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COHOMOLOGY OF LOOP SPACES OF THE SYMMETRIC SPACE EI

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(*communicated by J.P.C. Greenlees*)

Abstract

We determine the mod p cohomology of the loop space and the double loop space of the symmetric space of exceptional type EI exploiting the Serre spectral sequence and the Eilenberg–Moore spectral sequence.

1. Introduction

The 1-connected irreducible symmetric spaces have been classified by E. Cartan. Among them, there are twelve 1-connected compact irreducible symmetric spaces of exceptional type besides of the compact simple Lie group cases. Unlike the classical cases, the cohomology of loop spaces of exceptional types is almost unknown, except for easy cases such as $EIV = E_6/F_4$ and $FII = F_4/Spin(9)$ [**4, 5**].

In this paper we study the mod p cohomology of the loop space and the double loop space of the symmetric space of exceptional type EI. As a homogeneous space, EI is expressed by $E_6/PSp(4)$ where $PSp(n)$ is the projective symplectic group. The cohomology of EI is determined in [**7, 8**]. In fact, the cohomology of EI has two torsion, but it is odd torsion free [**8**].

This paper is organized as follows: In Section 2, we collect some known facts which will be used in the next section. In Section 3, we calculate the mod 2 cohomology of the loop space of EI exploiting the Serre spectral sequence and the Eilenberg–Moore spectral sequence going to the same destination space. In [**2**], Bott and Samelson asked whether the cohomology of loop spaces of symmetric spaces has only two torsion. So, in Section 4, we calculate the mod p cohomology of the loop space of EI for odd primes p and prove that the cohomology of the loop space of EI has only two torsion. In Section 5, we apply the results in Sections 4 and 5 to determine the mod p (co)homology of the double loop space of EI.

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2. Preliminaries

Let $E(x)$ be the exterior algebra on x and $\Gamma(x)$ the divided power algebra on x, which is free over $\gamma_i(x)$ as a $\mathbb{F}_p$-module with product

$$\gamma_i(x)\gamma_j(x) = \binom{i+j}{j}\gamma_{i+j}(x).$$

We have homology operations, Dyer–Lashof operations, $Q_{i(p-1)}$ on the $(n+1)$-loop space $\Omega^{n+1}X$

$$Q_{i(p-1)}\colon H_q(\Omega^{n+1}X;\mathbb{F}_p) \to H_{pq+i(p-1)}(\Omega^{n+1}X;\mathbb{F}_p)$$

for $0 \leqslant i \leqslant n$ when $p = 2$, and for $0 \leqslant i \leqslant n$ and $i + q$ even when $p > 2$. They are natural with respect to $(n+1)$-loop maps. In particular, we have $Q_0x = x^p$. The iterated power Q_i^a denotes the composition of Q_i a times. These operations satisfy the following properties:

Theorem 2.1 ([**6**]). *In the path-loop fibration*

$$\Omega^{n+2}X \to P\Omega^{n+1}X \to \Omega^{n+1}X,$$

we have the following:

1. *If $x \in H_*(\Omega^{n+1}X;\mathbb{F}_p)$ is transgressive in the Serre spectral sequence, then so is Q_ix and $\tau \circ Q_{i(p-1)}x = Q_{(i+1)(p-1)} \circ \tau x$ for each i, $0 \leqslant i \leqslant n$ where τ is the transgression.*
2. *For $p > 2$ and $n \geqslant 1$,*
$$d^{2q(p-1)}(x^{p-1} \otimes \tau(x)) = -\beta Q_{(p-1)}\tau(x) \text{ if } x \in H_{2q}(\Omega^{n+1}X;\mathbb{F}_2).$$
3. *For $p = 2$, $Sq_*^1 Q_ix = Q_{i-1}x$ if $x \in H_q(\Omega^{n+1}X;\mathbb{F}_2)$ and $q + i$ is even.*

We denote the primitives and the indecomposables of $H^*(X;\mathbb{F}_p)$ by $PH^*(X;\mathbb{F}_p)$ and $QH^*(X;\mathbb{F}_p)$, respectively. In the Eilenberg–Moore spectral sequence associated with the path-loop fibration converging to $H^*(\Omega X;\mathbb{F}_p)$, we have a map

$$\sigma\colon QH^*(X;\mathbb{F}_p) \cong \operatorname{Tor}^{-1,*}_{H^*(X;\mathbb{F}_p)}(\mathbb{F}_p,\mathbb{F}_p) = E_2^{-1,*} \to E_\infty^{-1,*} \subset H^{*-1}(\Omega X;\mathbb{F}_p).$$

Since the elements of $\mathrm{Tor}^{-1,*}_{\mathrm{H^*(X;\mathbb{F}_p)}}(\mathbb{F}_\mathrm{p},\mathbb{F}_\mathrm{p})$ are primitive and permanent cycles in the Eilenberg–Moore spectral sequence, the above map induces the suspension homomorphism $\sigma\colon QH^*(X;\mathbb{F}_p) \to PH^{*-1}(\Omega X;\mathbb{F}_p)$.

