Homology, Homotopy and Applications, vol. 15(1), 2013, pp.137–149

HOMOTOPY TYPE OF SPACE OF MAPS INTO A $K(G, n)$

JAKA SMREKAR

(*communicated by Ronald Brown*)

Abstract

Let X be a connected CW complex and let $K(G, n)$ be an Eilenberg-Mac Lane CW complex where G is abelian. As $K(G, n)$ may be taken to be an abelian monoid, the *weak* homotopy type of the space of continuous functions $X \to K(G, n)$ depends only upon the homology groups of X. The purpose of this note is to prove that this is true for the *actual* homotopy type. Precisely, the space $\mathrm{map}_*\big(X, K(G, n)\big)$ of pointed continuous maps $X \to K(G, n)$ is shown to be homotopy equivalent to the Cartesian product

$$\prod_{i \leqslant n} \mathrm{map}_*\big(M_i, K(G, n)\big).$$

Here, M_i is a Moore complex of type $M\big(H_i(X), i\big)$. The spaces of functions are equipped with the compact open topology.

1. Introduction

Let G be an abelian group, let Y be a CW complex of type $K(G, n)$, and let X be any CW complex. Then Y and consequently $\mathrm{map}_*(X, Y)$ are topological abelian monoids (if G is uncountable then in the category of compactly generated spaces). As has been observed by Thom [**21**] (see also Brown [**2**] and Federer [**6**]), it follows that all Postnikov invariants of (a CW approximation of) $\mathrm{map}_*(X, Y)$ vanish, and $\mathrm{map}_*(X, Y)$ is a *weak* product of Eilenberg-Mac Lane spaces. Thus the weak homotopy type of $\mathrm{map}_*(X, Y)$ is determined by homology groups of X. We show that the actual homotopy type of $\mathrm{map}_*(X, Y)$ is determined by homology groups of X.

Theorem 1.1. *Let X be a connected* CW *complex and let Y be a $K(G, n)$ complex. Set $M_i = M(H_i X, i)$. The space* $\mathrm{map}_*(X, Y)$ *is homotopy equivalent to the product*

$$\mathrm{map}_*(M_1, Y) \times \cdots \times \mathrm{map}_*(M_n, Y).$$

In particular, if X' is another CW *complex and $H_*(X) \cong H_*(X')$ (abstractly), then the spaces* $\mathrm{map}_*(X, Y)$ *and* $\mathrm{map}_*(X', Y)$ *are homotopy equivalent.*

The author was supported in part by the Slovenian Research Agency program No. P1-0292-0101-04 and project No. J1-6128-0101-04.
Received August 6, 2012, revised November 6, 2012; published on April 21, 2013.
2000 Mathematics Subject Classification: 55P15, 54C35, 55P20.
Key words and phrases: function space, homotopy type, Eilenberg-Mac Lane space, abelian monoid.
Article available at `http://intlpress.com/HHA/v15/n1/a8` and `doi:10.4310/HHA.2013.v15.n1.a8`

We emphasize that the spaces $\mathrm{map}_*(X,Y)$ and $\mathrm{map}_*(M_i,Y)$ need not have CW homotopy type. In fact, straightforward criteria for a function space to have CW homotopy type are difficult to obtain in general, and this served as the author's principal motivation (the interested reader is referred to papers [**9, 15, 16, 17**] for more on CW homotopy type of function spaces). The following criterion provides a handy, if somewhat surprising, reduction to less complicated function spaces:

Corollary 1.2. *Given the assumptions of Theorem 1.1,* $\mathrm{map}_*(X,Y)$ *has the homotopy type of a* CW *complex if and only if the spaces* $\mathrm{map}_*(M_i,Y)$*, for* $i \leqslant n$*, have the homotopy type of a* CW *complex.*

Proof. Use Theorem 1.1 as well as Theorem 2 and Proposition 3 of Milnor [**13**]. □

Corollary 1.3. *Given the assumptions of Theorem 1.1, the space of unpointed maps* $\mathrm{map}(X,Y)$ *is homotopy equivalent to* $Y \times \mathrm{map}_*(M_1,Y) \times \cdots \times \mathrm{map}_*(M_n,Y)$.

(Theorem 1.1 and Corollary 1.3 will be proved below.)

Corollary 1.3 can be thought of as a considerable generalization of Corollary 1.4 of Kahn [**9**]. It gives the best possible result for a general CW complex X as domain.

In [**18**], Spanier considered cofunctors $X \mapsto \mathrm{map}\big(X,Y\big)$ with Y a space of type $K(\mathbb{Z},n)$. He investigated only weak homotopy type although one of his original questions (see [**18**, Introduction]) concerned the actual homotopy type. Our results provide some insight into the latter.

Conventions

The terms *map* and *continuous function* will be used synonymously. If X and Y are topological spaces, then $\mathrm{map}(X,Y)$ denotes the space of maps $X \to Y$ with the compact open topology. A fibration is a map with the homotopy lifting property for all spaces. Dually, a cofibration is a map whose image is closed with the homotopy extension property for all spaces. A homotopy equivalence is a map $f\colon X \to Y$ which admits a homotopy inverse, i.e., a map $g\colon Y \to X$ such that the composites gf and fg are homotopic to their respective identities. If $\mathrm{map}(X,Y)$ contains a homotopy equivalence, then X and Y are called homotopy equivalent which we denote $X \simeq Y$. If $X \simeq \{*\}$ then X is called contractible. By Strøm [**20**], this defines a closed model category structure on the category of topological spaces and continuous maps. Hence Theorem 1.1 can be interpreted as a statement within that closed model category.

We topologize $\mathrm{map}\big((X,A),(Y,B)\big) = \{f \in \mathrm{map}(X,Y) \mid f(A) \subset B\}$ as a subspace of $\mathrm{map}(X,Y)$. Taking $A = *$ and $B = *$ yields the space of pointed maps denoted by $\mathrm{map}_*(X,Y)$. If X is a pointed space, then we let SX denote its reduced suspension with the obvious base point. Dually, if Y is a pointed space, then $\Omega Y = \mathrm{map}_*(S^1,Y)$ is the loop space with the constant loop as base point.

