POWER OPERATIONS IN ORBIFOLD TATE K-THEORY

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(*communicated by J. Daniel Christensen*)

Abstract
We formulate the axioms of an orbifold theory with power operations. We define orbifold Tate K-theory, by adjusting Devoto's definition of the equivariant theory, and proceed to construct its power operations. We calculate the resulting symmetric powers, exterior powers and Hecke operators and put our work into context with orbifold loop spaces, level structures on the Tate curve and generalized Moonshine.

1. Introduction

Tate K-theory,

$$K_{\text{Tate}}(X) = K(X)[[q]]$$

is a form of elliptic cohomology. The corresponding generalized elliptic curve is the Tate curve, and the corresponding elliptic genus is the Witten genus [**4**]. The most conceptual definition of the Tate curve is probably the characterization in [**13**, Thm. VII.2.1]: if $\mathcal{M}_{\text{ell}}$ is the algebraic stack of (nice enough) generalized elliptic curves,[1] then the completion of $\mathcal{M}_{\text{ell}}$ at infinity,

$$\text{spf } \mathbb{Z}[\![q]\!] \longrightarrow \mathcal{M}_{\text{ell}},$$

classifies a formal projective curve. This possesses an algebraization, living over $\text{spec } \mathbb{Z}[\![q]\!]$, which we will call $\text{Tate}(q)$. A more pedestrian, but equivalent, formulation of this definition of $\text{Tate}(q)$ can be found in [**4**].

Out of all known elliptic cohomology theories, K_{Tate} might be the one where the conjectural relationship to string theory is best understood, and Witten's original definition of his genus involves loop spaces.

The definition of G-equivariant Tate K-theory for finite G goes back to Devoto [**14**] and is modeled on the loop space of a global quotient orbifold. Our definition of

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[1]We write $\mathcal{M}_{\text{ell}}$ for the stack denoted $\mathcal{M}_1$ in [**13**, III.2.6].

orbifold Tate K-theory is a mild modification of Devoto's. Devoto's work also already makes the connection to level structures on elliptic curves.

Guided by the ideas in [**15**], we define power operations on K_{Tate}. This project grew out of the desire for a more explicit link between the findings in [**16**] and the string theoretic picture in [**15**] and was inspired by the similarities between [**15**] and [**24**, **34**].

The story in [**16**] is one of convergence: there, we saw how the purely homotopy theoretic formalism of power operations in elliptic cohomology (the theory there is E_2) results in a formula much resembling that of Dijkgraaf, Moore, Verlinde and Verlinde.

Here, we take a more traditional approach to mathematical physics, modeling the mathematical definitions on the calculations of physicists. The resulting power operations are closely linked to level structures and isogenies on the Tate curve, suggesting that this is the 'right' definition of elliptic power operations on K_{Tate}. To make precise exactly what is meant by 'right' in this context, one has to revisit the work of Matthew Ando, his power operations on K_{Tate}, and the definition of *elliptic power operations* in [**5**]. All this was pioneered in the work of Andrew Baker, who was the first to define Hecke operators on elliptic cohomology [**8**, **7**] and [**9**].

The formalism of power operations leads to a number of internal operations, acting on the Tate K-theory of any space. In particular, it yields *symmetric powers*

$$\text{Sym}_{\text{Tate}}^n \colon K_{\text{Tate}}(X) \longrightarrow K_{\text{Tate}}(X).$$

We will write

$$S_t^{\text{Tate}}(x) := \sum_{n=o}^{\infty} \text{Sym}_{\text{Tate}}^n(x)\, t^n$$

for the *total symmetric power*. Here t is a formal variable. On the coefficients of K_{Tate}, the total symmetric power satisfies the identity

$$S_t^{\text{Tate}}(x) = \exp\left(\sum_{m \geqslant 1} T_m(x) t^m\right), \tag{1}$$

where the T_m are the equivariant Hecke operators of generalized Moonshine defined in [**17**, **29**] and prominent in Carnahan's proof of the generalized Moonshine Conjectures [**12**]. The identity (1) gives rise to a new formulation of the replicability condition, although the latter remains mysterious. On the other hand, the total symmetric power of a vector bundle is the stable exponential characteristic class

$$S_t^{\text{Tate}}(V) = \bigotimes_{k \geqslant 1} S_{t^k}(V), \tag{2}$$

used to define the Witten genus. Here S_t is the total symmetric power in K-theory. The Witten genus is, of course, an important player in Moonshine, because of the Hirzebruch conjecture [**21**, **25**].

In the context of replicability, the variable t in (1) lives a peculiar and poorly understood double life as a formal variable on one hand and as the modular variable $e^{2\pi i\tau}$ on the other. In the context of (2), the same double life occurs, and a compelling explanation for this phenomenon, involving the boundary of a Krichever

style[2] construction of the relevant moduli space, goes back to unpublished work of Looijenga [**10**].

Acknowledgements

2.   Orbifold Tate K-theory

We will work in the 2-category Gpd whose objects are the (small) topological groupoids and with

$$1\mathrm{Hom}(X,Y) = \mathrm{Fun}(X,Y)$$

the groupoid of continuous functors from X to Y and continuous natural isomorphisms between them. We do not emphasize the 2-category point of view. For all practical purposes, we could as well work in the category

$$\mathrm{Gpd} = \mathrm{Gpd}\,/\cong,$$

obtained from Gpd by identifying naturally isomorphic 1-morphisms, or even in Moerdijk's orbifold category Orb (see [**28**]). The *center* of a groupoid X is the group

$$\mathrm{Center}(X) := 2\mathrm{Hom}\,(\mathrm{Id}_X, \mathrm{Id}_X) = \mathrm{Nat}\,(\mathrm{Id}_X, \mathrm{Id}_X),$$

of natural transformations from Id_x to Id_x. We will also need the 2-category $\mathrm{Gpd}^{\mathrm{cen}}$ whose objects are pairs (X,ξ) with ξ a center element of X and with

$$1\mathrm{Hom}((X,\xi),(Y,\upsilon)) \subset \mathrm{Fun}(X,Y)$$

the full subcategory of functors f satisfying

$$f\xi = \upsilon f.$$

Lemma 2.1. *If $\eta\colon f \cong f'$ is a natural isomorphism and $f\xi = \upsilon f$, then $f'\xi = \upsilon f'$.*

Proof. For any $x \in \mathrm{ob}(X)$, we have

$$\upsilon_{f'(x)} = \eta_{f(x)} \circ \upsilon_{f(x)} \circ \eta_{f(x)}^{-1} = \eta_{f(x)} \circ f(\xi_x) \circ \eta_{f(x)}^{-1} = f'(\xi_x). \qquad \square$$

For simplicity of exposition, all center elements are assumed to be of finite order. This is automatically the case if X is an orbifold groupoid in the sense of Moerdijk. If G is a finite group and M is a G-space, then we write $M/\!\!/G$ for the corresponding translation groupoid.

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[2]Cutting along the circle $[0,1]$ inside $\mathbb{C}/\langle\tau,1\rangle$, instead of Krichever's small circle as in [**31**, 8.11].

Example 2.2. The group

$$\mathrm{Center}(\mathrm{pt}/\!/G) = \mathrm{Center}(G)$$

is the center of the group G.

Example 2.3. The inertia groupoid

$$\Lambda X = \mathrm{Fun}(\mathrm{pt}/\!/\mathbb{Z}, X)$$

of X admits a group homomorphism

$$\mathbb{Z} \longrightarrow \mathrm{Center}(\Lambda X)$$
$$k \longmapsto \left(\xi^k \colon f \mapsto f(k) \right).$$

Viewing objects of ΛX as pairs (x, g) with $x \in \mathrm{ob}(X)$ and $g \in \mathrm{aut}(x)$, we have

$$\xi^k_{(x,g)} = g^k$$

(arrow in ΛX).

For any $k \in \mathbb{Z}$, we then have the 2-functor

$$\mathrm{Gpd} \longrightarrow \mathrm{Gpd}^{\mathrm{cen}}$$
$$X \longmapsto (\Lambda X, \xi^k).$$

Example 2.4. In the global quotient case, we have

$$\Lambda(M/\!/G) \simeq \coprod_{[g]} M^g /\!/ C_g \quad \text{and} \quad \xi^k|_{M^g} = g^k.$$

Here $[g]$ is the conjugacy class, M^g is the fixed point locus, and C_g is the centralizer of g in G.

Definition 2.5. The kth power map

$$\Pi^k \colon (\Lambda X, \xi^k) \longrightarrow (\Lambda X, \xi^1)$$

sends the object (x, g) to (x, g^k) and the arrow $(x, g) \xrightarrow{h} (hx, hgh^{-1})$ to

$$(x, g^k) \xrightarrow{h} (hx, hg^k h^{-1}).$$

We may also interpret Π^k as a 1-morphism from $(\Lambda X, \xi^{km})$ to $(\Lambda X, \xi^m)$.

Definition 2.6. Let X be a topological groupoid, let ξ be an element of its center, and let k be an integer. Then we define the groupoid

$$X[\xi^{\frac{1}{k}}] := (\mathrm{pt}/\!/\mathbb{Z}) \times \Lambda(X)/\sim,$$

where the equivalence relation $\sim$ is generated by $k \sim \xi$.

Definition 2.7. For an object (X, ξ) of $\mathrm{Gpd}^{\mathrm{cen}}$, the subring

$$K_{\mathrm{rot}}(X, \xi) \subset K_{\mathrm{orb}}(X)[\![q^{\frac{1}{|\xi|}}]\!]$$

is the Grothendieck group of formal power series

$$F(q) = \sum_{a \in \mathbb{Q}_{\geqslant 0}} V_a q^q \tag{3}$$

satisfying the *rotation condition* with respect to ξ:

for each $a \in \mathbb{Q}_{\geqslant 0}$ the coefficient V_a is an $e^{2\pi i a}$-eigenbundle of ξ.

