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SPACES OF TOPOLOGICAL COMPLEXITY ONE

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Abstract

We prove that a space whose topological complexity equals 1 is homotopy equivalent to some odd-dimensional sphere. We prove a similar result, although not in complete generality, for spaces X whose higher topological complexity $\mathsf{TC}_n(X)$ is as low as possible, namely $n-1$.

1. Introduction

The *topological complexity* of a space is a numerical homotopy invariant, of Lusternik-Schnirelmann type, introduced by Farber [**6**] and motivated by the motion planning problem in the field of topological robotics.

Here, we use $\mathrm{cat}(X)$ to denote the Lusternik-Schnirelmann (L-S) category of X (normalized, so that $\mathrm{cat}(S^n) = 1$), and we use $\mathrm{secat}(p)$ to denote the sectional category of a fibration p (normalized, so that $\mathrm{secat}(p) = 0$ when p admits a section). Then we define $\mathsf{TC}(X)$, the *topological complexity* of X, to be the sectional category $\mathrm{secat}\big(P_2\big)$ of the fibration $P_2\colon PX \to X \times X$, which evaluates a (free) path in X at its initial and final points. See the next section for a review of terminology and precise, more verbose definitions.

The basic inequalities that relate $\mathrm{cat}(-)$ and $\mathsf{TC}(-)$ are

$$\mathrm{cat}(X) \leqslant \mathsf{TC}(X) \leqslant \mathrm{cat}(X \times X). \tag{1}$$

It follows from the definition that we have $\mathrm{cat}(X) = 0$ exactly when X is contractible. It is also easy to show that $\mathsf{TC}(X) = 0$ exactly when X is contractible. In this paper, we consider the next step, namely when these invariants equal 1. As is well-known, $\mathrm{cat}(X) = 1$ corresponds to the case in which X is a co-H-space. This is a large class of spaces which includes all suspensions. In addition there are well-known examples of co-H-spaces that are not suspensions. By contrast, we find that the class of spaces with $\mathsf{TC}(X) = 1$ is very restrictive. By inequality (1), if $\mathsf{TC}(X) = 1$, then X must be a co-H-space, i.e., we must have $\mathrm{cat}(X) = 1$. Further, we show the following:

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Theorem 1.1 (Corollary 3.5)**.** *Let X be a path-connected CW complex of finite type. If $\mathsf{TC}(X) = 1$, then X is homotopy equivalent to some sphere S^{2r+1} of odd dimension, $r \geqslant 0$.*

The converse of Theorem 1.1 is also true: for $r \geqslant 0$, we have $\mathsf{TC}(S^{2r+1}) = 1$ [**6**, Th. 8] (note that [**6**] uses "un-normalized" $\mathsf{TC}(-)$, which is one more than our $\mathsf{TC}(-)$).

If we also assume that X is a closed manifold, then Theorem 1.1, together with the positive solution to the topological Poincaré conjecture, yield the following:

Corollary 1.2. *If X is a closed manifold with $\mathsf{TC}(X) = 1$, then X is homeomorphic to some sphere of odd dimension.*

We also consider the "higher analogues" of topological complexity introduced by Rudyak in [**12**] (see also [**1**] and [**13**]). This notion may also be motivated by a motion planning problem of a constrained type (see [**12**, Rem. 3.2.7]). For $n \geqslant 3$, we define $\mathsf{TC}_n(X)$, the *higher topological complexity* of X, as the sectional category $\mathrm{secat}(P_n)$ of the fibration $P_n \colon PX \to X^n$, which evaluates a (free) path in X not only at its initial and final points, but also at $(n-2)$ equally timed intermediate points as well. Again, see the next section for full definitions. As a matter of notation, we may write $\mathsf{TC}(X) = \mathsf{TC}_2(X)$.

