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EFFECTIVE DESCENT MORPHISMS IN STAR-REGULAR CATEGORIES

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(*communicated by George Janelidze*)

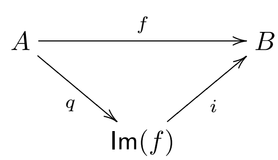
Abstract

In this article a sufficient condition on a star-regular category is introduced guaranteeing that regular epimorphisms are effective descent morphisms. This condition is satisfied by any category with a good theory of ideals (thus, in particular, by any ideal determined category), by any almost abelian category (for instance, by the categories of torsion abelian groups, torsion-free abelian groups, normed vector spaces, Banach spaces, locally compact abelian groups, etc.) and by any category of topological Mal'tsev algebras (in particular, by the category of topological groups).

Introduction

A finitely complete category $\mathbb{C}$ is *regular* when

(a) Any arrow $f: A \to B$ in $\mathbb{C}$ can be factorised as $f = iq$



with q a regular epimorphism and i a monomorphism;

(b) These factorisations are pullback-stable in $\mathbb{C}$.

In the category of sets, and, more generally, in any variety of universal algebras, an arrow is a regular epimorphism precisely when it is surjective. Accordingly, the factorisation in (a) is the usual one of a function (or a homomorphism) as a surjection $q: A \to \mathsf{Im}(f)$ onto its direct image $\mathsf{Im}(f) = \{f(a) \mid a \in A\}$ followed by the inclusion $i: \mathsf{Im}(f) \to B$ of its image in its codomain B. Furthermore, surjective functions (and surjective homomorphisms) are clearly pullback-stable, so that these categories are

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all regular. For many purposes the notion of regular epimorphism provides a very suitable abstraction of the notion of *quotient* in the context of regular categories.

However, although regular epimorphisms are known to be *descent* morphisms in any regular category, they fail to be *effective descent* morphisms, in general [**18**]. This is due to the fact that not all equivalence relations in a regular category $\mathbb{C}$ are *effective*, namely they do not necessarily occur as the *kernel pair* of a morphism in $\mathbb{C}$. The recent work on descent theory (see [**17**], for instance, and the references therein) has made it clear that the right notion of "good quotient" in a general category is precisely provided by the notion of effective descent morphism.

The present paper deals with the problem of finding a simple condition guaranteeing that regular epimorphisms and effective descent morphisms coincide in a regular category. This is obviously the case when the regular category is *exact* [**1**] (as the category of sets or any variety of algebras is, for instance), since all equivalence relations are effective. However, this is also the case for many (not necessarily exact) regular categories: G. Janelidze and M. Sobral call such categories *almost exact* [**16**]. The condition we propose in this article is expressed in the realm of star-regular categories recently introduced in [**10**], which are a special kind of regular categories equipped with an ideal of morphisms (in the sense of [**7**]), a concept that we recall in Section 2. We call *semi-effective* the star-regular categories satisfying this suitable condition (see Definition 2.11) which guarantees, under a mild further condition, that regular epimorphisms are effective descent morphisms (Theorem 3.2). Several interesting categories turn out to have this property, many of which are not exact: any "efficiently regular category" in the sense of D. Bourn [**4**], any "almost abelian" category in the sense of W. Rump [**24**] (see also [**22**]), and any "category with a good theory of ideals" in the sense of Z. Janelidze, A. Ursini and the first author [**10**]. In the last part of our work we analyse several further examples, some of which have been studied in the interesting article [**8**] by T. Everaert. A difference with that article is that here we adopt the "elementary approach" to descent theory (recalled here below) instead of the "monadic approach" used in [**8**].

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1. Elementary descent theory

Let $\mathbb{C}$ be a category with pullbacks. For a morphism $p\colon E \to B$ in $\mathbb{C}$, we denote by

$$Eq(p) \underset{p_2}{\overset{p_1}{\rightrightarrows}} E$$

the equivalence relation in $\mathbb{C}$ determined by the kernel pair of p. Given such an equivalence relation in $\mathbb{C}$, we write $\mathsf{DiscFib}(Eq(p))$ for the category of discrete fibrations of equivalence relations over $Eq(p)$ and natural morphisms: recall that an object in this category is an internal functor $(\varphi_0, \varphi_1)\colon R \to Eq(p)$ as in the commutative diagram

$$\begin{array}{ccc} R & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & D \\ {\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle \varphi_0} \\ Eq(p) & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & E \end{array} \tag{1}$$

having the property that the square involving the second projections is a pullback (this implies that the square involving the first projections is a pullback). Let

$$F^p\colon (\mathbb{C} \downarrow B) \to \mathsf{DiscFib}(Eq(p))$$

be the functor sending an object (A, α) in the comma category $(\mathbb{C} \downarrow B)$ to the discrete fibration (C) in the commutative diagram

$$\begin{array}{ccccc} Eq(q) & \rightrightarrows & E \times_B A & \xrightarrow{q} & A \\ {\scriptstyle \overline{\sigma}}\downarrow & (C) & \downarrow{\scriptstyle \sigma} & (D) & \downarrow{\scriptstyle \alpha} \\ Eq(p) & \rightrightarrows & E & \xrightarrow[p]{} & B, \end{array}$$

where (D) is the pullback of p and α, $Eq(q)$ is the kernel pair of q, and $\overline{\sigma}$ the induced arrow making the two left squares commutative.

A morphism $p\colon E \to B$ in $\mathbb{C}$ is said to be:

(a) a *descent morphism* if the functor F^p is full and faithful,

(b) an *effective descent morphism* if the functor F^p is a category equivalence.

Remark that a morphism $p\colon E \to B$ is an effective descent morphism as defined above exactly when the change-of-base functor $p^*\colon (\mathbb{C} \downarrow B) \to (\mathbb{C} \downarrow E)$ is *monadic*, as explained in [**17**], for instance. The following theorem will also be needed (see [**8, 16, 17**]).

Theorem 1.1. *Let $\mathbb{C}$ be a finitely complete regular category. Then*

(1) $p\colon E \to B$ in $\mathbb{C}$ is a descent morphism if and only if it is a regular epimorphism;

(2) A regular epimorphism $p\colon E \to B$ in $\mathbb{C}$ is an effective descent morphism if and only if for any discrete fibration as in (1) *over the kernel pair $Eq(p)$ of p, the equivalence relation R is effective.*

2. Star-regular categories

In this section we recall some basic aspects of the theory of "star relations" in a regular "multi-pointed category", as introduced by Z. Janelidze, A. Ursini and the first author. We refer the reader to [**10**] for further details.