Here we are using Moerdijk's definition of orbifold K-theory $K_{\mathrm{orb}}(X)$; see [**28**, 5.4].

Power series satisfying the rotation condition may be thought of as (infinite dimensional) vector bundles over

$$\mathbb{R} \ltimes_\xi X := (\mathrm{pt}/\!\!/\mathbb{R}) \times X/\sim$$

with $\sim$ generated by $1 \sim \xi$.

Definition 2.8. Let X be a topological groupoid. Then the Tate K-theory of X is defined as

$$K_{\mathrm{Tate}}(X) := K_{\mathrm{rot}}(\Lambda(X), \xi^1).$$

2.1. Motivation
2.1.1. The Tate curve
Let $S = \mathrm{spec}(\mathbb{Z}[\![q]\!])$. Then there is an isomorphism of formal schemes over S

$$\mathrm{spf}\left(K_{\mathrm{Tate}}\left(\mathbb{C}P^\infty\right)\right) = \mathrm{Tate}(q)\widehat{S},$$

making K_{Tate} an elliptic cohomology theory; cf. [**4**, 2.6]. Let $C_k := \mathbb{Z}/k\mathbb{Z}$ be the cyclic group with k elements. Then the complex representation ring of C_k is

$$R(C_k) \cong \mathbb{Z}[\zeta_k],$$

where the kth root ζ_k of one is identified with the representation where 1 acts as $e^{2\pi i/k}$. So,

$$\mathrm{spec}\, R(C_k) \cong \mu_k$$

is the scheme of kth roots of unity over $\mathbb{Z}$, where ζ_k is the irreducible representation with $\zeta_k(1) = e^{2\pi i/k}$, and one expects[3]

$$\mathrm{spec}(K_{\mathrm{Tate}}(\mathrm{pt}/\!\!/C_k)) = \mathrm{Tate}(q)[k], \qquad (4)$$

where the right-hand side is the scheme of points of order k in $\mathrm{Tate}(q)$. By [**13**, VII(1.13)],

$$\mathrm{Tate}(q)[k] = \mathrm{spec}(D_k),$$

where

$$D_k := \bigoplus_{j=0}^{k-1} \mathbb{Z}[\![q]\!][x, x^{-1}]/x^k - q^j.$$

After inverting q, we have the map

$$J \colon \mathbb{Z}(\!(q)\!)[x] \longrightarrow \mathbb{Z}[\zeta_k](\!(q^{\frac{1}{k}})\!)$$
$$q \longmapsto q$$
$$x \longmapsto \zeta_k q^{\frac{j}{k}}.$$

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[3]See [**3, 14, 17, 19, 20**].

The map J identifies the jth summand of D_k with the Laurent series satisfying the rotation condition for the group element j. So, (4) holds over the locus $\operatorname{spec}\mathbb{Z}((q))$, where the Tate curve is non-singular. The ring D_k is closely related to isogenies into $\operatorname{Tate}(q)$ and plays a key role in Ando's definition of power operations on K_{Tate}; see [2, p. 26 ff].

We would also like to draw the reader's attention to Rezk's [32], where a much deeper connection between the constructions in [13] and our Definition 2.8 is discussed.

2.1.2. Constant loops

The original motivation for Definition 2.8 came from the theory of orbifold loop spaces. In [23], Lupercio and Uribe identify the inertia groupoid as the full subcategory

$$\Lambda X \subseteq \operatorname{Fun}(\mathbb{R}/\!\!/\mathbb{Z}, X)$$

of functors that are constant on objects. Viewing the right-hand side as part of the loop groupoid

$$LX = \operatorname{Orb}\left(\mathbb{S}^1, X\right),$$

then leads them to the identification

$$\Lambda X = (LX)^{\mathbb{R}}$$

of the inertia groupoid with the orbifold loops that are fixed by the rotation action. Under this identification, ξ^k agrees with the rotation action of $k \in \mathbb{Z}$ on (constant) orbifold loops.

Example 2.9 (Global quotients)*.* By [23, 4.1],

$$L\left(M/\!\!/G\right) \simeq \coprod_{[g]} \mathcal{L}_g M /\!\!/ C_g$$

is the union of the *"twisted loop spaces"* of M, i.e., of paths in M from x to gx, acted upon by the centralizers C_g. This is what motivated Devoto's definition of equivariant Tate K-theory [14, pp. 5f].

In this context, the rotation condition is motivated by the following lemma. Let V be a finite dimensional, real orbifold vector bundle on X, the V may itself be viewed as a groupoid (see, e.g., [28]), and we can form LV.

Lemma 2.10. *In this situation, we have a dense isomorphism*

$$LV|_{\Lambda(X)} \cong V_0 \oplus \bigoplus_{a \in \mathbb{Q}_+} V_a^{\mathbb{C}} q^a.$$

Here V_0 is the summand of $V|_{\Lambda(X)}$ fixed by ξ^1, and for $a \in \mathbb{Q}_+$, the bundle $V_a^{\mathbb{C}}$ is the $e^{2\pi i a}$-eigenbundle of ξ^1 inside the complexification $V_{\Lambda(X)}^{\mathbb{C}}$, while q^a indicates that $\mathbb{R}$ acts on this summand via $t \mapsto e^{2\pi i a t}$.

Proof. Let (x, g) be an object in $\Lambda(X)$. By [**23**, 4.1.1], we may identify $(LV)_{(x,g)}$ with the space of loops

$$\{\gamma \colon \mathbb{R} \to V_x \mid \gamma(t+1) = g\gamma(t)\}.$$

So,

$$LV_{(x,g)} \subseteq \mathrm{maps}\left(\mathbb{R}/|g|\mathbb{Z}, V_x\right).$$

The Fourier expansion principle gives a dense isomorphism

$$\mathrm{maps}\left(\mathbb{R}/|g|\mathbb{Z}, V_x\right) \cong V_{x,0} \oplus \bigoplus_{n \geqslant 1} V_x^{\mathbb{C}} q^{\frac{n}{|g|}},$$

under which $LV_{(x,g)}$ maps to the submodule specified in the lemma. $\qquad\square$

2.1.3. Moonshine

Let M be the monster, and consider the central extension $\widetilde{\Lambda}_\alpha\left(\mathrm{pt} /\!\!/ M\right)$ of the inertia groupoid $\Lambda\left(\mathrm{pt} /\!\!/ M\right)$ classified by the Moonshine cocycle

$$\alpha \in H^3\left(BM, \mathrm{U}(1)\right)$$

(see [**26**, Meta Thm. p. 29]). Let $\widetilde{\xi}$ be a lift of ξ^1 to an element of the center of $\widetilde{\Lambda}_\alpha\left(\mathrm{pt} /\!\!/ M\right)$. Then

$$\widetilde{\Lambda}_\alpha\left(\mathrm{pt} /\!\!/ M\right) \simeq \coprod_{[g]} \mathrm{pt} /\!\!/ \widetilde{C}_g,$$

where the $\widetilde{C}_g$ are central extensions of the centralizers in M. A choice of $\widetilde{\xi}$ amounts to a choice of lift $\widetilde{g} \in \widetilde{C}_g$ for each g in a system of representatives for the conjugacy classes of M. A Laurent series

$$F \in \bigoplus_{n \geqslant 0} \mathrm{Rep}\left(\widetilde{C}_g\right)\left(\!\left(q^{\frac{1}{|g|}}\right)\!\right)$$

satisfies the rotation condition if and only if its character, a function $F\left(g, h; q^{\frac{1}{|g|}}\right)$ in $h \in \widetilde{C}_g$, satisfies

$$F\left(g, \widetilde{g}h; q^{\frac{1}{|g|}}\right) = F\left(g, h; e^{\frac{2\pi i}{|g|}} q^{\frac{1}{|g|}}\right).$$

Interpreting the coefficients as characters of projective representations of C_g, this equation becomes

$$F(g, gh; \tau) = \zeta \cdot F(g, h; \tau + 1),$$

where ζ is a root of unity and $q = e^{2\pi i \tau}$. These are Condition (3) and half of Condition (1) of Norton's generalized Moonshine conjecture [**30**].

2.2. Properties

Tate K-theory is an *orbifold theory with transfers*. In other words, it satisfies the following list of properties:

Orbifold theory. Let

$$f \colon (X, \xi) \longrightarrow (Y, \upsilon)$$

be a 1-morphisms in $\mathrm{Gpd}^{\mathrm{cen}}$. Then f induces a ring map

$$f^* \colon K_{\mathrm{rot}}(Y, \upsilon) \longrightarrow K_{\mathrm{rot}}(X, \xi).$$

If f and g are naturally isomorphic then $f^* = g^*$, and if f is an equivalence of groupoids then f^* is an isomorphism. It follows that the analogous statements hold for K_{Tate}, and that the latter is a well-defined functor on Moerdijk's orbifold category.

Sums and products. We have natural isomorphisms

$$K_{\mathrm{Tate}}(\emptyset) \cong \{0\},$$
$$K_{\mathrm{Tate}}(\mathrm{pt}) \cong \mathbb{Z}[\![q]\!],$$
$$K_{\mathrm{Tate}}(X \sqcup Y) \cong K_{\mathrm{Tate}}(X) \oplus K_{\mathrm{Tate}}(Y),$$

and natural maps

$$K_{\mathrm{Tate}}(X) \otimes K_{\mathrm{Tate}}(Y) \longrightarrow K_{\mathrm{Tate}}(X \times Y),$$

making K_{Tate} a lax rig functor from Gpd to abelian rings.