The inequalities (1) extend to the following ([**1**, Cor. 3.3], [**10**]):

$$\mathrm{cat}(X^{n-1}) \leqslant \mathsf{TC}_n(X) \leqslant \mathrm{cat}(X^n) \tag{2}$$

for $n \geqslant 2$. Now, if X is not contractible, then $\mathrm{cat}(X^{n-1}) \geqslant n-1$ [**2**, Th. 1.47]. Therefore, the next step, in the spirit of our first result, is to consider the case in which $\mathsf{TC}_n(X) = n-1$. For $n \geqslant 3$, there are some subtleties that arise, because of the non-straightforward way in which $\mathrm{cat}(-)$ may behave with respect to products. We prove a result (Theorem 3.4) that substantially handles this situation and in a way that very naturally extends Theorem 1.1. For instance, our result includes the following:

Theorem 1.3. *Let X be a path-connected CW complex of finite type. Suppose that we have $\mathsf{TC}_n(X) = n-1$ for some $n \geqslant 3$. If X is simply connected, then X is homotopy equivalent to some odd-dimensional sphere S^{2r+1}, $r \geqslant 1$. If $\pi_1(X) \neq \{e\}$, and X is a nilpotent space, then X is homotopy equivalent to the circle S^1.*

Conversely, it is known that, for $r \geqslant 0$, we have $\mathsf{TC}_n(S^{2r+1}) = n-1$ ([**12**, Sec.4]—note that un-normalized $\mathsf{TC}(-)$ is used there). This completely describes the situation in the nilpotent case. Notice that our results imply that, for X nilpotent, if $\mathsf{TC}_n(X) = n-1$ for some $n \geqslant 2$, then we have $\mathsf{TC}_n(X) = n-1$ *for all* $n \geqslant 2$. Our actual results do give partial information about the general, non-nilpotent situation (see Theorem 3.4 for details). Once more, if we also assume that X is a closed manifold, then we may replace "homotopy equivalent" in the conclusions of Theorem 1.3 by "homeomorphic."

From our results, we precisely identify how (higher) topological complexity behaves for co-H-spaces:

Corollary 1.4 (Corollary 3.6)**.** *Let X be a non-contractible, path-connected CW complex of finite type. If X is a co-*H*-space, then either (a) X is of the homotopy type of some odd-dimensional sphere, and we have $\mathsf{TC}_n(X) = n-1$ for all $n \geqslant 2$; or (b) we have $\mathsf{TC}_n(X) = n$ for all $n \geqslant 2$.*

Possibility (b) of this result includes, for example, any even-dimensional sphere ([**6**, Th. 8], [**12**, Sec. 4]), and any simply connected Moore space of type $M(G,k)$ with G a finite abelian group. Also, with some exceptions, wedges of co-H-spaces, or suspensions of finite-type CW complexes provide many more examples.

The paper is organized as follows: In Section 2 we review basic definitions and vocabulary and establish two intermediate results: Proposition 2.1 is basic for what follows; Proposition 2.2 gives an interesting lower bound for $\mathrm{cat}(X^n)$. In Section 3, we prove our main result about $\mathsf{TC}_n(X) = n-1$, from which we conclude our result about $\mathsf{TC}(X) = 1$.

2. Definitions and preliminary results

We refer to [**2**] for a general introduction to L-S category and related topics such as sectional category. Here, we recall that $\mathrm{cat}(X)$ is the smallest n for which there is an open covering $\{U_0, \dots, U_n\}$ by $(n+1)$ open sets, each of which is contractible in X. The *sectional category* of a fibration $p\colon E \to B$, denoted by $\mathrm{secat}(p)$, is the smallest number n for which there is an open covering $\{U_0, \dots, U_n\}$ of B by $(n+1)$ open sets, for each of which there is a local section $s_i\colon U_i \to E$ of p, so that $p \circ s_i = j_i\colon U_i \to B$, where j_i denotes the inclusion.