Remarks

Let X be a CW complex. Since $Y = K(G,n)$ is homotopy equivalent to $\Omega K(G,n+1)$, the function space $\mathrm{map}_*(X,Y)$ is homotopy equivalent to the space $\mathrm{map}_*\big(X,\Omega K(G,n+1)\big)$ (see Lemma 2.1 below), which in turn is homeomorphic with $\mathrm{map}_*\big(SX,K(G,n+1)\big)$ (see Lemma 2.3 of [**16**]). For the purpose of Theorem 1.1, therefore, $M(A,1)$ can be any CW complex M with $H_1(M) \cong A$ and $\tilde{H}_k(M) = 0$ for $k \neq 1$. In fact it is sufficient to prove Theorem 1.1 when X is simply connected (and $n \geqslant 2$).

If we choose Y to be the geometric realization of a simplicial $K(G,n)$ (see Milnor [**12**]), then Y is a group with continuous inverse $\iota\colon Y \to Y$ and a multiplication $\mu\colon Y \times Y \to Y$ that is continuous on all products $C \times Y$ (and $Y \times C$) for compact subspaces C of Y. It is not difficult to check that in this case $\operatorname{map}_*(X,Y)$ is a group that has an induced continuous inverse and multiplication M, which is continuous on products $\Gamma \times \operatorname{map}_*(X,Y)$ for compact Γ. The standard argument of topological groups applies to deduce that we may arrange for a pointed homotopy equivalence in Theorem 1.1.

Since $\pi_k\big(\operatorname{map}_*(X, K(G,n)), *\big)$ is isomorphic with $\tilde{H}^{n-k}(X;G)$, it follows that the weak homotopy type of $\operatorname{map}_*(X, K(G,n))$ is determined by the cohomology groups $\tilde{H}^j(X;G)$. However, the *actual homotopy type* is not determined by the cohomology groups $\tilde{H}^j(X;G)$, as shown by the following counterexample:

Example

Let G be the product of cyclic groups $\prod_{p\in\mathbb{P}} \mathbb{Z}/p$, where p ranges over the set of all primes $\mathbb{P}$, and let $Y = K(G,n)$. Then $\operatorname{Hom}(\mathbb{Q}/\mathbb{Z}, G) = 0$ since the quotient $\mathbb{Q}/\mathbb{Z} \cong \oplus_{p\in\mathbb{P}}\mathbb{Z}_{p^\infty}$ is divisible and G is reduced. On the other hand, $\operatorname{Ext}(\mathbb{Q}/\mathbb{Z}, G) \cong \prod_{p\in\mathbb{P}} \operatorname{Ext}(\mathbb{Q}/\mathbb{Z}, G) \cong \prod_{p\in\mathbb{P}} \mathbb{Z}/p = G$. Hence, if we take $2 \leqslant m \leqslant n-1$ and let $X = M(\mathbb{Q}/\mathbb{Z}, m-1)$, then we have $\tilde{H}^j(X;G) \cong \tilde{H}^j(S^m;G)$ for all j. However, by Theorem 3 of Milnor [**13**], $\operatorname{map}_*(S^m, Y)$ has CW homotopy type, while the space $\operatorname{map}_*(X,Y)$ does not have CW homotopy type. If it did, then by [**16**, Proposition 4.6] (take $P = \mathbb{P}$), all but finitely many of the spaces $Z_p = \operatorname{map}_*\big(M(\mathbb{Z}_{p^\infty}, m-1), Y\big)$ would be contractible. But none are contractible as $\pi_{n-m}(Z_p,*) \cong \tilde{H}^m(M(\mathbb{Z}_{p^\infty}, m-1); G) \cong \mathbb{Z}/p$. Hence $\operatorname{map}_*(S^m,Y)$ and $\operatorname{map}_*(X,Y)$ are not homotopy equivalent.

Discussion of (im)possible generalizations

Generally, even the weak homotopy type of a mapping space $\operatorname{map}_*(X,Y)$ is not determined merely by homology groups of X. For a striking example of this, the space of pointed maps from $\mathbb{R}P^\infty$ to S^2 is weakly contractible by celebrated Miller's theorem [**11**], while if we take $X = \bigvee_{n=1}^\infty M(\mathbb{Z}/2, 2n-1)$, the mapping space $\operatorname{map}_*(X, S^2)$ which is homeomorphic with the Cartesian product $\prod_{n=1}^\infty \operatorname{map}_*(M(\mathbb{Z}/2, 2n-1), S^2)$ has plenty of nontrivial homotopy groups as there is plenty of 2-torsion in the homotopy groups of S^2.

Clearly, Theorem 1.1 implies the analogous result for Y a finite product of Eilenberg-Mac Lane spaces (of abelian groups). On the other hand, I do not know if the result can be extended to CW complexes (simple, say) with trivial Postnikov invariants but infinitely many nontrivial homotopy groups.

It seems difficult to achieve results about $\operatorname{map}(X,Y)$ of the same strength and at the same level of generality as in Theorem 1.1 even when Y is a two-stage Postnikov space. Namely, if $k\colon K(A,r) \to K(B,n+1)$ is the k-invariant in question, then the problem of identifying the homotopy class of the induced map $k_\#\colon \operatorname{map}(X, K(A,r)) \to \operatorname{map}(X, K(B,n+1))$ is not an easy one if X is an infinite complex.

If X is a finite complex, however, then by Theorem 3 of Milnor [**13**], $\operatorname{map}(X, Y)$ is homotopy equivalent to a CW complex. In that case, its homotopy type is determined by its weak homotopy type, which in turn can be recovered from the simplicial mapping space model, and hence the technique of Brown [**2**] may be used to compute the homotopy type of $\operatorname{map}(X, Y)$ from the Postnikov invariants of Y. It would be interesting to know if the results of of [**2**] could be used for determination of the actual homotopy type of $\operatorname{map}(X, Y)$ from the Postnikov decomposition of Y when X is infinite.

Another direction would be to try to generalize Theorem 1.1 to spaces of maps $\operatorname{map}(X, BC)$, where BC is the classifying space of a crossed complex as in Brown and Higgins [**4**], viewing Theorem A therein itself a generalization of Thom's result [**21**] to more general classifying spaces.