Exactness. Fix a finite group G. Then the functor

$$M \longmapsto K_{\mathrm{Tate}}(M /\!\!/ G)$$

sends cofiber sequences of G-spaces to exact sequences.

Proof. Indeed, for any element $g \in G$, the functor

$$G\text{-spaces} \longrightarrow C_g\text{-spaces}$$
$$X \longmapsto X^g$$

preserves cofiber sequences.   □

2.3. Transfers

In this section all groupoids are assumed to have finite stabilizer groups. Given two maps of groupoids

$$X \xrightarrow{\;u\;} Z \xleftarrow{\;a\;} Y,$$

we can form the *fiber* square

$$
\begin{array}{ccc}
X \times_Z Y & \xrightarrow{\;v\;} & Y \\
{\scriptstyle b}\big\downarrow & & \big\downarrow{\scriptstyle a} \\
X & \xrightarrow{\;u\;} & Z,
\end{array}
\tag{5}
$$

as in [**28**, 2.3], commuting up to a natural isomorphism η (part of the data) and universal with respect to this property. We will refer to the groupoid $X \times_Z Y$ as the

fibred product of X and Y over Z. Its objects are triples $(x, y, u(x) \xrightarrow{g} a(y))$, consisting of an object of each X and Y and an arrow between their images in Z. Arrows between two such triples (x, y, g) and (x', y', g') are pairs of arrows $(x \xrightarrow{h} x', y \xrightarrow{k} y')$ in X and Y satisfying $g'u(h) = a(k)g$. Examples are the *comma category* groupoids $y \uparrow f$ and $Y \uparrow f$, defined by the fiber squares

$$
\begin{array}{ccc}
y \uparrow f & \xrightarrow{\ p_X\ } & X \\
\downarrow & & \downarrow f \\
\mathrm{pt} & \xrightarrow{\ y\ } & Y,
\end{array}
\qquad
\begin{array}{ccc}
Y \uparrow f & \xrightarrow[\sim]{\ P_X\ } & X \\
\downarrow{\scriptstyle P_Y} & & \downarrow f \\
Y & =\!=\!= & Y.
\end{array}
$$

Definition 2.11. We say that $f \colon X \to Y$ is *essentially a finite cover* if for each object $y \in Y$, there is a neighborhood U of y in $\mathrm{ob}Y$ such that

$$U \times_Y X \simeq U \times (y \uparrow f),$$

and $y \uparrow f$ is equivalent to a finite groupoid. Here we have viewed U as a groupoid with only identity morphisms, so that we can make sense of $U \times_Y X$ as in (5).

This condition ensures that pull-back of vector bundles possesses a right-adjoint

$$\mathrm{RKan}_f \colon \mathrm{Vect}_{\mathbb{C}}(X) \longrightarrow \mathrm{Vect}_{\mathbb{C}}(Y),$$

the *right Kan extension* along f, with

$$\left(\mathrm{RKan}_f V\right)_y = \varprojlim{}_{y \uparrow f}\left(p_X^* V\right).$$

Definition 2.12. Let $f \colon X \to Y$ be essentially a finite cover. Then we write

$$f_! \colon K_{\mathrm{orb}}(X) \longrightarrow K_{\mathrm{orb}}(Y)$$

For the map induced by RKan_f. We will refer to $f_!$ as the *transfer along* f.

Lemma 2.13. *Let* $f \colon (X, \xi) \longrightarrow (Y, \upsilon)$ *be a 1-morphism in* $\mathrm{Gpd}^{\mathrm{cen}}$. *If* ξ *acts with eigenvalue* $e^{2\pi i a}$ *on the vector bundle* V *on* X, *then* υ *acts with the same eigenvalue on* $\mathrm{RKan}_f V$.

Proof. The action of υ on

$$\left(\mathrm{RKan}_f V\right)_y \subseteq \bigoplus_{(x,g)} V_x$$

is by the permutation of the summands sending the object (x, g) of $y \uparrow f$ to $(x, \upsilon_y g)$. The limit condition applied to the arrow $\xi_x \colon (x, \upsilon_y g) \to (x, g)$ in $y \uparrow f$ forces an element of the limit to satisfy

$$v_{(x,g)} = \xi_x\big(v_{(x,\upsilon_y g)}\big) = e^{2\pi i a} \cdot v_{(x,\upsilon_y g)}.$$

This proves the claim. $\qquad\square$

Corollary 2.14. *We have transfers along essentially finite covers in* K_{rot} *and in* K_{Tate}.

Proof. The statement follows immediately from the fact that Λ preserves essentially finite covers. $\qquad\square$

We will denote these transfers also by $f_!$. So, $f_!$ in K_{Tate} stands for $(\Lambda f)_!$ in K_{rot}.

Proposition 2.15. *Given a fiber square as in* (5), *assume that the map a is essentially a finite cover. Then $b_!$ is defined, and we have*

$$u^* a_! = b_!\, v^*.$$

Proof. Let x be an object of X. Then we have an equivalence of groupoids

$$\begin{aligned}
x \uparrow b &= \text{pt} \times_X (X \times_Z Y) \\
&\simeq \text{pt} \times_Z Y \\
&= u(x) \uparrow a.
\end{aligned}$$

Consider the canonical natural transformation of functors $\text{Vect}_{\mathbb{C}} Y \to \text{Vect}_{\mathbb{C}} X$

$$u^* \, \text{RKan}_a \Longrightarrow \text{RKan}_b\, v^*.$$

Restricted to the fibers, this is the composition of isomorphisms

$$\begin{aligned}
(\text{RKan}_b\, v^* V)_x &= \varprojlim{}_{x \uparrow b} \left(p^*_{(X \times_Z Y)} v^* V \right) \\
&\cong \varprojlim{}_{u(x) \uparrow a} (p^*_Y V) \\
&= (\text{RKan}_a)_{u(x)}.
\end{aligned}$$

Further, Λ preserves fiber squares. $\qquad\square$

Corollary 2.16. *Orbifold Tate K-theory, restricted to the 2-category Gpd^{fin} of finite groupoids, is a global Mackey functor.*

Here we are using the definition of global Mackey functor spelled out in [**18**]. It is well known that, in the same manner, orbifold K-theory gives the Mackey functor sending a finite group(oid) to its representation ring.

2.4.   Characters

Unlike most of this paper, this section only applies to finite groupoids.

2.4.1. n-class functions

Let X be a finite groupoid, let R be a ring, and let $n \geqslant 0$ be a natural number.

Definition 2.17. An n-class function on X with values in R is an R-valued map

$$\chi \colon [\Lambda^n X] \longrightarrow R,$$

defined on the set of isomorphism classes of the n-fold inertia groupoid of X. We denote the ring of all such maps by

$$n\text{-class}(X, R) \cong H^0(\text{Borel}(\Lambda^n X); R).$$

Explicitly, a 0-class function is a function on the isomorphism classes of X, and for $n \geqslant 1$, an n-class function is defined on n-tuples of commuting automorphisms of X and satisfies

$$\chi(g_1, \ldots, g_n) = \chi(sg_1 s^{-1}, \ldots, sg_n s^{-1}).$$

Here all the g_i are automorphisms of the same object x. It is well known that 0-class$(-, R)$ is a global Mackey functor, namely group cohomology, with transfers along faithful maps $f \colon X \to Y$. It follows that n-class$(-, R)$ is also a Mackey functor, whose transfers along faithful maps are given by

$$f_!(\chi)(g_1, \ldots, g_n) = \sum_{[\underline{h}] \mapsto [\underline{g}]} \frac{|\mathrm{aut}(\underline{g})|}{|\mathrm{aut}(\underline{h})|} \cdot \chi(h_1, \ldots, h_n). \tag{6}$$

Here $\underline{g}$ abbreviates $(g_1, \ldots, g_n)$, and $[\underline{g}]$ and $\mathrm{aut}(\underline{g})$ are, respectively, its isomorphism class and its automorphism group in $\Lambda^n Y$, and similarly for $\underline{h}$.

If R is a $\mathbb{Q}$-algebra, then (6) makes sense for all f and extends n-class$(-; R)$ to a Mackey functor with all transfers.

Example 2.18. Let $j \colon H \to G$ be an inclusion of groups and write j also for the corresponding map of groupoids $j \colon \mathrm{pt} /\!\!/ H \to \mathrm{pt} /\!\!/ G$. Then

$$j_!(\chi)(g_1, \ldots, g_n) = \frac{1}{|H|} \sum_{s\underline{g}s^{-1} \in H^n} \chi(sg_1 s^{-1}, \ldots, sg_n s^{-1}).$$

Example 2.19. Let G be a finite group, and let $\varepsilon_G \colon \mathrm{pt} /\!\!/ G \to \mathrm{pt}$ be the unique map. Then

$$(\varepsilon_G)_!(\chi)(1) = \frac{1}{|G|} \sum_{\underline{g}} \chi(g_1, \ldots, g_n),$$

where the sum runs over all n-tuples of commuting elements of G.

Note that, unlike equation (6), the sums in these examples are over elements, rather than conjugacy classes. This is consistent, because various $|\mathrm{aut}(\cdot)|$ factors cancel out.

2.4.2. Character theory
Write Gpd^f for the 2-category of finite groupoids, and assume that E is an orbifold theory with transfers as in Section 2.2.

