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CO-REPRESENTABILITY OF THE GROTHENDIECK GROUP OF ENDOMORPHISMS FUNCTOR IN THE CATEGORY OF NONCOMMUTATIVE MOTIVES

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(*communicated by Daniel Grayson*)

Abstract

In this article we prove that the additive invariant co-represented by the noncommutative motive $\mathbb{Z}[r]$ is the Grothendieck group of endomorphisms functor $K_0\mathrm{End}$. Making use of Almkvist's foundational work, we then show that the ring $\mathrm{Nat}(K_0\mathrm{End}, K_0\mathrm{End})$ of natural transformations (whose multiplication is given by composition) is naturally isomorphic to the direct sum of $\mathbb{Z}$ with the ring $W_0(\mathbb{Z}[r])$ of fractions of polynomials with coefficients in $\mathbb{Z}[r]$ and constant term 1.

1. Introduction and statement of results

A *differential graded (=dg) category* is a category enriched over complexes of abelian groups; see §2.2. Dg categories enhance and solve many of the technical problems inherent to triangulated categories; see Keller's ICM address [**14**]. In *noncommutative algebraic geometry* in the sense of Drinfeld, Kaledin, Kontsevich, Orlov, Van den Bergh, and others (see [**4, 5, 7, 8, 13, 17, 18, 19, 20**]), dg categories are considered as dg enhancements of bounded derived categories of coherent sheaves on a hypothetic noncommutative space. Let dgcat denote the category of dg categories.

All the classical invariants such as cyclic homology (and its variants), algebraic K-theory, and even topological cyclic homology, extend naturally from rings to dg categories. In order to study all these invariants simultaneously the author introduced in [**25**] the notion of *additive invariant*; see Definition 4.1. Roughly speaking, these are functors $E\colon \mathsf{dgcat} \to \mathsf{D}$ with values in additive categories which invert *Morita equivalences* (i.e., dg functors inducing an equivalence on the associated derived categories) and satisfy additivity. Thanks to the work of Blumberg-Mandell [**3**], Keller [**15, 16**], and Quillen [**22**], all the mentioned invariants are additive. In [**25**] the *universal additive invariant*

$$\mathcal{U}_{\mathsf{A}}\colon \mathsf{dgcat} \longrightarrow \mathsf{Hmo}_0$$

was constructed; see Theorem 4.3. Given any additive category D, we have an induced

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equivalence of categories

$$(\mathcal{U}_{\mathsf{A}})^* : \mathrm{Fun}_{\mathsf{add}}(\mathsf{Hmo}_0, \mathsf{D}) \xrightarrow{\sim} \mathrm{Fun}_{\mathsf{A}}(\mathsf{dgcat}, \mathsf{D}), \tag{1}$$

where the left-hand side denotes the category of additive functors and the right-hand side the category of additive invariants. Because of this universal property, which is a reminiscence of motives, the category Hmo_0 is called the *category of noncommutative motives*. A fundamental problem in the theory of noncommutative motives is the description of the additive invariants associated to noncommutative motives. An example is given by the Grothendieck group functor K_0. It is an additive invariant, and hence by equivalence (1) it gives rise to an additive functor $\overline{K_0}$. As proved in [**25**], it becomes co-representable in Hmo_0 by the noncommutative motive $\mathcal{U}_{\mathsf{A}}(\underline{\mathbb{Z}})$, where $\underline{\mathbb{Z}}$ is the dg category with one object and with $\mathbb{Z}$ as the dg algebra of endomorphisms. In this article we study the additive invariant associated to the noncommutative motive $\mathcal{U}_{\mathsf{A}}(\underline{\mathbb{Z}[r]})$, where $\mathbb{Z}[r]$ is the ring of polynomials in the variable r.

Given a ring A, let $\mathbf{P}(A)$ be the category of finitely generated projective right A-modules and $\mathrm{End}(\mathbf{P}(A))$ the associated category of endomorphisms. This latter category inherits naturally from $\mathbf{P}(A)$ an exact structure in the sense of Quillen, and so we can consider its Grothendieck group $K_0\mathrm{End}(\mathbf{P}(A))$. We then obtain a well-defined functor

$$\mathsf{Ring} \longrightarrow \mathsf{Ab} \qquad A \mapsto K_0\mathrm{End}(\mathbf{P}(A)) \tag{2}$$

from the category of (not necessarily commutative) rings to the category of abelian groups; see §3.1 for details. Moreover, it comes equipped with classical natural transformations such as the *Frobenius operations* (F_n), the *Verschiebung operations* (V_n), and the *substitution operations* $(S_{p(r)})$ (where $p(r)$ is a given polynomial); see [**1**, page 319]. As explained in §3, the above functor (2) (as well as the natural transformations) extends naturally to a functor

$$K_0\mathrm{End}\colon \mathsf{dgcat} \longrightarrow \mathsf{Ab} \tag{3}$$

defined on the category of dg categories. Our first main result is the following:

Theorem 1.1. *The functor* $K_0\mathrm{End}$ *is an additive invariant and there is a natural isomorphism of additive functors*

$$\mathsf{Hom}_{\mathsf{Hmo}_0}(\mathcal{U}_{\mathsf{A}}(\underline{\mathbb{Z}[r]}), -) \simeq \overline{K_0\mathrm{End}}\,, \tag{4}$$

where $\overline{K_0\mathrm{End}}$ *is the additive functor associated to* $K_0\mathrm{End}$ *via the equivalence* (1)*.*

By combining Theorem 1.1 with the universal property of $\mathcal{U}_{\mathsf{A}}$ and with the (enriched) Yoneda lemma, we obtain the following result:

Proposition 1.2. *There is a natural isomorphism of abelian groups*

$$\mathrm{Nat}(K_0\mathrm{End}, K_0\mathrm{End}) \simeq K_0\mathrm{End}(\mathbb{Z}[r])\,,$$

where Nat *stands for the abelian group of natural transformations (with group structure given by objectwise addition).*