Outline of proof of Theorem 1.1

To a simply connected CW complex X, we associate a *homology decomposition*, i.e., a sequence of subcomplex inclusions $\{*\} = X_1 \leqslant X_2 \leqslant X_3 \leqslant \cdots$, where each inclusion $X_{i-1} \to X_i$ is a principal cofibration with homotopy cofibre of type $M(H_i X, i)$, and the union (colimit) complex $\cup_i X_i$ is homotopy equivalent to X. For any complex Y, the mapping space $\operatorname{map}_*(X, Y)$ may be viewed as the limit space of the induced inverse sequence $\cdots \to \operatorname{map}_*(X_3, Y) \to \operatorname{map}_*(X_2, Y) \to \operatorname{map}_*(X_1, Y)$. We prove Theorem 1.1 by showing that if Y is a $K(G, n)$, then, in fact, $\operatorname{map}_*(X, Y)$ is homotopy equivalent to $\operatorname{map}_*(X_n, Y)$ and, in addition, the fibrations

$$\operatorname{map}_*(X_i, Y) \to \operatorname{map}_*(X_{i-1}, Y) \tag{1}$$

are all fibre homotopically trivial, i.e., homotopy equivalent to product fibrations.

2. Principal fibration induced by cone adjunction

We first recall that the homotopy type of $\operatorname{map}(X, Y)$ (respectively, of $\operatorname{map}_*(X, Y)$) depends only upon the homotopy types (respectively, pointed homotopy types) of X and Y. Next, we carefully describe the properties of the principal fibration of mapping spaces $\operatorname{map}(C_\varphi, Y) \to \operatorname{map}(L, Y)$, where C_φ is the mapping cone of a map $\varphi\colon A \to L$.

Lemma 2.1. *Let $\varphi\colon (A, A_1) \to (X, X_1)$ and $\psi\colon (Y, Y_1) \to (B, B_1)$ be maps of pairs and define $\Phi\colon \operatorname{map}(X, Y) \to \operatorname{map}(A, B)$ by $f \mapsto \psi f \varphi$. If φ and ψ are homotopy equivalences of pairs, then $\Phi\colon \operatorname{map}\big((A, A_1), (X, X_1)\big) \to \operatorname{map}\big((B, B_1), (Y, Y_1)\big)$ is a homotopy equivalence.*

Proof. See Maunder [**10**, Theorem 6.2.25 and its proof]. □

Proposition 2.2. *Let $\varphi\colon A \to L$ be a cellular map of* CW *complexes (with respect to some decomposition of L) and let $C_\varphi = CA \sqcup_\varphi L$ denote the mapping cone of φ (reduced or unreduced). Pick $a_0 \in A$ and set $x_0 = \varphi(a_0)$. Let (Y, y_0) be any pointed* CW *complex.*

1. *The following is a pullback diagram:*

$$\begin{array}{ccc} \operatorname{map}\big((C_\varphi, x_0),(Y,y_0)\big) & \longrightarrow & \operatorname{map}\big((CA, a_0),(Y,y_0)\big) \\ \big\downarrow R & & \big\downarrow r \\ \operatorname{map}\big((L, x_0),(Y,y_0)\big) & \longrightarrow & \operatorname{map}\big((A, a_0),(Y,y_0)\big). \end{array} \tag{2}$$

   Vertical arrows R and r are fibrations and $\operatorname{map}\big((CA, a_0),(Y,y_0)\big)$ *is contractible. This renders R a principal fibration with all fibres either empty or homotopy equivalent to the loop space* $\Omega\big(\operatorname{map}\big((A,a_0),(Y,y_0)\big), \operatorname{const}_{y_0}\big)$ *which is homeomorphic with* $\operatorname{map}\big((SA,*),(Y,y_0)\big)$.

2. *If (X, L) is a* CW *pair and φ induces a homotopy equivalence $C_\varphi \to X$, then*

$$\operatorname{map}\big((C_\varphi, x_0),(Y,y_0)\big) \to \operatorname{map}\big((X, x_0),(Y,y_0)\big) \tag{3}$$

   is a fibre homotopy equivalence over $\operatorname{map}\big((L,x_0),(Y,y_0)\big)$*. Also, the restriction map* $\operatorname{map}(C_\varphi, Y) \to \operatorname{map}(X,Y)$ *is a fibre homotopy equivalence over* $\operatorname{map}(L,Y)$.

Proof. It is not difficult to check that the square is a pullback since it is dual to the pushout diagram of the mapping cone adjunction of a cellular map. The vertical arrows are fibrations because they are restrictions to cofibred subspaces (see Lemma A.2 of [**16**], for example). The space $\operatorname{map}\big((CA, a_0),(Y,y_0)\big)$ is contractible by Lemma 2.1, hence (2) is the diagram of a principal fibration. (It is equivalent to the 'standard one' when we are pulling back the path fibration $P\operatorname{map}\big((A,a_0),(Y,y_0)\big) \to \operatorname{map}\big((A,a_0),(Y,y_0)\big)$ by the coglueing theorem of Brown and Heath [**3**].) Therefore, every nonempty fibre of R is homotopy equivalent to the fibre of r over $\operatorname{const}_{y_0}$, that is, precisely, $\operatorname{map}\big((CA, A),(Y,y_0)\big)$. The claim about the fibres now follows from Lemma 2.3 of [**16**].

By Lemma 2.1, the map (3) is a homotopy equivalence, hence it is a fibre homotopy equivalence by [**3**, Corollary 3.7] (similarly for the unpointed version). □

Remark 2.3. The harmless looking Proposition 2.2 deserves a comment. In our most important application (end of the proof of Lemma 5.1 below, fibre homotopy equivalence $F^{\#}$) the base function space $\operatorname{map}_*(L,Y)$ will not be assumed to have CW homotopy type. In fact, the homotopy type of a mapping space can be rather gruesome. (See [**16**] for examples.) Hence it is important that no assumptions on the base space be necessary for the application of Corollary 3.7 of [**3**] (or, equivalently, Theorem 6.1 of Dold [**5**]) as opposed to, for example, Theorem 6.3 of [**5**].

3. Minimal decompositions

To show that the fibrations (1) are fibre homotopy trivial, we associate to the domain complex X a homotopy equivalent CW complex with the least possible number of cells and, equally important, with gluing maps whose images meet 'only those cells which they are supposed to meet'. Such a representative for X is called, by abuse of language, a minimal decomposition. The corresponding homology decomposition will be defined in Definition 3.2 below.

Lemma 3.1 (Minimal decomposition)**.** *Given a simply connected* CW *complex X and a specific free presentation of each of its homology groups $H_n(X) \cong \langle S_n; \Sigma_n \rangle$ ($n \geqslant 2$), there are a* CW *complex Z and a homotopy equivalence $f\colon Z \to X$ such that each cell of Z is either:*

(1) A 'generator' n-cell e^n_α, which is a cycle in cellular homology mapped by f to a cellular cycle representing the specified generator $\alpha \in S_n$; or

(2) A 'relator' $(n+1)$-cell e^{n+1}_β with cellular boundary corresponding to the specified relator $\beta \in \Sigma_{n+1}$.

