*Homology, Homotopy and Applications, vol. 13(2), 2011, pp.263–292*

# LIE COALGEBRAS AND RATIONAL HOMOTOPY THEORY, I: GRAPH COALGEBRAS

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(*communicated by Gunnar Carlsson*)

*Abstract*

We give a new presentation of the Lie cooperad as a quotient of the graph cooperad, a presentation which is not linearly dual to any of the standard presentations of the Lie operad. We use this presentation to explicitly compute duality between Lie algebras and coalgebras, to give a new presentation of Harrison homology, and to show that Lyndon words yield a canonical basis for cofree Lie coalgebras.

## 1. Introduction

In this paper we develop a new, computationally friendly approach to Lie coalgebras through graph coalgebras, and we apply this approach to Harrison homology. There are two standard presentations of a Lie algebra through "simpler" algebras. One is as a quotient of a non-associative binary algebra by Jacobi and anti-commutativity identities. Another presentation is as embedded as Hopf algebra primitives in an associative universal enveloping algebra. The standard presentation of Lie coalgebras in the literature is dual to the second of these – as a quotient of the associative coenveloping coalgebra, namely the Hopf algebra indecomposables [**12, 17**]. We describe an approach to Lie coalgebras indigenous to the realm of coalgebras, dual to neither of these. We define a new kind of coalgebra structure, namely anti-commutative graph coalgebras, and we show that Lie coalgebras are quotients of these graph coalgebras.

Our approach through graph coalgebras gives a presentation for Lie coalgebras which works better than the classical presentation in two respects. First, cofree graph coalgebras come with a simple and easily computable pairing with free binary nonassociative algebras which passes to Lie coalgebras and algebras, making duality not just a theoretical statement but an explicitly computable tool. Secondly, the quotient used to create Lie coalgebras from graph coalgebras is a locally defined relation. The quotient creating Lie coalgebras from associative coalgebras is the shuffle relation, which causes global changes to an expression. As a result, proofs in the realm of Lie

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The first author is supported in part by NSF-DMS 1006819.
The second author was supported in part by the Mittag-Leffler Institute (Djursholm, Sweden), and the Max-Planck Institute, Bonn.
Received September 16, 2009, revised February 9, 2010; published on December 6, 2011.
2000 Mathematics Subject Classification: 55P62; 16E40, 55P48.
Key words and phrases: Lie coalgebras, rational homotopy theory, graph cohomology.
Article available at http://intlpress.com/HHA/v13/n2/a16 and doi:10.4310/HHA.2011.v13.n2.a16

coalgebras are often simpler to give through graph coalgebras than through associative coalgebras, and for some important statements we have only found proofs in the graph coalgebra setting. For applications, we investigate the word problem for Lie coalgebras, show that Lyndon words define a canonical basis for the cofree Lie coalgebra on a vector space with an ordered basis, and we revisit Harrison homology. In the sequel to this paper, we use this presentation to model fibrations in rational homotopy theory. The category of graph coalgebras, and the graph cooperad on which it is based, may also be of intrinsic interest. The graph cooperad is not binary, but could play a similar role in some natural category of cooperads as is played by the tree operad for binary operads.

The plan of the paper is to first define the graph cooperad and pair it with the tree operad to give rise to a pairing of cofree and free algebras over them. We then show that upon quotienting by the kernels of the pairing, it descends to a pairing between cofree Lie coalgebras and free Lie algebras. This graphical model for the cofree Lie coalgebra on a vector space $V$ cleanly determines how that model pairs with the free Lie algebra on a linear dual of $V$. Moreover, we can deduce a formula for the linear duality between Michaelis's Lie coalgebra model [**12**] and the tree/bracket model for free Lie algebras. We are also able to shed new light on the structure of cofree Lie coalgebras, for example viewing them as what one gets when one starts with a graph or associative coalgebra and "kills the kernel of the cobracket."

We then lift the André-Quillen construction on a differential graded commutative algebra (DGCA) from the category of differential graded Lie coalgebras (DGLC) to anti-commutative differential graded graph coalgebras (DGGC). The Harrison model for this bar construction passes through the category of associative coalgebras, but our factorization through graph coalgebras is needed for example in developing algebraic models for fibrations in the Lie coalgebraic formulation of rational homotopy theory. Such a result is critical in the sequel to this paper, where we define generalized Hopf invariants and show from first principals that they give a complete set of homotopy functionals in the simply-connected setting. Indeed it was an investigation of generalized Hopf invariants, which we found to be naturally indexed by graphs, which led us to the framework of this paper.

Finally, we combine these results to shed new light on Quillen's seminal work on rational homotopy theory [**15**]. Quillen produced a pair of adjoint functors $\mathcal{L}$ and $\mathcal{C}$ between the categories of dg-commutative coalgebras (DGCC) and dg-Lie algebras (DGLA). In the linearly dual setting, there previously were two avenues towards understanding the functors between DGCA and DGLC. One would be a formal application of linear duality to Quillen's functors. The other way to go from DGCA to DGLC explicitly was to use the Harrison complex, which from [**17**] has the structure of a Lie coalgebra dual to Quillen's Lie algebraic functor. Our techniques allow us to explicitly calculate the linear duality between Harrison homology of a differential graded commutative algebra and Quillen's functor $\mathcal{L}$ on the corresponding linearly dual coalgebra, unifying these approaches.

In our appendices, we give a spectral sequence for rational homotopy groups of a simply connected space, we explicitly define model structures, and we discuss minimal models.