Now, recall from [**1, 2**] Almkvist's foundational computation of $K_0\mathrm{End}(\mathbb{Z}[r])$. Let

$$W_0(\mathbb{Z}[r]) := \left\{ \frac{1+p_1(r)t+\cdots+p_n(r)t^n}{1+q_1(r)t+\cdots+q_m(r)t^m} \mid p_i(r), q_j(r) \in \mathbb{Z}[r] \right\}$$

be the multiplicative abelian group of fractions of polynomials (in the variable t) with coefficients in $\mathbb{Z}[r]$ and constant term 1. Almkvist proved that there is a natural isomorphism of abelian groups $K_0\mathrm{End}(\mathbb{Z}[r]) \simeq \mathbb{Z} \oplus W_0(\mathbb{Z}[r])$. We then obtain the following corollary:

Corollary 1.3. *There is a natural isomorphism of abelian groups*

$$\mathrm{Nat}(K_0\mathrm{End}, K_0\mathrm{End}) \simeq \mathbb{Z} \oplus W_0(\mathbb{Z}[r]) . \tag{5}$$

Moreover, under this natural isomorphism, we have the following identifications:

$$F_n \leftrightarrow (1, 1+r^n t) \quad V_n \leftrightarrow (n, 1+rt^n) \quad S_{p(r)} \leftrightarrow (1, 1+p(r)t) .$$

Corollary 1.3 shows us that all the information concerning a natural transformation of the functor (3) can be completely encoded in an integer and in a fraction between polynomials with constant term 1. Note also that the substitution operations are the "simplest ones" with respect to the variable t, while the Verschiebung operations are the "simplest ones" with respect to the variable r; note that $F_n = S_{r^n}$. The former ones correspond to the unit of $\mathbb{Z}$ and to the polynomials of degree one (in the variable t), while the latter ones correspond to the natural numbers in $\mathbb{Z}$ and to those polynomials concentrated in degree n (in the variable t) whose coefficient is simply the polynomial r.

The abelian group $\mathrm{Nat}(K_0\mathrm{End}, K_0\mathrm{End})$ naturally carries a (noncommutative) ring structure given by composition of natural transformations. Via the isomorphism (5), we then obtain a composition law on the abelian group $\mathbb{Z} \oplus W_0(\mathbb{Z}[r])$. In order to describe it, we now introduce some notation. Given a square matrix B and a polynomial $p(r) = a_0 + a_1 r + \cdots + a_n r^n \in \mathbb{Z}[r]$, we write $p(B)$ for the matrix $a_0 \,\mathrm{Id} + a_1 B + \cdots + a_n B^n$. Given an integer $k \geqslant 1$ and polynomials $p_1(r), \ldots, p_k(r) \in \mathbb{Z}[r]$, let

$$M(p_1(r), \ldots, p_k(r)) := \begin{bmatrix} 0 & \cdots & \cdots & 0 & (-1)^{k+1} p_k(r) \\ 1 & \ddots & & \vdots & \vdots \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -p_2(r) \\ 0 & \cdots & 0 & 1 & p_1(r) \end{bmatrix}_{(k \times k)} .$$

Under these notations, our second main result is the following:

Theorem 1.4. *Let $\mathcal{P} = 1 + p_1(r)t + \cdots + p_k(r)t^k$ and $\mathcal{H} = 1 + h_1(r)t + \cdots + h_l(r)t^l$ be two arbitrary polynomials of $W_0(\mathbb{Z}[r])$. Then, the assignment*

$$(\mathcal{P}, \mathcal{H}) \mapsto \mathcal{P} \star \mathcal{H} := \det(\mathrm{Id} + M_{\mathcal{P};\mathcal{H}} t) , \tag{6}$$

where $M_{\mathcal{P};\mathcal{H}}$ is the block-matrix

$$\begin{bmatrix} 0 & \cdots & \cdots & 0 & (-1)^{l+1}h_l(M(p_1(r),\ldots,p_k(r))) \\ \mathrm{Id} & \ddots & & \vdots & \vdots \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -h_2(M(p_1(r),\ldots,p_k(r))) \\ 0 & \cdots & 0 & \mathrm{Id} & h_1(M(p_1(r),\ldots,p_k(r))) \end{bmatrix}_{(k\times l)\times(k\times l)},$$

determines a unique (noncommutative) ring structure $\star$ on the abelian group $W_0(\mathbb{Z}[r])$. The $\star$-unit is the polynomial $1+rt$. Moreover, under this ring structure on $W_0(\mathbb{Z}[r])$, the above isomorphism (5) *is a ring isomorphism.*

Remark 1.5. As the proof of Theorem 1.4 shows, the ring $\mathrm{Nat}(\mathrm{K}_0\mathrm{End}, K_0\mathrm{End})$ of natural transformations can be identified with the ring of endomorphisms of the noncommutative motive $\mathcal{U}_{\mathsf{A}}(\underline{\mathbb{Z}[r]})$. Moreover, the multiplication law in the latter ring is induced by the tensor product of bimodules. As the anonymous referee kindly informed the author, the tensor product of bimodules was also used by Stienstra [**24**] in his construction of operations on the K-theory of endomorphisms.

Informally speaking, Theorem 1.4 shows us that the ring of natural transformations decomposes as the direct sum of a "commutative piece" (the ring $\mathbb{Z}$) and a "noncommutative piece" (the ring $(W_0(\mathbb{Z}[r]),\cdot,\star)$). The ring $\mathbb{Z}$ corresponds to the behavior of the natural transformations on the underlying modules, while the ring $(W_0(\mathbb{Z}[r]),\cdot,\star)$ corresponds to the behavior of the natural transformations on the endomorphisms per se.

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2. Preliminaries

2.1. Notations

Given categories $\mathcal{C}$ and $\mathcal{D}$, we will denote by $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ the category of functors from $\mathcal{C}$ to $\mathcal{D}$. Whenever F and G are two objects of $\mathrm{Fun}(\mathcal{C},\mathcal{D})$, we will write $\mathrm{Nat}(F,G)$ for the set of natural transformations from F to G. The adjunctions will be displayed vertically with the left (resp. right) adjoint on the left- (resp. right-) hand side.