Let PX denote the space of (free) paths on a space X. There is a fibration $P_2\colon PX \to X \times X$, which evaluates a path at initial and final point: for $\alpha \in PX$, we have $P_2(\alpha) = \big(\alpha(0), \alpha(1)\big)$. This is a fibrational substitute for the diagonal map $\Delta\colon X \to X \times X$. We define the *topological complexity* $\mathsf{TC}(X)$ of X to be the sectional category $\mathrm{secat}\big(P_2\big)$ of this fibration. That is, $\mathsf{TC}(X)$ is the smallest number n for which there is an open cover $\{U_0, \dots, U_n\}$ of $X \times X$ by $(n+1)$ open sets, for each of which there is a local section $s_i\colon U_i \to PX$ of P_2, i.e., for which $P_2 \circ s_i = j_i\colon U_i \to X \times X$, where j_i denotes the inclusion.

More generally, let $n \geqslant 2$ and consider the fibration

$$P_n\colon PX \to X \times \cdots \times X = X^n,$$

defined by dividing the unit interval $I = [0,1]$ into $(n-1)$ subintervals of equal length, with n subdivision points $t_0 = 0, t_1 = 1/(n-1), \dots, t_{n-1} = 1$ (thus $(n-2)$ subdivision points interior to the interval), and then evaluating at each of the n subdivision points. Thus,

$$P_n(\alpha) = \big(\alpha(0), \alpha(t_1), \dots, \alpha(t_{n-2}), \alpha(1)\big)$$

for $\alpha \in PX$. This is a fibrational substitute for the n-fold diagonal $\Delta_n\colon X \to X^n$. Then the *higher topological complexity* $\mathsf{TC}_n(X)$ is defined as $\mathsf{TC}_n(X) = \mathrm{secat}(P_n)$.

Let $H_*(X)$, respectively $\widetilde{H}_*(X)$, denote homology, respectively reduced homology, with integer coefficients. By $\dim_{\Bbbk}\big(\widetilde{H}^*(X;\Bbbk)\big)$, we mean $\sum_{i\geqslant 1} \dim_{\Bbbk}\big(H^i(X;\Bbbk)\big)$ for path-connected X. In this paper, by an *integral homology sphere*, we mean a space X with integral homology isomorphic to that of S^n for some $n \geqslant 1$. (Note that, here, we do not implicitly assume that X is a manifold.) By a CW complex of *finite type*, we mean one that has finitely many cells of each dimension. Note that a CW complex of finite type has the integral homology group $H_i(X)$ a finitely generated abelian group, for each i. In the proof of the following result, and in the sequel, we make use of the universal coefficient theorem for cohomology (UCT), as given in [**8**, Th. 3.2], for instance.

Proposition 2.1. *Let X be a path-connected CW complex of finite type. Suppose that $\dim_{\Bbbk}\big(\widetilde{H}^*(X;\Bbbk)\big) \leqslant 1$ for all choices of field $\Bbbk$. Then either X is acyclic, or X is an integral homology sphere.*

Proof. If X is acyclic, then $\widetilde{H}_*(X) = 0$ and hence $\widetilde{H}^*(X;\Bbbk) = 0$ for all choices of field $\Bbbk$. So suppose that X is not acyclic, and let $H_r(X)$ be the first non-trivial homology group of X, $r \geqslant 1$. Since X is of finite type, we may write

$$H_r(X) \cong \mathbb{Z}^n \qquad \text{or} \qquad H_r(X) \cong \mathbb{Z}^n \oplus \mathbb{Z}/p^k \oplus \mathbb{Z}/p_1^{k_1} \oplus \cdots \oplus \mathbb{Z}/p_\ell^{k_\ell}$$

for some rank $n \geqslant 0$, primes $p \leqslant p_1 \leqslant \cdots \leqslant p_\ell$ and natural numbers $k, k_1, \ldots, k_\ell$.