Our work throughout is over a field of characteristic zero. We are adding a finiteness hypothesis, namely that our algebras and coalgebras are finite-dimensional in each positive degree, for the sake of linear duality theorems. Under this hypothesis the category of chain complexes is canonically isomorphic to that of cochain complexes, and by abuse we denote both categories by DG. To clarify when possible, we have endeavored to use $V$ to denote a chain complex and $W$ to denote a cochain complex.

We further restrict our work to 1-connected objects both to mirror the classical constructions of [**15**] and to allow ourselves to cleanly express our cofree Lie coalgebras as coinvariants rather than invariants. We plan to remove the finiteness and 1-connectivity hypotheses in the third paper in this series. In Sullivan's rational homotopy theory it is fairly typical to quickly move to the nilpotent setting, but this step requires a significant change to foundations of our work. The first author is currently writing a general theory of coalgebras over cooperads [**25**] so that we may proceed with such a program, where it looks like we can extend even beyond the nilpotent setting.

While we start by giving operadic definitions, we work more explicitly at the algebra and coalgebra level in later sections. One reason for this change in emphasis is a desire for explicit formulae. But the change in emphasis is necessary, since we have yet to find a purely operadic argument for the existence of the lift of the bar construction on a commutative algebra from the category of Lie coalgebras to the category of graph coalgebras. We are applying graph cooperad and graph coalgebras in current work, and hope to understand them more generally as well. We have yet to fully understand even what general (that is, not cofree) graph coalgebras are in explicit algebraic terms.

## 2. The graph cooperad and the configuration pairing

We begin with constructions on the level of operads and cooperads, to give more fundamental understanding (to readers familiar with operads) and provide a general road-map for the following sections. A reader solely interested in Lie algebras and coalgebras could skip most of this section, with the exceptions of the definitions of graphs (2.1), the configuration pairing (2.12), and the quotients defining Lie coalgebras (2.15).

**Definition 2.1.** The graph symmetric sequence is defined as follows.

1. Let $S$ be a finite set. An $S$-graph is a connected oriented acyclic graph with vertices labelled by the set $\mathrm{Vert}(G) = S$.
2. For each $S$, let $\mathcal{G}r(S)$ be the vector space freely generated by $S$-graphs. Write $\mathcal{G}r$ for the associated symmetric sequence of vector spaces, and write $\mathcal{G}r(n)$ for $\mathcal{G}r(\{1, \dots, n\})$.
3. If $G \in \mathcal{G}r(S)$, define $|G|$ to be the cardinality of $S$, which we call the weight of $G$.

We outline the basic properties of the graph cooperad. For proofs and more detailed discussion, see the examples section of [**25**] where a more convenient notation for cooperads is developed.

**Definition 2.2.** A graph quotient $\phi\colon G \twoheadrightarrow K$ maps vertices of $G$ to vertices of $K$ such that edges of $G$ are mapped to either edges of $K$ (with the same orientation) or vertices of $K$, and the inverse image of each vertex of $K$ is a non-empty connected subgraph of $G$.

**Proposition 2.3.** *The symmetric sequence $\mathcal{G}r$ has a cooperad structure induced by the map*

$$G \longmapsto \sum_{\phi\colon G \twoheadrightarrow K} K \bigotimes \Big( \bigotimes_{k \in \mathrm{Vert}(K)} \phi^{-1}(k) \Big),$$

*where $\sum_\phi$ is a formal sum over all graph quotient maps and $\phi^{-1}(k)$ is the connected subgraph of $G$ mapping to vertex $k$.*

*Example 2.4.* The image of the graph ${}_a{\nearrow}^{b}{\nwarrow}_c$ in $\mathcal{G}r(2) \otimes \mathcal{G}r^{\otimes 2}$ under this structure is

$$ {}_1{\nearrow}^2 \bigotimes \Big( a \otimes {}^b{\nwarrow}_c + c \otimes {}_a{\nearrow}^b \Big) + {}^1{\nwarrow}_2 \bigotimes \Big( {}^b{\nwarrow}_c \otimes a + {}_a{\nearrow}^b \otimes c \Big).$$

The cooperad structure above is non-associative in the sense that the 2-arity structure maps

$$\Big( {}_1{\nearrow}^2 \Big)^{*} \colon G \longmapsto \sum_{\phi\colon G \twoheadrightarrow {}_1{\nearrow}^2} \phi^{-1}(1) \otimes \phi^{-1}(2)$$

are not (co-)associative. We may make an associative structure from this by inserting terms reversing arrows in either a commutative or anti-commutative manner. The anti-commutative cooperad structure on $\mathcal{G}r$ is given as follows.

**Definition 2.5.** Let $E \subset \mathrm{Edge}(K)$. Define $\mathrm{rev}_E(K)$ to be the graph resulting from reversing the orientations of the edges $E$ of $K$.