Definition 2.1. Let $\mathbb{C}$ be a category, $\mathcal{N}$ a class of morphisms of $\mathbb{C}$ that forms an *ideal* in the sense of C. Ehresmann [**7**]: for any composable pair of morphisms f, g of $\mathbb{C}$ the composite gf belongs to $\mathcal{N}$ whenever either f or g belongs to $\mathcal{N}$. A category $\mathbb{C}$ equipped with an ideal $\mathcal{N}$ of morphisms is called a *multi-pointed category* [**10**].

The following examples will be the guiding ones in this article:

Example 2.2.

(1) Any category $\mathbb{C}$ can be seen as a multi-pointed category by choosing for $\mathcal{N}$ the class of all morphisms in $\mathbb{C}$: this situation is usually referred to as the *total context*.
(2) Any pointed category, with zero object 0, can be thought of as a multi-pointed category by choosing for $\mathcal{N}$ the class of zero morphisms: this is the *pointed context*.

Convention. From now on we shall assume that the category $\mathbb{C}$ is finitely complete.

A pair of parallel morphisms $\sigma = (\sigma_1, \sigma_2) \colon S \rightrightarrows X$ is called a *star* when $\sigma_1 \in \mathcal{N}$; it is called a monic star when the pair (σ_1, σ_2) is jointly monomorphic. A morphism $k \colon K \to X$ is the $\mathcal{N}$-*kernel* of a morphism $f \colon X \to Y$ if the composite fk belongs to $\mathcal{N}$ and, for any other morphism $g \colon L \to X$ such that $fg \in \mathcal{N}$, there exists a unique morphism $u \colon L \to K$ such that $ku = g$:

$$\begin{array}{ccccc} K & \xrightarrow{k} & X & \xrightarrow{f} & Y. \\ {\scriptstyle u}\uparrow & \nearrow_{g} & & & \\ L & & & & \end{array}$$

Note that such a k is always a monomorphism. In the pointed context, the $\mathcal{N}$-*kernel* of a morphism $f \colon X \to Y$ is the classical kernel of this morphism; in the total context, the $\mathcal{N}$-*kernel* of $f \colon X \to Y$ is simply the identity morphism 1_X.

For a relation $\varrho = (\varrho_1, \varrho_2) \colon R \rightrightarrows X$ on an object X, we denote by ϱ^* the largest subrelation of ϱ which is a monic star. Such a star exists whenever the $\mathcal{N}$-kernels exist, since in this case it is given by $\varrho^* = (\varrho_1 k, \varrho_2 k) \colon K \rightrightarrows X$, where $k \colon K \to R$ is the $\mathcal{N}$-kernel of ϱ_1. In particular, if $\Delta_X \colon X \rightrightarrows X$ is the discrete equivalence relation on X, we have $\Delta_X^* = (k_X, k_X) \colon K \rightrightarrows X$, where $k_X \colon K \to X$, is the $\mathcal{N}$-kernel of 1_X.

A *kernel star* (or a *star-kernel*) of a morphism $f \colon X \to Y$ is a star $\kappa = (\kappa_1, \kappa_2)$: $K \rightrightarrows X$ such that $f\kappa_1 = f\kappa_2$ and, for any other star $\kappa' = (\kappa'_1, \kappa'_2) \colon K' \rightrightarrows X$ such that $f\kappa'_1 = f\kappa'_2$, there exists a unique morphism $u \colon K' \to K$ such that $\kappa u = \kappa'$:

$$\begin{array}{ccccc} K & \overset{\kappa}{\rightrightarrows} & X & \xrightarrow{f} & Y\ . \\ {\scriptstyle u}\uparrow & \nearrow\!\!\nearrow_{\kappa'} & & & \\ K' & & & & \end{array}$$

It is easy to see that, in the presence of $\mathcal{N}$-kernels, the kernel star of an arrow $f \colon X \to Y$ is given by $Eq(f)^* \rightrightarrows X$.

In the pointed context, the notion of a kernel star of a morphism becomes the classical notion of a *kernel* of a morphism, while in the total context it gives the notion of a *kernel pair* of a morphism. Let us then consider a commutative diagram

$$\begin{array}{ccc} S & \overset{\sigma}{\rightrightarrows} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ K & \underset{\kappa}{\rightrightarrows} & Y \end{array}$$

of stars and morphisms: $f\sigma = \kappa g$ means that $f\sigma_1 = \kappa_1 g$ and $f\sigma_2 = \kappa_2 g$. Such a diagram is called a *star-pullback* when, given a star $\sigma' : S' \rightrightarrows X$ and a morphism $g' \colon S' \to K$ such that $f\sigma' = \kappa g'$, there exists a unique morphism $h \colon S' \to S$ such that $\sigma h = \sigma'$ and $gh = g'$.

Definition 2.3 ([**10**])**.** A *star-regular* category is a regular multi-pointed category $(\mathbb{C}, \mathcal{N})$ with $\mathcal{N}$-kernels in which every regular epimorphism is a coequaliser of a star.

In the total context, a star-regular category is simply a regular category. In the pointed context, a star-regular category is the same as a normal category in the sense of [**19**], i.e., a regular category in which any regular epimorphism is a normal epimorphism.

The following lemma provides a characterisation of those arrows that are monomorphisms in terms of their kernel star:

Lemma 2.4. *In a star-regular category $\mathbb{C}$, the following conditions are equivalent for a morphism $f \colon X \to Y$:*

(1) $f \colon X \to Y$ is a monomorphism;

(2) $Eq(f)^* = \Delta_X^*$;

(3) The projections $p_1 \colon Eq(f)^* \to X$ and $p_2 \colon Eq(f)^* \to X$ are equal.

Proof. It is obvious that that (1) implies (2), and let us prove that (2) $\Rightarrow$ (1). For this, we observe that the correspondence $\phi\colon$ KernelPairs $\longrightarrow$ KernelStars mapping any kernel pair $Eq(f)$ in a star-regular category $\mathbb{C}$ to its corresponding star $Eq(f)^*$ is a bijection, since any regular epimorphism in $\mathbb{C}$ is the coequaliser of its kernel star. Accordingly, $Eq(f)^* = \Delta_X^*$ implies that $Eq(f) = \Delta_X$, and $f \colon X \to Y$ is then a monomorphism. Finally, the implication (2) $\Rightarrow$ (3) is clear, whereas (3) $\Rightarrow$ (2) easily follows from the universal properties of the kernel stars $Eq(f)^*$ and Δ_X^*. ☐

In the total context, Lemma 2.4 says, in particular, that f is a monomorphism if and only if $Eq(f) = \Delta_X$; in the pointed context, it says that f is a monomorphism if and only if its kernel $\mathsf{ker}\,(f)$ is trivial: $\mathsf{ker}\,(f) = 0$ (see [**5**] for the case of normal categories).