Theorem 2.2 ([**3**]). *Let X be a path connected H-space. Then the following is true:*

1. *The Eilenberg–Moore spectral sequence collapses at E_2 if and only if $\ker\sigma = 0$.*
2. *The suspension $\sigma\colon QH^{\text{odd}}(X;\mathbb{F}_p) \to PH^{\text{even}}(\Omega X;\mathbb{F}_p)$ is injective.*
3. *The quotient $PH^{\text{even}}(\Omega X;\mathbb{F}_p)/\sigma(QH^{\text{odd}}(X;\mathbb{F}_p))$ is obtained by transpotence.*
4. *The elements in $\ker\sigma$ are dual to elements in the image of the homology transpotence.*

Throughout this paper, the subscript of an element means the degree of that element.

3. The mod 2 cohomology of the loop space of EI

The space EI is simply connected and $\pi_2(EI) = Z_2$, so we have the following fibration:

$$\widetilde{EI} \to EI \to K(Z_2, 2),$$

where $\widetilde{EI}$ is the 2-connected cover of EI. Then we have the following morphisms of fibrations:

$$\begin{array}{ccccc}
Sp(4) & \longrightarrow & PSp(4) & \longrightarrow & K(Z_2,1) \\
\downarrow & & \downarrow & & \downarrow \\
E_6 & \longrightarrow & E_6 & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{EI} & \longrightarrow & EI & \longrightarrow & K(Z_2,2).
\end{array}$$

Consider the following fibrations:

$$\Omega\widetilde{EI} \longrightarrow \ * \ \longrightarrow \widetilde{EI}, \tag{1}$$

$$\Omega E_6 \longrightarrow \Omega\widetilde{EI} \longrightarrow Sp(4). \tag{2}$$

Recall the following fact from [**7**]:

Theorem 3.1. $H^*(\widetilde{EI}; \mathbb{F}_2) = E(x_5, Sq^1x_5, x_8, Sq^4x_5, Sq^4x_8, Sq^8Sq^4x_5) \otimes \mathbb{F}_2[x_{16}]$.

First of all, we study the mod 2 cohomology of the loop space of $\widetilde{EI}$.

Proposition 3.2. *The mod 2 cohomology of the loop space of $\widetilde{EI}$ is as follows:*

$$H^*(\Omega\widetilde{EI}; \mathbb{F}_2) = \mathbb{F}_2[y_{2^{i+2}}; i \geqslant 0]/(y_{2^{i+2}}^8) \otimes \Gamma(Sq^1y_4, y_7, Sq^4y_7) \otimes E(z_{15}).$$

Proof. Consider the Eilenberg–Moore spectral sequence of the path-loop fibration (1) converging to $H^*(\Omega\widetilde{EI}; \mathbb{F}_2)$ with

$$\begin{aligned}
E_2 &= \mathrm{Tor}_{H^*(\widetilde{EI};\mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2) \\
&= \mathrm{Tor}_{E(x_5,Sq^1x_5,x_8,Sq^4x_5,Sq^4x_8,Sq^8Sq^4x_5)\otimes\mathbb{F}_2[x_{16}]}(\mathbb{F}_2, \mathbb{F}_2) \\
&= \Gamma(y_4, Sq^1y_4, y_7, Sq^4y_4, Sq^4y_7, Sq^8Sq^4y_4) \otimes E(z_{15}) \\
&= \mathbb{F}_2[y_{2^{i+2}}; i \geqslant 0]/(y_{2^{i+2}}^8) \otimes \Gamma(Sq^1y_4, y_7, Sq^4y_7) \otimes E(z_{15}).
\end{aligned} \tag{3}$$

We also consider the Serre spectral sequence converging to $H^*(\Omega\widetilde{EI}; \mathbb{F}_2)$ for the fibration (2) with

$$\begin{aligned}
E_2 = {} & \mathbb{F}_2[u_2]/(u_2^{16}) \otimes (\otimes_{i\geqslant 0}\mathbb{F}_2[\gamma_{2^i}(u_8)]/(\gamma_{2^i}(u_8)^8)) \\
& \otimes \Gamma(u_{10}, u_{14}, u_{22}) \otimes E(c_3, c_7, c_{11}, c_{15}),
\end{aligned}$$

where

$$H^*(\Omega E_6; \mathbb{F}_2) = \mathbb{F}_2[u_2]/(u_2^{16}) \otimes (\otimes_{i\geqslant 0}\mathbb{F}_2[\gamma_{2^i}(u_8)]/(\gamma_{2^i}(u_8)^8)) \otimes \Gamma(u_{10}, u_{14}, u_{22})$$

in [**5**] and $H^*(Sp(4); \mathbb{F}_2) = E(c_3, c_7, c_{11}, c_{15})$.