First suppose that the torsion part of $H_r(X)$ is non-trivial, so that at least the summand $\mathbb{Z}/p^k$ is non-zero. By the UCT we have

$$\begin{aligned} H^r(X;\mathbb{Z}/p) &\cong \mathrm{Hom}(H_r(X),\mathbb{Z}/p) \cong \mathbb{Z}/p \oplus S, \\ H^{r+1}(X;\mathbb{Z}/p) &\supseteq \mathrm{Ext}(H_r(X),\mathbb{Z}/p) \cong \mathbb{Z}/p \oplus T, \end{aligned}$$

where S and T are some finite $\mathbb{Z}/p$-vector spaces. It then follows that we have $\dim_{\mathbb{Z}/p}\big(\widetilde{H}^*(X;\mathbb{Z}/p)\big) \geqslant 2$, which contradicts our assumption. Thus $H_r(X)$ is torsion-free.

Now suppose that $H_r(X) \cong \mathbb{Z}^n$ with rank $n \geqslant 2$. Then $H^r(X;\mathbb{Q}) \cong \mathbb{Q}^n$ is of dimension at least 2, which again contradicts our assumption. We conclude that $H_r(X) \cong \mathbb{Z}$.

Now consider homology groups in higher degrees, starting with $H_{r+1}(X)$. Because $H_r(X) \cong \mathbb{Z}$, a similar argument to the above, using the UCT and then rational coefficients, shows that $H_{r+1}(X) = 0$. Then, arguing inductively, one sees that $H_i(X) = 0$ for all $i > r$, and thus X is an integral homology r-sphere. □

Our next result seems of interest in its own right, as a general statement about the L-S category of products. In its proof, we use the notion of the *category weight* of a cohomology class, which is commonly used to obtain lower bounds on L-S category. A general discussion of this notion is given in [**2**, Sec. 2.7, Sec. 8.3].

Proposition 2.2. *Let X be a path-connected CW complex whose fundamental group has a non-trivial element of finite order. Then we have $\mathrm{cat}(X^n) \geqslant 2n$, for each $n \geqslant 1$.*

Proof. By assumption, $\pi_1(X)$ has an element of prime order. Hence, we may choose a cover Y of X whose fundamental group is the cyclic group $\mathbb{Z}/p$, with p a prime. Note that we then also have $H_1(Y) \cong \mathbb{Z}/p$. We look at the long exact cohomology sequence associated to the short exact sequence of coefficients

$$0 \longrightarrow \mathbb{Z}/p \xrightarrow{\times p} \mathbb{Z}/p^2 \xrightarrow{r_p} \mathbb{Z}/p \longrightarrow 0$$

in which r_p denotes reduction mod p (see, e.g. [**8**, Sec. 3.E]). From the UCT, we have $H^1(Y;\mathbb{Z}/p^2) \cong \mathrm{Hom}(\mathbb{Z}/p,\mathbb{Z}/p^2) \cong \mathbb{Z}/p$ (no finiteness assumptions on Y are required here), and likewise $H^1(Y;\mathbb{Z}/p) \cong \mathbb{Z}/p$.

The map $H^1(Y;\mathbb{Z}/p^2) \to H^1(Y;\mathbb{Z}/p)$ induced by r_p is zero. Therefore the Bockstein $\beta\colon H^1(Y;\mathbb{Z}/p) \to H^2(Y;\mathbb{Z}/p)$ is injective. Set $y = \beta(x) \in H^2(Y;\mathbb{Z}/p)$, where $x \in H^1(Y;\mathbb{Z}/p)$ is a generator. By [**11**, Cor. 4.7] (see also [**5**]), the class y has (strict, or essential) category weight at least 2. (Actually, [**11**, Cor. 4.7] is stated for odd primes p. But if $p = 2$, then we have $\beta(x) = Sq^1(x) = x \cup x$, which is certainly of weight 2.)