Also, we may assume that all cells are attached along based maps of spheres, and that

(2′) The closure of an n-cell e^n meets an $(n-1)$-cell e^{n-1} if and only if the incidence number $[e^n : e^{n-1}]$ is non-zero. These are necessarily only generator $(n-1)$-cells. In particular, each generator n-cell e^n_α is attached along a (based) gluing map $\varphi^n_\alpha\colon S^{n-1} \to Z^{(n-2)}$ of the $(n-1)$-sphere into the $(n-2)$-skeleton.

Proof. Theorem 4C.1 of Hatcher [**8**] states (1) and (2) for X of finite type. The finite type restriction is unnecessary, and the generalization to (2′) is easy. (Compare also Rutter [**14**, Lemma 2.1].) □

Definition 3.2. A CW decomposition with the properties of Z as in Lemma 3.1 will be called *minimal*. When speaking about a minimal decomposition of a simply connected CW complex X (with some free presentations of its homology groups) it will be tacitly assumed that X has been replaced with the homotopy equivalent complex Z guaranteed by Lemma 3.1.

To a minimal decomposition of a complex X, we may associate a homology filtration (homology decomposition)

$$\{*\} = X_1 \leqslant X_2 \leqslant X_2 \leqslant X_3 \leqslant \cdots$$

by letting X_i be the union of the i-skeleton $X^{(i)}$ of X and all the relator $(i+1)$-cells. Then the inclusion induced morphism $H_j(X_i) \to H_j(X)$ is bijective for $j \leqslant i$, and $H_j(X_i) = 0$ for $j > i$. Moreover, if we set $H_i = H_i(X)$, then the quotient X_i/X_{i-1} is a Moore complex $M(H_i, i)$, and there exists a map $\varphi\colon M(H_i, i-1) \to X_{i-1}$ which induces a homotopy equivalence $C_\varphi \to X_i$. Here C_φ is the homotopy cofibre of φ.

4. Reduction to a finite dimensional domain

Let X be any CW complex and let Y be a CW complex with $\pi_k(Y)$ trivial for $k > n$. The purpose of this section is to show that if A is any subcomplex of X containing the n-skeleton $X^{(n)}$, then the restriction fibration $\mathrm{map}_*(X, Y) \to \mathrm{map}_*(A, Y)$ is a homotopy equivalence onto image. This is used to show that if X_n is the n-th stage of the homology decomposition of X, then the mapping $\mathrm{map}_*(X, K(G, n)) \to \mathrm{map}_*(X_n, K(G, n))$ is actually a homotopy equivalence.

Lemma 4.1. *Let Y be a connected* CW *complex with $\pi_i(Y) = 0$ for $i \geqslant n$, where $n \geqslant 1$. The function space $\mathrm{map}_*(S^n, Y)$ is homotopy equivalent to a discrete space, while, for $m > n$, the space $\mathrm{map}_*(S^m, Y)$ is contractible.*

Proof. Since $Z_m = \operatorname{map}_*(S^m, Y)$ is an H-group, all path-components of Z_m are homotopy equivalent. By Theorem 3 of Milnor [**13**], the space Z_m has the homotopy type of a CW complex. Hence all path-components of Z_m are open, and each has the homotopy type of a CW complex. We consider the path-component of the constant loop const: $S^m \to \{*\} \subset Y$. Note that the homotopy group $\pi_i(Z_m, \text{const}) = \big[(S^i, *), \big(\operatorname{map}_*(S^m, Y), \text{const}\big)\big]$ can be identified with the set of path-components of $\operatorname{map}_*\big(S^i, \operatorname{map}_*(S^m, Y)\big)$. By the pointed version of the exponential law, the latter function space is homeomorphic with $\operatorname{map}_*(S^i \wedge S^m, Y) = \operatorname{map}_*(S^{i+m}, Y)$ (the proof of Lemma 2.3 of [**16**] can be used almost verbatim). Hence $\pi_i(Z_m, \text{const}) \cong \pi_{i+m}(Y, *)$. An application of Whitehead's theorem concludes the proof. □

Proposition 4.2. *Let X be a* CW *complex with a single* 0*-cell x_0. Assume that all cells are based at x_0, i.e., each n-cell is attached along a based map $(S^{n-1}, *) \to (X^{(n-1)}, x_0)$. Let A be a subcomplex of X such that the relative* CW *complex (X, A) has cells of dimension at least k. Finally, let Y be a connected* CW *complex such that $\pi_i(Y) = 0$ for $i \geqslant n+1$. The restriction fibration*

$$R\colon \operatorname{map}_*(X, Y) \to \operatorname{map}_*(A, Y)$$

is a homotopy equivalence onto image if $k \geqslant n+1$. It is surjective if $k \geqslant n+2$.

Proof. Set $X_A^{(m)} = A \cup X^{(m)}$ and note that, by assumption, $X_A^{(k-1)} = A$. For each m, the relative skeleton $X_A^{(m)}$ is the reduced mapping cone of a based map $\varphi_m\colon \vee_{\lambda\in\Lambda_m} S_\lambda^{m-1} \to X_A^{(m-1)}$. By Proposition 2.2, the induced fibration $R_m\colon \operatorname{map}_*\big(X_A^{(m)}, Y\big) \to \operatorname{map}_*\big(X_A^{(m-1)}, Y\big)$ is equivalent to the homotopy fibre of

$$\varphi_m^{\#}\colon \operatorname{map}_*\big(X_A^{(m-1)}, Y\big) \to \operatorname{map}_*\big(\vee_{\lambda\in\Lambda_m} S_\lambda^{m-1}, Y\big).$$

Denote $W_m = \operatorname{map}_*\big(\vee_{\lambda\in\Lambda_m} S_\lambda^{m-1}, Y\big)$.