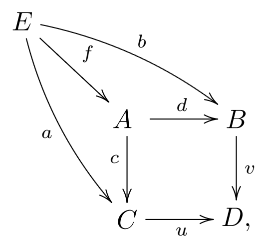
Corollary 2.5. *A span $Y \xleftarrow{f} X \xrightarrow{g} Z$ in a star-regular category $\mathbb{C}$ is a relation if and only if $Eq(f)^* \wedge Eq(g)^* = \Delta_X^*$.*

Proof. This follows from Lemma 2.4 and the fact that "starring" preserves meets, i.e., $Eq(f)^* \wedge Eq(g)^* = (Eq(f) \wedge Eq(g))^*$ (see Lemma 2.6 in [**10**]). ☐

The following proposition extends to the star-regular context some properties well known in the total and in the pointed contexts, and are needed for our work.

Proposition 2.6. *In a star-regular category $\mathbb{C}$, the following conditions hold:*

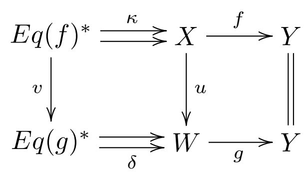
(1) Consider a commutative diagram



where (c, d) is jointly monomorphic. Then

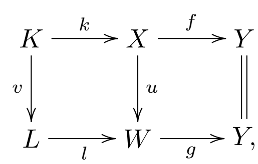
$$Eq(f)^* = Eq(a)^* \wedge Eq(b)^*.$$

(2) Given a commutative diagram



of stars and morphisms, the left-hand side diagram is a star-pullback.

(3) For a commutative diagram



where k and l are the $\mathcal{N}$-kernels of f and g, respectively, the left square is a pullback.

In order to prove the main result of this article, we need an additional assumption on the star-regular category. This property is referred to as *having enough trivial objects* in [**9**] (see also [**12**], and the references therein, for the related notion of a closed ideal of morphisms). There are several equivalent conditions defining when a category has enough trivial objects (see Proposition 3.5 in [**9**]). For the purpose of the present article, the following will be the most suitable one:

Definition 2.7 ([**9**])**.** Let $(\mathbb{C}, \mathcal{N})$ be a regular multi-pointed category with $\mathcal{N}$-kernels. $\mathbb{C}$ has *enough trivial objects* when, for any relation $(r_1, r_2) \colon R \rightrightarrows X$ in $\mathbb{C}$ and any arrow $f \colon K \to R$ such that $r_1 f, r_2 f \in \mathcal{N}$, one then has that $f \in \mathcal{N}$.

Example 2.8. It is clear that both in the *total context* and in the *pointed context* $\mathbb{C}$ has enough trivial objects. Besides the pointed and the total contexts, one can also consider the so-called *proto-pointed context* introduced in [**10**]: this is the situation of a regular multi-pointed category $(\mathbb{C}, \mathcal{N})$ with $\mathcal{N}$-kernels, where the class $\mathcal{N}$ consists of the morphisms $f\colon X \to Y$ whose regular image is the smallest subobject of Y. Recall also that a finitely complete category is *quasi-pointed* [**3**] if it has an initial object 0, a terminal object 1, and the unique arrow $0 \to 1$ is a monomorphism. As explained in [**9**] any quasi-pointed category provides an example of proto-pointed context: it suffices to choose for $\mathcal{N}$ the class of morphisms that factor through the initial object. Also, in the quasi-pointed context, $\mathbb{C}$ clearly has enough trivial objects.

Proposition 2.9. *Let $\mathbb{C}$ be a star-regular category. If the following diagram*

$$\begin{array}{ccc} A & \xrightarrow{(\sigma_1,\sigma_2)} & B \times B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f\times f} \\ C & \xrightarrow[(\kappa_1,\kappa_2)]{} & D \times D \end{array}$$

is a pullback with $\sigma_1, \kappa_1 \in \mathcal{N}$, then the following commutative diagram

$$\begin{array}{ccc} A & \overset{\sigma}{\rightrightarrows} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ C & \underset{\kappa}{\rightrightarrows} & D \end{array}$$

of stars and morphisms is a star-pullback. The converse is true when f is a monomorphism and $\mathbb{C}$ has enough trivial objects.

Lemma 2.10. *Let $(r_1, r_2)\colon R \rightrightarrows X$ be a reflexive relation on an object X in a star-regular category $\mathbb{C}$, and let $(r_1k, r_2k)\colon R^* \rightrightarrows X$ be the star associated with R. An arrow $q\colon X \to Y$ is the coequaliser of (r_1k, r_2k) if and only if q is the coequaliser of (r_1, r_2).*

Proof. In order to prove the result it suffices to show that an arrow $q\colon X \to Y$ coequalises (r_1, r_2) whenever it coequalises (r_1k, r_2k):

$$R^* \xrightarrow{k} R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X \xrightarrow{q} Y.$$

Assume then that $qr_1k = qr_2k$, where k is the $\mathcal{N}$-kernel of r_1. Since r_1 is a split epimorphism in a star regular category, r_1 is then the coequaliser of its kernel star $(\mu_1, \mu_2)\colon Eq(r_1)^* \rightrightarrows R$. Since $r_1\mu_1 = r_1\mu_2 \in \mathcal{N}$ there are unique morphisms θ_i such that $k\theta_i = \mu_i$ for $i \in \{1, 2\}$ as in the diagram

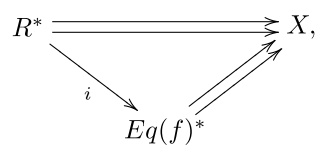
$$\begin{array}{ccccc} & & R^* & & \\ & {\scriptstyle \theta_1}\nearrow\!\!\nearrow{\scriptstyle \theta_2} & & \searrow{\scriptstyle k} & \\ Eq(r_1)^* & & \underset{\mu_2}{\overset{\mu_1}{\rightrightarrows}} & & R \xrightarrow[r_1]{} X. \end{array}$$

Consequently, we have

$$\begin{aligned}(qr_2)\mu_1 &= qr_2k\theta_1\\ &= qr_1k\theta_1 \quad \text{(since } q \text{ coequalises } (r_1k, r_2k))\\ &= qr_1\mu_1\\ &= qr_1\mu_2 \quad \text{(since } r_1 \text{ coequalises } (\mu_1, \mu_2))\\ &= qr_1k\theta_2\\ &= qr_2k\theta_2\\ &= (qr_2)\mu_2.\end{aligned}$$

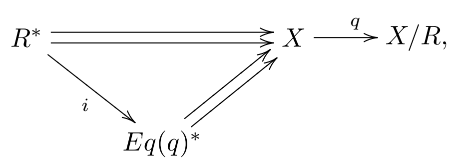
It follows that there exists a unique $t\colon X \to Y$ such that $tr_1 = qr_2$. Since the relation R is reflexive, there is an arrow $\delta\colon X \to R$ such that $r_1\delta = 1_X = r_2\delta$, and this implies that $t = tr_1\delta = qr_2\delta = q$, as desired. □

Definition 2.11. A star-regular category $\mathbb{C}$ is said to be *semi-effective star-regular* when any equivalence relation $R \rightrightarrows X$ in $\mathbb{C}$ has the following property: if the star R^* associated with R is a subobject of a kernel star $Eq(f)^*$



with i a split monomorphism in $\mathbb{C}$, then R^* is itself a kernel star.