Now we determine all differentials in the above Serre spectral sequence. Since this Serre spectral sequence is a spectral sequence of Hopf algebras, it is enough to determine the following transgressions:

$$\tau(u_2), \tau(u_{10}), \tau(u_{14}).$$

We have two spectral sequences going to the same destination space $H^*(\Omega\widetilde{EI};\mathbb{F}_2)$. Comparing these two spectral sequences as a graded vector space, the E_2-term of the Eilenberg–Moore spectral sequence implies the following transgressions of the Serre spectral sequence:

$$\begin{aligned} \tau(u_2) &= c_3 \\ \tau(u_{10}) &= 0 \\ \tau(u_{14}) &= 0. \end{aligned}$$

If $\tau(u_2)$ were trivial, then u_2 would survive permanently. But there is no 2-dimensional element in the E_2-term (3) of the Eilenberg–Moore spectral sequence. So $\tau(u_2)$ should be non-trivial. If $\tau(u_{10}) \neq 0$, or $\tau(u_{14}) \neq 0$, then these would imply the following differentials in the Eilenberg–Moore spectral sequence:

$$\begin{aligned} d(\gamma_2(Sq^1 y_4)) &= Sq^4 y_7, \\ d(\gamma_2(y_7)) &= z_{15}, \end{aligned}$$

but these are impossible because of the following bidegree reason:

$$|\gamma_2(Sq^1 y_4)| = (-2, 12), |Sq^4 y_7| = (-1, 12), |\gamma_2(y_7)| = (-2, 16), |z_{15}| = (-1, 16).$$

Letting $u_2^2 = u_4$ and $c_3 u_2 = c_5$, we obtain that

$$\begin{aligned} E_\infty = {} & \mathbb{F}_2[u_4]/(u_4^8) \otimes (\otimes_{i\geqslant 0}\mathbb{F}_2[\gamma_{2^i}(u_8)]/(\gamma_{2^i}(u_8)^8)) \\ & \otimes \Gamma(u_{10}, u_{14}, u_{22}) \otimes E(c_5, c_7, c_{11}, c_{15}). \end{aligned}$$

As a graded vector space, the size of this E_∞-term is the same as the E_2-term (3) of the Eilenberg–Moore spectral sequence. This means that the above Eilenberg–Moore spectral sequence collapses at the E_2-term, and, on the other side, the E_∞-term of the Serre spectral sequence has the following extensions:

$$c_5^2 = u_{10}, \quad c_7^2 = u_{14}, \quad c_{11}^2 = u_{22}.$$

So we obtain that

$$H^*(\Omega\widetilde{EI};\mathbb{F}_2) = \mathbb{F}_2[y_{2^i+2}; i \geqslant 0]/(y_{2^i+2}^8) \otimes \Gamma(Sq^1 y_4, y_7, Sq^4 y_7) \otimes E(z_{15}). \qquad \square$$

Since the Eilenberg–Moore spectral sequence converging to $H^*(\Omega\widetilde{EI};\mathbb{F}_2)$ collapses at E_2, where $E_2 = \mathrm{Tor}_{H^*(\widetilde{EI};\mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2)$, the Eilenberg–Moore spectral sequence converging to $H_*(\Omega\widetilde{EI};\mathbb{F}_2)$ also collapses at E^2, where

$$\begin{aligned} E^2 &= \mathrm{Cotor}^{H_*(\widetilde{EI};\mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2) \\ &\cong \mathrm{Ext}_{H^*(\widetilde{EI};\mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2) \\ &= \mathbb{F}_2[\nu_4, \nu_5, \nu_7, \nu_9, \nu_{11}, \nu_{16}] \otimes E(\omega_{15}). \end{aligned}$$

Hence we get the following:

Corollary 3.3. $H_*(\Omega\widetilde{EI};\mathbb{F}_2) = \mathbb{F}_2[\nu_4, \nu_5, \nu_7, \nu_9, \nu_{11}, \nu_{16}] \otimes E(\omega_{15})$.