Now note that the cross product $y \times y = p_1^*(y) \cup p_2^*(y) \in H^4(Y \times Y; \mathbb{Z}/p)$ has category weight at least 4. This follows from standard properties of category weight, as summarized, for example, in [**2**, Prop. 8.22]: Here, $p_1, p_2 \colon Y \times Y \to Y$ denote the projections onto either factor, and we denote by $\mathrm{wgt}(u)$ the (strict, or essential) category weight of a cohomology class u. Then we have

$$\begin{aligned}\mathrm{wgt}(y \times y) &= \mathrm{wgt}\big(p_1^*(y) \cup p_2^*(y)\big) \geqslant \mathrm{wgt}\big(p_1^*(y)\big) + \mathrm{wgt}\big(p_2^*(y)\big)\\ &\geqslant \mathrm{wgt}(y) + \mathrm{wgt}(y) = 4.\end{aligned}$$

That the cross product $y \times y$ is nonzero follows from [**3**, VII.Ex. 7.15(1)] since $\mathbb{Z}/p$ is a field. By an easy inductive argument, we also have that the n-fold cross product $y \times \cdots \times y \in H^{2n}(Y^n; \mathbb{Z}/p)$ is nonzero and has category weight at least $2n$.

Therefore, by [**2**, Prop. 8.22], we have $\mathrm{cat}(Y^n) \geqslant 2n$. Hence, since Y^n covers X^n, and therefore $\mathrm{cat}(Y^n) \leqslant \mathrm{cat}(X^n)$ (see [**2**, Cor. 1.45]), we have $\mathrm{cat}(X^n) \geqslant 2n$ also. □

Example 2.3. Let P denote the Poincaré homology 3-sphere and P^* denote its 2-skeleton. Then P is an integral homology 3-sphere, and P^* is an acyclic space. One might wonder whether $\mathsf{TC}_n(P) = n - 1$ or $\mathsf{TC}_n(P^*) = n - 1$ for some n. However, $\pi_1(P) \cong \pi_1(P^*)$ is not torsion-free—it is a finite group of order 120, in fact. Hence Proposition 2.2 implies that both $\mathrm{cat}\big((P)^{n-1}\big)$ and $\mathrm{cat}\big((P^*)^{n-1}\big)$ are at least $2n - 2$, for each $n \geqslant 2$, and thus $\mathsf{TC}_n(P), \mathsf{TC}_n(P^*) \geqslant 2n - 2 > n - 1$, for each $n \geqslant 2$.

3. Spaces of lowest possible (higher) topological complexity

We begin by recalling the standard cohomological lower bound for $\mathsf{TC}_n(-)$, which will be used in the sequel.

Definition 3.1. Let $\Bbbk$ be a field. The homomorphism induced on cohomology with coefficients in $\Bbbk$ by the n-fold diagonal $\Delta_n \colon X \to X^n$ (and thus by $P_n \colon PX \to X^n$, which is a fibrational substitute for it) may be identified with the n-fold cup product homomorphism

$$\cup_n(X) \colon H^*(X; \Bbbk) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} H^*(X; \Bbbk) \to H^*(X; \Bbbk).$$

The *ideal of n-fold zero divisors* is $\ker \cup_n (X)$, the kernel of $\cup_n(X)$. The *n-fold zero-divisors cup-length* is $\mathrm{nil}\big(\ker \cup_n (X)\big)$, the nilpotency of this ideal, which is to say the number of factors in the longest non-trivial product of elements from this ideal.

Proposition 3.2 ([**1**, Th. 3.9], [**6**, Th. 7], [**12**, Prop. 3.4])**.** *For any field $\Bbbk$, we have $\mathrm{nil}\big(\ker \cup_n (X)\big) \leqslant \mathsf{TC}_n(X)$.* □

For an element $a \in \widetilde{H}^*(X; \Bbbk)$, we write $\bar{a} = a \otimes 1 - 1 \otimes a \in H^*(X; \Bbbk) \otimes H^*(X; \Bbbk)$. Clearly, $\bar{a}$ is a non-zero element in the ideal of 2-fold zero divisors. We adopt notation from the proof of [**1**, Th. 3.14] to describe certain n-fold zero divisors. For $i = 1, \ldots, n$, let $p_i \colon X^n \to X$ denote projection on the nth factor. Then we write $a_i = (p_i)^*(a) \in H^*(X^n; \Bbbk)$, which we regard as an element of $H^*(X; \Bbbk)^{\otimes n}$ under the identification $H^*(X^n; \Bbbk) \cong H^*(X; \Bbbk)^{\otimes n}$, namely, the Künneth theorem. Then we have the $n - 1$

elements $\{a_1 - a_2, a_1 - a_3, \ldots, a_1 - a_n\}$, each of which is an n-fold zero divisor in $H^*(X)^{\otimes n}$.