Remark 2.12. If the category $\mathbb{C}$ has coequalisers of equivalence relations, then $\mathbb{C}$ is a semi-effective star-regular category if and only if for any equivalence relation $R \rightrightarrows X$ on an object X and q the coequaliser of the equivalence relation R



with i a split monomorphism in $\mathbb{C}$, the star R^* associated with R is the kernel star of q, i.e., $R^* \cong Eq(q)^*$.

Example 2.13. In the *total context*, it is obvious that any Barr-exact category is semi-effective star-regular. More generally, any "efficiently regular category" in the sense of [**4**] is a semi-effective star-regular category, since any split monomorphism is a regular monomorphism. Examples of efficiently regular categories are provided by the category of topological groups and, more generally, by any category of topological models of a Mal'tsev algebraic theory (see Section 4.5).

In the *pointed context*, among the examples of semi-effective star-regular categories there are also any "category with a good theory of ideals" in the sense of [**10**], and any "almost abelian category" in the sense of [**24**]. These examples, and many other ones, will be examined in Section 4.

Birkhoff subcategories

By a *regular-epireflective subcategory* $\mathbb{D}$ of a regular category $\mathbb{C}$, we mean a full replete reflective subcategory

$$\mathbb{C} \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbb{D}$$

with the property that the A-component $\eta_A \colon A \to UF(A)$ of the unit of the adjunction is a regular epimorphism for each $A \in \mathbb{C}$. It is well known that the last requirement is equivalent to the fact that $\mathbb{D}$ is closed in $\mathbb{C}$ under subobjects. $\mathbb{D}$ is a *Birkhoff subcategory* of $\mathbb{C}$ if, moreover, $\mathbb{D}$ is closed in $\mathbb{C}$ under regular quotients.

Lemma 2.14. *Let $\mathbb{D}$ be a Birkhoff subcategory of a semi-effective star-regular category $(\mathbb{C}, \mathcal{N}_{\mathbb{C}})$. Then $\mathbb{D}$ is a semi-effective star-regular category as well.*

Proof. It is well known that if $\mathbb{C}$ is regular, then $\mathbb{D}$ is regular as well. This essentially follows from the fact that the regular epi-mono factorisation in $\mathbb{C}$ of an arrow in $\mathbb{D}$ is still the regular epi-mono factorisation of this arrow in $\mathbb{D}$. Since $\mathbb{D}$ is a full subcategory of $\mathbb{C}$, we can choose the ideal of morphisms of $\mathbb{D}$ induced by the ideal of morphisms in $\mathbb{C}$, so that, for any X,Y in $\mathbb{D}$, $\mathcal{N}_{\mathbb{D}}(X,Y) = \mathcal{N}_{\mathbb{C}}(X,Y) = \mathcal{N}(X,Y)$. The fact that $\mathbb{D}$ is closed in $\mathbb{C}$ under subobjects implies that the $\mathcal{N}$-kernel of an arrow in $\mathbb{D}$ is computed in the same way in the categories $\mathbb{C}$ and $\mathbb{D}$.

The category $(\mathbb{D}, \mathcal{N})$ is a regular multi-pointed category. Since it is a regular-epireflective subcategory of $\mathbb{C}$, any regular epimorphism in $\mathbb{D}$ is a regular epimorphism in $\mathbb{C}$, and $(\mathbb{D}, \mathcal{N})$ is a star-regular category.

Consider then the following diagram in $\mathbb{D}$:

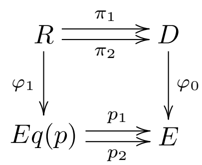
$$\begin{array}{ccccccc}
R^* & \xrightarrow{k} & R & \overset{r_1}{\underset{r_2}{\rightrightarrows}} & A & \xrightarrow{q} & E, \\
{\scriptstyle p}\uparrow\downarrow{\scriptstyle s} & & & \nearrow\!\!\nearrow {\scriptstyle q_1, q_2} & & & \\
Eq(q)^* & \xrightarrow[l]{} & Eq(q) & & & &
\end{array}$$

where $R \rightrightarrows A$ is an equivalence relation in $\mathbb{D}$ and $Eq(q)^* \rightrightarrows A$ is a kernel star of its coequaliser q (in $\mathbb{D}$), k and l are the $\mathcal{N}$-kernels of r_1 and q_1, respectively, and s is a split monomorphism with $ps = 1_{R^*}$, and $q_i ls = r_i k$ for $i \in \{1, 2\}$. If we look at this diagram in $\mathbb{C}$, then q is still the coequaliser of (q_1, q_2) in $\mathbb{C}$, since $\mathbb{D}$ is stable in $\mathbb{C}$ under quotients. The category $\mathbb{C}$ is semi-effective star-regular, so that $(r_1 k, r_2 k) \colon R^* \rightrightarrows A$ is a kernel star in $\mathbb{C}$ of its coequaliser $q' \colon A \to E'$ in $\mathbb{C}$. Moreover, $\mathbb{D}$ is stable in $\mathbb{C}$ under quotients, so that E' lies in $\mathbb{D}$, and $(r_1 k, r_2 k) \colon R^* \rightrightarrows A$ is a kernel star in $\mathbb{D}$. □

3. Main result

The following lemma will be needed to prove the main result of this article:

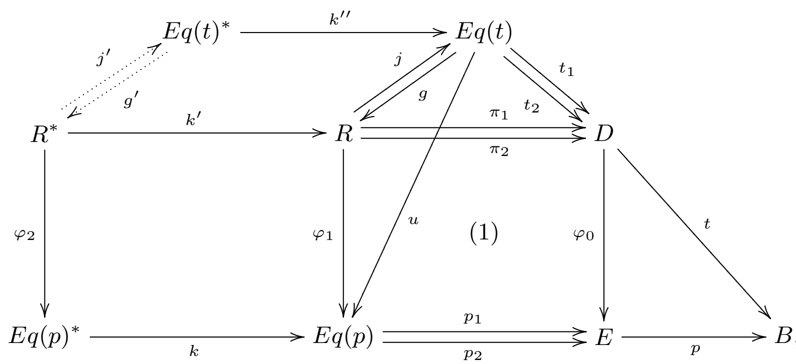
Lemma 3.1. *Let $\mathbb{C}$ be a semi-effective star-regular category. Then, given a discrete fibration of equivalence relations*



(1)

over the kernel pair of p*, the coequaliser of* π_1 *and* π_2 *exists.*

Proof. Let us build the following diagram:



where

(a) $t = p\varphi_0$;

(b) the morphisms k, k' and k'' are the $\mathcal{N}$-kernels of p_1, π_1 and t_1, respectively;

(c) u is the unique arrow such that $\varphi_0 t_l = p_l u$, for $l \in \{1; 2\}$;

(d) g is the unique arrow such that $\pi_1 g = t_1$ and $\varphi_1 g = u$.

There is a unique morphism j such that $t_l j = \pi_l$, for $l \in \{1; 2\}$. Since (φ_1, π_1) is jointly monomorphic and $uj = \varphi_1$, we have the equality $gj = 1_R$. By assumption $R^* \rightrightarrows D$ is then a kernel star, since $Eq(t)^* \rightrightarrows D$ is a kernel star, and the induced arrow j' a split monomorphism. This implies that the coequaliser $q \colon D \to D/R^*$ of $\pi_1 k'$ and $\pi_2 k'$ exists, and q is also the coequaliser of π_1 and π_2 by Lemma 2.10. □

Theorem 3.2. *Let* $\mathbb{C}$ *be a semi-effective star-regular category with enough trivial objects. Then the following conditions for an arrow* $p \colon E \to B$ *are equivalent:*

(1) p *is an effective descent morphism;*

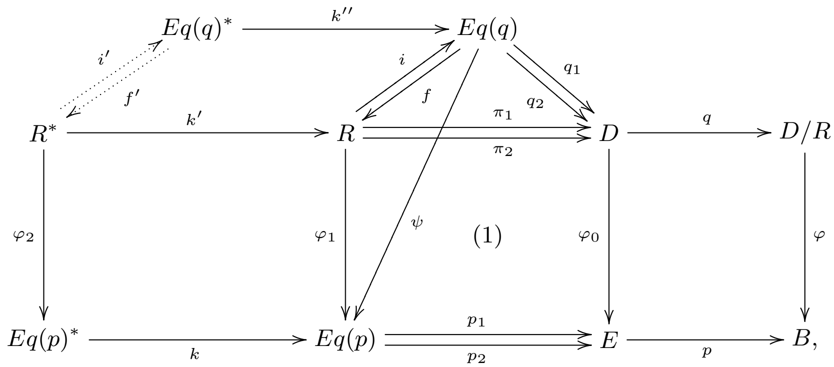
(2) p *is a descent morphism;*

(3) p *is a regular epimorphism.*

Proof. The implication (1) ⇒ (2) is trivial, whereas (2) ⇒ (3)) is true in any finitely complete regular category (see Theorem 1.1).

We are now going to prove that (3) ⇒ (1). Assume that $p \colon E \to B$ is a regular epimorphism, and consider the commutative diagram (1) as in Lemma 3.1. We would like to prove that the equivalence relation $R \rightrightarrows D$ is effective.

Let us then build the commutative diagram



where

(a) q is the coequaliser of π_1 and π_2 (which exists by Lemma 3.1);

(b) i is the unique arrow such that $q_h i = \pi_h$, for $h \in \{1; 2\}$;

(c) k, k' and k'' are the $\mathcal{N}$-kernels of p_1, π_1 and q_1, respectively;

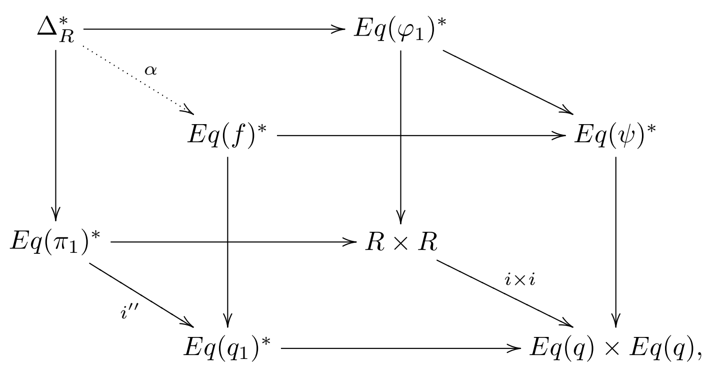
(d) ψ is the unique arrow such that $\varphi_0 q_h = p_h \psi$, for $h \in \{1; 2\}$;

(e) f is the unique arrow such that $\pi_1 f = q_1$ and $\varphi_1 f = \psi$.

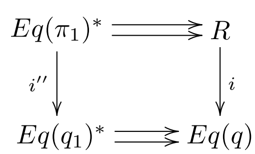
A similar argument to the one used in Lemma 3.1 (to show that $gj = 1_R$) implies that $fi = 1_R$. We also know that the induced split monomorphism $i' \colon R^* \to Eq(q)^*$ is an isomorphism, thanks to Lemma 3.1 and to Remark 2.12.

It will suffice to show that the split epimorphism $f \colon Eq(q) \to R$ is a monomorphism, and this will imply that $i \colon R \to Eq(q)$ will be an isomorphism, as desired.