**Proposition 2.6.** *The symmetric sequence $\mathcal{G}r$ has an anti-commutative cooperad structure induced by*

$$G \longmapsto \sum_{\substack{\phi\colon G \twoheadrightarrow K \\ E \subset \mathrm{Edge}(K)}} (-1)^{|E|} \ \mathrm{rev}_E(K) \bigotimes \Big( \bigotimes_{k \in \mathrm{Vert}(K)} \phi^{-1}(k) \Big),$$

*where $\phi$ and $\phi^{-1}(k)$ are as above.*

*Example 2.7.* The image of the graph ${}_a{\nearrow}^{b}{\nwarrow}_c$ in $\mathcal{G}r(2) \otimes \mathcal{G}r^{\otimes 2}$ under the anti-commutative structure is

$$\begin{aligned} & {}_1{\nearrow}^2 \bigotimes \Big( a \otimes {}^b{\nwarrow}_c - {}^b{\nwarrow}_c \otimes a + c \otimes {}_a{\nearrow}^b - {}_a{\nearrow}^b \otimes c \Big) \\ & \qquad + {}^1{\nwarrow}_2 \bigotimes \Big( {}^b{\nwarrow}_c \otimes a - a \otimes {}^b{\nwarrow}_c + {}_a{\nearrow}^b \otimes c - c \otimes {}_a{\nearrow}^b \Big). \end{aligned}$$

**Definition 2.8.** The anti-commutative graph cooperad, denoted $\mathcal{ACG}r$, is given by the symmetric sequence $\mathcal{G}r$ equipped with the anti-commutative cooperad structure of Proposition 2.6.

The non-associative graph cooperad, denoted $\mathcal{NAG}r$, is given by the symmetric sequence $\mathcal{G}r$ equipped with the non-associative cooperad structure of Proposition 2.3.

In the language of operads, the standard approaches to Lie algebras can be summarized by a sequence of operad maps $\mathcal{T}r \to \mathcal{L}ie \to \mathcal{A}s$. Recall that the associative operad has $\mathcal{A}s(n)$ of rank $n!$, naturally spanned by monomials in $n$ variables with no repetition, and $\mathcal{T}r$ is the tree operad whose structure maps are defined by grafting and which governs non-associative binary algebras (see 2.11 below). Our Lie coalgebra model follows from fitting the anti-commutative graph cooperad into the linearly dual sequence of cooperads as $\mathcal{T}r^\vee \leftarrow \mathcal{L}ie^\vee \leftarrow \mathcal{ACG}r \leftarrow \mathcal{A}s^\vee$. The following propositions are easily verified by direct calculation.

**Proposition 2.9.** *The associative cooperad $\mathcal{A}s^\vee$ maps to the non-associative graph cooperad $\mathcal{NAG}r$ by sending the monomial $x_1 x_2 \cdots x_n$ to the graph* ${}_{x_1}\nearrow^{x_2}\searrow_{\cdots}\nearrow^{x_n}$ .

**Proposition 2.10.** *The non-associative graph cooperad $\mathcal{NAG}r$ maps to the anti-commutative graph cooperad $\mathcal{ACG}r$ via the map*

$$G \longmapsto \frac{1}{2^{\#(\mathrm{Edge}(G))}} \sum_{E \subset \mathrm{Edge}(G)} (-1)^{|E|} \mathrm{rev}_E(G).$$

Coalgebras over these graph cooperads have not, to our knowledge, been studied before. It seems that such coalgebras arise naturally in an approach to detecting when an element of a finitely presented group lies in the $n$th commutator subgroup (as the Magnus expansion does for free groups), which we are currently investigating. Such graph coalgebras are not binary coalgebras. For example, the binary cooperad structure map on $\mathcal{G}r(3)$ maps a twelve-dimensional vector space to $\mathcal{G}r(2) \otimes (\mathcal{G}r(2) \otimes \mathcal{G}r(1)) \oplus \mathcal{G}r(2) \otimes (\mathcal{G}r(2) \otimes \mathcal{G}r(1))$, which is eight-dimensional, so it cannot be injective.

Next, we develop the configuration pairing between graphs and trees, which allows us to explicitly relate graph coalgebras to associative and Lie algebras, in particular defining a map $\mathcal{ACG}r \to \mathcal{T}r^\vee$.

**Definition 2.11.** Let $S$ be a finite set. An $S$-tree is an isotopy class of acyclic graphs embedded in the upper half plane with all vertices either trivalent or univalent. Trivalent vertices are called internal vertices. One univalent vertex is distinguished as the root and embedded at the origin. The other univalent vertices are called leaves and are equipped with a labeling ismorphism $\ell\colon \mathrm{Leaves} \xrightarrow{\cong} S$. We will standardly conflate leaves with their labels.

Let $\mathcal{T}r(S)$ be the vector space generated by $S$-trees, $\mathcal{T}r$ be the associated symmetric sequence of vector spaces, and write $\mathcal{T}r(n)$ for $\mathcal{T}r(\{1, \dots, n\})$.

See II.1.9 in [**10**] for a precise definition of the operad structure maps of $\mathcal{T}r$ through grafting. The pairing between $\mathcal{G}r(n)$ and $\mathcal{T}r(n)$ was developed in [**19**], and arises in the study of configuration spaces. Let the height of a vertex in a tree be the number of edges between that vertex and the root, and let $gcv(a, b)$ be the vertex of greatest height which lies beneath leaves labelled $a$ and $b$.

**Definition 2.12.** Fix a finite set $S$. Given an $S$-graph $G$ and an $S$-tree $T$, define the map

$$\beta_{G,T}\colon \big\{\text{edges of } G\big\} \longrightarrow \big\{\text{internal vertices of } T\big\}$$

by sending the edge ${}_a\nearrow^b$ in $G$ to the vertex $gcv(a, b)$ in $T$. The configuration pairing of $G$ and $T$ is

$$\langle G, T \rangle = \begin{cases} \displaystyle\prod_{\substack{e \text{ an edge} \\ \text{of } G}} \operatorname{sgn}\big(\beta_{G,T}(e)\big) & \text{if } \beta \text{ is surjective,} \\ 0 & \text{otherwise,} \end{cases}$$

where given an edge $\underset{a}{\nearrow}{}^{b}$ of $G$, $\operatorname{sgn}\left(\beta\left(\underset{a}{\nearrow}{}^{b}\right)\right) = 1$ if leaf $a$ is to the left of leaf $b$ under the planar embedding of $T$; otherwise it is $-1$.