For the next step we consider the following fibrations:

$$\Omega\widetilde{EI} \longrightarrow \Omega EI \longrightarrow K(Z_2, 1) \tag{4}$$

$$\Omega E_6 \longrightarrow \Omega EI \longrightarrow PSp(4). \tag{5}$$

Theorem 3.4. *The mod 2 cohomology of the loop space of EI is as follows:*

$$H^*(\Omega EI; \mathbb{F}_2) = \mathbb{F}_2[x_1]/(x_1^{16}) \otimes \mathbb{F}_2[y_{2^{i+2}}; i \geqslant 0]/(y_{2^{i+2}}^8) \otimes \Gamma(Sq^1 y_4, y_7, Sq^4 y_7).$$

Proof. Consider two Serre spectral sequences converging to $H^*(\Omega EI; \mathbb{F}_2)$ for the fibration (4) with

$$E_2 = \mathbb{F}_2[y_{2^{i+2}}; i \geqslant 0]/(y_{2^{i+2}}^8) \otimes \Gamma(Sq^1 y_4, y_7, Sq^4 y_7) \otimes E(z_{15}) \otimes \mathbb{F}_2[x_1],$$

where $H^*(K(Z_2, 1); \mathbb{F}_2) = \mathbb{F}_2[x_1]$, and for the fibration (5) with

$$\begin{aligned} E_2 = \mathbb{F}_2[u_2]/(u_2^{16}) &\otimes (\otimes_{i\geqslant 0} \mathbb{F}_2[\gamma_{2^i}(u_8)]/(\gamma_{2^i}(u_8)^8)) \otimes \Gamma(u_{10}, u_{14}, u_{22}) \\ &\otimes \mathbb{F}_2[v_1]/(v_1^{16}) \otimes E(b_3, b_7, b_{11}), \end{aligned}$$

where $H^*(PSp(4); \mathbb{F}_2) = \mathbb{F}_2[v_1]/(v_1^{16}) \otimes E(b_3, b_7, b_{11})$ [**1**].

Since the Serre spectral sequences for the fibration (4) are spectral sequences of Hopf algebras, the source of the first non-trivial differential is an indecomposable element, and its target is a primitive element. Then by dimensional reason, there are only two possible non-trivial differentials as follows: $\tau(y_7) = x_1^8$, $\tau(z_{15}) = x_1^{16}$. Now the 7-dimensional generator b_7 is a permanent cycle because 6-dimensional elements are all decomposables in the Serre spectral sequence for (5). Comparing two Serre spectral sequences for (4) and (5) as graded vector spaces, we can deduce that $\tau(y_7)$ is trivial. However, $\tau(z_{15}) = x_1^{16}$ since there is a truncation of 1-dimensional generator v_1 in the E_2-term of the Serre spectral sequences for (5).

So we conclude that the mod 2 cohomology of the loop space of EI is

$$\mathbb{F}_2[x_1]/(x_1^{16}) \otimes \mathbb{F}_2[y_{2^{i+2}}; i \geqslant 0]/(y_{2^{i+2}}^8) \otimes \Gamma(Sq^1 y_4, y_7, Sq^4 y_7). \qquad \square$$

Corollary 3.5. *The cohomology of the loop space of EI has two torsion.*

4. The mod p cohomology of the loop space of EI

Recall the following fact from [**7**]:

Theorem 4.1. *For odd primes p we have*

$$H^*(EI; \mathbb{F}_p) = \mathbb{F}_p[e_8]/(e_8^3) \otimes E(e_9, e_{17}).$$

Note that $\beta e_8 = 0$, where β is the Bockstein homomorphism because the cohomology of EI is odd torsion free [**8**].

We also recall the following facts from [**5**]:

$$H^*(\Omega E_6; \mathbb{F}_3) = \mathbb{F}_3[y_2]/(y_2^9) \otimes \Gamma(y_8, y_{10}, \mathcal{P}^1 y_{10}, y_{16}, y_{18}, y_{22}),$$
$$H^*(\Omega E_6; \mathbb{F}_p) = \Gamma(y_2, y_8, y_{10}, y_{14}, y_{16}, y_{22}), \quad p \geqslant 5.$$

We will determine the mod p cohomology of the loop space of EI exploiting the following fibrations:

$$\Omega EI \longrightarrow \quad * \quad \longrightarrow EI, \tag{6}$$

$$\Omega E_6 \longrightarrow \Omega EI \longrightarrow PSp(4). \tag{7}$$

Theorem 4.2. *For odd primes p, the mod p cohomology of the loop space of EI is as follows:*

$$H^*(\Omega EI;\mathbb{F}_p) = E(x_7)\otimes\Gamma(y_8,y_{16},y_{22}).$$

Proof. Consider the Eilenberg–Moore spectral sequence for the fibration (6) converging to $H^*(\Omega EI;\mathbb{F}_3)$ with

$$\begin{aligned} E_2 &= \mathrm{Tor}_{H^*(EI;\mathbb{F}_3)}(\mathbb{F}_3,\mathbb{F}_3)\\ &= \mathrm{Tor}_{\mathbb{F}_3[e_8]/(e_8^3)\otimes E(e_9,e_{17})}(\mathbb{F}_3,\mathbb{F}_3)\\ &= E(a_7)\otimes\Gamma(a_8,a_{16},a_{22}), \end{aligned}$$

and also consider the Serre spectral sequence converging to $H^*(\Omega EI;\mathbb{F}_3)$ for the fibration (7) with

$$E_2 = E(x_3,\mathcal{P}^1x_3,x_{11},\mathcal{P}^1x_{11})\otimes\mathbb{F}_3[y_2]/(y_2^9)\otimes\Gamma(y_8,y_{10},\mathcal{P}^1y_{10},y_{16},y_{18},y_{22}),$$

where $H^*(PSp(4);\mathbb{F}_3) = E(x_3,\mathcal{P}^1x_3,x_{11},\mathcal{P}^1x_{11})$.