For this consider the following commutative cube:



where the back square is a pullback by Corollary 2.5, the front square is a pullback by Proposition 2.6(1), and the dotted arrow α is induced by the universal property of this latter pullback. By Proposition 2.6(2), the commutative diagram



is a star-pullback, and by Proposition 2.9 the bottom square of the cube above is a pullback. The same arguments are applied to conclude that the right square is a pullback as well, so that the left square is a pullback. The arrow i'' is an isomorphism: indeed, this follows from Proposition 2.6(2) and (3), the fact that $i' \colon R^* \to Eq(q)^*$ is an isomorphism, and the assumption that $\mathbb{C}$ has enough trivial objects. Accordingly, the dotted arrow α is an isomorphism as well. This implies that the projections $f_1 \colon Eq(f)^* \to Eq(q)$ and $f_2 \colon Eq(f)^* \to Eq(q)$ of the kernel star of f are equal, as one can see from the commutativity of the following diagram:

$$\begin{array}{ccc} \Delta_R^* & \overset{k_R}{\underset{k_R}{\rightrightarrows}} & R \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle i} \\ Eq(f)^* & \overset{f_1}{\underset{f_2}{\rightrightarrows}} & Eq(q), \end{array}$$

where k_R is the $\mathcal{N}$-kernel of 1_R. From Lemma 2.4(c) we conclude that f is a monomorphism, as desired. □

4. Examples

4.1. Categories with a good theory of ideals

In any regular multi-pointed category $\mathbb{C}$ there is a natural notion of *ideal*, that extends the one coming from universal algebra considered in [**21, 25**]. A star-relation $(r_1, r_2) \colon R \rightrightarrows X$ in $\mathbb{C}$ is an ideal if it is the regular image of a kernel star: this means that there exists a kernel star $(k_1, k_2) \colon K \rightrightarrows Y$ and a regular epi $f \colon Y \to X$ with the property that $f(K) = R$. This categorical notion of ideal, introduced in the context of regular multi-pointed categories in [**10**], extends the one defined and studied in regular categories in [**14, 15**]. It is obvious that any kernel star is an ideal, but the converse is not true, in general, even for varieties of algebras. When the classes of ideals and of kernel stars coincide in a star-regular category $\mathbb{C}$, one says that $\mathbb{C}$ is a *category with a good theory of ideals* [**10**]. The results in [**10**] show that, in the total context, the categories with a good theory of ideals are exactly the exact Goursat categories [**6**], whereas, in the pointed context, they are the so-called ideal determined categories introduced in [**13**] (in the presence of finite colimits as required in [**13**]).

Let us then observe that, in the pointed context, the star $(r_1 k, r_2 k) \colon R^* \rightrightarrows X$ associated with an equivalence relation $(r_1, r_2) \colon R \rightrightarrows X$ on X is the star whose first component is the zero arrow, and the second one the *normal subobject* in the sense of Bourn [**3**] associated with R, i.e., the 0-class of R.

In any category $\mathbb{C}$ with a good theory of ideals, it is possible to show that any star $(r_1 k, r_2 k) \colon R^* \rightrightarrows X$ associated with an equivalence relation $(r_1, r_2) \colon R \rightrightarrows X$ on X is necessarily a kernel star. Indeed, any such equivalence relation determines a canonical discrete fibration $(r_2, \sigma) \colon Eq(r_1) \to R$ as in the diagram

$$\begin{array}{ccc} Eq(r_1) & \xrightarrow{\sigma} & R \\ {\scriptstyle p_1}\downarrow\downarrow{\scriptstyle p_2} & & {\scriptstyle r_1}\downarrow\downarrow{\scriptstyle r_2} \\ R & \xrightarrow[r_2]{} & X, \end{array}$$

where σ is the arrow (internally) sending an element $((x,y),(x,z))$ of $Eq(r_1)$ to the element (y,z) of R. The following pushout

$$\begin{array}{ccc} R & \xrightarrow{r_2} & X \\ {\scriptstyle r_1}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow[q]{} & X/R \end{array}$$

exists in $\mathbb{C}$, since r_1 and r_2 are regular epimorphisms (see [**10**]). Thanks to the characterisation of the categories with a good theory of ideals given in Theorem 3.8 in [**10**], it follows that the induced composite arrow $Eq(r_1)^* \to R^* \to Eq(q)^*$ is a regular epi. This implies that the canonical monomorphism $R^* \to Eq(q)^*$ is also a regular epi, thus an isomorphism.

Accordingly, *any category $\mathbb{C}$ with a good theory of ideals is semi-effective star-regular*. By Theorem 3.2, when $\mathbb{C}$ also has enough trivial objects, any regular epimorphism in $\mathbb{C}$ is then an effective descent morphism.

The particular case of ideal determined categories has been considered by T. Everaert [**8**], who arrived at the conclusion that regular epimorphisms therein are effective for descent by using a completely different approach. Observe that our Theorem 3.2 applies to any star-regular category with enough trivial objects for which the class of "stars of equivalence relations" coincides with the class of "kernels stars". This latter condition is weaker than the one asserting that the class of "ideals" coincide with the class of 'kernel stars".

The property of being a category with a good theory of ideals is stable under Birkhoff subcategories. More precisely, one has the following

Proposition 4.1. *Let $\mathbb{D}$ be a regular-epireflective subcategory of a category $\mathbb{C}$ with a good theory of ideals. Then the following conditions are equivalent:*

(1) $\mathbb{D}$ is a Birkhoff subcategory of $\mathbb{C}$;

(2) $\mathbb{D}$ is a category with a good theory of ideals;

(3) For any span of regular epimorphisms $C \xleftarrow{g} A \xrightarrow{f} B$ in $\mathbb{D}$ their pushout (P, f', g') in $\mathbb{C}$ is also their pushout in $\mathbb{D}$.

Proof of (1) $\Rightarrow$ (2). By Lemma 2.14 we know that the category $\mathbb{D}$ is star-regular. Consider then the diagram

$$\begin{array}{ccc} K & \xrightarrow{g} & I \\ {\scriptstyle \lambda}\downarrow\downarrow & & \downarrow\downarrow{\scriptstyle \mu} \\ A & \xrightarrow[f]{} & B, \end{array}$$

where f is a regular epimorphism in $\mathbb{D}$, λ a kernel star in $\mathbb{D}$ and μg the factorisation (regular epi)-(monic-star) of $f\lambda$ in $\mathbb{C}$. One clearly has that $I \in \mathbb{D}$. By assumption μ is then the kernel star of its coequaliser $q\colon B \to Q$ in $\mathbb{C}$. The category $\mathbb{D}$ is stable in $\mathbb{C}$ under quotients, and this implies that $q\colon B \to Q$ is also the coequaliser of μ in $\mathbb{D}$. Accordingly, μ is a kernel star in $\mathbb{D}$.

(2) ⇒ (3). Let $f\colon A \to B$ and $g\colon A \to C$ be regular epimorphisms lying in $\mathbb{D}$. In the category $\mathbb{C}$ with a good theory of ideals the pushout square

$$\begin{array}{ccc} A & \xrightarrow{\;f\;} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ C & \xrightarrow[\;f'\;]{} & P' \end{array}$$

always exists: it is obtained by considering the coequaliser $g'\colon B \to P'$ of the regular image $f(Eq(g)^*) \rightrightarrows B$ of the kernel star $Eq(g)^* \rightrightarrows A$ along f (by Theorem 3.8 in [**10**]). The assumption that $\mathbb{D}$ is a category with a good theory of ideals implies that the pushout (P'', f'', g'') of f and g exists also in $\mathbb{D}$, and g'' is the coequaliser of $f(Eq(g)^*) \rightrightarrows B$ in $\mathbb{D}$. The canonical comparison $\eta\colon P' \to P''$ such that $\eta g' = g''$ is a regular epimorphism. The fact that both g' and g'' are regular epimorphisms in $\mathbb{C}$ with the same kernel star implies that the arrow $\eta\colon P' \to P''$ is an isomorphism.