Note that, for any family of based CW complexes $\{(T_\lambda, *_\lambda) \mid \lambda\}$, the evident function from $\operatorname{map}_*(\vee_\lambda T_\lambda, Y)$ to the Cartesian product $\prod_\lambda \operatorname{map}_*(T_\lambda, Y)$ is a homeomorphism. Using this observation together with Lemma 4.1, it follows that for $m \geqslant n+2$, the space W_m is contractible. Therefore, R_m is a homotopy equivalence if $m \geqslant n+2$. Next, the space W_{n+1} is homotopy equivalent to a totally disconnected space (precisely, to a Cartesian product of discrete spaces) all of whose path-components are contractible. This means that the image C_{n+1} of $\operatorname{map}_*\big(C(\vee_{\lambda\in\Lambda_{n+1}} S_\lambda^n), Y\big) \to W_{n+1}$, which is exactly the path-component of the constant map in W_{n+1}, is contractible. As R_{n+1} is a principal fibration, it follows that $R_{n+1}\colon \operatorname{map}_*\big(X_A^{(n+1)}, Y\big) \to \operatorname{im}(R_{n+1})$ is equivalent to the homotopy fibre of $\operatorname{im}(R_{n+1}) \to C_{n+1}$. Hence R_{n+1} is a homotopy equivalence onto image. By Geoghegan [**7**], the restriction fibration $\operatorname{map}_*(X, Y) \to \operatorname{map}_*(X^{(n+1)}, Y)$ is a homotopy equivalence, being a canonical projection corresponding to the inverse sequence of fibrations R_m, for $m \geqslant n+2$, that are homotopy equivalences. This concludes the proof of the proposition. □

Lemma 4.3. *Let Y be an Eilenberg-Mac Lane space $K(G, n)$ and let X be a simply connected* CW *complex with homology filtration $X_2 \leqslant X_3 \leqslant \cdots$ associated to a minimal decomposition. Then $\operatorname{map}_*(X, Y) \to \operatorname{map}_*(X_n, Y)$ is a homotopy equivalence.*

Proof. By Proposition 4.2 we know that $\mathrm{map}_*(X,Y) \to \mathrm{map}_*\big(X^{(n+1)},Y\big)$ is a homotopy equivalence and that

$$\mathrm{map}_*\big(X^{(n+1)},Y\big) \to \mathrm{map}_*(X_n,Y) \tag{$*$}$$

is a homotopy equivalence onto image. It suffices to prove that $(*)$ is surjective. By assumption, $X^{(n+1)}$ is obtained from X_n by attaching generator $(n+1)$-cells, that is, $X^{(n+1)}$ is the cofibre of a map

$$\varphi\colon \bigvee_\lambda S^n \to X^{(n-1)} \hookrightarrow X_n.$$

Restriction $(*)$ is the homotopy fibre of the induced function $\varphi^\#\colon \mathrm{map}_*(X_n,Y) \to \mathrm{map}_*\big(\vee_\lambda S^n,Y\big)$, which factors through $\mathrm{map}_*(X^{(n-1)},Y)$. Since Y is a $K(G,n)$, on the group of path components $\varphi^\#$ transforms as

$$\tilde{H}^n(X_n;G) \to \tilde{H}^n(X^{(n-1)};G) \to \tilde{H}^n(\vee_\lambda S^n;G).$$

But $\tilde{H}^n(X^{(n-1)};G)$ is trivial; hence the homotopy fibre $(*)$ is surjective. □

5. Proof of Theorem 1.1 and Corollary 1.3

Proof of Theorem 1.1. We assume a minimal decomposition for X and take the associated homology filtration $X_2 \leqslant X_3 \leqslant \cdots$. By Lemma 4.3 we know that the restriction $\mathrm{map}_*(X,Y) \to \mathrm{map}_*(X_n,Y)$ is a homotopy equivalence, and we may replace X by its n-th homology stage X_n. Hence it suffices to prove Lemma 5.1 below. □

Lemma 5.1. *Let $Y = K(G,n)$ and $3 \leqslant i \leqslant n$. The restriction fibration $\mathrm{map}_*(X_i,Y) \to \mathrm{map}_*(X_{i-1},Y)$ is fibre homotopy trivial.*

The following step is crucial.

Lemma 5.2. *Given the assumptions of Lemma 5.1, let (L',L) be the adjunction of an i-cell e attached along a based map*

$$\varphi\colon (S^{i-1},*) \to (L^{(i-2)},x_0)$$

of the $(i-1)$-sphere to the $(i-2)$-skeleton of L. Denote by K the smallest subcomplex of L that contains the image of φ. Note that K is finite and contained in $L^{(i-2)}$. Set $K' = K \cup e$.

Then the induced map $\varphi^\#\colon \mathrm{map}_*(K,Y) \to \mathrm{map}_*(S^{i-1},Y)$ is nullhomotopic as a pointed map. Hence the restriction fibration $\mathrm{map}_*(K',Y) \to \mathrm{map}_*(K,Y)$ is fibre-homotopy trivial and consequently so is $\mathrm{map}_*(L',Y) \to \mathrm{map}_*(L,Y)$. There are sections $s\colon \mathrm{map}_*(L,Y) \to \mathrm{map}_*(L',Y)$ and $s_K\colon \mathrm{map}_*(K,Y) \to \mathrm{map}_*(K',Y)$ giving rise to a commutative diagram as follows:

$$\begin{array}{ccc} \mathrm{map}_*(L', Y) & \longrightarrow & \mathrm{map}_*(K \cup e, Y) \\ {\scriptstyle s}\uparrow & & \uparrow{\scriptstyle s_K} \\ \mathrm{map}_*(L, Y) & \longrightarrow & \mathrm{map}_*(K, Y). \end{array}$$

In other words, if $f, g\colon L \to Y$ are continuous functions with $f|_K = g|_K$, then the extensions $s(f)$ and $s(g)$ restrict to the same function on $K \cup e$.

Proof. Proposition 2.2 yields the following pullback diagram:

$$\begin{array}{ccc} \mathrm{map}_*(L', Y) & \longrightarrow & \mathrm{map}_*(B^i, Y) \\ \downarrow & & \downarrow \\ \mathrm{map}_*(L, Y) & \xrightarrow{\varphi^{\#}} & \mathrm{map}_*(S^{i-1}, Y). \end{array}$$

This is to say that the fibration $\mathrm{map}_*(L', Y) \to \mathrm{map}_*(L, Y)$ is the principal fibration obtained as the homotopy fibre of $\varphi^{\#}$. The latter factors as

$$\mathrm{map}_*(L, Y) \to \mathrm{map}_*(K, Y) \xrightarrow{\varphi^{\#}} \mathrm{map}_*(S^{i-1}, Y).$$