2.2. Differential graded categories

A *differential graded (=dg) category* is a category enriched over (unbounded) cochain complexes of abelian groups in such a way that the composition operation fulfills the Leibniz rule: $d(f\circ g)=d(f)\circ g+(-1)^{\deg(f)}f\circ d(g)$. For a survey article on dg categories we invite the reader to consult Keller's ICM address [**14**]. The category of dg categories will be denoted by dgcat.

Let $\mathcal{A}$ be a dg category. Its *opposite* dg category $\mathcal{A}^{\mathsf{op}}$ has the same objects and complexes of morphisms given by $\mathcal{A}^{\mathsf{op}}(x,y) := \mathcal{A}(y,x)$. A *right dg $\mathcal{A}$-module* (or simply a $\mathcal{A}$-module) is a dg functor $\mathcal{A}^{\mathsf{op}} \to \mathcal{C}_{\mathsf{dg}}(\mathbb{Z})$ with values in the dg category of complexes of abelian groups. We will write $\mathcal{C}(\mathcal{A})$ for the category of $\mathcal{A}$-modules. Recall from [**14**, §3] that $\mathcal{C}(\mathcal{A})$ carries a standard *projective* model structure with weak equivalences and fibrations defined objectwise. Let $\mathcal{D}(\mathcal{A})$ be the derived category of $\mathcal{A}$, i.e., the localization of $\mathcal{C}(\mathcal{A})$ with respect to the class of quasi-isomorphisms. Given a dg functor $F\colon \mathcal{A} \to \mathcal{B}$, we have a *restriction/extension of scalars* Quillen adjunction (on the left-hand side)

$$\begin{array}{ccc} \mathcal{C}(\mathcal{B}) & \qquad\qquad & \mathcal{D}(\mathcal{B}) \\ F_! \uparrow \downarrow F^* & & \mathbb{L}F_! \uparrow \downarrow F^* \\ \mathcal{C}(\mathcal{A}) & & \mathcal{D}(\mathcal{A}) \end{array}$$

which can be naturally derived (on the right-hand side). We will say that F is a *Morita equivalence* if the derived functor $\mathbb{L}F_!\colon \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ is an equivalence of (triangulated) categories; see [**14**, §4.6].

3. K-theory of endomorphisms

In this section we recall from [**1, 2, 10, 11**] the classical K-theory of endomorphisms and extend it to the setting of dg categories.

3.1. Classical setting

Recall from [**22**, §2] that given a ring A, the category $\mathbf{P}(A)$ of finitely generated projective right A-modules carries a natural exact structure in the sense of Quillen. Given a ring homomorphism $A \to B$, the extension of scalars functor restricts to an exact functor $- \otimes_A B\colon \mathbf{P}(A) \to \mathbf{P}(B)$. Hence, we obtain a well-defined functor $\mathbf{P}\colon \mathsf{Ring} \to \{\text{Exact}\}$ from the category of (not necessarily commutative) rings to the category of exact categories. The category $\mathrm{End}(\mathbf{P}(A))$ of endomorphisms in $\mathbf{P}(A)$ has as objects the pairs (M,α), where $M \in \mathbf{P}(A)$ and α is an endomorphism of M. Its morphisms $(M,\alpha) \to (M',\alpha')$ are the ring homomorphisms $M \to M'$ verifying the equality $f\alpha = \alpha' f$. By declaring a sequence in $\mathrm{End}(\mathbf{P}(A))$ to be exact if the underlying sequence in $\mathbf{P}(A)$ is exact, the category $\mathrm{End}(\mathbf{P}(A))$ inherits naturally from $\mathbf{P}(A)$ an exact structure. By combining the above constructions we then obtain a well-defined functor

$$\mathsf{Ring} \xrightarrow{\mathbf{P}} \{\text{Exact}\} \xrightarrow{\mathrm{End}} \{\text{Exact}\} \xrightarrow{K_0} \mathsf{Ab} \qquad A \mapsto K_0\mathrm{End}(\mathbf{P}(A))\,,$$

where K_0 stands for the Grothendieck group functor of exact categories as defined in [**22**, §2].

3.2. Extension to dg categories

Given a dg category $\mathcal{A}$, let $\mathsf{perf}(\mathcal{A})$ be the full subcategory of $\mathcal{C}(\mathcal{A})$ consisting of those $\mathcal{A}$-modules which are cofibrant in the projective model structure and which become compact (see [**21**, Def. 4.2.7]) in the derived category $\mathcal{D}(\mathcal{A})$. As explained in [**9**, Example 3.5], the category $\mathsf{perf}(\mathcal{A})$ carries a Waldhausen structure (see [**26**, §1.2]) whose cofibrations and weak equivalences are those of the projective model structure on $\mathcal{C}(\mathcal{A})$. Given a dg functor $F\colon \mathcal{A} \to \mathcal{B}$, the extension of scalars $F_!\colon \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{B})$ is a left Quillen functor; see [**12**]. Hence, it preserves cofibrant objects, cofibrations, weak equivalences, and pushouts. Since it preserves, moreover, modules which become compact in the derived category, it restricts to an exact functor

$$\mathsf{perf}(F)\colon \mathsf{perf}(\mathcal{A}) \to \mathsf{perf}(\mathcal{B}).$$

We then obtain a well-defined functor $\mathsf{perf}\colon \mathsf{dgcat} \to \{\text{Wald.}\}$ with values in the category of Waldhausen categories. By declaring a morphism in $\mathrm{End}(\mathsf{perf}(\mathcal{A}))$ to be a cofibration (resp. a weak equivalence) if the underlying morphism in $\mathsf{perf}(A)$ is a cofibration (resp. a weak equivalence), we observe that the category $\mathrm{End}(\mathsf{perf}(\mathcal{A}))$ inherits from $\mathsf{perf}(\mathcal{A})$ a Waldhausen structure. Note that the gluing axiom follows automatically from the fact that pushouts in $\mathrm{End}(\mathsf{perf}(\mathcal{A}))$ are computed on the underlying category $\mathsf{perf}(\mathcal{A})$. By combining the above constructions we then obtain a well-defined functor

$$K_0\mathrm{End} : \mathsf{dgcat} \xrightarrow{\mathsf{perf}} \{\text{Wald.}\} \xrightarrow{\mathrm{End}} \{\text{Wald.}\} \xrightarrow{K_0} \mathsf{Ab}\,,$$

where K_0 stands for the Grothendieck group functor of Waldhausen categories as defined in [**26**, §1.3].