(3) ⇒ (1). Let $f\colon A \to B$ be a regular epimorphism in $\mathbb{C}$, with $A \in \mathbb{D}$, and consider the kernel pair $(f_1, f_2)\colon Eq(f) \rightrightarrows A$ of f, which lies in $\mathbb{D}$, since $\mathbb{D}$ is stable in $\mathbb{C}$ under subobjects. The projections $f_1\colon Eq(f) \to A$ and $f_2\colon Eq(f) \to A$ are regular epimorphisms in $\mathbb{D}$, so that (B, f, f) is their pushout in $\mathbb{D}$ by the assumption. This shows that $B \in \mathbb{D}$. □

Observe that Proposition 4.1 is useful to find examples of semi-effective star-regular categories which are not categories with a good theory of ideals.

This is the case, for instance, for the category $\mathsf{Ab}_{t.f.}$ of torsion-free abelian groups. Indeed, $\mathsf{Ab}_{t.f.}$ is obviously a normal category, and it is also semi-effective star-regular (see Example 4.4); however, by Proposition 4.1, it does not have a good theory of ideals. This is due to the fact that, although $\mathsf{Ab}_{t.f.}$ is a regular-epireflective subcategory of the category Ab of abelian groups, it is not stable in Ab under quotients.

4.2. Regular epimorphisms in a category with a good theory of ideals

When $\mathbb{C}$ is a category, we denote by $\mathsf{RegEpi}(\mathbb{C})$ the category of regular epimorphisms in $\mathbb{C}$: an object in $\mathsf{RegEpi}(\mathbb{C})$ is a regular epi $a\colon A_1 \to A_0$ in $\mathbb{C}$, and a morphism $f\colon a \to b$ in $\mathsf{RegEpi}(\mathbb{C})$ is a pair (f_0, f_1) of morphisms in $\mathbb{C}$ with $f_0\colon A_0 \to B_0$ and $f_1\colon A_1 \to B_1$ such that $f_0 a = b f_1$.

Lemma 4.2. *Let $\mathbb{C}$ be a category with a good theory of ideals and enough trivial objects. Then $\mathsf{RegEpi}(\mathbb{C})$ is star-regular with enough trivial objects.*

Proof. The category $\mathsf{RegEpi}(\mathbb{C})$ is finitely complete, and it also has coequalisers of effective equivalence relations since $\mathbb{C}$ has pushouts of regular epimorphisms by regular epimorphisms (see also [**8**]). Furthermore, a regular epi in $\mathsf{RegEpi}(\mathbb{C})$ is simply given by a pair of regular epimorphisms $(f_0, f_1)\colon a \to b$ determining a pushout in $\mathbb{C}$. The assumption that $\mathbb{C}$ has enough trivial objects and the characterisation of the categories with a good theory of ideals given in Theorem 3.8 in [**10**] allow one to prove that regular epimorphisms in $\mathsf{RegEpi}(\mathbb{C})$ are stable under pullbacks, and $\mathsf{RegEpi}(\mathbb{C})$ is then a regular category. Let us then denote by $\mathcal{M}$ the class of morphisms $f = (f_0, f_1)$ in $\mathsf{RegEpi}(\mathbb{C})$ defined by $f = (f_0, f_1) \in \mathcal{M}$ if and only if $f_1 \in \mathcal{N}$. This class $\mathcal{M}$ is an ideal of morphisms in $\mathsf{RegEpi}(\mathbb{C})$ (since $\mathcal{N}$ is an ideal in $\mathbb{C}$), so that $(\mathsf{RegEpi}(\mathbb{C}), \mathcal{M})$

is a regular multi-pointed category. The $\mathcal{M}$-kernel of a morphism $f = (f_0, f_1)\colon a \to b$ in $\mathsf{RegEpi}(\mathbb{C})$ is constructed as follows: one takes the $\mathcal{N}$-kernel k_1 of f_1, and then the factorisation k_0c of ak_1 as a regular epimorphism c followed by a monomorphism k_0. The arrow (k_0, k_1) is the $\mathcal{M}$-kernel of f in $\mathsf{RegEpi}(\mathbb{C})$. Since any morphism in $(\mathbb{C}, \mathcal{N})$ has an $\mathcal{N}$-kernel by assumption, then any morphism in $\mathsf{RegEpi}(\mathbb{C})$ has an $\mathcal{M}$-kernel.

To see that any regular epimorphism in $\mathsf{RegEpi}(\mathbb{C})$ is the coequaliser of its kernel star, consider a regular epimorphism $(e_0, e_1)\colon a \to b$ in $\mathsf{RegEpi}(\mathbb{C})$. The induced arrow $c\colon Eq(e_1)^* \to Eq(e_0)^*$ is a regular epimorphism (again by Theorem 3.8 in [**10**]). If $((m_1, \kappa_1), (m_2, \kappa_2))\colon c \rightrightarrows a$ is the kernel star of (e_0, e_1), then (e_0, e_1) is the coequaliser of $((m_1, \kappa_1), (m_2, \kappa_2))\colon c \rightrightarrows a$, and $\mathsf{RegEpi}(\mathbb{C})$ is a star-regular category. If $\mathbb{C}$ has enough trivial objects, it is easy to verify that $\mathsf{RegEpi}(\mathbb{C})$ has enough trivial objects as well. □

Remark 4.3. It is not true, in general, that $\mathsf{RegEpi}(\mathbb{C})$ has a good theory of ideals when $\mathbb{C}$ is a category with a good theory of ideals (and enough trivial objects). For instance, in the total context, even when $\mathbb{C}$ is abelian, the category $\mathsf{RegEpi}(\mathbb{C})$ is regular but not *exact* Goursat (= with a good theory of ideals, in the total context).

From Lemma 4.2, and Corollary 2.4 in [**8**], we get

Corollary 4.4. *Let $\mathbb{C}$ be a category with a good theory of ideals and enough trivial objects. Then regular epimorphisms are effective descent morphisms in* $\mathsf{RegEpi}(\mathbb{C})$.