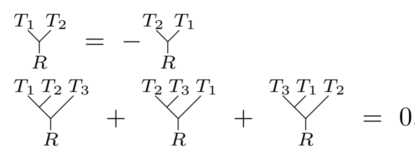
*Example 2.13.* Following is the map $\beta_{G,T}$ for a single graph $G$ and two different trees $T$.


In the first example, $\operatorname{sgn}\big(\beta(e_1)\big) = -1$ and $\operatorname{sgn}\big(\beta(e_2)\big) = 1$. In the second example, $\operatorname{sgn}\big(\beta(e_1)\big) = 1$ and $\operatorname{sgn}\big(\beta(e_2)\big) = -1$. The graph and tree of the first example pair to $-1$, and those in the second example pair to 0.

**Definition 2.14.** $\mathcal{L}ie(n)$ is the quotient of $\mathcal{T}r(n)$ by the anti-symmetry and Jacobi relations:

(anti-symmetry)

(Jacobi)



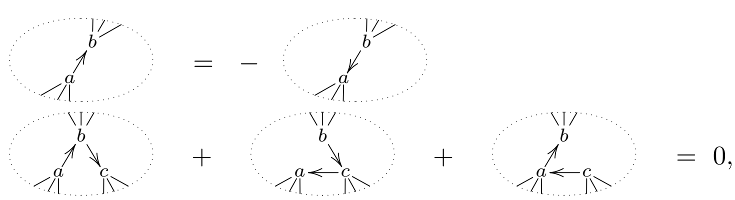

where $R$, $T_1$, $T_2$, and $T_3$ stand for arbitrary (possibly trivial) subtrees which are not modified in these operations.

The configuration pairing respects anti-symmetry and Jacobi relations among trees. There is a similar set of relations which the configuration pairing respects among graphs.

**Definition 2.15.** Let $\mathcal{E}il(n)$ be the quotient of $\mathcal{G}r(n)$ by the relations

(arrow-reversing)

(Arnold)




where $a$, $b$, and $c$ stand for vertices in the graph which could possibly have other connections to other parts of the graph which are not modified in these operations. We emphasize that $a$, $b$, $c$ are *vertices*, not subgraphs.

The first author's paper [**19**] establishes the following theorem, which was first proven independently by Tourtchine [**24**] and, in the odd setting, Melancon and Reutenauer [**11**].

**Theorem 2.16.** *The configuration pairing $\langle G, T\rangle$ between $\mathcal{G}r(n)$ and $\mathcal{T}r(n)$ descends to a perfect equivariant pairing between $\mathcal{E}il(n)$ and $\mathcal{L}ie(n)$.*

*There is an isomorphism of symmetric sequences $\mathcal{E}il(n) \cong \mathcal{L}ie^{\vee}(n)$.*

The theorem is proven by first showing that the pairing vanishes on Jacobi and anti-symmetry combinations of trees as well as on arrow-reversing and Arnold combinations of graphs. These relations allow one to reduce to generating sets of "tall" trees and "long" graphs – as in the figure below. The pairing is a Kronecker pairing on these generating sets.


Figure 1: Tall trees and long graphs

**Proposition 2.17.** *The subcomplex of graph expressions generated by arrow-reversing and Arnold expressions of graphs is a coideal [**7**, §2.1] of $\mathcal{ACG}r$.*

**Corollary 2.18.** *The symmetric sequence $\mathcal{E}il$ inherits an anti-commutative cooperad structure from $\mathcal{ACG}r$.*

By abuse, we use $\mathcal{E}il$ to denote the cooperad induced by quotienting $\mathcal{ACG}r$ by the Arnold and arrow-reversing identites.

**Proposition 2.19.** *The cooperad structure of $\mathcal{ACG}r$ is compatible with the operad structure of $\mathcal{T}r$ via the configuration pairing.*

**Corollary 2.20.** *The cooperad structure on $\mathcal{E}il$ is compatible with the operad structure of $\mathcal{L}ie$ (inherited from that of $\mathcal{T}r$) via the configuration pairing.*

**Theorem 2.21.** *As cooperads, $\mathcal{E}il \cong \mathcal{L}ie^{\vee}$. Quotienting by Arnold and arrow-reversing identities gives a surjection of cooperads from $\mathcal{ACG}r$ to $\mathcal{L}ie^{\vee}$.*

Since we would rather emphasize free and cofree algebras than the operads defining them, we will reserve the computations required for Propositions 2.17 and 2.19 for the proofs of Propositions 3.7 and 3.14 which are the analogous statements on the level of coalgebras and algebras. A short duality computation (which we leave for the reader) now completes our operadic picture.

**Proposition 2.22.** *The following duality diagram of operads and cooperads commutes.*

$$\begin{tikzcd}
\mathcal{L}ie \arrow[rrr] \arrow[d, leftrightarrow, "*"] & & & \mathcal{A}s \arrow[d, leftrightarrow, "*"] \\
\mathcal{E}il & \mathcal{ACG}r \arrow[l, two heads] & \mathcal{NAG}r \arrow[l] & \mathcal{A}s^{\vee} \arrow[l, hook']
\end{tikzcd}$$

Algebra level consequences of this duality are discussed in Section 3.3 on coenveloping graph coalgebras.