Since K is finite, $\mathrm{map}_*(K, Y)$ is globally well-pointed (see [**15**, Lemmas 3.2–3.4]) and has the homotopy type of a CW complex. Therefore, $\mathrm{map}_*(K, Y)$ has the pointed homotopy type of a CW complex for any choice of base point. Thus we compute

$$[\mathrm{map}_*(K, Y), \mathrm{map}_*(S^{i-1}, Y)]_* \cong \tilde{H}^{n-i+1}\big(\mathrm{map}_*(K, Y); G\big).$$

Note that $\pi_k\big(\mathrm{map}_*(K, Y), \mathrm{const}\big) \cong \tilde{H}^{n-k}(K; G)$ is trivial for $n - k \geqslant (i-2) + 1$ since K is $(i-2)$-dimensional. By the Hurewicz theorem and universal coefficients, $\tilde{H}^{n-i+1}\big(\mathrm{map}_*(K, Y); G\big)$ is trivial. Therefore, $\varphi^{\#}$ is nullhomotopic as a pointed map. Let $h\colon \mathrm{map}_*(K, Y) \times I \to \mathrm{map}_*(S^{i-1}, Y)$ denote a homotopy between the constant map and $\varphi^{\#}$. Then a section $s_K\colon \mathrm{map}_*(K, Y) \to \mathrm{map}_*(K', Y)$ is given by

$$s_K(f)|_K = f, \quad s_K(f)(\phi[\zeta, t]) = h(f, t)(\zeta).$$

Here, $[\_, \_]$ denotes the quotient map $S^{i-1} \times I \to S^{i-1} \times I / S^{i-1} \times 0 = B^i$, and $\phi\colon B^i \to K \cup e$ denotes the characteristic map of the cell e.

The pre-composition of h with restriction $\mathrm{map}_*(L, Y) \to \mathrm{map}_*(K, Y)$ gives a pointed trivialization of $\varphi^{\#}\colon \mathrm{map}_*(L, Y) \to \mathrm{map}_*(S^{i-1}, Y)$. The corresponding section $s\colon \mathrm{map}_*(L, Y) \to \mathrm{map}_*(L', Y)$, defined by

$$s(f)|_L = f, \;\; s(f)\big(\phi[\zeta, t]\big) = h(f|_K, t)(\zeta),$$

lifts the section s_K, as claimed. □

Proof of Lemma 5.1. We construct X_i from X_{i-1} in two stages. First, we attach all the generator i-cells via an attaching map

$$\varphi = \vee_\lambda \varphi_\lambda\colon \bigvee_\lambda S^{i-1} \to X_{i-1}^{(i-2)} \hookrightarrow X_{i-1}$$

to obtain $X^{(i)}$. By Proposition 2.2 the map φ induces the pullback diagram below:

$$\begin{array}{ccccc}
\operatorname{map}_*(X^{(i)},Y) & \longrightarrow & \operatorname{map}_*(\vee_\lambda B^i,Y) & \xrightarrow{=} & \prod_\lambda \operatorname{map}_*(B^i,Y) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\operatorname{map}_*(X_{i-1},Y) & \xrightarrow{\varphi^\#} & \operatorname{map}_*(\vee_\lambda S^{i-1},Y) & \xrightarrow{=} & \prod_\lambda \operatorname{map}_*(S^{i-1},Y).
\end{array}$$

The induced map $\varphi^\#$ maps into the product $\prod_\lambda \operatorname{map}_*(S^{i-1},Y)$. The composition with the λ-th projection $\operatorname{pr}_\lambda \circ \varphi^\#$ equals $\varphi_\lambda^\#$, which is nullhomotopic as a pointed map by Lemma 5.2. Hence $\varphi^\#$ is nullhomotopic by a product homotopy. Let K_λ denote the smallest subcomplex of X that contains $\operatorname{im}\varphi_\lambda$, and fix homotopies $h_\lambda\colon \operatorname{map}_*(K_\lambda,Y)\times I \to \operatorname{map}_*(S^{i-1},Y)$ between $*$ and $\varphi_\lambda^\#$. Then we can define a section $s\colon \operatorname{map}_*(X_{i-1},Y) \to \operatorname{map}_*(X^{(i)},Y)$ by

$$s(f)|_{X_{i-1}} = f, \;\; s(f)\big(\phi_\lambda[\zeta,t]\big) = h_\lambda(f|_{K_\lambda},t)(\zeta).$$

As above, B^i is identified with the quotient $S^{i-1}\times I/S^{i-1}\times 0$, and $\phi_\lambda\colon B^i \to X^{(i)}$ are characteristic maps of the attached cells.

Now we attach the relator $(i+1)$-cells to $X^{(i)}$ via an attaching map

$$\psi = \vee_\mu \psi_\mu\colon \bigvee_\mu S^i \to X^{(i)}.$$

We obtain a pullback diagram similar to the one above:

$$\begin{array}{ccc}
\operatorname{map}_*(X_i,Y) & \longrightarrow & \operatorname{map}_*(\bigvee_\mu B^{i+1},Y) \\
\big\downarrow & & \big\downarrow \\
\operatorname{map}_*(X^{(i)},Y) & \xrightarrow{\psi^\#} & \operatorname{map}_*(\bigvee_\mu S^i,Y).
\end{array}$$

We claim that we can lift s to a map (necessarily a section) $\sigma\colon \operatorname{map}_*(X_{i-1},Y) \to \operatorname{map}_*(X_i,Y)$. To prove this we observe projections of the composite

$$\operatorname{map}_*(X_{i-1},Y) \xrightarrow{s} \operatorname{map}_*(X^{(i)},Y) \xrightarrow{\psi^\#} \operatorname{map}_*(\bigvee_\mu S^i,Y) = \prod_\mu \operatorname{map}_*(S^i,Y)$$

onto the factors. Let e_μ be an $(i+1)$-relator cell with attaching map $\psi_\mu\colon S^i \to X^{(i)}$, and let L denote the smallest subcomplex containing the image of ψ_μ. Note that L contains only *generator* i-cells; by property **b.'** of Lemma 3.1. (On $L\cap X^{(i-1)}$ there are no particular restrictions.) We number the i-cells $e_1,\dots,e_r$ and let K denote $L\setminus\{e_1,\dots,e_r\}$. Then K is a finite subcomplex contained in $X^{(i-1)}\subset X_{i-1}$. By construction, the following diagram is commutative:

$$\begin{array}{ccccc} \mathrm{map}_*(X^{(i)},Y) & \longrightarrow & \mathrm{map}_*(L,Y) & \xrightarrow{\psi_\mu^{\#}} & \mathrm{map}_*(S^i,Y) \\ \uparrow{\scriptstyle s} & & \uparrow{\scriptstyle s_K} & & \\ \mathrm{map}_*(X_{i-1},Y) & \longrightarrow & \mathrm{map}_*(K,Y). & & \end{array}$$

Here, s_K is defined by $s_K(g)|_K = g$, and $s_K(\phi_\lambda[\zeta,t]) = h_\lambda(g|_{K_\lambda},t)(\zeta)$ whenever e_λ is one of the $e_1, e_2, \ldots, e_r$ in which case, tautologically, $K_\lambda \subset K$.