Under the assumptions of Corollary 4.4 it can be shown that, more generally, the categories $\mathsf{RegEpi}(\mathbb{C})^n$ of n-fold regular epimorphisms in $\mathbb{C}$ are star-regular with enough trivial objects (see also Proposition 3.1 in [**8**]).

4.3. Monomorphisms in a semi-effective star-regular category

Let us denote by $\mathsf{Mono}(\mathbb{C})$ the category of monomorphisms in a category $\mathbb{C}$. An object in $\mathsf{Mono}(\mathbb{C})$ is a monomorphism $m\colon M_1 \to M_0$ in $\mathbb{C}$ and a morphism $f\colon m \to n$ in $\mathsf{Mono}(\mathbb{C})$ is a pair (f_0, f_1) of morphisms of $\mathbb{C}$ with $f_0\colon M_0 \to N_0$ and $f_1\colon M_1 \to N_1$ such that $f_0 m = n f_1$. When $(\mathbb{C}, \mathcal{N})$ is a regular multi-pointed category, let us denote by $\mathcal{M}$ the class of morphisms $k = (k_0, k_1)$ of $\mathsf{Mono}(\mathbb{C})$ such that k_0 and k_1 are in $\mathcal{N}$. In this way, $(\mathsf{Mono}(\mathbb{C}), \mathcal{M})$ becomes a regular multi-pointed category as well. The reader will find it easy to verify the following

Lemma 4.5. *Let $(\mathbb{C}, \mathcal{N})$ be a semi-effective star-regular category with enough trivial objects. Then $\mathsf{Mono}(\mathbb{C})$ is a semi-effective star-regular category with enough trivial objects.*

As a consequence, under the assumptions of the lemma above, the regular epimorphisms are effective descent morphisms in $\mathsf{Mono}(\mathbb{C})$ (see also Example 4.4 in [**8**]).

4.4. Almost abelian categories

Another class of examples to which Theorem 3.2 applies is provided by the so-called "almost abelian categories" in the sense of W. Rump [**24**], also called "Raikov semi-abelian" [**22**] in the literature. An almost abelian category can be defined as an additive category with kernels and cokernels with the property that normal epimorphisms are pullback-stable and normal monomorphisms are pushout-stable. As explained in [**23**], G. Janelidze has observed that a category $\mathbb{C}$ is almost abelian if and only if it is both homological (in the sense of F. Borceux and D. Bourn [**2**]) and co-homological. It is well known that, in an almost abelian category, any arrow $f\colon A \to B$ has a canonical factorisation

$$\begin{array}{ccc} A & \xrightarrow{\;f\;} & B \\ {\scriptstyle \mathsf{coker}(\mathsf{ker}(f))}\downarrow & & \uparrow{\scriptstyle \mathsf{ker}(\mathsf{coker}(f))} \\ \mathsf{Coim}(f) & \xrightarrow[\;\overline{f}\;]{} & \mathsf{Im}(f) \end{array} \tag{2}$$

with $\overline{f}\colon \mathsf{Coim}(f) \to \mathsf{Im}(f)$ a bimorphism, i.e., an arrow which is at the same time monic and epic. Any almost abelian category $\mathbb{C}$ is normal (it is even homological), and we are now going to prove that it is also semi-effective star-regular (with respect to the class $\mathcal{N}$ of zero arrows). For this, consider a commutative diagram

$$\begin{array}{ccc} R^* & \xrightarrow{\quad n \quad} & X, \\ & {\scriptstyle i}\searrow \quad \nearrow{\scriptstyle k} & \\ & \mathsf{Im}(n) & \end{array}$$

where $n\colon R^* \to X$ is the (Bourn-)normal monomorphism yielding the 0-class of an equivalence relation R, $k = \mathsf{ker}(\mathsf{coker}(n))$ is the kernel of the cokernel $\mathsf{coker}(n)$ of n, and i is a split monomorphism. Then, by factorising $n = k\overline{n}\mathsf{coker}(\mathsf{ker}(n))$ as in diagram (2), we see that the arrow $i = \mathsf{coker}(\mathsf{ker}(n))\overline{n}$ is an epimorphism, as a composite of two epimorphisms. It follows that i is an isomorphism, and $\mathbb{C}$ is a semi-effective star-regular category, as desired.

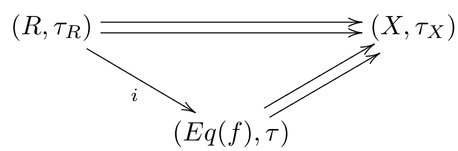
We observe that, more generally, any normal category $\mathbb{C}$ such that any arrow in $\mathbb{C}$ has a factorisation as an epimorphism followed by a normal monomorphism is semi-effective star-regular.

It is explained in [**24**] that any *torsion-free* subcategory of an abelian category $\mathbb{C}$ is necessarily almost abelian, as is any *torsion* subcategory of $\mathbb{C}$. Further examples of almost abelian categories are given, for instance, by the categories of real (or complex) *normed vector spaces*, *Banach spaces* (with bounded linear maps as morphisms), and also by the category of *locally compact abelian groups*. By Theorem 3.2 the regular epimorphisms are then effective descent morphisms in all these categories.

4.5. Categories of topological Mal'tsev algebras

Consider $\mathbb{T}$ a Mal'tsev theory, i.e., an algebraic theory containing a ternary term $p(x, y, z)$ satisfying the identities $p(x, x, y) = y$ and $p(x, y, y) = x$. The category $\mathbb{T}(\mathsf{Top})$ of topological models of such a theory (i.e., models in the category Top of topological spaces) is called a category of *topological Mal'tsev algebras*. The category $\mathbb{T}(\mathsf{Top})$ is a regular category, as shown in [**20**]. We now prove that $\mathbb{T}(\mathsf{Top})$ is semi-effective star-regular (thinking of $(\mathbb{T}(\mathsf{Top}), \mathcal{N})$ as a star-regular category with $\mathcal{N}$ the ideal of all morphisms).

When $(R, \tau_R) \rightrightarrows (X, \tau_X)$ is an equivalence relation in $\mathbb{T}(\mathsf{Top})$, and we consider a commutative diagram



with the property that i is a split monomorphism, then the topology of (R, τ_R) is the one induced by the topology of the product $(X \times X, \tau_{X \times X})$. Accordingly, the equivalence relation (R, τ_R) is the kernel pair of its coequaliser. Hence, the category $\mathbb{T}(\mathsf{Top})$ is semi-effective star-regular and, by Theorem 3.2, every regular epimorphism is an effective descent morphism in $\mathbb{T}(\mathsf{Top})$ (this result is known, see [**11**] for instance, although the proof presented here is different).

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