*Remark 2.23.* This construction of coalgebras is over cooperads rather than over operads. It is common in the literature (such as [**22**]) to largely eschew the use of cooperads when discussing coalgebras, instead defining coalgebras over operads briefly as follows. Recall the endomorphism operad $\mathrm{End}(V)$ of an object $V$ in a closed symmetric monoidal category. The endomorphism operad of $V$ in the opposite category is called its coendomorphism operad $\mathrm{Coend}(V)(n) = \mathrm{Hom}(V, V^{\otimes n})$. If $\mathcal{P}$ is an operad then a $\mathcal{P}$-algebra structure on $V$ is an operad map $\mathcal{P} \to \mathrm{End}(V)$, and a $\mathcal{P}$-coalgebra structure on $V$ is an operad map $\mathcal{P} \to \mathrm{Coend}(V)$.

This relates to coalgebras over a cooperad in the following manner. A map $\mathcal{P} \to \mathrm{Coend}(V)$ consists of equivaraint maps $\mathcal{P}(n) \to \mathrm{Hom}(V, V^{\otimes n})$. If $\mathcal{P}(n)$ is dualizable then these are the same as equivariant maps $V \to \mathcal{P}(n)^* \otimes V^{\otimes n}$, which because $V$ has trivial action are simply maps from $V$ to the $\Sigma_n$-invariants of the right side. If $\mathcal{P}(n)$ is dualizable then the $\mathcal{P}(n)^*$ form a cooperad, and the structure above defines a coalgebra over this cooperad. This construction is immediately dual to the structure maps $\mathcal{P}(n) \otimes V^{\otimes n} \to V$ defining algebras over an operad. We write $\mathcal{P}^\vee$ for the cooperad $\mathcal{P}^\vee(n) = \mathcal{P}(n)^*$.

For more information about a general approach to cooperads and coalgebras over cooperads in their own terms, see [**25**].

*Remark 2.24.* While tree operad $\mathcal{T}r$ governs binary non-associative algebras, the graph cooperads cannot govern non-associative binary coalgebras. The configuration pairing between $\mathcal{T}r$ and $\mathcal{G}r$ is not perfect, nor could there be a different pairing which is perfect. For example, $\mathcal{T}r(n)$ has dimension $\frac{n(n-1)}{2} - 1$ as a $\mathbb{Q}[\Sigma_n]$-module (for $n > 1$). But as a $\mathbb{Q}[\Sigma_3]$-module $\mathcal{G}r(3)$ is of dimension 3, and as a $\mathbb{Q}[\Sigma_4]$-module $\mathcal{G}r(4)$ is of dimension 8. It is not clear what either the linear or Koszul-Moore duals (in the sense of [**13**]) of graph cooperads are.

## 3. The pairing between free tree algebras and cofree graph coalgebras

Constructing our graphical model for Lie coalgebras, we are interested in coalgebras over the anti-commutative graph cooperad $\mathcal{ACG}r$. Though we may occasionally write "anti-commutative graph coalgebra" for emphasis, in general we will write simply "graph coalgebra" to mean a coalgebra over the cooperad $\mathcal{ACG}r$. We explicitly develop only the quadratic structure of graph coalgebras since that is all that we require to understand Lie coalgebras. For an alternate development of graph coalgebras and the configuration pairing, see [**26**].

### 3.1. Basic manipulations of cofree graph coalgebras

A first step in the theory of operads is the construction of free algebras. We use underlying symmetric sequences to build co-Schur functors associated to $\mathcal{G}r$ and $\mathcal{E}il$ (dual to the Schur functors of [**6**]) and construct models for the vector spaces underlying Lie coalgebras as quotients of anti-commutative graph coalgebras. We then explicitly place a coproduct on our vector spaces and show that this coproduct is dual to the Lie algebra product.

**Definition 3.1.** Let $W$ be a vector space. Define the vector spaces $\overline{\mathbb{G}}(W)$ and $\overline{\mathbb{E}}(W)$

as follows.

$$\begin{aligned} \overline{\mathbb{G}}(W) &\cong \bigoplus_n \left(\mathcal{G}r(n)\otimes W^{\otimes n}\right)_{\Sigma_n} \\ \overline{\mathbb{E}}(W) &\cong \bigoplus_n \left(\mathcal{G}r(n)\otimes W^{\otimes n}\right)\big/_{\sim,\ \Sigma_n} = \overline{\mathbb{G}}(W)\big/_{\sim}, \end{aligned}$$

where $\sim$ is the relation induced by arrow-reversing and Arnold on $\mathcal{G}r(n)$.

There is a difficulty in defining general cofree graph and Lie coalgebras similar to that of defining general cofree associative coalgebras. Recall that the cotensor coalgebra does not give cofree associative coalgebras, since in particular it is always cofinite (that is, a finite iteration of the coproduct will reduce any element to primitives). Trying to remedy this by replacing colimits by limits usually does not yield a coalgebra, since this would require the tensor product to commute with infinite products. Using results of Smith [**21**], a cofree graph coalgebra is given in general by the largest coalgebra contained in $\prod_n \mathcal{G}r(n)\otimes^{\Sigma_n} W^{\otimes n}$. For our current work, we appeal to the time-honored tradition of restricting to 1-reduced (that is, trivial in grading zero and below) coalgebras. In this category, all coalgebras are cofinite, the cotensor coalgebra models cofree associative coalgebras, and we have the following.