Definition 2.20. A *character theory* for E *of height n* consists of a ring R and a natural transformation of Mackey functors

$$\chi \colon E|_{\mathrm{Gpd}^f} \Longrightarrow n\text{-class}(-, R).$$

Here R is assumed to be either a $\mathbb{Q}$-algebra or torsion free with the understanding that some transfers in n-class$(-, R)$ are, *a priori*, only defined over $\mathbb{Q} \otimes R$. Naturality of χ then implies integrality of transfers in the image of χ.

Example 2.21. K-theory has a character theory of height 1 with

$$R = \varinjlim_k \mathbb{Z}[\zeta_k],$$

where ζ_k is a kth root of 1. Note that the universal level k-structure on the multiplicative group has its home over $\mathrm{spec}(\mathbb{Z}[\zeta_k])$. The integrality statement above amounts

to the well-known fact that

$$\frac{1}{|G|} \sum_{g \in G} \chi(g) \in \mathbb{Z}$$

when χ is the character of a representation.

Example 2.22. Tate K-theory has a height 2 character theory, taking values in

$$\varinjlim_k \left(\mathbb{Z}[\zeta_k][\![q^{\frac{1}{k}}]\!] \right);$$

compare to Section 2.1.1. By [**13**, VII.2.4,VII.1.16.4]),

$$\operatorname{spec} \mathbb{Z}[\zeta_k][\![q^{\frac{1}{k}}]\!] \longrightarrow (\mathcal{M}_k)$$

is the completion at infinity of the level k stack $\mathcal{M}_k$ defined in [**13**, p. 36, IV.3.2,3.5].

Example 2.23. Fix a prime p, and let E_n be Borel equivariant Morava-Lubin-Tate theory. This has a height n character theory ([**16**, 7.9], [**22**, Thms. C&D]), taking values in

$$L(E_n) = S^{-1} \varinjlim_k E_n(B(\mathbb{Z}/p^k \mathbb{Z})^n),$$

where the multiplicative set S consists of the Euler classes of non-trivial representations. The ring

$$D_k := \operatorname{im} \left(E_n(B(\mathbb{Z}/p^k \mathbb{Z})^n) \to S^{-1} E_n(B(\mathbb{Z}/p^k \mathbb{Z})^n) \right)$$

is the home of the universal level k-structure on the formal group of E_n (see [**1**, Thm. 3.3.2]).

The character theory in [**22**] is a lot more powerful than our summary here suggests. It would be interesting to formulate Hopkins-Kuhn-Ravenel theory for orbifolds and to see exactly how much of the Hopkins-Kuhn-Ravenel story can be told for K_{Tate}. It seems reasonable to expect that K_{Tate} fits into Stapleton's framework of transchromatic character maps [**33**].

2.4.3.   A formula for induction

Let $H \subseteq G$ be an inclusion of finite groups, M a G-manifold, and let

$$j \colon M /\!\!/ H \longrightarrow M /\!\!/ G$$

be the inclusion. Then the transfer $j_!$ in K_{orb} is

$$\operatorname{ind}_H^G \colon K_H(M) \longrightarrow K_G(M).$$

We will write

$$I_H^G \colon K_{\mathrm{Tate}}(M /\!\!/ H) \longrightarrow K_{\mathrm{Tate}}(M /\!\!/ G)$$

for the transfer $j_!$ in K_{Tate}. Let a be an element of $K_{\mathrm{Tate}}(M /\!\!/ H)$. Then $I_H^G(a)$ is an element of

$$\bigoplus_{[g]} K_{C_G(g)}(M^g)[\![q^{\frac{1}{|g|}}]\!].$$

Fix $g \in G$, choose representatives $h_1, \dots, h_k$ for the H-conjugacy classes contained in $[g]_G$, and for $1 \leqslant i \leqslant k$, choose $s_i \in G$ with $s_i h_i s_i^{-1} = g$. Write $C_i := C_H(h_i)$, and

$C_G := C_G(g)$, and view the former as a subgroup of the latter via the inclusion

$$C_i \hookrightarrow C_G$$
$$t \mapsto s_i t s_i^{-1}.$$

Proposition 2.24. *With the above notation, the $[g]$th summand of $I_H^G(a)$ equals*

$$I_H^G(a)_{[g]} = \bigoplus_{i=1}^{k} \mathrm{ind}_{C_i}^{C_G}(a_i),$$

where $a_i = a_{[h_i]}$

Proof. Because of the commuting diagram

$$
\begin{array}{ccc}
M /\!\!/ H & \xrightarrow{\quad j \quad} & M /\!\!/ G \\
{\scriptstyle \sim} \big\uparrow & \nearrow{\scriptstyle \mathrm{pr}_2} & \\
(G \times_H M) /\!\!/ G, & &
\end{array}
$$

I_H^G may be identified with the G-equivariant (Atiyah) transfer in K_{rot} along $\Lambda(\mathrm{pr}_2)$. This may be viewed as the map of G-spaces

$$G \times_H \mathrm{ob}(\Lambda(M /\!\!/ H)) \longrightarrow \mathrm{ob}(\Lambda(M /\!\!/ G)$$
$$(s, (x, h)) \longmapsto (sx, shs^{-1}).$$

Hence we have

$$I_H^G(a)_{[g]} = \sum_{r^{-1} g r \in H} r \cdot a_{[r^{-1} g r]}, \tag{7}$$

where the sum runs over $r \in \mathcal{R}$, a system of representatives of G/H. Without loss of generality, the set of those representatives contributing to the sum may be chosen of the form

$$s_1 \cdot \mathcal{R}_1 \sqcup \cdots \sqcup s_k \cdot \mathcal{R}_k,$$

with $\mathcal{R}_i$ a system of representatives for $C_G(h_i)/C_H(h_i)$. Then (7) becomes

$$\bigoplus_{i=1}^{k} s_i \cdot \bigoplus_{r \in \mathcal{R}_i} r a_{[h_i]} = \bigoplus_{i=1}^{k} s_i \cdot \mathrm{ind}_{C_H(h_i)}^{C_G(h_i)} a_{[h_i]},$$

as claimed. □

3. Power operations

3.1. Symmetric powers of orbifolds

Recall that Gpd is a symmetric bimonoidal category ("rig" category) with the monoidal structures given by $(\sqcup, \emptyset)$ and $(\times, \mathrm{pt})$. Every groupoid $X \in \mathrm{ob}(\mathrm{Gpd})$ is a monoid with respect to $(\sqcup, \emptyset)$, via the fold map

$$d: X \sqcup X \longrightarrow X.$$

A groupoid X is a monoid with respect to $(\times, \mathrm{pt})$ if and only if X is endowed with the structure of a symmetric monoidal category.

Definition 3.1. Then nth symmetric power of $X \in \mathrm{ob}(\mathrm{Gpd})$ is the groupoid $S_n \wr X$ obtained from X^n by adding the additional arrows

$$(x_1, \ldots, x_n) \xrightarrow{\ \sigma\ } (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}) \qquad \sigma \in S_n,$$

composing with the arrows in X^n as follows:

$$\sigma \circ (g_1, \ldots, g_n) = (g_{\sigma^{-1}(1)}, \ldots, g_{\sigma^{-1}(n)}) \circ \sigma.$$

The total symmetric power of X is

$$S(X) := \coprod_{n \geqslant 0} S_n \wr X.$$

The endofunctor S of *Gpd* is exponential in the sense that it is a monoidal functor

$$S \colon (\mathrm{Gpd}, \sqcup, \emptyset) \longrightarrow (\mathrm{Gpd}, \times, \mathrm{pt}).$$

It follows that $S(X)$ comes equipped with a symmetric monoidal structure

$$* := S(d),$$

which turns out to be concatenation. The unit of $*$ is the unique object $()$ of $S_0 \wr X$. The triple

$$(S(X), *, ())$$

may be viewed as the free symmetric monoidal category on X. More precisely, the functor

$$X \longmapsto (S(X), *, ())$$

is the left adjoint to the forgetful functor from the category of monoids in Gpd to Gpd. In particular, S is a monad.

Definition 3.2. We will write

$$\mu \colon S^2 \Longrightarrow S$$

and

$$\iota \colon \mathrm{Id} \Longrightarrow S$$

for the structure maps of the monad S. Explicitly, these are given by the inclusions $S_n \wr S_m \subseteq S_{nm}$ and by the inclusion of $X = S_1 \wr X$ inside $S(X)$.

3.2. The inertia groupoid of $S(X)$

Objects of $\Lambda(S(X))$ are given by quadruples $(n, \underline{x}, \underline{g}, \sigma)$, where $n \geqslant 0$ is a natural number, $\underline{x} \in \mathrm{ob}(X^n)$, and $\sigma \in S_n$, while $\underline{g}$ is a system of arrows $x_i \xrightarrow{g_i} x_{\sigma(i)}$. Note that this notation is somewhat redundant, all the information is contained in the pair $(\sigma; \underline{g})$. The inertia groupoid inherits a monoidal structure from $S(X)$, given by

$$(\sigma; g_1, \ldots, g_n) * (\tau; h_1, \ldots, h_m) = (\sigma \sqcup \tau; g_1, \ldots, g_n, h_1, \ldots, h_m).$$

An object $(\sigma, \underline{g})$ of $\Lambda(S(X))$ is indecomposable with respect to $*$ if and only if $n > 0$ and σ acts on $\{1, \ldots, n\}$ with a single orbit (i.e., σ is a long cycle). Otherwise, the cycle decomposition of σ yields a decomposition of $(\sigma, \underline{g})$ into indecomposables.