Lemma 3.3. *Suppose we have $a, b \in H^*(X)$. With the above notation, for $n \geqslant 2$, we have*

$$(a_1 - a_2)(a_1 - a_3)\cdots(a_1 - a_n) \equiv (-1)^n\big(a \otimes 1 - 1 \otimes a\big) \otimes a \otimes \cdots \otimes a$$

modulo terms in the ideal of $H^*(X)^{\otimes n}$ generated by the elements $a^2 \otimes 1 \otimes \cdots \otimes 1$ and $a \otimes \cdots \otimes a \otimes 1$. Consequently, we have

$$(b_1 - b_2)(a_1 - a_2)(a_1 - a_3)\cdots(a_1 - a_n) \\ \equiv (-1)^{n+1}\big(b \otimes a + (-1)^{|a||b|} a \otimes b\big) \otimes a \otimes \cdots \otimes a$$

modulo terms in the ideal of $H^*(X)^{\otimes n}$ generated by the elements $a^2 \otimes 1 \otimes \cdots \otimes 1$, $ba \otimes 1 \otimes \cdots \otimes 1$, and $1 \otimes ba \otimes 1 \otimes \cdots \otimes 1$.

Proof. We proceed by induction, with the induction hypothesis that, for $2 \leqslant k \leqslant n$, we have

$$(a_1 - a_2)(a_1 - a_3)\cdots(a_1 - a_k) \equiv (-1)^k\big(a \otimes 1 - 1 \otimes a\big) \otimes a \otimes \cdots \otimes a \otimes 1 \otimes \cdots \otimes 1$$

modulo terms in the ideal I_k of $H^*(X)^{\otimes n}$ generated by the elements $a^2 \otimes 1 \otimes \cdots \otimes 1$ and $a \otimes \cdots \otimes a \otimes 1 \cdots \otimes 1$, where we have $(k-1)$ occurrences of a in each term of the displayed element and in the latter ideal generator. Induction starts with $k = 2$, where there is nothing to prove. For the induction step, we use the induction hypothesis to write $\big((a_1 - a_2)(a_1 - a_3)\cdots(a_1 - a_k)\big)(a_1 - a_{k+1})$ as

$$\equiv (-1)^k\big((a \otimes 1 - 1 \otimes a) \otimes a \otimes \cdots \otimes a \otimes 1 \otimes \cdots \otimes 1\big)\big(a \otimes 1 \otimes \cdots \otimes 1\big) \\ - (-1)^k\big((a \otimes 1 - 1 \otimes a) \otimes a \otimes \cdots \otimes a \otimes 1 \otimes \cdots \otimes 1\big)\big(1 \otimes \cdots \otimes 1 \otimes a \otimes 1 \otimes \cdots \otimes 1\big).$$

The first part of this expression contributes

$$\pm(a^2 \otimes 1 \pm a \otimes a) \otimes a \otimes \cdots \otimes a \otimes 1 \otimes \cdots \otimes 1,$$

which is in the ideal I_{k+1}. The second part contributes

$$(-1)^{k+1}\big(a \otimes 1 - 1 \otimes a\big) \otimes a \otimes \cdots \otimes a \otimes 1 \otimes \cdots \otimes 1,$$

with k occurrences of a in each term. This completes the induction step, and the first assertion follows.