**Proposition 3.2.** *If $W$ is* 1*-reduced, then $\overline{\mathbb{G}}(W)$ is the vector space which underlies the cofree graph coalgebra on $W$ and $\overline{\mathbb{E}}(V)$ underlies the cofree Lie coalgebra on $V$.*

We now explicitly develop the graph and Lie coalgebra structures referred to in the previous proposition. In the ungraded case, $\overline{\mathbb{G}}(W)$ is generated by oriented, connected, acyclic graphs (of possibly infinite size) whose vertices are labeled by elements of $W$ modulo multilinearity in the labels. Cutting a single edge separates graphs in $\overline{\mathbb{G}}(W)$, so we may define a coproduct by a summation cutting each edge in turn and tensoring the resulting graphs in the order determined by the direction of the edge which was cut – this is the coproduct encoded by $\mathcal{NAG}r$. In order to descend to the Lie coalgebra cobracket (see Corollary 3.15) we add a twisted term to the above coproduct with signs to make the result anti-cocommutative – this is the coproduct encoded by $\mathcal{ACG}r$. Explicitly, $]G[\, = \sum_{e\in G}(G_1^{\hat{e}}\otimes G_2^{\hat{e}} - G_2^{\hat{e}}\otimes G_1^{\hat{e}})$, where $e$ ranges over the edges of $G$, and $G_1^{\hat{e}}$ and $G_2^{\hat{e}}$ are the connected components of the graph obtained by removing $e$, which points from $G_1^{\hat{e}}$ to $G_2^{\hat{e}}$.

Unfortunately, graded graph coalgebras are more complicated to represent due to the presence of Koszul signs. For example, $\begin{smallmatrix} & b & \\ \swarrow & & \searrow \\ a & & c\end{smallmatrix}$ could mean either

$$\left[\begin{smallmatrix} & 1 & \\ \swarrow & & \searrow \\ 2 & & 3\end{smallmatrix} \bigotimes b\otimes a\otimes c\right] \quad\text{or}\quad \left[\begin{smallmatrix} & 1 & \\ \swarrow & & \searrow \\ 3 & & 2\end{smallmatrix} \bigotimes b\otimes c\otimes a\right],$$

which differ by a sign of $(-1)^{|a||c|}$. The same difficulty arises when defining graded Lie algebras via the $\mathcal{L}ie$ operad (or non-associative algebras via the $\mathcal{T}r$ operad), but the simple convention there is to choose the equivalence class representative whose $\mathcal{L}ie(n)$ component has the ordering of its leaves consistent with the planar ordering. Because there is no general canonical choice for representatives of $\Sigma_n$-equivalence classes in $\mathcal{G}r(n)$, we are forced to write elements of $\overline{\mathbb{G}}(W)$ explicitly via representatives in $\mathcal{G}r(n)\otimes W^{\otimes n}$.

We define the graded anti-commutative graph cobracket as follows.

**Definition 3.3.** The anti-commutative graph cobracket
$$]\cdot[\ :\overline{\mathbb{G}}(W)\to\overline{\mathbb{G}}(W)\otimes\overline{\mathbb{G}}(W)$$
is given by
$$\begin{aligned}\Big]G\bigotimes w_1\otimes\cdots\otimes w_n\Big[\ =\sum_{e\in G}&(-1)^{\kappa_1}\big(G_1^{\hat e}\bigotimes w_{\sigma^{\hat e}(1)}\otimes\cdots\otimes w_{\sigma^{\hat e}(|G_1^{\hat e}|)}\big)\\ &\qquad\bigotimes\big(G_2^{\hat e}\bigotimes w_{\sigma^{\hat e}(|G_1^{\hat e}|+1)}\otimes\cdots\otimes w_{\sigma^{\hat e}(n)}\big)\\ &-(-1)^{\kappa_2}\big(G_2^{\hat e}\bigotimes w_{\sigma^{\hat e}(|G_1^{\hat e}|+1)}\otimes\cdots\otimes w_{\sigma^{\hat e}(n)}\big)\\ &\qquad\bigotimes\big(G_1^{\hat e}\bigotimes w_{\sigma^{\hat e}(1)}\otimes\cdots w_{\sigma^{\hat e}(|G_1^{\hat e}|)}\big),\end{aligned}$$
where $e$ ranges over the edges of $G$ and points from the connected subgraph $G_1^{\hat e}$ to the connected subgraph $G_2^{\hat e}$, $\sigma^{\hat e}$ is the unshuffling of vertex labels induced by separating $G$ into $G_1^{\hat e}$ and $G_2^{\hat e}$, and $(-1)^{\kappa_1}$, $(-1)^{\kappa_2}$ are the Koszul signs due to reordering the $w_i$'s.

**Proposition 3.4.** *The anti-commutative graph cobracket* $]\cdot[$ *on* $\overline{\mathbb{G}}(W)$ *coincides with the binary coproduct arising from the* 2*-arity cooperad structure map of* $\mathcal{ACG}r$.

Rather than apply this proposition, we prefer to give a more direct proof that the cobracket given above is dual to Lie algebra brackets, which will aid in the work which follows.

**Definition 3.5.** Let $\mathbb{G}(W)$ denote the cofree anti-commutative graph coalgebra on $W$, whose binary structure is thus given by $\overline{\mathbb{G}}(W)$ with anti-commutative graph cobracket $]\cdot[$ . Similarly, let $\mathbb{E}(W)$ denote the cofree Lie coalgebra on $W$.

For horizontal brevity we will generally write $\underset{w_1\otimes\cdots\otimes w_n}{G}:=G\bigotimes w_1\otimes\cdots\otimes w_n$ for all graphs except for the trivial one: $G=\bullet^1$.