We compute

$$[\mathrm{map}_*(K,Y), \mathrm{map}_*(S^i,Y)]_* \cong \tilde{H}^{n-i}\big(\mathrm{map}_*(K,Y);G\big).$$

Note that $\pi_k\big(\mathrm{map}_*(K,Y),\mathrm{const}\big) \cong \tilde{H}^{n-k}(K;G)$ is trivial for $n-k \geqslant (i-1)+1$, i.e., $k \leqslant n-i$, since K is $(i-1)$-dimensional. By the Hurewicz theorem and universal coefficients, the group $[\mathrm{map}_*(K,Y), \mathrm{map}_*(S^i,Y)]_*$ is trivial, and therefore $\psi_\mu^{\#} \circ s_K$ is nullhomotopic (as a pointed map). Hence so is the pre-composite with restriction $\mathrm{map}_*(X_{i-1},Y) \to \mathrm{map}_*(K,Y)$, which equals $\psi_\mu^{\#} \circ s\colon \mathrm{map}_*(X_{i-1},Y) \to \mathrm{map}_*(S^i,Y)$ by commutativity. Hence the map into the product $\psi^{\#} \circ s$ is also nullhomotopic, and s may be lifted to $\sigma\colon \mathrm{map}_*(X_{i-1},Y) \to \mathrm{map}_*(X_i,Y)$, which is automatically a section. There exists a (pointed cellular) map $k\colon M(H_iX, i-1) \to X_{i-1}$ such that the inclusion $X_{i-1} \to X_i$ extends to a homotopy equivalence $F\colon C_k \to X_i$ with C_k the mapping cone of k. Proposition 2.2 gives the following commutative diagram, where the square is a pullback and $F^{\#}$ is a fibre homotopy equivalence:

$$\begin{array}{ccccc} \mathrm{map}_*(X_i,Y) & \xrightarrow{F^{\#}} & \mathrm{map}_*(C_k,Y) & \longrightarrow & \mathrm{map}_*(CM(H_iX,i-1),Y) \\ & \searrow & \downarrow & & \downarrow \\ & & \mathrm{map}_*(X_{i-1},Y) & \xrightarrow{k^{\#}} & \mathrm{map}_*(M(H_iX,i-1),Y). \end{array}$$

The section of $\mathrm{map}_*(X_i,Y) \to \mathrm{map}_*(X_{i-1},Y)$ yields a section of $\mathrm{map}_*(C_k,Y) \to \mathrm{map}_*(X_{i-1},Y)$. Since $\mathrm{map}_*(CM(H_iX,i-1),Y)$ is contractible, it follows that $k^{\#}$ is nullhomotopic. Hence $\mathrm{map}_*(C_k,Y) \to \mathrm{map}_*(X_{i-1},Y)$ is fibre homotopy equivalent to a product fibration. Since $F^{\#}$ is a fibre homotopy equivalence, $\mathrm{map}_*(X_i,Y) \to \mathrm{map}_*(X_{i-1},Y)$ is fibre homotopy equivalent to a product fibration with fibre $\mathrm{map}_*(SM(H_iX,i-1),Y) \simeq \mathrm{map}_*(M(H_iX,i),Y)$, as claimed. □

Proof of Corollary 1.3. Let $\mu\colon Y \times Y \to Y$ be the multiplication on $Y = K(G,n)$, and let $M\colon \mathrm{map}(X,Y) \times \mathrm{map}(X,Y) \to \mathrm{map}(X,Y)$ be the induced multiplication on $\mathrm{map}(X,Y)$, defined by $M(f,g)\colon x \mapsto \mu(f(x),g(x))$. We write $\mu(y_1,y_2) = y_1 \cdot y_2$ and $M(f,g) = f \cdot g$. Next, we use y^{-1} for the inverse, and note that $y \mapsto y^{-1}$ is always continuous. If $y \in Y$ and $g \in \mathrm{map}(X,Y)$, then $y \cdot g$ is shorthand for $\mathrm{const}_y \cdot g$, i.e., the continuous map $x \mapsto y \cdot g(x)$.

If G is countable, then μ, and hence M are continuous and the assignment $\Phi_X\colon \mathrm{map}(X,Y) \to Y \times \mathrm{map}_*(X,Y)$, $f \mapsto \big(f(x_0), f(x_0)^{-1} \cdot f\big)$ is a homeomorphism with inverse $\Psi_X\colon (y,g) \mapsto y \cdot g$. The subscript X is employed to indicate the evident functorial dependence on X.

If G is not countable, then μ and M are certainly continuous on compact subsets. To exploit that, we will make use of the 'convenient category' introduced by Brown [**1**] and popularized by Steenrod [**19**].

There exists a pointed homotopy equivalence $\alpha\colon Y \to Y'$ with pointed homotopy inverse $\beta\colon Y' \to X$, where Y' is a simplicial complex with the metric topology (see Theorem 2 of [**13**]). If L is any subcomplex of X, then the associated maps $\beta^{\#}\colon \operatorname{map}(L, Y') \to \operatorname{map}(L, Y)$ and $\alpha^{\#}\colon \operatorname{map}(L, Y) \to \operatorname{map}(L, Y')$ are mutually inverse homotopy equivalences that are clearly natural in L. In addition, the 'standard' homotopies $\beta^{\#}\alpha^{\#} \simeq \mathrm{id}$ and $\alpha^{\#}\beta^{\#} \simeq \mathrm{id}$ (see Maunder [**10**, Proof of 6.2.25]) are also natural in L. The analogous remark holds for the pointed version.