For the second assertion, observe that we may write

$$(b_1 - b_2)(a_1 - a_2)(a_1 - a_3)\cdots(a_1 - a_n),$$

from the first part, as congruent to

$$(-1)^n\big((b \otimes 1 - 1 \otimes b)(a \otimes 1 - 1 \otimes a)\big) \otimes a \otimes \cdots \otimes a$$

modulo terms in the ideal $\big((b \otimes 1 - 1 \otimes b) \otimes 1 \otimes \cdots \otimes 1\big)I_n$. But now it is clear that the only contribution outside the ideal generated by $a^2 \otimes 1 \otimes \cdots \otimes 1$, $ba \otimes 1 \otimes \cdots \otimes 1$, and $1 \otimes ba \otimes 1 \otimes \cdots \otimes 1$ is as asserted. □

Theorem 3.4. *Let X be a path-connected CW complex of finite type. If $\mathsf{TC}_n(X) = n - 1$, for some $n \geqslant 2$, then $\pi_1(X)$ is torsion-free and either X is acyclic or X is an odd-dimensional integral homology sphere. Furthermore, we have:*

(A) If X is simply connected, then for some $r \geqslant 1$ we have $X \simeq S^{2r+1}$;

(B) If $\pi_1(X) \neq \{e\}$ and if X is a nilpotent space, then we have $X \simeq S^1$; and

(C) If $\pi_1(X) \neq \{e\}$, and if X is a co-H-space, then we have $X \simeq S^1$.

Proof. By combining Proposition 2.2 with the first inequality of (2), we conclude that $\pi_1(X)$ must be torsion-free.

Next, we show that X satisfies the hypotheses of Proposition 2.1. For this, we argue by contradiction. Suppose that, for some field $\Bbbk$, we have $a \in H^r(X;\Bbbk)$ and $b \in H^s(X;\Bbbk)$ with $r, s > 0$, and $\{a, b\}$ linearly independent over $\Bbbk$. By Lemma 3.3, the n-fold product of n-fold zero-divisors

$$(b_1 - b_2)(a_1 - a_2)(a_1 - a_3)\cdots(a_1 - a_n)$$

is congruent to

$$(-1)^{n+1}\big(b \otimes a + (-1)^{|a||b|} a \otimes b\big) \otimes a \otimes \cdots \otimes a$$

modulo terms in the ideal of $H^*(X)^{\otimes n}$ generated by the elements $a^2 \otimes 1 \otimes \cdots \otimes 1$, $ba \otimes 1 \otimes \cdots \otimes 1$, and $1 \otimes ba \otimes 1 \otimes \cdots \otimes 1$. It follows that this term is nonzero, as a and b are linearly independent, and so we have $\mathsf{TC}_n(X) \geqslant n$ by Proposition 3.2, which is a contradiction.

Therefore, X must satisfy the hypothesis of Proposition 2.1, and either X is acyclic, or X is an integral homology sphere. If X is an even-dimensional integral homology sphere, however, then once again by Lemma 3.3, the n-fold product of n-fold zero-divisors

$$(a_1 - a_2)(a_1 - a_2)(a_1 - a_3)\cdots(a_1 - a_n)$$

is congruent to

$$(-1)^{n+1}\big(a \otimes a + (-1)^{|a||a|} a \otimes a\big) \otimes a \otimes \cdots \otimes a = (-1)^{n+1}\, 2\, a \otimes \cdots \otimes a$$

modulo terms in the ideal of $H^*(X)^{\otimes n}$ generated by the elements $a^2 \otimes 1 \otimes \cdots \otimes 1$ and $1 \otimes a^2 \otimes 1 \otimes \cdots \otimes 1$. We may take rational coefficients here, for example, and then we have $\mathsf{TC}_n(X) \geqslant n$ by Proposition 3.2, which is again a contradiction. The only possibilities that remain, then, are that X is acyclic or X is an odd-dimensional integral homology sphere.

We treat the remaining cases separately:

(A) Assume that X is simply connected. Then X cannot be acyclic. Indeed, Whitehead's Theorem would then imply that X is contractible, and hence we would have $\mathsf{TC}_n(X) = 0$. Therefore, X is an odd-dimensional integral homology sphere. But any simply connected integral homology sphere is of the homotopy type of the sphere (of the same dimension), by the theorems of Hurewicz and Whitehead.