*Example 3.6.* The anti-commutative graph coalgebra element $\underset{a\otimes b\otimes c}{\begin{smallmatrix} & 2 & \\ \nearrow & & \nwarrow \\ 1 & & 3\end{smallmatrix}}$ has cobracket:
$$\begin{aligned}\Bigg]\underset{a\otimes b\otimes c}{\begin{smallmatrix} & 2 & \\ \nearrow & & \nwarrow \\ 1 & & 3\end{smallmatrix}}\Bigg[\ =(-1)^{(|a|+|b|)|c|}&\left(c\ \otimes\ \underset{a\otimes b}{\begin{smallmatrix} & 2 \\ \nearrow & \\ 1 & \end{smallmatrix}}\right)-\left(\underset{a\otimes b}{\begin{smallmatrix} & 2 \\ \nearrow & \\ 1 & \end{smallmatrix}}\ \otimes\ c\right)\\ &+\left(a\ \otimes\ \underset{b\otimes c}{\begin{smallmatrix} & 1 \\ \nearrow & \\ 2 & \end{smallmatrix}}\right)-(-1)^{|a|(|b|+|c|)}\left(\underset{b\otimes c}{\begin{smallmatrix} & 1 \\ \nearrow & \\ 2 & \end{smallmatrix}}\ \otimes\ a\right).\end{aligned}$$

**Proposition 3.7.** *Let* $\mathrm{Arn}(W)$ *be the vector subspace of* $\overline{\mathbb{G}}(W)$ *generated by arrow-reversing and Arnold expressions of graphs (2.15). Then* $\mathrm{Arn}(W)$ *is a coideal of* $\mathbb{G}(W)$*. That is*
$$\big]\mathrm{Arn}(W)\big[\ \subset\mathrm{Arn}(W)\otimes\mathbb{G}(W)+\mathbb{G}(W)\otimes\mathrm{Arn}(W).$$
*Thus the cobracket descends to a well-defined operation* $]\cdot[\ :\overline{\mathbb{E}}(W)\to\overline{\mathbb{E}}(W)\otimes\overline{\mathbb{E}}(W)$.

*Proof.* Due to the local definition of arrow-reversing and Arnold, it is enough to check the behaviour of the cobracket on an expression reversing the arrow of a graph with only two vertices and on an Arnold expression for a graph with only three vertices.

The arrow-reversing check, neglecting Koszul signs, is:

$$\left]\ \underset{a\otimes b}{{}_1\overset{2}{\nearrow}} + \underset{a\otimes b}{{}_1\overset{2}{\swarrow}}\ \right[ \;=\; (a\otimes b - b\otimes a) + (b\otimes a - a\otimes b) \;=\; 0.$$

Modulo arrow-reversing, all graphs with only three vertices are long graphs, so it suffices to compute that

$$\left]\ \underset{a\otimes b\otimes c}{{}_1\nearrow\overset{2}{}\searrow{}_3}\ \right[ \;=\; \underset{a\otimes b}{{}_1\overset{2}{\nearrow}} \otimes c \;+\; a \otimes \underset{b\otimes c}{{}_1\overset{2}{\nearrow}} \;-\; c \otimes \underset{a\otimes b}{{}_1\overset{2}{\nearrow}} \;-\; \underset{b\otimes c}{{}_1\overset{2}{\nearrow}} \otimes a$$

and then check that the sum of these terms, cyclically permuted and with Koszul signs, cancel. $\square$

*Remark 3.8.* For long $n$-graphs, we can make a canonical choice of $\Sigma_n$-representatives, namely so that the ordering of vertices is consistent with the direction of arrows. In this case we use "bar" notation

$$a_1|a_2|\cdots|a_n := \left[\ {}_1\nearrow\overset{2}{}\searrow{}_3\nearrow\overset{4}{}\cdots{}_{n-1}\nearrow\overset{n}{}\ \bigotimes a_1\otimes a_2\otimes\cdots\otimes a_n\right].$$

Because long $n$-graphs span $\mathcal{E}il(n)$, the bar classes above span $\overline{\mathbb{E}}(W)$. For example, if $a$, $b$, $c$ and $d$ are all in even degree,

$$\left[\ \underset{a\otimes b\otimes c\otimes d}{\begin{smallmatrix} & 4 & \\ & \uparrow & \\ 1\nearrow & 2 & \searrow 3\end{smallmatrix}}\ \right] \;=\; \left[\ \underset{a\otimes b\otimes c\otimes d}{{}_4\nearrow\overset{2}{}\searrow{}_3\nearrow\overset{1}{}}\ \right] - \left[\ \underset{a\otimes b\otimes c\otimes d}{{}_3\nearrow\overset{1}{}\searrow{}_2\nearrow\overset{4}{}}\ \right] \;=\; d|b|c|a - c|a|b|d.$$

For the bar generators of $\overline{\mathbb{E}}(W)$, the cobracket given in Proposition 3.7 is essentially the anti-cocommutative coproduct (i.e., $]\cdot[\; = \Delta - \tau\Delta$ where $\tau$ is the twisting map $\tau(x\otimes y) = y\otimes x$). This recovers the approach taken by Michaelis [**12**] and Schlessinger-Stasheff [**17**]. We elaborate in Section 3.3.

### 3.2. Duality of free algebras and cofree coalgebras

As in the previous section, we start with underlying vector spaces and then move on to product and coproduct structures.