Comparing these E_2-terms of two spectral sequences as a graded vector space, we can obtain the following differentials in the Serre spectral sequence:

$$\begin{aligned} \tau(y_2) &= x_3,\\ d_3(y_2^2) &= x_3\otimes y_2,\\ \tau(y_2^3) &= \mathcal{P}^1x_3,\\ d_7(y_2^6) &= \mathcal{P}^1x_3\otimes y_2^3,\\ d_7(y_{18}) &= \mathcal{P}^1x_3\otimes y_2^6,\\ \tau(y_{10}) &= x_{11},\\ d_{11}(\gamma_{i+1}(y_{10})) &= x_{11}\otimes\gamma_i(y_{10}), i\geqslant 1,\\ \tau(\mathcal{P}^1y_{10}) &= \mathcal{P}^1x_{11},\\ d_{15}(\gamma_{i+1}(\mathcal{P}^1y_{10})) &= \mathcal{P}^1x_{11}\otimes\gamma_i(\mathcal{P}^1y_{10}), i\geqslant 1. \end{aligned}$$

Note that $d_3(y_2^3)$ cannot be $x_3\otimes y_2^2$ because $d_3(y_2^3) = 3x_3\otimes y_2^2 = 0$. Hence $x_3\otimes y_2^2$ is a permanent cycle because there is no element of dimension 8 in the base space $H^*(PSp(4);\mathbb{F}_3)$. Then we obtain

$$E_{16} = E(x_3\otimes y_2^2)\otimes\Gamma(y_8,y_{16},y_{22}).$$

Now by bidegree reason, there are no more non-trivial differentials. Note that $|x_3\otimes y_2^2| = (3,4)$, $|y_8| = (0,8)$. So $E_{16} = E_\infty$. Then letting $x_3\otimes y_2^2 = x_7$, we get the conclusion for the odd prime 3.

Next we consider cases of odd primes $p\geqslant 5$. Similarly, we consider the Eilenberg–Moore spectral sequence of the fibration (6) converging to $H^*(\Omega EI;\mathbb{F}_p)$ with

$$\begin{aligned} E_2 &= \mathrm{Tor}_{H^*(EI;\mathbb{F}_p)}(\mathbb{F}_p,\mathbb{F}_p)\\ &= E(a_7)\otimes\Gamma(a_8,a_{16},a_{22}), \end{aligned}$$

and the Serre spectral sequence converging to $H^*(\Omega EI;\mathbb{F}_p)$ for the fibration (7) with

$$E_2 = E(x_3,x_7,x_{11},x_{15})\otimes\Gamma(y_2,y_8,y_{10},y_{14},y_{16},y_{22}).$$

Then comparing these E_2-terms of two spectral sequences as a graded vector space, we can obtain the following transgressions in the Serre spectral sequence for (7):

$$\begin{aligned}\tau(y_2) &= x_3,\\ d_3(\gamma_{i+1}(y_2)) &= x_3 \otimes \gamma_i(y_2), i \geqslant 1,\\ \tau(y_{10}) &= x_{11},\\ d_{11}(\gamma_{i+1}(y_{10})) &= x_{11} \otimes \gamma_i(y_{10}), i \geqslant 1,\\ \tau(y_{14}) &= x_{15},\\ d_{15}(\gamma_{i+1}(y_{14})) &= x_{15} \otimes \gamma_i(y_{14}), i \geqslant 1.\end{aligned}$$

Hence we get

$$E_{16} = E(x_7) \otimes \Gamma(y_8, y_{16}, y_{22}).$$

Then by bidegree reason, there are no more non-trivial differentials. So $E_{16} = E_\infty$, and we get the conclusion for odd primes $p \geqslant 5$. □

Corollary 4.3. $H_*(\Omega EI; \mathbb{F}_p) = E(\omega_7) \otimes \mathbb{F}_p[\nu_8, \nu_{16}, \nu_{22}]$.