(B) Suppose that X is a nilpotent space with $\pi_1(X) \neq \{e\}$. Since $\pi_1(X)$ is nilpotent, we cannot have $H_1(X) = 0$. Therefore, the only possibility is that X is an integral homology circle. So let $j\colon S^1 \to X$ be a generator of $\pi_1(X)$ that, under the Hurewicz homomorphism $h\colon \pi_1(X) \to H_1(X)$, is mapped to the generator $1 \in H_1(X) \cong \mathbb{Z}$. Then we have $j_*\colon H_1(S^1) \to H_1(X)$ is an isomorphism, since both groups are isomorphic to $\mathbb{Z}$. However, $H_i(S^1)$ and $H_i(X)$ are both zero for $i \geqslant 2$, and thus $j\colon S^1 \to X$ is an integral homology equivalence. As both S^1 and X are nilpotent spaces, it follows from [**4**] (see also [**7**]) that $j\colon S^1 \to X$ is a homotopy equivalence.

(C) Finally, suppose that X is a co-H-space with $\pi_1(X) \neq \{e\}$. We claim that $\pi_1(X)$ must be isomorphic to $\mathbb{Z}$. For, as a co-H-space, X must have free fundamental group. Since X is of finite type, $\pi_1(X)$ must be a finitely-generated free group, and hence isomorphic to a free product of k copies of $\mathbb{Z}$, for some positive integer k. If $k \geqslant 2$, then the rational cohomology group $H^1(X;\mathbb{Q}) \cong \mathrm{Hom}(H_1(X);\mathbb{Q}) \cong \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Q}) \cong \mathbb{Q}^n$ has dimension at least 2, which contradicts the first part of the present theorem. Thus we have $\pi_1(X) \cong \mathbb{Z}$. Now, since X is a co-H-space that is a homology circle, it follows directly from [**9**, Corollary 2.2] that X is homotopy equivalent to the circle. Note that [**9**, Corollary 2.2] is stated for X finite, but it is a corollary to [**9**, Theorem 2.1] , which in turn is deduced as an immediate consequence of Theorem 4.4 of the same paper, which only requires X of finite type. □

Corollary 3.5. *Let X be a path-connected CW complex of finite type. If $\mathsf{TC}(X) = 1$, then X is homotopy equivalent to some sphere of odd dimension.*

Proof. The inequalities $\mathrm{cat}(X) \leqslant \mathsf{TC}(X) = 1$ imply that X is a co-H-space. The result follows from parts (A) and (C) of Theorem 3.4. □

Corollary 3.6. *Let X be a path-connected, non-contractible CW complex of finite type. If X is a co-*H*-space, then either (a) X is of the homotopy type of some odd-dimensional sphere, and we have $\mathsf{TC}_n(X) = n - 1$ for all $n \geqslant 2$; or (b) we have $\mathsf{TC}_n(X) = n$ for all $n \geqslant 2$.*

Proof. Recall the basic inequalities (2) from the introduction. From the remark immediately below them, we have $n - 1 \leqslant \mathsf{TC}_n(X)$. At the other end, we have $\mathsf{TC}_n(X) \leqslant \mathrm{cat}(X^n) \leqslant n\,\mathrm{cat}(X)$, from the usual product inequality for L-S category [**2**, Th. 1.37]. We assume that X is a co-H-space, so $\mathrm{cat}(X) = 1$ and, for each $n \geqslant 2$, we have

$$n - 1 \leqslant \mathsf{TC}_n(X) \leqslant n.$$

Either $\mathsf{TC}_n(X) = n - 1$ for some n, in which case we conclude as in (a) from parts (A) and (C) of Theorem 3.4, or we conclude as in (b). □

Remark 3.7. The combination of Proposition 2.2 and the first inequality of (2) actually implies the following: If $n - 1 \leqslant \mathsf{TC}_n(X) \leqslant 2n - 3$, then $\pi_1(X)$ is torsion-free. This fact suggests that merely requiring a small value of $\mathsf{TC}_n(X)$—as opposed to requiring that it equal the lower bound from the first inequality of (2)—already entails strong restrictions on the topology of a space.

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