Definition 3.3. Let $\varsigma_k \in S_k$ be the cycle $\varsigma_k = (1, \ldots, k)$, and let $\Phi_k(X)$ be the full subgroupoid of $\Lambda(S(X))$ with objects $(\varsigma_k, \underline{x})$. Let $\Phi(X)$ be the groupoid

$$\Phi(X) := \coprod_{k \geqslant 1} \Phi_k(X).$$

Further, let $\varphi \in \mathrm{Center}(\Phi)$ be the restriction of ξ^1 to $\Phi(X)$.

The essential image of $\Phi(X)$ inside $\Lambda(SX)$ is the subgroupoid of indecomposable objects. The functor Φ is additive, i.e., $\Phi(\emptyset) = \emptyset$, and Φ preserves $\sqcup$.

Lemma 3.4. *We have an equivalence of monoidal groupoids*

$$Q \colon S(\Phi(X)) \longrightarrow \Lambda(S(X)),$$

which is natural in X and satisfies

$$QS(\varphi) = \xi^1 Q.$$

Proof. Let I be the inclusion

$$I \colon \Phi(X) \hookrightarrow \Lambda(S(X)),$$

and let Q be the composite

$$Q \colon S(\Phi(X)) \xrightarrow{\ S(I)\ } S(\Lambda(S(X))) \xrightarrow{\ \varepsilon\ } \Lambda(S(X)),$$

where the second map is the counit of the adjunction $(S, *, ()) \dashv \mathrm{forget}$. Then Q is monoidal. Since the essential image of I consists exactly of the indecomposable objects of $\Lambda(S(X))$, it follows that Q is essentially surjective. One checks that Q is also fully faithful. (This boils down to the fact that σ and τ are conjugate in S_n if and only if their cycle decompositions are congruent.) $\square$

Definition 3.5. For each $k \geqslant 1$, we define a map of groupoids

$$E_k \colon \Phi_k(X) \longrightarrow \Lambda(X)[\xi^{\frac{1}{k}}]$$
$$((x_1, \ldots, x_k), (\varsigma_k; g_1, \ldots, g_k)) \longmapsto (x_1, g_k \cdots g_1) \qquad \text{on objects, and}$$
$$(\mathrm{id}; h_1, \ldots, h_k)) \longmapsto h_1 \qquad \text{and}$$
$$\varphi \longmapsto \xi^{\frac{1}{k}} \qquad \text{on morphisms.}$$

Lemma 3.6. *The map E_k is a well-defined equivalence of groupoids.*

In order to make the notation more manageable, we will simply write g for the object of $\Lambda(X)$ previously denoted (x, g).

Proof. It is clear that E_k is surjective on objects. Let $(\varsigma_k; \underline{g})$ be an object of $\Phi_k(X)$, and write $(\mathrm{id}; \widehat{\underline{g}})$ for its image under Π^k. In other words, $\widehat{g}_i \in \mathrm{ob}(\Lambda(X))$ is the composite

$$\widehat{g}_i \colon x_i \xrightarrow{g_i} x_{i+1} \xrightarrow{g_{i+1}} x_{i+2} \xrightarrow{g_{i+2}} \cdots x_{i-1} \xrightarrow{g_{i-1}} x_i.$$

Then we have

$$\varphi^k_{(\underline{x},\underline{g})} = (\mathrm{id}, \widehat{\underline{g}}). \tag{8}$$

Let $(\varsigma_k^m, \underline{h})$ be an arrow in $\Phi_k(X)$. Then $(\varsigma_k^m, \underline{h})$ can be factored as

$$(\mathrm{id}, \underline{h}') \circ \varphi^m = \varphi^m \circ (\mathrm{id}, \underline{h}')$$

with $m \in \mathbb{Z}$, and this presentation is unique up to the relation (8).

Consider an arrow of the form $(\mathrm{id}, \underline{h})$ from $(\underline{x}, \underline{g})$ to $(\underline{x}', \underline{g}')$ in $\Phi_k(X)$. Then the h_i fit into a commuting diagram

$$
\begin{array}{ccccccccc}
x_1 & \xrightarrow{g_1} & x_2 & \xrightarrow{g_2} & x_3 & \longrightarrow \cdots \longrightarrow & x_k & \xrightarrow{g_k} & x_1 \\
\downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} & & \downarrow{\scriptstyle h_k} & & \downarrow{\scriptstyle h_1} \\
x_1' & \xrightarrow{g_1'} & x_2' & \xrightarrow{g_2'} & x_3' & \longrightarrow \cdots \longrightarrow & x_k' & \xrightarrow{g_k'} & x_1'.
\end{array}
\tag{9}
$$

In particular,

$$h_1 \widehat{g}_1 h_1^{-1} = \widehat{g}_1' \tag{10}$$

(and, similarly for i). Hence E_k is well defined. Further, the h_i are uniquely determined by h_1, and any h_1 satisfying (10) gives rise a to system $\underline{h}$ fitting into (9). Hence E_k is fully faithful. $\square$

Definition 3.7. Let F_k be the quasi-inverse of E_k that sends the object g to

$$(\varsigma_k; g, \mathrm{id}, \dots, \mathrm{id})$$

($k - 1$ times id).

All the above constructions are Morita invariant.

3.2.1.   Comparison to Dijkgraaf, Moore, Verlinde and Verlinde

Let X be the groupoid of a global quotient orbifold $M /\!\!/ G$, where G is a finite group. Then the loop space of $S_n \wr X$,

$$L(S_n \wr X) \simeq \mathrm{Fun}(\mathbb{R} /\!\!/ \mathbb{Z}, S_n \wr X),$$

has objects $(\sigma; \underline{g}, \underline{\gamma})$, where $(\sigma; \underline{g}) \in S_n \wr G$ and $\underline{\gamma}$ is an n-tuple of paths

$$\gamma_i \colon \mathbb{R} \longrightarrow M$$

satisfying

$$g_i \gamma_i(t) = \gamma_{\sigma(i)}(t + 1). \tag{11}$$

Example 3.8. Assume that $\sigma = \varsigma_k$. Then we have

$$\gamma_1(t + k) = g_k \cdots g_1 \gamma_1(t)$$
$$\gamma_2(t + k) = g_1 g_k \cdots g_2 \gamma_2(t)$$

and so on, and each γ_i determines the others via (11). Hence

$$\gamma_i \in \mathrm{Fun}(\mathbb{R} /\!\!/ k\mathbb{Z}, X)$$

may be thought of as a loop of length k in $M /\!\!/ G$.

Let $L_k X \subseteq L(S_n \wr X)$ be the full subgroupoid whose objects have $\sigma = \varsigma_k$. Then our groupoid Φ_k can be identified with $L_k X^{\mathbb{R}}$, the part of $L_k X$ that is invariant under the rotation action by $\mathbb{R}$. So, $\Phi_k X$ may be viewed as the groupoid of constant loops of length k in $M /\!\!/ G$, and φ is identified with the rotation action by $1 \in \mathbb{Z}$ on these long loops. The key argument in [**15**] is summarized by the equivalence of groupoids

$$LS(X) \simeq S\left(\coprod_{k \geqslant 1} L_k X \right) \tag{12}$$

(see also [**34**]). Our Lemma 3.4 above follows from (12) by restricting to constant loops on both sides.
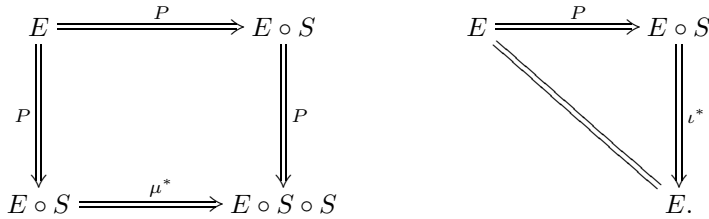
3.3. Power operations in orbifold theories

Let E be an orbifold theory with products.

Definition 3.9. A *total power operation* for E is a (non-linear) natural transformation

$$P \colon E \Longrightarrow E \circ S$$

satisfying the following properties:

Comodule property. P makes E a comodule over the comonad $(-) \circ S$, i.e., the following diagrams commute:

Exponentiality: The map

$$P \colon E(\emptyset) \longrightarrow E(\mathrm{pt})$$

sends 0 to 1, and

$$P \colon E(X \sqcup Y) \longrightarrow E(SX \times SY)$$

sends (a, b) to the external product $P(a) \otimes P(b)$.

We can write P as $P = (P_n)_{n \geqslant 0}$, with $P_n := \iota_n^* P$. Then P_n is called the nth power operation of P.

Example 3.10 (Atiyah power operations). Let V be an orbifold vector bundle on X. Then we have the Atiyah Power Operations

$$P_n(V) := S_n \ltimes V^{\otimes n}$$

of [**6**]. We will see below how to extend this definition to virtual vector bundles. For now we note that $P = (P_n)_{n \geqslant 0}$ satisfies the axioms of a total power operation, whenever it is defined.

3.3.1. Consequences of the definition
Proposition 3.11.

1. We have $P_1(a) = a$.

2. More generally,

    $$i_k^* P_k(a) = a^{\otimes k}$$

    (external product), where i_k is the inclusion of X^k in $S_k \wr X$.

3. We have

    $$(PS) \circ P = \mu^* \circ P,$$

    or, equivalently,

    $$P_n(P_m(a)) = \operatorname{res}^{S_{nm}}_{S_n \wr S_m} (P_{nm}(a)).$$

4. $P_0(a) = 1$.

5. The external product of $a \in E(X)$ and $b \in E(Y)$ is given by

    $$a \otimes b = i^* P(a, b),$$

    where $(a, b) \in E(X \sqcup Y)$, and

    $$i := (\iota_X \times \iota_Y) \colon X \times Y \hookrightarrow SX \times SY$$

    is the product of the canonical inclusions (see Definition 3.2).