In [**2**], Bott and Samelson mentioned that they do not know whether the cohomology of the loop space of a symmetric space has only two torsion. Now we determine whether the cohomology of the loop space of EI has only two torsion. Rationally we have

$$\begin{aligned}H^*(\Omega E_6; \mathbb{Q}) &= \mathbb{Q}[y_2, y_8, y_{10}, y_{14}, y_{16}, y_{22}],\\ H^*(PSp(4); \mathbb{Q}) &= E(x_3, x_7, x_{11}, x_{15}).\end{aligned}$$

Then, similar to the mod p case, comparing the Eilenberg–Moore spectral sequence for the fibration (6) converging to $H^*(\Omega EI; \mathbb{Q})$ and the Serre spectral sequence converging to $H^*(\Omega EI; \mathbb{Q})$ for the fibration (7), we can determine the rational cohomology of the loop space of EI as follows:

$$H^*(\Omega EI; \mathbb{Q}) = E(x_7) \otimes \mathbb{Q}[y_8, y_{16}, y_{22}].$$

Note that for a space X

$$\begin{aligned}\dim_{\mathbb{F}_p}(H^*(X; \mathbb{F}_p)) &\geqslant \dim \mathbb{F}_p(H^*(X; Z)/\text{torsion} \otimes \mathbb{F}_p)\\ &= \dim_{\mathbb{Q}}(H^*(X; \mathbb{Q})).\end{aligned}$$

So if $\dim_{\mathbb{F}_p}(H^*(X; \mathbb{F}_p)) = \dim_{\mathbb{Q}}(H^*(X; \mathbb{Q}))$, then the Bockstein spectral sequence converging to $(H^*(X; Z)/\text{torsion}) \otimes \mathbb{F}_p$ collapses at E_1 where $E_1 = H^*(X; \mathbb{F}_p)$. This implies that $H^*(X; Z)$ is p-torsion free.

Since $\dim_{\mathbb{F}_p}(H^*(\Omega EI; \mathbb{F}_p)) = \dim_{\mathbb{Q}}(H^*(\Omega EI; \mathbb{Q}))$, the cohomology of the loop space of EI is odd torsion free. Hence with Corollary 3.5, we get the following conclusion:

Corollary 4.4. *The cohomology of the loop space of EI has only two torsion.*

5. The (co)homology of the double loop space of EI

We determine the mod p (co)homology of the double loop space of EI by applying the results in Sections 4 and 5. Since $\pi_2(EI) = Z_2$, $\pi_0(\Omega^2 EI) = Z_2$. Let $\Omega^2_0 EI$ be the zero component of $\Omega^2 EI$. By looping the fibration twice, $\widetilde{EI} \to EI \to K(Z_2, 2)$, we get $\Omega_0^2 EI \simeq \Omega^2 \widetilde{EI}$. Now we compute $H^*(\Omega^2 \widetilde{EI}; \mathbb{F}_2)$ instead of $H^*(\Omega_0^2 EI; \mathbb{F}_2)$.

Lemma 5.1. *The cohomology suspension map*

$$\sigma \colon QH^*(\Omega \widetilde{EI}; \mathbb{F}_2) \to PH^{*-1}(\Omega^2 \widetilde{EI}; \mathbb{F}_2)$$

is injective.

Proof. From Theorem 2.2, if $x \in \ker \sigma$ for $x \in QH^*(\Omega \widetilde{EI}; \mathbb{F}_2)$, the degree of x should be of the form $n \times 2^k + 2$ for $n \geqslant 1$ and $k \geqslant 2$. There are $y_{2^{i+2}}$, $i \geqslant 0$, $\gamma_{2^i}(Sq^1 y_4)$, $i \geqslant 1$, $\gamma_{2^i}(y_7)$, $i \geqslant 1$, $\gamma_{2^i}(Sq^1 y_7)$, $i \geqslant 1$ in $QH^{\text{even}}(\Omega \widetilde{EI}; \mathbb{F}_2)$, but degrees of these elements cannot be of the form $n \times 2^k + 2$ for $k \geqslant 2$. Hence the suspension

$$\sigma \colon QH^*(\Omega \widetilde{EI}; \mathbb{F}_2) PH^{*-1}(\Omega^2 \widetilde{EI}; \mathbb{F}_2)$$

is injective. □

By Theorem 2.2 we obtain the following corollary:

Corollary 5.2. *The Eilenberg–Moore spectral sequence converging to $H^*(\Omega^2 \widetilde{EI}; \mathbb{F}_2)$ collapses at E_2, where*

$$E_2 = \mathrm{Tor}_{H^*(\Omega \widetilde{EI}; \mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2).$$

Corollary 5.3. *The Eilenberg–Moore spectral sequence converging to $H_*(\Omega^2 \widetilde{EI}; \mathbb{F}_2)$ collapses at E^2, where*

$$E^2 = \mathrm{Ext}_{H^*(\Omega \widetilde{EI}; \mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Cotor}^{H_*(\Omega \widetilde{EI}; \mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2).$$

Theorem 5.4. *The mod 2 homology of the double loop space of EI is as follows:*

$$\begin{aligned} H_*(\Omega_0^2 EI; \mathbb{F}_2) = {} & E(Q_1^a z_3 : a \geqslant 0) \otimes \mathbb{F}_2[\beta_3 Q_1^{a+3} z_3 : a \geqslant 0] \otimes \mathbb{F}_2[w_{14}] \\ & \otimes \mathbb{F}_2[Q_1^a z_4 : a \geqslant 0] \otimes \mathbb{F}_2[Q_1^a z_6 : a \geqslant 0] \otimes \mathbb{F}_2[Q_1^a z_{10} : a \geqslant 0], \end{aligned}$$

where β_3 is the tertiary homology Bockstein operator.