Let $F_L\colon \operatorname{map}(L, Y') \to Y' \times \operatorname{map}_*(L, Y')$ denote the composite

$$\operatorname{map}(L, Y') \xrightarrow{\beta^{\#}} \operatorname{map}(L, Y) \xrightarrow{\Phi_L} Y \times \operatorname{map}_*(L, Y) \xrightarrow{\alpha\times\alpha^{\#}} Y' \times \operatorname{map}_*(L, Y'),$$

and let $G_L\colon Y' \times \operatorname{map}_*(L, Y') \to \operatorname{map}(L, Y')$ denote the composite

$$Y' \times \operatorname{map}_*(L, Y') \xrightarrow{\beta\times\beta^{\#}} Y \times \operatorname{map}_*(L, Y) \xrightarrow{\Psi_L} \operatorname{map}(L, Y) \xrightarrow{\alpha^{\#}} \operatorname{map}(L, Y').$$

On the nose, F_L and G_L are continuous in the category of compactly generated Hausdorff spaces, where products and function spaces are equipped with the compactly generated refinements of, respectively, the Cartesian product and the compact open topology, by results of [**19**]. However, as Y' is metrizable, so is $\operatorname{map}(L, Y')$ if L is finite. This implies that for finite L, the spaces $\operatorname{map}(L, Y')$ and $Y' \times \operatorname{map}_*(L, Y')$ are already compactly generated. Thus, F_L and G_L are continuous as they are whenever L is finite. Finally, note that $\operatorname{map}(X, Y')$ is the inverse limit of the system consisting of spaces $\{\operatorname{map}(L, Y) \mid L \text{ finite}\}$ together with restriction maps. Also, $Y' \times \operatorname{map}_*(X, Y')$ is the inverse limit of the system $\{Y' \times \operatorname{map}_*(L, Y') \mid L \text{ finite}\}$ together with the obvious maps. As the functions F_L and G_L are natural in L, they uniquely define continuous functions $F\colon \operatorname{map}(X, Y') \to Y' \times \operatorname{map}_*(X, Y')$ and $G\colon Y' \times \operatorname{map}_*(X, Y') \to \operatorname{map}(X, Y')$.

By using naturality of homotopies $\beta^{\#}\alpha^{\#} \simeq \mathrm{id}$ and $\alpha^{\#}\beta^{\#} \simeq \mathrm{id}$, one proceeds similarly as above to obtain continuous homotopies $H_L\colon \operatorname{map}(L, Y') \times I \to \operatorname{map}(L, Y')$ between $G_L F_L$ and the identity, as well as homotopies $K_L\colon Y' \times \operatorname{map}_*(L, Y') \times I \to Y' \times \operatorname{map}_*(L, Y')$ between $F_L G_L$ and the identity, all natural in L. Hence, the functions H_L and K_L define homotopies $H\colon \operatorname{map}(X, Y') \times I \to \operatorname{map}(X, Y')$ and $K\colon Y' \times \operatorname{map}_*(X, Y') \times I \to Y' \times \operatorname{map}_*(X, Y')$. By uniqueness, $H\colon GF \simeq \mathrm{id}$ and $K\colon FG \simeq \mathrm{id}$, as claimed. The statement of Corollary 1.3 now follows by another application of the homotopy equivalences $\alpha, \beta, \alpha^{\#}, \beta^{\#}$, and Theorem 1.1. □

References

[**1**] R. Brown, Function spaces and product topologies, *Quart. J. Math. Oxford Ser. (2)* **15** (1964), no. 1, 238–250.

[**2**] R. Brown, On Künneth suspensions, *Proc. Cambridge Philos. Soc.* **60** (1964), no. 4, 713–720.

[**3**] R. Brown and P.R. Heath, Coglueing homotopy equivalences, *Math. Z.* **113** (1970), 313–325.

[**4**] R. Brown and P.J. Higgins, The classifying space of a crossed complex, *Math. Proc. Cambridge Philos. Soc.* **110** (1991), no. 1, 95–120.

[**5**] A. Dold, Partitions of unity in the theory of fibrations, *Ann. of Math. (2)* **78** (1963), no. 2, 223–255.

[**6**] H. Federer, A study of function spaces by spectral sequences, *Trans. Amer. Math. Soc.* **82** (1956), no. 2, 340–361.

[**7**] R. Geoghegan, The inverse limit of homotopy equivalences between towers of fibrations is a homotopy equivalence-a simple proof, *Topology Proc.* **4** (1979), no. 1, 99–101.

[**8**] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

[**9**] P.J. Kahn, Some function spaces of CW type, *Proc. Amer. Math. Soc.* **90** (1984), no. 4, 599-607.

[**10**] C.R.F. Maunder, *Algebraic topology*, Dover Publications, Inc., Mineola, New York, 1996.

[**11**] H. Miller, The Sullivan conjecture on maps from classifying spaces, *Ann. of Math. (2)* **120** (1984), no. 1, 39–87.

[**12**] J. Milnor, The geometric realization of a semis-simplicial complex, *Ann. of Math. (2)* **65** (1957), no. 2, 357–362.

[**13**] J. Milnor, On spaces having the homotopy type of a CW-complex, *Trans. Amer. Math. Soc.* **90** (1959), no. 2, 272–280.

[**14**] J.W. Rutter, On skeleton preserving homotopy self-equivalences of CW complexes, in *Groups of self-equivalences and related topics* (Montreal, PQ, 1988), 147–156, Lecture Notes in Math. **1425**, Springer-Verlag, New York, 1990.

[**15**] J. Smrekar, Compact open topology and CW homotopy type, *Topol. Appl.* **130** (2003), no. 3, 291–304.

[**16**] J. Smrekar, Homotopy type of mapping spaces and existence of geometric exponents, *Forum Math.* **22** (2010), no. 3, 433–456.

[**17**] J. Smrekar and A. Yamashita, Function spaces of CW homotopy type are Hilbert manifolds, *Proc. Amer. Math. Soc.* **137** (2009), no. 2, 751–759.

[**18**] E.H. Spanier, Infinite symmetric products, function spaces, and duality, *Ann. of Math. (2)* **69** (1959), no. 1, 142–198.

[**19**] N.E. Steenrod, A convenient category of topological spaces, *Michigan Math. J.* **14** (1967), no. 2, 133–152.

[**20**] A. Strøm, The homotopy category is a homotopy category, *Arch. Math. (Basel)* **23** (1972), no. 1, 435–441.

[**21**] R. Thom, L'homologie des espaces fonctionnels, *Colloque de topologie algébrique* (Louvain, 1956), 29–39. Georges Thone, Liège; Masson & Cie, Paris, 1957.

Jaka Smrekar `jaka.smrekar@fmf.uni-lj.si`

Fakulteta za matematiko in fiziko, Univerza v Ljubljani, Jadranska ulica 19, SI-1111 Ljubljana, Slovenia