6. We have

    $$P(a \otimes b) = j^* (P(a) \otimes P(b))$$

    (external products), where

    $$j \colon S(X \times Y) \longrightarrow S(X) \times S(Y)$$

    is defined as

    $$j = (S(\pi_X), S(\pi_Y)),$$

    with π_X and π_Y the projections to the respective factors.

7. The map P, and equivalently all the P_n, preserves interior products:

    $$P_n(ab) = P_n(a)P_n(b),$$

    for $a, b \in E(X)$.

8. For each n, we have $P_n(1) = 1$.

Proof. The Comodule Property translates into Points 1. and 3. To prove Point 2., we apply naturality of P to the fold map $f\colon X \sqcup \cdots \sqcup X \to X$, noting that $S(f)$ is the concatenation product $*$. This leads to a commuting diagram

$$
\begin{array}{ccc}
E(X) & \xrightarrow{\ \ P\ \ } & E(SX) \xrightarrow{\ (\iota_k i_k)^*\ } E(X^k) \\
\downarrow{\scriptstyle f^*} & & \quad\ \downarrow{\scriptstyle (*)^*} \nearrow{\scriptstyle (\iota_1^k)^*} \\
E(X \sqcup \cdots \sqcup X) & \xrightarrow{\ \ P\ \ } & E((SX)^k),
\end{array}
$$

where $f^*(a) = (a, \ldots, a)$. By the Exponential Property, the bottom composite sends $(a, \ldots, a)$ to $P_1(a)^{\otimes k}$, hence 2.

For Point 4., we apply naturality of P to the unique map $e\colon \emptyset \to X$, arriving at

$$
\iota_0^* \circ P|_X = P|_\emptyset \circ e^*.
$$

The left-hand side of this is P_0. By the first part of the Exponential Property, the right-hand side is the constant map 1.

Point 5. is an immediate consequence of the Exponential Property and 1.

To prove Point 6., one checks that $j = \mu \circ S(i)$ and considers the diagram

$$
\begin{array}{ccc}
E(X \sqcup Y) & \xrightarrow{\ \ P\ \ } E(SX \times SY) \xrightarrow{\ \ i^*\ \ } & E(X \times Y) \\
\downarrow{\scriptstyle P} & \downarrow{\scriptstyle P} & \downarrow{\scriptstyle P} \\
E(SX \times SY) & \xrightarrow[\ \ \mu^*\ \ ]{} E(S^2(X \sqcup Y)) \xrightarrow[\ (Si)^*\ ]{} & E(S(X \times Y)),
\end{array}
$$

whose left square commutes by 3., and whose right square commutes by naturality of P. By Point 5., the composite on the top row sends (a, b) to $a \otimes b$. By the Exponential Property, the leftmost vertical arrow sends (a, b) to $P(a) \otimes P(b)$.

Point 7. follows from 6. by considering the special case $X = Y$: Let $\delta\colon X \to X \times X$ be the diagonal map. Then we have

$$
\begin{aligned}
P(ab) &= P\left(\delta^*\left(a \otimes b\right)\right) \\
&= (S\delta)^* P(a \otimes b) \\
&= (S\delta)^* j^* (P(a) \otimes P(b)) \\
&= \delta^*_{S(X)} \left(P(a) \otimes P(b)\right) \\
&= P(a)P(b).
\end{aligned}
$$

Finally, we show Point 8. In the special case $X = \mathrm{pt}$, this follows from 3. and the first part of the Exponential Property. The general case follows by applying naturality of P to the unique map from X to pt. $\qquad\square$

3.4. The graded ring $E(S(X))$

From now on, we assume that E possesses transfers along (faithful) essentially finite covers. Then transfer along the concatenation product

$$
*\colon S(X) \times S(X) \longrightarrow S(X)
$$

defines a second multiplication $\bullet$ on the ring $E(S(X))$, making it a graded ring with respect to the grading

$$E(S(X)) \cong \bigoplus_{n \geqslant 0}^{\wedge} E(S_n \wr X)$$

with unit $\mathbf{1}$ in degree zero.

Proposition 3.12. *Assume that P is natural also with respect to transfers, and let $a, b \in E(X)$. Then we have*

$$P(0) = \mathbf{1}$$

and

$$P(a + b) = P(a) \bullet P(b).$$

Proof. The first equality follows by applying naturality of P to the transfer along the inclusion of $\emptyset$ in X. The second follows by applying naturality of P to the transfer along the fold map $X \sqcup X \to X$. $\square$

Assume now that E is an orbifold theory such that, for each finite group G, the functor $E_G(M) = E(M /\!\!/ G)$ is not just exact, but actually the degree zero part of a G-equivariant cohomology theory. In this situation, we demand that P be natural also with respect to maps in the G-equivariant stable category. Spanier-Whitehead duality implies that each E_G possesses transfers along G-equivariant finite covers and our assumption on P implies that P is natural with respect to these transfers. Note that by the proof of Proposition 2.24, this class of transfers includes transfers along the maps

$$M /\!\!/ H \longrightarrow M /\!\!/ G,$$

induced by the inclusion of a subgroup $H \subseteq G$. In particular, P is natural with respect to the map

$$*\colon SX \times SX \to SX$$

whenever $X /\!\!/ G$ is a global quotient orbifold.

It seems reasonable to expect that, in a suitably defined category of 'orbispectra', a similar argument would make the demand for naturality along faithful transfers redundant and that all K-theoretic transfers above should be induced by maps in a K_{orb}-localization of this orbispectra category.

Corollary 3.13. *The Atiyah power operations of Example 3.10 can be extended, in a unique way, to virtual vector bundles.*

Proof. Indeed, P takes values in $\mathbf{1} + K_{\mathrm{orb}}(SX)$ and hence $P([V])$ is invertible with respect to $\bullet$ for any vector bundle V. $\square$

Similarly, one argues that the definition of Atiyah power operations extends to K_{rot}, giving

$$P\colon K_{\mathrm{rot}}(X, \xi) \longrightarrow K_{\mathrm{rot}}(SX, S\xi).$$

3.5. Power operations in Tate K-theory

Recall from Lemma 3.4 that we have

$$K_{\mathrm{Tate}}(S(X)) \cong K_{\mathrm{rot}}\left(S(\Phi(X)), S(\varphi)\right).$$

This becomes a graded isomorphism if, on the right-hand side, elements supported on $S_n \wr \Phi_k(X)$ are given degree nk.

Definition 3.14. Let X be an orbifold groupoid, let ξ be an element of its center, and let $k \geqslant 1$ be a natural number. Then we define the map

$$s_k : K_{\mathrm{rot}}(X) \longrightarrow K_{\mathrm{rot}}(X[\xi^{\frac{1}{k}}])$$
$$\left[\sum V_a q^a\right] \longmapsto \left[\sum V_a q^{\frac{a}{k}}\right],$$

where $\xi^{\frac{1}{k}}$ acts on the coefficient V_a by $e^{2\pi i a/k}$.

The map s_k commutes with the Atiyah power operations in the following sense: let (X, ξ) be as in Definition 3.14, recall the notation $X^{(k)}$ for $X[\xi^{\frac{1}{k}}]$ and let

$$\delta : (SX)^{(k)} \hookrightarrow S(X^{(k)})$$

be the canonical inclusion (i.e., δ sends $(S\xi)^{\frac{1}{k}}$ to $S(\xi^{\frac{1}{k}})$). Then we have

$$s_k \circ P = \delta^* P s_k.$$

Definition 3.15. Let

$$\theta : K_{\mathrm{Tate}}(X) \longrightarrow K_{\mathrm{rot}}(\Phi(X), \varphi)$$

be the additive operation whose kth component is $E_k^* \circ s_k$. Here E_k is the equivalence defined in Lemma 3.6.

Definition 3.16. The total power operation in K_{Tate} is defined as the composite

$$P^{\mathrm{Tate}} : K_{\mathrm{Tate}}(X) \xrightarrow{\ \theta\ } K_{\mathrm{rot}}(\Phi X, \varphi) \xrightarrow{\ P\ } K_{\mathrm{rot}}(S\Phi X, S\varphi) \xrightarrow{\ (Q^*)^{-1}\ } K_{\mathrm{Tate}}(SX).$$

Theorem 3.17. *This P^{Tate} satisfies the axioms of a total power operation.*

Proof. The Exponential Property follows immediately from that of P and from additivity of θ,

$$\theta : K_{\mathrm{Tate}}(X \sqcup Y) \longrightarrow K_{\mathrm{rot}}(\Phi X \sqcup \Phi Y)$$
$$(a, b) \longmapsto (\theta a, \theta b).$$

The following lemma is the first step towards proving the Comodule Property:

Lemma 3.18. *Let the natural transformation ϑ be defined as the composite*

$$\vartheta_X : \Phi SX \xrightarrow{\ I_{SX}\ } \Lambda S^2 X \xrightarrow{\ \Lambda \mu_X\ } \Lambda SX.$$

(Note that ϑ and θ denote two different things.) Then we have

$$(\theta S) \circ P^{\mathrm{Tate}} = \vartheta^* P^{\mathrm{Tate}}.$$

In other words, the diagram

$$
\begin{array}{ccc}
K_{\mathrm{Tate}}(X) & \xrightarrow{\;P^{\mathrm{Tate}}\;} & K_{\mathrm{Tate}}(SX) = K_{\mathrm{rot}}(\Lambda SX, \xi) \\
\downarrow{\scriptstyle P^{\mathrm{Tate}}} & & \downarrow{\scriptstyle \vartheta^*} \\
K_{\mathrm{Tate}}(SX) & \xrightarrow{\quad\theta\quad} & K_{\mathrm{rot}}(\Phi SX, \varphi)
\end{array}
$$

commutes.

