}$ is an E-epic if and only if the induced map $\hom_{\mathcal{C}}(P, M) \to \hom_{\mathcal{C}}(P, N)$ of abelian groups is an epimorphism for all E-projectives P.*

References

[**1**] Hyman Bass. Injective dimension in Noetherian rings. *Trans. Amer. Math. Soc.*, 102:18–29, 1962.

[**2**] Marcel Bökstedt and Amnon Neeman. Homotopy limits in triangulated categories. *Compositio Math.*, 86(2):209–234, 1993.

[3] Carl Faith. Embedding modules in projectives: A report on a problem. In *Advances in Non-Commutative Ring Theory*, volume 951, pages 21–40. Lecture Notes in Mathematics, 1982.

[4] P. Gabriel and A. V. Roiter. *Representations of finite-dimensional algebras*. Springer-Verlag, Berlin, 1997. Translated from the Russian, With a chapter by B. Keller, Reprint of the 1992 English translation.

[5] P. J. Hilton and D. Rees. Natural maps of extension functors and a theorem of R. G. Swan. *Proc. Cambridge Philos. Soc.*, 57:489–502, 1961.

[6] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[7] Saunders Mac Lane. *Homology*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1975 edition.

[8] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[9] H. R. Margolis. *Spectra and the Steenrod algebra*, volume 29 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1983. Modules over the Steenrod algebra and the stable homotopy category.

[10] F. Oort. A note on natural maps of higher extension functors. *Proc. Cambridge Philos. Soc.*, 59:283–286, 1963.

[11] Juan Rada and Manuel Saorín. On two open problems about embedding of modules in free modules. *Comm. Algebra*, 27(1):105–118, 1999.

[12] Douglas C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*, volume 121 of *Pure and Applied Mathematics*. Academic Press Inc., Orlando, FL, 1986.

[13] Andrew Salch. Functorial injective resolutions. *In preparation*.

[14] Andrew Salch. Relative homological algebra, Waldhausen K-theory, and quasi-Frobenius conditions. Preprint, `arXiv:1303.3672`, 2013.

[15] Friedhelm Waldhausen. Algebraic K-theory of spaces. In *Algebraic and geometric topology (New Brunswick, N.J., 1983)*, volume 1126 of *Lecture Notes in Math.*, pages 318–419. Springer, Berlin, 1985.

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