3.3. Agreement

Notation 3.1. Given a ring A, we will denote by $\underline{A}$ the dg category with a single object $*$ and with A as the dg algebra of endomorphisms (concentrated in degree zero). Note that this gives rise to a fully-faithful functor $\mathsf{Ring} \to \mathsf{dgcat}, A \mapsto \underline{A}$.

Recall from [**26**, §1.9] that by declaring the cofibrations to be the admissible monomorphisms and the weak equivalences to be the isomorphisms, every exact category in the sense of Quillen becomes a Waldhausen category. Given a ring A, we then have a well-defined exact functor $\mathbf{P}(A) \to \mathsf{perf}(\underline{A})$ which maps a A-module to the associated complex of $\underline{A}$-modules concentrated in degree zero.

Proposition 3.2. *For every ring A, the exact functor $\mathbf{P}(A) \to \mathsf{perf}(\underline{A})$ gives rise to an abelian group isomorphism*

$$K_0\mathrm{End}(\mathbf{P}(A)) \xrightarrow{\sim} K_0\mathrm{End}(\mathsf{perf}(\underline{A}))\,. \tag{7}$$

Proof. Let $(M^\bullet, \alpha)$ be an object of $\mathrm{End}(\mathsf{perf}(\underline{A}))$. By definition, $M^\bullet \in \mathsf{perf}(\underline{A})$ and α is an endomorphism of $M^\bullet$. Note that the A-linear endomorphisms $\alpha_n\colon M^n \to M^n$ give rise to A-linear endomorphisms $H^n(\alpha)\colon H^n(M^\bullet) \to H^n(M^\bullet)$. Since $M^\bullet$ belongs to $\mathsf{perf}(\underline{A})$, the A-modules $H^n(M^\bullet)$ are not only projective of finite type but, moreover, they vanish for $|n| \gg 0$. Hence, the assignment

$$(M^\bullet, \alpha) \mapsto \sum_{n\in\mathbb{Z}} (-1)^n [(H^n(M^\bullet), H^n(\alpha))] \in K_0\mathrm{End}(\mathbf{P}(A)) \tag{8}$$

is well-defined. By definition of $K_0\mathrm{End}(\mathbf{P}(A))$ and $K_0\mathrm{End}(\mathsf{perf}(\underline{A}))$, we then conclude that (8) gives rise to the inverse abelian group homomorphism of (7). □

4. Noncommutative motives

In this section we recall from [**25**] the construction of the category of *noncommutative motives* and describe in Theorem 4.3 its precise universal property. This category will play a key role in the proof of Theorems 1.1 and 1.4.

Let $\mathcal{A}$ be a dg category. Consider the dg category $T(\mathcal{A})$ whose objects are the pairs (i, x), where $i \in \{1, 2\}$ and x is an object of $\mathcal{A}$. The complex of morphisms in $T(\mathcal{A})$, from (i, x) to (i', x'), is given by $\mathcal{A}(x, x')$ if $i' \geqslant i$ and is 0 otherwise. Composition is induced by the one on $\mathcal{A}$; see [**25**, §4] for details. Note that we have two natural inclusion dg functors

$$i_1 \colon \mathcal{A} \longrightarrow T(\mathcal{A}), \qquad i_2 \colon \mathcal{A} \longrightarrow T(\mathcal{A})\,.$$

Definition 4.1. Let $E \colon \mathsf{dgcat} \to \mathsf{D}$ be a functor with values in an additive category. We say that E is an *additive invariant* if it satisfies the following two conditions:

(i) It maps the Morita equivalences (see §2.2) to isomorphisms;

(ii) For every dg category $\mathcal{A}$, the inclusion dg functors induce an isomorphism[1]

$$[E(i_1)\ \ E(i_2)] \colon E(\mathcal{A}) \oplus E(\mathcal{A}) \xrightarrow{\sim} E(T(\mathcal{A}))\,.$$

A *morphism* of additive invariants is a natural transformation of functors. We will denote by $\mathrm{Fun}_{\mathsf{A}}(\mathsf{dgcat}, \mathsf{D})$ the category of additive invariants (with values in D).

As in the case of rings, we can form the tensor product $\mathcal{A} \otimes \mathcal{B}$ of dg categories. This tensor product admits a natural derived version $\mathcal{A} \otimes^{\mathbb{L}} \mathcal{B}$ with respect to Morita equivalence; see [**14**, §2.3]. The category Hmo_0 of *noncommutative motives* is defined as follows: its objects are the dg categories and its abelian groups of morphisms are given by

$$\mathsf{Hom}_{\mathsf{Hmo}_0}(\mathcal{A}, \mathcal{B}) := K_0\mathsf{rep}(\mathcal{A}, \mathcal{B})\,.$$

Here, $K_0\mathsf{rep}(\mathcal{A}, \mathcal{B})$ stands for the Grothendieck group (see [**21**, Def. 4.5.8]) of the full triangulated subcategory $\mathsf{rep}(\mathcal{A}, \mathcal{B})$ of $\mathcal{D}(\mathcal{A}^{\mathsf{op}} \otimes^{\mathbb{L}} \mathcal{B})$ spanned by those bimodules X such that for every object x in $\mathcal{A}$ the $\mathcal{B}$-module $X(-, x)$ belongs to $\mathsf{perf}(\mathcal{B})$. Composition in Hmo_0 is induced by the (derived) tensor product of bimodules; see [**25**, §6] for details. The category Hmo_0 is additive and there is a natural functor

$$\mathcal{U}_{\mathsf{A}} \colon \mathsf{dgcat} \longrightarrow \mathsf{Hmo}_0, \tag{9}$$

which is the identity on objects and which maps a dg functor $F \colon \mathcal{A} \to \mathcal{B}$ to the class in the Grothendieck group $K_0\,\mathsf{rep}(\mathcal{A}, \mathcal{B})$ of the bimodule in $\mathsf{rep}(\mathcal{A}, \mathcal{B})$ naturally associated to F.