Proof. It suffices to prove this equality after restricting both sides to $\Phi_k S$. Consider the commuting diagram

$$
\begin{array}{ccccc}
\Lambda^{(k)}S & \xrightarrow[\simeq]{F_k S} & \Phi_k S & \xrightarrow{\vartheta} & \Lambda S \\
\uparrow{\scriptstyle Q^{(k)}}{\simeq} & & & & \uparrow{\scriptstyle Q}{\simeq} \\
(S\Phi)^{(k)} & \xrightarrow{\delta} & S(\Phi^{(k)}) & \xrightarrow[\simeq]{Su} & S\Phi,
\end{array}
$$

where the equivalence u sends the object $(\varsigma_l; \underline{g})$ of $\Phi_l^{(k)})$ to the object

$$
(\varsigma_{kl}; \underbrace{\mathrm{id}, \ldots, \mathrm{id}}_{k-1}, g_1, \underbrace{\mathrm{id}, \ldots, \mathrm{id}}_{k-1}, g_2, \mathrm{id}, \ldots, \ldots, \mathrm{id}, g_l),
$$

of Φ_{kl}, sends a morphism of the form $(\mathrm{id}, \underline{h})$ to

$$
(\mathrm{id}; \underbrace{h_1, \ldots, h_1}_{k}, \underbrace{h_2, \ldots, h_2}_{k}, \ldots, \ldots, \underbrace{h_l, \ldots, h_l}_{k}),
$$

and $\varphi^{\frac{1}{k}}$ to φ. We have

$$
u^*\theta = s_k\theta,
$$

and

$$
(Q^{(k)})^* s_k = s_k Q^*
$$

where s_k us as in Definition 3.14. Hence

$$
\begin{aligned}
(Q^{(k)})^* \circ F_k^* \circ \vartheta^* \circ P^{\mathrm{Tate}} &= \delta^* \circ (Su)^* \circ P \circ \theta \\
&= \delta^* \circ P \circ u^* \circ \theta \\
&= \delta^* \circ P \circ s_k \circ \theta \\
&= s_k \circ P \circ \theta \\
&= (Q^{(k)})^* \circ s_k \circ P^{\mathrm{Tate}} \\
&= (Q^{(k)})^* \circ F_k^* \circ E_k^* \circ s_k \circ P^{\mathrm{Tate}} \\
&= (Q^{(k)})^* \circ F_k^* \circ \theta \circ s_k \circ P^{\mathrm{Tate}}
\end{aligned}
$$

Since $(F_k \circ Q^{(k)})^*$ is an isomorphism, the claim of the lemma follows. □

To complete the proof of the theorem, we consider the commuting diagram

$$
\begin{array}{ccccc}
S\Phi S & \xrightarrow{\ S\vartheta\ } & S\Lambda S & \xleftarrow[\simeq]{\ SQ\ } & S^2\Phi \\
{\scriptstyle QS}\downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\mu\Phi} \\
\Lambda S^2 & \xrightarrow{\ \Lambda\mu\ } & \Lambda S & \xleftarrow[\simeq]{\ Q\ } & S\Phi,
\end{array}
$$

where the middle map, ε, is the counit of the adjunction $(S,*,())\dashv$ forget (compare to the definition of Q in the proof of Lemma 3.4). This yields

$$
\begin{aligned}
(P^{\mathrm{Tate}}S)\circ P^{\mathrm{Tate}} &= ((QS)^*)^{-1}\circ(P\Phi S)\circ\vartheta^*\circ(Q^*)^{-1}\circ(P\Phi)\circ\theta \\
&= ((QS)^*)^{-1}\circ(S\vartheta)^*\circ((SQ)^*)^{-1}\circ(PS\Phi)\circ(P\Phi)\circ\theta \\
&= ((QS)^*)^{-1}\circ(S\vartheta)^*\circ((SQ)^*)^{-1}\circ(\mu\Phi)^*\circ(P\Phi)\circ\theta \\
&= (\Lambda\mu)^*(Q^*)^{-1}\circ(P\Phi)\circ\theta \\
&= (\Lambda\mu)^* P^{\mathrm{Tate}}.
\end{aligned}
$$
$\qquad\square$

3.6. Symmetric and exterior powers

Let E be a theory with transfers and power operations. Write

$$\varepsilon\colon S(\mathrm{pt})\longrightarrow\mathbb{N}$$

for the augmentation map, sending $S_n\wr\mathrm{pt}$ to n. We may identify $E(\mathbb{N}\times X)$ with the formal power series ring $E(X)[\![t]\!]$.

Definition 3.19. The total E-theoretic symmetric power S_t is the composite

$$S_t\colon EX\xrightarrow{\ P\ }ESX\xrightarrow{\ \delta^*\ }E(S(\mathrm{pt})\times X)\xrightarrow{\ \varepsilon_!\ }E(X)[\![t]\!],$$

where δ is the diagonal map, sending the arrow (σ,g) to the arrow $(\sigma;g,\ldots,g)$ with n copies of g if $\sigma\in S_n$.

Proposition 3.20. *In the definition of S_t, the maps δ^* and $\varepsilon_!$ are maps of graded rings.*

Corollary 3.21. *The total symmetric power is exponential:*

$$S_t(a+b)=S_t(a)\cdot S_t(b).$$

Proof. We now prove the proposition. The ring multiplication on $E(X)[\![t]\!]$ corresponds to

$$E(\mathbb{N}\times X\times\mathbb{N}\times X)\xrightarrow{\ \delta_X^*\ }E(\mathbb{N}\times X\times\mathbb{N})\xrightarrow{\ m_!\ }E(\mathbb{N}\times X),$$

where $\delta_X\colon X\to X\times X$ is the diagonal map and $m\colon\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ is multiplication. Similarly, one defines the graded multiplication on $E(S(\mathrm{pt})\times X)$ as the push-pull

$$E\left((S(\mathrm{pt})\times X)^2\right)\xrightarrow{\ \delta_X^*\ }E\left(S(\mathrm{pt})^2\times X\right)\xrightarrow{\ *_!\ }E(S(\mathrm{pt})\times X).$$

The claim then follows by a tedious but straightforward iterated application of Proposition 2.15. $\qquad\square$

3.7.   Symmetric powers in Tate K-theory

Definition 3.22. For a positive natural number k, we define the operator

$$\beta_k \colon K_{\text{Tate}}(X) \longrightarrow K_{\text{Tate}}(X)$$
$$F \longmapsto \left[\Pi_k^* F(q^{\frac{1}{k}})\right]^{\mathbb{Z}/k\mathbb{Z}},$$

where Π_k is the kth power map of Definition 2.5, and the generator of $\mathbb{Z}/k\mathbb{Z}$ acts on the coefficient $\Pi_k^*(V_a)$ of $q^{\frac{a}{k}}$ by $\xi^1 \cdot e^{-2\pi i \frac{a}{k}}$. So, the $\mathbb{Z}/k\mathbb{Z}$-invariant part consists of the largest possible subspaces of the coefficients such that the result satisfies the rotation condition for ξ^1.

The action of the β_k should be compared to the action of $\widehat{\mathbb{Z}}^\times$ in [**22**].

Say the order of ξ^1 is r and we are given an element $F(q^{\frac{1}{r}}) \in K_{\text{Tate}}(X)$. A priori, it looks like $\beta_k(F)$ is a power series in $q^{\frac{1}{kr}}$, but the rotation condition implies that it is in fact a power series in $q^{\frac{1}{r}}$.

Example 3.23. If $X = M$ is a manifold, then F is of the form $F = \sum_{n=0}^{\infty} V_n q^n$, and

$$\beta_k(F) = \sum_{m=1}^{\infty} V_{km} q^m.$$

This should be compared to [**15**, (2.13)].

Theorem 3.24. *The total symmetric power in Tate K-theory is described by the formula*

$$S_t^{\text{Tate}}(F) = \bigotimes_{k=1}^{\infty} S_{t^k}(\beta_k(F)). \tag{13}$$

Here S_t stands for the total Atiyah symmetric power on K_{Tate}.