Proof. Consider the Eilenberg–Moore spectral sequence converging to $H_*(\Omega^2 \widetilde{EI}; \mathbb{F}_2)$ with

$$E^2 = \mathrm{Ext}_{H^*(\Omega \widetilde{EI}; \mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2) = \mathrm{Cotor}^{H_*(\Omega \widetilde{EI}; \mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2).$$

Then by Theorem 2.1, Corollary 3.3, and Corollary 5.3,

$$\begin{aligned} E^\infty &= E^2 \\ &= E(Q_1^a z_3 : a \geqslant 0) \otimes \mathbb{F}_2[w_{2^{i+5}-2} : i \geqslant 0] \otimes \mathbb{F}_2[w_{14}] \\ &\quad \otimes \mathbb{F}_2[Q_1^a z_4 : a \geqslant 0] \otimes \mathbb{F}_2[Q_1^a z_6 : a \geqslant 0] \otimes \mathbb{F}_2[Q_1^a z_{10} : a \geqslant 0]. \end{aligned} \tag{8}$$

Consider the Serre spectral sequence associated with the fibration

$$\Omega^2 E_6 \to \Omega^2 \widetilde{EI} \to \Omega Sp(4).$$

It is easy to compute that $H_*(\Omega Sp(4); \mathbb{F}_2) = \mathbb{F}_2[a_2, a_6, a_{10}, a_{14}]$.

Recall the following in [**5**]:

$$H_*(\Omega^2 E_6;\mathbb{F}_2) = E(z_1)\otimes E(Q_1^a z_7 : a\geqslant 0)\otimes \mathbb{F}_2[\beta_3 Q_1^{a+2} z_7 : a\geqslant 0] \otimes \mathbb{F}_2[Q_1^a z_9 : a\geqslant 0]\otimes \mathbb{F}_2[Q_1^a z_{13} : a\geqslant 0]\otimes \mathbb{F}_2[Q_1^a z_{21} : a\geqslant 0]. \quad (9)$$

As a graded vector space, the E^∞-term of the Serre spectral sequence should have the same size in every total degree as the above E^∞-term of the Eilenberg–Moore spectral sequence. Then we obtain the following transgression: $\tau(a_2) = z_1$, and $a_2\otimes z_1$ survives permanently. Let $a_2\otimes z_1 = z_3$. Then by (8) and (9),

$$H_*(\Omega^2 \widetilde{EI};\mathbb{F}_2) = E(Q_1^a z_3 : a\geqslant 0)\otimes \mathbb{F}_2[\beta_3 Q_1^{a+3} z_3 : a\geqslant 0]\otimes \mathbb{F}_2[w_{14}] \otimes \mathbb{F}_2[Q_1^a z_4 : a\geqslant 0]\otimes \mathbb{F}_2[Q_1^a z_6 : a\geqslant 0]\otimes \mathbb{F}_2[Q_1^a z_{10} : a\geqslant 0]. \qquad \square$$

Corollary 5.5. *The mod 2 cohomology of the double loop space of EI is as follows:*

$$H^*(\Omega_0^2 EI;\mathbb{F}_2) = E(e_3,e_7,e_{15})\otimes E(\beta^3\alpha_{2^{i+5}-2} : i\geqslant 0)\otimes \Gamma(\alpha_{2^{i+5}-2} : i\geqslant 0)\otimes \Gamma(\alpha_{14}) \otimes \Gamma(e_{2^i\times 5-1} : i\geqslant 0)\otimes \Gamma(e_{2^i\times 7-1} : i\geqslant 0)\otimes \Gamma(e_{2^i\times 11-1} : i\geqslant 0),$$

where β^3 is the tertiary Bockstein operator.