Notation 4.2. Since the functor $\mathcal{U}_{\mathsf{A}}$ is the identity on objects we will write simply $\mathcal{A}$ instead of $\mathcal{U}_{\mathsf{A}}(\mathcal{A})$ in what follows.

Theorem 4.3. *The functor (9) is the universal additive invariant, i.e., given any additive category* D*, we have an induced equivalence of categories*

$$(\mathcal{U}_{\mathsf{A}})^* \colon \mathrm{Fun}_{\mathsf{add}}(\mathsf{Hmo}_0, \mathsf{D}) \xrightarrow{\sim} \mathrm{Fun}_{\mathsf{A}}(\mathsf{dgcat}, \mathsf{D}), \tag{10}$$

where $\mathrm{Fun}_{\mathsf{add}}(\mathsf{Hmo}_0, \mathsf{D})$ *denotes the category of additive functors (with values in* D*).*

[1] Condition (ii) can be equivalently formulated in terms of a general semi-orthogonal decomposition in the sense of Bondal-Orlov; see [**25**, Thm. 6.3(4)].

Proof. Let E be an object of $\mathrm{Fun}_{\mathsf{A}}(\mathsf{dgcat}, \mathsf{D})$, i.e., an additive invariant with values in D. Thanks to [**25**, Thms. 5.3 and 6.3] the functor E factors uniquely through $\mathcal{U}_{\mathsf{A}}$, giving rise to an additive functor $\overline{E}\colon \mathsf{Hmo}_0 \to \mathsf{D}$. If η is an element of $\mathrm{Nat}(E, E')$, i.e., a morphism $\eta\colon E \Rightarrow E'$ of additive invariants, then $\overline{\eta}\colon \overline{E} \Rightarrow \overline{E'}$, with $\overline{\eta}(\mathcal{A}) := \eta(\mathcal{A})$ for every dg category $\mathcal{A}$, is a natural transformation of additive functors. We then obtain a well-defined functor

$$\overline{(-)}\colon\ \mathrm{Fun}_{\mathsf{A}}(\mathsf{dgcat}, \mathsf{D}) \longrightarrow \mathrm{Fun}_{\mathsf{add}}(\mathsf{Hmo}_0, \mathsf{D})\,. \tag{11}$$

Making use of [**25**, Thms. 5.3 and 6.3], we observe that the functors (10) and (11) are (quasi-)inverse of each other. This concludes the proof. □

Proof of Theorem 1.1

Recall that we are using Notation 4.2. The functor $\mathsf{Hom}_{\mathsf{Hmo}_0}(\underline{\mathbb{Z}[r]}, -)$ is clearly additive. Hence, by (10), its pre-composition with $\mathcal{U}_{\mathsf{A}}$ gives rise to an additive invariant $\mathsf{Hom}_{\mathsf{Hmo}_0}(\underline{\mathbb{Z}[r]}, \mathcal{U}_{\mathsf{A}}(-))$. The proof will consist on showing that we have a natural isomorphism of functors

$$K_0\mathrm{End} \simeq \mathsf{Hom}_{\mathsf{Hmo}_0}(\underline{\mathbb{Z}[r]}, \mathcal{U}_{\mathsf{A}}(-))\,. \tag{12}$$

Note that such an isomorphism implies automatically that the functor $K_0\mathrm{End}$ is an additive invariant. Moreover, since the functors (10) and (11) are (quasi-)inverse of each other, it implies also the natural isomorphism (4). Let us then show the natural isomorphism (12) of functors. Given a dg category $\mathcal{A}$, we need to show that the abelian groups

$$K_0\mathrm{End}(\mathsf{perf}(\mathcal{A})) \qquad \mathsf{Hom}_{\mathsf{Hmo}_0}(\underline{\mathbb{Z}[r]}, \mathcal{U}_{\mathsf{A}}(\mathcal{A})) := K_0\mathsf{rep}(\underline{\mathbb{Z}[r]}, \mathcal{A})$$

are naturally isomorphic. Since the underlying abelian group of $\mathbb{Z}[r]$ is torsionfree [**27**, Cor. 3.1.5] implies that the ring $\mathbb{Z}[r]$ (and hence the dg category $\underline{\mathbb{Z}[r]}$) is flat. As a consequence, $\underline{\mathbb{Z}[r]} \otimes^{\mathbb{L}} \mathcal{A} \simeq \mathbb{Z}[r] \otimes \mathcal{A}$, and so $\mathsf{rep}(\underline{\mathbb{Z}[r]}, \mathcal{A})$, identifies with the full triangulated subcategory of $\mathcal{D}(\underline{\mathbb{Z}[r]}^{\mathsf{op}} \otimes \mathcal{A})$ spanned by those $\underline{\mathbb{Z}[r]}$-$\mathcal{A}$-bimodules X such that the $\mathcal{A}$-module $X(-, *)$ belongs to $\mathsf{perf}(\mathcal{A})$. Note that such a bimodule X consists of the same data as an object in the category $\mathrm{End}(\mathsf{perf}(\mathcal{A}))$. Therefore, if we denote by $\mathrm{rep}(\underline{\mathbb{Z}[r]}, \mathcal{A})$ the full subcategory of $\mathcal{C}(\underline{\mathbb{Z}[r]}^{\mathsf{op}} \otimes \mathcal{A})$ with the same objects as those of $\mathsf{rep}(\underline{\mathbb{Z}[r]}, \mathcal{A})$, we have natural equivalences of categories

$$\mathrm{End}(\mathsf{perf}(\mathcal{A})) \simeq \mathsf{rep}(\underline{\mathbb{Z}[r]}, \mathcal{A}) \simeq \mathrm{rep}(\underline{\mathbb{Z}[r]}, \mathcal{A})[w^{-1}]\,,$$

where $\mathrm{rep}(\underline{\mathbb{Z}[r]}, \mathcal{A})[w^{-1}]$ denotes the localization of $\mathrm{rep}(\underline{\mathbb{Z}[r]}, \mathcal{A})$ with respect to the class of quasi-isomorphisms. As a consequence, we obtain an isomorphism

$$K_0\mathrm{End}(\mathsf{perf}(\mathcal{A}))[w^{-1}] \simeq K_0\mathsf{rep}(\underline{\mathbb{Z}[r]}, \mathcal{A})\,.$$