Corollary 3.25. *If $X = M$ is a manifold and $F = V$ is a vector bundle on M, then $S_t^{\text{Tate}}(V)$ is Witten's stable exponential characteristic class,*

$$S_t^{\text{Tate}}(V) = \bigotimes_{k=1}^{\infty} S_{t^k}(V)$$

Proof of Theorem 3.24. We will construct a commuting diagram:

$$
\begin{array}{ccccccc}
K(\Lambda X) & =\!= & K(\Lambda X) \\
\downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle(\beta_k)} \\
K(\Phi(X)) \xrightarrow{d^*} K(\Phi \times \Lambda X) \xrightarrow{(a\times\mathrm{id})_!} K(\mathbb{N}_{\geqslant 1}\times\Lambda X) =\!=\!\overset{\sim}{=\!=}\!= \bigoplus_{k\geqslant 1} K(\Lambda X) \\
\downarrow{\scriptstyle P} \quad\quad \downarrow{\scriptstyle P} \quad\quad \downarrow{\scriptstyle P} \quad\quad \downarrow{\scriptstyle\otimes P} \\
K(S\Phi(X)) \xrightarrow{(Sd)^*} K(S(\Phi\times\Lambda X)) \xrightarrow{S(a\times\mathrm{id})_!} K(S(\mathbb{N}_{\geqslant 1}\times\Lambda X)) \longleftarrow \bigotimes_{k\geqslant 1} K(S\Lambda X) \\
\downarrow{\scriptstyle Q_!} \quad\quad \downarrow{\scriptstyle\mathrm{diag}^*_{\Lambda X}} \quad\quad \downarrow{\scriptstyle\mathrm{diag}^*_{\Lambda X}} \quad\quad \downarrow{\scriptstyle\otimes\delta^*} \\
K(\Lambda SX) \quad\quad K(S\Phi\times\Lambda X) \xrightarrow{(Sa\times\mathrm{id})_!} K(S\mathbb{N}_{\geqslant 1}\times\Lambda X) \longleftarrow \bigotimes_{k\geqslant 1} K(S(\mathrm{pt})\times\Lambda X) \\
\downarrow{\scriptstyle\Lambda\delta^*} \nearrow{\scriptstyle(Q\times\mathrm{id})_!} \quad\quad \downarrow{\scriptstyle(e\times\mathrm{id})_!} \quad\quad \downarrow{\scriptstyle\otimes(\varepsilon\times\mathrm{id})_!} \\
K(\Lambda S(\mathrm{pt})\times\Lambda X) \xrightarrow{(\Lambda\varepsilon\times\mathrm{id})_!} K(\mathbb{N}_0\times\Lambda X) \xleftarrow{(s\times\mathrm{id})_!} K(\Pi\times\Lambda X) =\!=\!\overset{\sim}{=\!=}\!= \bigotimes_{k\geqslant 1} K(\mathbb{N}_0\times\Lambda X) \\
\| {\scriptstyle\wr} \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| {\scriptstyle\wr} \\
K(\Lambda X)\llbracket t\rrbracket \xleftarrow{\quad t^k\leftarrow\!\shortmid\, t_k\quad} \bigotimes_{k\geqslant 1} K(\Lambda X)\llbracket t_k\rrbracket.
\end{array}
$$

Again, K is short for K_{rot}. The composite of the six red arrows (along the left edge of the diagram, starting in the upper-left corner and ending at $K(\Lambda X)\llbracket t\rrbracket$) gives the left-hand side of (13), that of the six blue arrows (the composite of the right-most vertical arrows and the bottom arrow) gives the right-hand side. The other arrows are defined as follows: abbreviating $\Phi(\mathrm{pt})$ as Φ, we let

$$d\colon \Phi\times\Lambda X\to\Phi(X)$$

be the restriction of $\Lambda\delta$ to the source of d. So, the kth component d_k of d sends the object g to $(\varsigma_k; g, \ldots, g)$ and the morphism (ς_k^m, h) to $(\varsigma_k^m; h, \ldots, h)$. Let

$$a\colon \Phi\to\mathbb{N}_{\geqslant 1}$$

be the map that sends $\Phi_k(\mathrm{pt})$ to $\{k\}$. Together with θ, these maps form the top square of the diagram.

The three commuting squares in the second row are almost immediate from the properties of P: the right-most follows, by induction over the degree, from the exponential property. The unlabeled horizontal maps out of the tensor products on the very right all are obtained as the (internal) product of projection maps of the form

$$S\left(\coprod_k Y_k\right)\longrightarrow S(Y_j),$$

mapping (y_i) to the list of entries from Y_j.

Next, we have the maps $\mathrm{diag}_{\Lambda X}$ defined as

$$
\begin{aligned}
\mathrm{diag}_{\Lambda X}\colon\ & S\mathbb{N}_{\geqslant 1}\times\Lambda X \longrightarrow S(\mathbb{N}_{\geqslant 1}\times\Lambda X) \\
& ((k_1,\ldots,k_n),g)\longmapsto ((k_1,g),\ldots,(k_n,g)) \qquad \text{on objects} \\
& (\sigma_n,h)\longmapsto (\sigma_n;h,\ldots,h)) \qquad\qquad\ \text{on morphisms},
\end{aligned}
$$

and similarly, with Φ in the role of $\mathbb{N}_{\geqslant 1}$.

Finally, we write Π for the set of isomorphism classes of $\Lambda S(\mathrm{pt})$ or, equivalently, $S\Phi$ or $S\mathbb{N}_{\geqslant 1}$. Elements of Π may be thought of as sequences $(n_k)_{k \geqslant 1}$ with all but finitely many entries equal to zero, and we have the degree map

$$s \colon \Pi \longrightarrow \mathbb{N}_0$$
$$(n_k)_k \longmapsto \sum_k k n_k,$$

and the canonical quotient map

$$e \colon S\mathbb{N}_{\geqslant 1} \longrightarrow \Pi,$$

fitting into the pentagon in the fourth row. The right square in the same row is obtained from the family of commuting squares

$$
\begin{array}{ccc}
S\mathbb{N}_{\geqslant 1} & \xrightarrow{\ \pi_k\ } & S\{k\} \\
\downarrow{\scriptstyle e} & & \downarrow{\scriptstyle \varepsilon} \\
\Pi & \xrightarrow{\ \overline{\pi}_k\ } & k\mathbb{N}_0,
\end{array}
$$

where $\overline{\pi}_k$ is the map of isomorphism classes induced by π_k. The fact that $e_! \circ \prod \pi_k^*$ equals $\bigotimes \varepsilon_!$ is not immediate. The argument boils down to the fact that, for a G-representation V and an H-representation W, the invariant part $(V \otimes W)^{G \times H}$ equals $V^G \otimes W^H$. $\square$

3.8. Adams operations and Hecke operators

The total symmetric power in K-theory has the well-known generating function

$$S_t(x) = \exp\left(\sum_{m \geqslant 1} \frac{\psi_m(x)}{m} t^m \right), \tag{14}$$

where the ψ_m are the *Adams operations*. In fact, (14) is often used as a definition of the Adams operators. As a consequence of Theorem 3.24, we obtain the following generating function for S_t^{Tate}:

$$S_t^{\mathrm{Tate}}(F) = \exp\left(\sum_{m \geqslant 1} T_m(F) t^m \right),$$

where the *Hecke operators* T_m are defined as

$$T_m(F) := \frac{1}{a} \sum_{ad=m} \psi_a\left(\beta_d(F) \right).$$

Example 3.26. It is well known that the Adams operations on the coefficient rings $K_G(\mathrm{pt}) = R(G)$ have the following effect on characters:

$$\psi_m(\chi)(g) = \chi(g^m).$$

Let F be an element of the coefficient ring

$$K_G^{\mathrm{Tate}}(\mathrm{pt}) \cong \bigoplus_{[g]} R(C_g)[\![q^{\frac{1}{|g|}}]\!].$$

As in Section 2.1.3, we view F as the q-expansion of a function $F(g, h; \tau)$, with $q^{\frac{1}{|g|}} = e^{2\pi i \tau / |g|}$. Then

$$\beta_d(F)(g, h; \tau) = \frac{1}{d} \sum_{0 \leqslant b < d} F\left(g^d, g^{-b}h; \frac{\tau + b}{d}\right),$$

and hence

$$T_m(F)(g, h; \tau) = \frac{1}{m} \sum_{ad=m} \sum_{0 \leqslant b < d} F\left(g^d, g^{-b}h^a; \frac{\tau + b}{d}\right).$$

These are the equivariant Hecke operators that play an important role in Moonshine; see [12, 17, 27] and [29].

The equivariant Hecke operators can be defined by an equivariant Hecke correspondence, and this is the sense in which our power operations on K_{Tate} are *'elliptic'*. A natural question is whether our notion of elliptic can be strengthened to resemble the definition of H_∞ *elliptic spectrum* in [5, Def. 16.4]. In other words, do our power operations define *descent data for level structures* on the Tate curve as in [5], and do our power operations specialize to the ones Ando defines in [2, 6.3]?

3.9. Exterior powers and replicability

The *total exterior power* Λ_t^{Tate} is defined by the equality

$$\Lambda_t^{\text{Tate}}(F) = \left(S_{-t}^{\text{Tate}}(F)\right)^{-1},$$

so

$$\Lambda_{-t}^{\text{Tate}}(F) = \exp\left(-\sum_{m \geqslant 1} T_m(F)t^m\right).$$

Let

$$F(q) \in R(G)((q))$$

be of the form

$$F(q) = q^{-1} + a_1 q + a_2 q^2 + \cdots.$$

From the Moonshine literature, such F are known as McKay-Thompson series. We recall[4] that the Faber polynomials $\Phi_{m,F}$ of F are defined by

$$-\sum_{m=1}^{\infty} \Phi_{m,F}(w)t^m = \log\left(t(F(t) - w)\right).$$

Hence $\Phi_{m,F}$ is a polynomial in w of degree m, which depends on the first m coefficients of F and is uniquely characterized by the fact that it is of the form

$$\Phi_{m,F}(F(q)) = q^{-m} + b_1 q + b_2 q^2 + \cdots.$$

---

[4]Compare, e.g., [35, (2.1)] with $b = 1$, $t = 1/z$ and $F(q) = g(z)$.

Definition 3.27. Let F be a McKay-Thompson series. We write $F^{(a)}$ for the ath Adams operations applied to F. We call F *replicable*, if for every natural number m, we have

$$\Phi_{m,F}(F(q)) = \sum_{ad=m} \sum_{0 \leqslant b < d} F^{(a)}\left(\frac{a\tau + b}{d}\right)$$

$$= m \cdot T_m(F)(q).$$

Here $q = e^{2\pi i \tau}$.

This appears to be the right notion of replicability of McKay-Thompson series; it is the one that turns up in [**11**]. It follows that a McKay-Thompson series F is replicable if and only if it satisfies

$$F(t) - F(q) = t^{-1} \cdot \Lambda_{-t}^{\text{Tate}}(F(q)).$$

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