For odd prime cases, we recall the following in [**9**]:

Theorem 5.6. *Let X be a 1-connected H-space of finite type and let $x\in \operatorname{Ker}\sigma$. Assume p is an odd prime. Then either there is an indecomposable class $u\in H^{2m+1}(x)$ such that $\beta_k\mathcal{P}^I$ for $I=(p^{k-l}m,\ldots,m)$ is defined and contains x, or else there is an indecomposable class $v\in H^{2s}(x)$ of height p^r such that $\beta_k\mathcal{P}^J\psi_r(v)$ for $J = (p^{k-l}(p^r s-1),\ldots,p(p^r s-1)))$ is defined and contains x.*

Then we get the following corollary:

Corollary 5.7. *Let X be a 1-connected H-space of finite type and p be an odd prime. Then if $H^*(X;Z)$ is odd torsion free, the Eilenberg–Moore spectral sequence converging to $H^*(\Omega X;\mathbb{F}_p)$ collapses at E_2, where*

$$E_2 = \operatorname{Tor}_{H^*(X;\mathbb{F}_p)}(\mathbb{F}_p,\mathbb{F}_p).$$

Localized at p for any odd primes, EI is 7-connected. Since the cohomology of the loop space of EI is odd torsion free by Corollary 4.4, we get the following:

Corollary 5.8. *The Eilenberg–Moore spectral sequence converging to $H^*(\Omega_0^2 EI;\mathbb{F}_p)$ collapses at E_2, where*

$$E_2 = \operatorname{Tor}_{H^*(\Omega EI;\mathbb{F}_p)}(\mathbb{F}_p,\mathbb{F}_p).$$

Corollary 5.9. *The Eilenberg–Moore spectral sequence converging to $H_*(\Omega_0^2 EI;\mathbb{F}_p)$ collapses at E^2, where*

$$E^2 = \operatorname{Ext}_{H^*(\Omega EI;\mathbb{F}_p)}(\mathbb{F}_p,\mathbb{F}_p)\cong \operatorname{Cotor}^{H_*(\Omega EI;\mathbb{F}_p)}(\mathbb{F}_p,\mathbb{F}_p).$$

Theorem 5.10. *For odd primes p, the mod p homology of the double loop space of EI is as follows:*

$$H_*(\Omega_0^2 EI;\mathbb{F}_p) = \mathbb{F}_p[z_6]\otimes E(Q_{(p-1)}^a z_7 : a\geqslant 0)\otimes \mathbb{F}_p[\beta Q_{(p-1)}^{a+1} z_7 : a\geqslant 0] \otimes E(Q_{(p-1)}^a z_{15} : a\geqslant 0)\otimes \mathbb{F}_p[\beta Q_{(p-1)}^{a+1} z_{15} : a\geqslant 0] \otimes E(Q_{(p-1)}^a z_{21} : a\geqslant 0)\otimes \mathbb{F}_p[\beta Q_{(p-1)}^{a+1} z_{21} : a\geqslant 0].$$

Proof. Consider the Eilenberg–Moore spectral sequence converging to $H_*(\Omega_0^2 EI;\mathbb{F}_p)$ with

$$E^2 = \mathrm{Ext}_{H^*(\Omega EI;\mathbb{F}_p)}(\mathbb{F}_p,\mathbb{F}_p) = \mathrm{Cotor}^{H_*(\Omega EI;\mathbb{F}_p)}(\mathbb{F}_p,\mathbb{F}_p).$$

Then by Theorem 2.1, Corollary 4.3, and Corollary 5.9,

$$\begin{aligned} E^\infty &= E^2 \\ &= \mathbb{F}_p[z_6] \otimes E(Q^a_{(p-1)} z_7 : a \geqslant 0) \otimes \mathbb{F}_p[\beta Q^{a+1}_{(p-1)} z_7 : a \geqslant 0] \\ &\quad \otimes E(Q^a_{(p-1)} z_{15} : a \geqslant 0) \otimes \mathbb{F}_p[\beta Q^{a+1}_{(p-1)} z_{15} : a \geqslant 0] \\ &\quad \otimes E(Q^a_{(p-1)} z_{21} : a \geqslant 0) \otimes \mathbb{F}_p[\beta Q^{a+1}_{(p-1)} z_{21} : a \geqslant 0]. \end{aligned}$$

□

Corollary 5.11. *For odd primes p, the mod p cohomology of the double loop space of EI is as follows:*

$$\begin{aligned} H^*(\Omega_0^2 EI;\mathbb{F}_p) &= \Gamma(\alpha_6) \otimes E(e_7, e_{15}, e_{21}) \otimes E(\beta\alpha_{8p^{i+1}-2} : i \geqslant 0) \otimes \Gamma(\alpha_{8p^{i+1}-2} : i \geqslant 0) \\ &\quad \otimes E(\beta\alpha_{16p^{i+1}-2} : i \geqslant 0) \otimes \Gamma(\alpha_{16p^{i+1}-2} : i \geqslant 0) \\ &\quad \otimes E(\beta\alpha_{22p^{i+1}-2} : i \geqslant 0) \otimes \Gamma(\alpha_{22p^{i+1}-2} : i \geqslant 0). \end{aligned}$$

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