By definition of the (different) Grothendieck groups, we observe that the localization functor $\mathrm{End}(\mathsf{perf}(\mathcal{A})) \to \mathrm{End}(\mathsf{perf}(\mathcal{A}))[w^{-1}]$ induces an isomorphism between $K_0\mathrm{End}(\mathsf{perf}(\mathcal{A}))$ and $K_0\mathrm{End}(\mathsf{perf}(\mathcal{A}))[w^{-1}]$. Hence, by concatenating all of the above arguments, we finally obtain a natural abelian group isomorphism

$$K_0\mathrm{End}(\mathsf{perf}(\mathcal{A})) \simeq K_0\mathsf{rep}(\underline{\mathbb{Z}[r]}, \mathcal{A})\,.$$

This concludes the proof.

Proof of Proposition 1.2

We start with the following result:

Lemma 4.4. *Let* $E\colon \mathsf{dgcat} \to \mathsf{Ab}$ *be an additive invariant with values in (the additive category of) abelian groups. Then, the canonical map*

$$\mathrm{Nat}(K_0\mathrm{End}, E) \xrightarrow{\sim} E(\underline{\mathbb{Z}[r]}) \qquad \eta \mapsto \eta(\underline{\mathbb{Z}[r]})([(\underline{\mathbb{Z}[r]}, \cdot r)]) \tag{13}$$

is an isomorphism of abelian groups. Here, the group structure on $\mathrm{Nat}(K_0\mathrm{End}, E)$ *is given by objectwise addition, and* $\cdot r$ *denotes the endomorphism of* $\underline{\mathbb{Z}[r]}$ *(considered as a module over itself) given by multiplication by* r.

Proof. Thanks to Theorem 1.1, the functor $K_0\mathrm{End}$ is an additive invariant. Hence, $K_0\mathrm{End}$ and E belong to the category $\mathrm{Fun}_{\mathsf{A}}(\mathsf{dgcat}, \mathsf{Ab})$. Using the equivalence (11), we then obtain an abelian group isomorphism

$$\mathrm{Nat}(K_0\mathrm{End}, E) \xrightarrow{\sim} \mathrm{Nat}(\overline{K_0\mathrm{End}}, \overline{E}) \qquad \eta \mapsto \overline{\eta}\,. \tag{14}$$

By Theorem 1.1, the additive functor $\overline{K_0\mathrm{End}}$ is co-representable in Hmo_0 by the dg categories $\underline{\mathbb{Z}[r]}$. Therefore, since every additive functor is a Ab-functor (see [**6**, Def. 6.2.3]), the enriched Yoneda lemma [**6**, Thm. 8.3.5] furnish us an abelian group isomorphism

$$\mathrm{Nat}(\overline{K_0\mathrm{End}}, \overline{E}) \xrightarrow{\sim} \overline{E}(\underline{\mathbb{Z}[r]}) \qquad \overline{\eta} \mapsto \overline{\eta}(\underline{\mathbb{Z}[r]})([\mathrm{id}_{\underline{\mathbb{Z}[r]}}])\,. \tag{15}$$

As explained in the proof of Theorem 1.1, there is a natural abelian group isomorphism

$$K_0\mathsf{rep}(\underline{\mathbb{Z}[r]}, \underline{\mathbb{Z}[r]}) \simeq K_0\mathrm{End}(\mathsf{perf}(\underline{\mathbb{Z}[r]}))$$

which identifies the class $[\mathrm{id}_{\underline{\mathbb{Z}[r]}}]$ of the bimodule associated to the identity of $\underline{\mathbb{Z}[r]}$ with the class $[(\underline{\mathbb{Z}[r]}, \cdot r)]$. Hence, since $\overline{E}(\underline{\mathbb{Z}[r]}) = E(\underline{\mathbb{Z}[r]})$ and $\overline{\eta}(\underline{\mathbb{Z}[r]}) = \eta(\underline{\mathbb{Z}[r]})$, we conclude that the canonical map (13) is the composition of the isomorphisms (14) and (15). This concludes the proof. ☐

Now, by combining Proposition 3.2, Theorem 1.1, and Lemma 4.4, we conclude that the canonical map

$$\mathrm{Nat}(K_0\mathrm{End}, K_0\mathrm{End}) \xrightarrow{\sim} K_0\mathrm{End}(\mathbf{P}(\mathbb{Z}[r])) \simeq K_0\mathrm{End}(\mathbb{Z}[r]), \tag{16}$$

sending a natural transformation η to $\eta(\mathbb{Z}[r])([(\mathbb{Z}[r], \cdot r)])$, is an isomorphism of abelian groups. This achieves the proof.

Proof of Corollary 1.3

Given a commutative ring A, let

$$W_0(A) := \left\{ \frac{1 + p_1 t + \cdots + p_n t^n}{1 + q_1 t + \cdots + q_m t^m} \mid p_i, q_j \in A \right\}$$

be the multiplicative abelian group of fractions of polynomials (in the variable t) with coefficients in A and constant term 1. Almkvist established in [**2**] the following natural isomorphism of abelian groups:

$$K_0\mathrm{End}(\mathbf{P}(A)) \xrightarrow{\sim} K_0(A) \oplus W_0(A) \qquad [(M,\alpha)] \mapsto ([M], \lambda_t(\alpha))\,, \tag{17}$$

where $\lambda_t(\alpha) := \det(\mathrm{Id} + \alpha t)$ denotes the characteristic polynomial (in the variable t) of the endomorphism α. The natural isomorphism (5) is then the following composition:

$$\mathrm{Nat}(K_0\mathrm{End}, \mathrm{K_0End}) \xrightarrow{(16)} K_0\mathrm{End}(\mathbf{P}(\mathbb{Z}[r])) \xrightarrow{\sim} \mathbb{Z} \oplus W_0(\mathbb{Z}[r])\,, \tag{18}$$

where the right-hand side isomorphism is obtained by combining (17) (with $A = \mathbb{Z}[r]$) with the natural isomorphism $K_0(\mathbb{Z}[r]) \simeq \mathbb{Z}$ (see [**23**, Cor. 3.2.13]).

Let us now show the identifications. Recall from [**1**, page 319] that the Frobenius operations are given by

$$F_n \colon K_0\mathrm{End} \Rightarrow K_0\mathrm{End} \qquad [(M,\alpha)] \mapsto [(M,\alpha^n)]\,,$$

that the Verschiebung operations are given by

$$V_n \colon K_0\mathrm{End} \Rightarrow K_0\mathrm{End} \qquad [(M,\alpha)] \mapsto [(M^{\oplus n}, V_n(\alpha))]\,,$$

where

$$V_n(\alpha) := \begin{bmatrix} 0 & \cdots & \cdots & 0 & (-1)^{n+1}\alpha \\ 1 & \ddots & & \vdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}_{(n\times n)}\,,$$

and that the substitution operations are given by

$$S_{p(r)} \colon K_0\mathrm{End} \Rightarrow K_0\mathrm{End} \qquad [(M,\alpha)] \mapsto [(M, p(\alpha))]\,,$$

where $p(r)$ is a given polynomial. Therefore, under the above composed isomorphism (18), we have the following identifications:

$$\begin{array}{lclcl} F_n & \leftrightarrow & [(\mathbb{Z}[r], \cdot r^n)] & \leftrightarrow & (1, 1 + r^n t) \\ V_n & \leftrightarrow & [(\mathbb{Z}[r]^{\oplus n}, V_n(\cdot r))] & \leftrightarrow & (n, 1 + rt^n) \\ S_{h(r)} & \leftrightarrow & [(\mathbb{Z}[r], \cdot p(r))] & \leftrightarrow & (1, 1 + p(r)t)\,. \end{array}$$

The equality

$$\lambda_t(V_n(\cdot r)) := \det(\mathrm{Id} + V_n(\cdot r)t) = 1 + rt^n\,,$$

which implies the identification $[(\mathbb{Z}[r]^{\oplus n}, V_n(\cdot r))] \leftrightarrow (n, 1 + rt^n)$, is a simple exercise which we leave for the reader.

5. Proof of Theorem 1.4

We start by reviewing the construction of the natural isomorphism (5). Thanks to Theorem 1.1, the functor $K_0\mathrm{End}$ is an additive invariant. Hence, by Theorem 4.3, it factors through Hmo_0 giving rise to an additive functor $\overline{K_0\mathrm{End}}$. Making use of equivalence (11), we then obtain a ring isomorphism

$$\mathrm{Nat}(K_0\mathrm{End}, K_0\mathrm{End}) \xrightarrow{\sim} \mathrm{Nat}(\overline{K_0\mathrm{End}}, \overline{K_0\mathrm{End}}) \qquad \eta \mapsto \overline{\eta}\,, \tag{19}$$

where multiplication is given by composition of natural transformations. By Theorem 1.1, the additive functor $\overline{K_0\mathrm{End}}$ is co-representable in Hmo_0 by the dg category $\underline{\mathbb{Z}[r]}$. As a consequence, the enriched Yoneda lemma [**6**, Thm. 8.3.5] furnish us a ring isomorphism

$$\mathrm{Nat}(\overline{K_0\mathrm{End}}, \overline{K_0\mathrm{End}}) \xrightarrow{\sim} \mathsf{Hom}_{\mathsf{Hmo}_0}(\underline{\mathbb{Z}[r]}, \underline{\mathbb{Z}[r]}) \qquad \overline{\eta} \mapsto \overline{\eta}(\underline{\mathbb{Z}[r]})([\mathrm{id}_{\underline{\mathbb{Z}[r]}}])\,, \tag{20}$$

where the multiplication on the right-hand side is given by composition. Recall from §4 that $\mathsf{Hom}_{\mathsf{Hmo}_0}(\underline{\mathbb{Z}[r]}, \underline{\mathbb{Z}[r]}) := K_0\mathsf{rep}(\underline{\mathbb{Z}[r]}, \underline{\mathbb{Z}[r]})$ and that the composition operation is induced by the tensor product of bimodules

$$K_0\mathsf{rep}(\underline{\mathbb{Z}[r]}, \underline{\mathbb{Z}[r]}) \otimes_{\mathbb{Z}} K_0\mathsf{rep}(\underline{\mathbb{Z}[r]}, \underline{\mathbb{Z}[r]}) \longrightarrow K_0\mathsf{rep}(\underline{\mathbb{Z}[r]}, \underline{\mathbb{Z}[r]}) \tag{21}$$

which maps $[X] \otimes [Y]$ to $[X \otimes_{\underline{\mathbb{Z}[r]}} Y]$. Now, making use of Proposition 3.2 and Theorem 1.1, we obtain a natural abelian group isomorphism

$$\mathsf{Hom}_{\mathsf{Hmo}_0}(\underline{\mathbb{Z}[r]}, \underline{\mathbb{Z}[r]}) = K_0\mathsf{rep}(\underline{\mathbb{Z}[r]}, \underline{\mathbb{Z}[r]}) \simeq K_0\mathrm{End}(\mathbf{P}(\mathbb{Z}[r]))\,. \tag{22}$$

Via this isomorphism, the composition operation on $\mathsf{Hom}_{\mathsf{Hmo}_0}(\underline{\mathbb{Z}[r]}, \underline{\mathbb{Z}[r]})$ then corresponds to a bilinear composition law $\star$ on $K_0\mathrm{End}(\mathbf{P}(\mathbb{Z}[r]))$. Note that the unit of $\star$ is given by the class $[(\mathbb{Z}[r], \cdot r)]$. Finally, using Almkvist's abelian group isomorphism

$$K_0\mathrm{End}(\mathbf{P}(\mathbb{Z}[r])) \xrightarrow{\sim} \mathbb{Z} \oplus W_0(\mathbb{Z}[r]) \qquad (M, \alpha) \mapsto ([M], \det(\mathrm{Id} + \alpha t)), \tag{23}$$

we transfer the bilinear composition law $\star$ to $\mathbb{Z} \oplus W_0(\mathbb{Z}[r])$; note that the unit corresponds to $(1, 1 + rt)$. Recall that the natural isomorphism (5) is the concatenation of the above isomorphisms (19), (20), (22) and (23).

Let us now study the composition law $\star$ on the direct summand $\mathbb{Z}$. Recall that the unit of the multiplicative group $W_0(\mathbb{Z}[r])$ is the polynomial 1. Given natural numbers $k, l \geqslant 1$, we have the following identifications:

$$[(\mathbb{Z}[r]^{\oplus k}, 0)] \leftrightarrow (k, 1) \qquad [(\mathbb{Z}[r]^{\oplus l}, 0)] \leftrightarrow (l, 1) \tag{24}$$

under the above isomorphism (23). Making use of isomorphism (22) and of the above description (21) of the composition operation, we observe that

$$[(\mathbb{Z}[r]^{\oplus k}, 0)] \star [(\mathbb{Z}[r]^{\oplus l}, 0)] = [(\mathbb{Z}[r]^{\oplus (k \times l)}, 0)]\,.$$

As a consequence, $(k, 1) \star (l, 1) = (k \times l, 1)$. Since the composition law $\star$ on $\mathbb{Z} \oplus W_0(\mathbb{Z}[r])$ is bilinear, we conclude not only that it restricts to the direct summand $\mathbb{Z}$ but, moreover, that its restriction to $\mathbb{Z}$ agrees with the ring of integers.

Let us now study the composition law $\star$ on the direct summand $W_0(\mathbb{Z}[r])$. Recall that the unit of the additive group $\mathbb{Z}$ is 0. Given arbitrary polynomials $\mathcal{P} = 1 + p_1(r)t + \cdots + p_k(r)t^k$ and $\mathcal{H} = 1 + h_1(r)t + \cdots + h_l(r)t^l$ of $W_0(\mathbb{Z}[r])$, a simple exercise (which we leave for the reader) shows us that, under the above isomorphism (23), we have the following identifications:

$$\mathcal{M}^k_{p_1,\ldots,p_k} := [(\mathbb{Z}[r]^{\oplus k}, M(p_1(r), \ldots, p_k(r)))] \leftrightarrow (k, \mathcal{P})$$

$$\mathcal{M}^l_{h_1,\ldots,h_l} := [(\mathbb{Z}[r]^{\oplus l}, M(h_1(r), \ldots, k_l(r)))] \leftrightarrow (l, \mathcal{H})\,.$$

Hence, making use of (24), we obtain the following identifications:

$$\big(\mathcal{M}^k_{p_1,\ldots,p_k} - [(\mathbb{Z}[r]^{\oplus k},0)]\big) \leftrightarrow (0,\mathcal{P}) \qquad \big(\mathcal{M}^l_{h_1,\ldots,h_l} - [(\mathbb{Z}[r]^{\oplus l},0)]\big) \leftrightarrow (0,\mathcal{H})\,.$$

Since the composition law $\star$ on $K_0\mathrm{End}(\mathbf{P}(\mathbb{Z}[r]))$ is bilinear, the element

$$\big(\mathcal{M}^k_{p_1,\ldots,p_k} - [(\mathbb{Z}[r]^{\oplus k},0)]\big) \star \big(\mathcal{M}^l_{h_1,\ldots,h_l} - [(\mathbb{Z}[r]^{\oplus l},0)]\big) \tag{25}$$

is the sum of $\left(\mathcal{M}^k_{p_1,\ldots,p_k} \star \mathcal{M}^l_{h_1,\ldots,h_l}\right)$ with the linear combination

$$-\big(\mathcal{M}^k_{p_1,\ldots,p_k} \star [(\mathbb{Z}[r]^{\oplus l},0)]\big) - \big([(\mathbb{Z}[r]^{\oplus k},0)] \star \mathcal{M}^l_{h_1,\ldots,h_l}\big) + \big([(\mathbb{Z}[r]^{\oplus k},0)] \star [(\mathbb{Z}[r]^{\oplus l},0)]\big)\ .$$

Making use of the isomorphism (22) and of the above description (21) of the composition operation, we observe that

$$\big(\mathcal{M}^k_{p_1,\ldots,p_k} \star \mathcal{M}^l_{h_1,\ldots,h_l}\big) = [(\mathbb{Z}[r]^{\oplus(k\times l)}, M_{\mathcal{P};\mathcal{H}})]$$

and that the three terms of the above linear combination are all equal to

$$[(\mathbb{Z}[r]^{\oplus(k\times l)},0)].$$

As a consequence, the above element (25) agrees with

$$[(\mathbb{Z}[r]^{\oplus(k\times l)}, M_{\mathcal{P};\mathcal{H}})] - [(\mathbb{Z}[r]^{\oplus(k\times l)},0)]\,.$$

Via the above isomorphism (23), we then obtain the equality

$$(0,\mathcal{P}) \star (0,\mathcal{H}) = (0, \mathcal{P}\star\mathcal{H} := \det(\mathrm{Id} + M_{\mathcal{P};\mathcal{H}}t))\,.$$

Therefore, since the composition law $\star$ on $\mathbb{Z}\oplus W_0(\mathbb{Z}[r])$ is bilinear, we conclude that it restricts to the direct summand $W_0(\mathbb{Z}[r])$. We then obtain a well-defined (noncommutative) ring $(W_0(\mathbb{Z}[r]),\cdot,\star)$ whose multiplication is completely characterized by the assignment (6).

Finally, the above arguments combined with the bilinearity of the composition law $\star$ on $\mathbb{Z}\oplus W_0(\mathbb{Z}[r])$ allow us to conclude that this (noncommutative) ring is the direct sum of $\mathbb{Z}$ with the ring $(W_0(\mathbb{Z}[r]),\cdot,\star)$. The fact that the abelian group isomorphism (5) is a ring isomorphism is now clear. This concludes the proof.

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