**Lemma 3.9.** *Let $G$ be a finite group, and let $V$ and $W$ be modules over a ring in which the order of $G$ is invertible. If $\langle -,-\rangle$ is an equivariant perfect pairing between $W$ and $V$, then the pairing defined between $W_G$ and $V_G$ by $\big\langle [w],[v]\big\rangle_G = \sum_{g\in G}\langle gw, v\rangle$ is also perfect.*

*Proof.* If $\big\langle [w],[v]\big\rangle_G = 0$ for all $[v]\in V_G$ then $\big\langle \sum_{g\in G} gw, v\big\rangle = 0$ for all $v\in V$. Because the pairing $\langle -,-\rangle$ is perfect, this means $\sum_{g\in G} gw = 0$ in $W$. Projecting to $W_G$ implies that $|G|\cdot[w] = 0$, which by our hypotheses means $[w]=0$. By equivariance we have $\big\langle [w],[v]\big\rangle_G = \sum_{g\in G}\langle w, gv\rangle$, so we may apply the same argument to show that there is no kernel for $\langle -,-\rangle_G$ in $V_G$ either, yielding the result. $\square$

Let $\mathbb{T}(V)$ be the free binary non-associative algebra on $V$, with underlying vector space $\overline{\mathbb{T}}(V)$ given by the Schur functor $\bigoplus_n (\mathcal{T}r(n) \otimes V^{\otimes n})_{\Sigma_n}$. Define $\mathbb{L}(V)$ and $\overline{\mathbb{L}}(V)$ similarly as the free Lie algebra on $V$ and its underlying vector space.

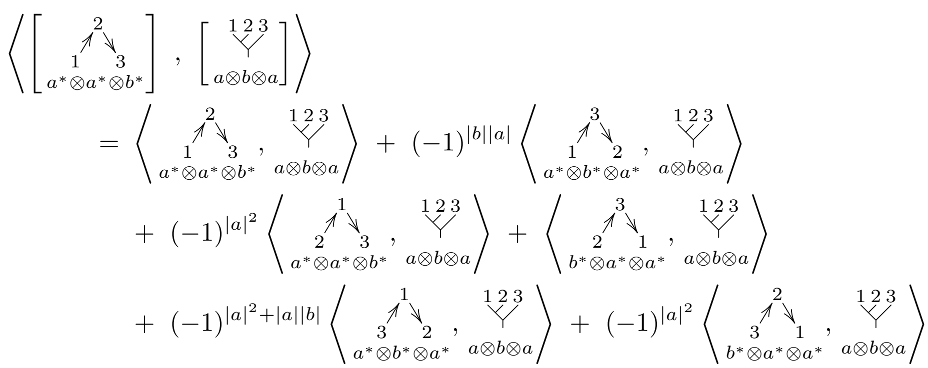
**Definition 3.10.** Given $W$ and $V$ vector spaces with a pairing $\langle -,- \rangle$, the configuration pairing between $\overline{\mathbb{G}}(W)$ and $\overline{\mathbb{T}}(V)$ is

$$\Big\langle [G \bigotimes w_1 \otimes \cdots \otimes w_n],\ [T \bigotimes v_1 \otimes \cdots \otimes v_n] \Big\rangle \;=\; \sum_{\sigma \in \Sigma_n} \left( \big\langle \sigma G,\, T \big\rangle \cdot \prod_{i=1}^{n} \langle w_{\sigma^{-1}(i)},\, v_i \rangle \right).$$

This descends also to a configuration pairing between $\overline{\mathbb{E}}(W)$ and $\overline{\mathbb{L}}(V)$ by Theorem 2.16. Applying Lemma 3.9 we have the following.

**Corollary 3.11.** *Over a field of characteristic zero, if $W$ and $V$ pair perfectly then the configuration pairing between $\overline{\mathbb{E}}(V)$ and $\overline{\mathbb{L}}(W)$ is perfect.*

*Example 3.12.* Consider the free Lie algebra on two letters, so that $V$ is spanned by $a$ and $b$. Then we have the following pairing.



$$= \; -(-1)^{|a|^2+|a||b|}$$

*Remark 3.13.* Melançon and Reutenauer [**11**] essentially showed that pairing with bar elements in $\mathbb{G}(V^*)$ defines functionals which can alternately be defined through coefficients of Lie polynomials (that is, coefficients of elements of $\mathbb{L}(V)$ in its standard embedding in the tensor algebra on $V$). It would be interesting to understand the functionals coming from other elements in $\mathbb{G}(V^*)$, such as those arising from Tourtchine's alternating trees [**24**], in a similar manner.

The configuration pairing further exhibits a duality between non-associative algebra multiplication and graph cobracket operations. This allows us to compute pairings inductively.

**Proposition 3.14.** *Non-associative algebra multiplication is dual to the anti-commutative graph cobracket in the configuration pairing. That is,*

$$\begin{aligned} \big\langle \gamma,\ (\tau_1\tau_2) \big\rangle &= \big\langle\, ]\gamma[\,,\ \tau_1 \otimes \tau_2 \big\rangle \\ &= \sum_e \big\langle \gamma_1^{\hat{e}},\, \tau_1 \big\rangle \big\langle \gamma_2^{\hat{e}},\, \tau_2 \big\rangle - \big\langle \gamma_2^{\hat{e}},\, \tau_1 \big\rangle \big\langle \gamma_1^{\hat{e}},\, \tau_2 \big\rangle, \end{aligned}$$

*where* $]\gamma[\; = \sum_e \big(\gamma_1^{\hat{e}} \otimes \gamma_2^{\hat{e}}\big) - \big(\gamma_2^{\hat{e}} \otimes \gamma_1^{\hat{e}}\big)$.

*Proof.* Recall that non-associative algebra multiplication is induced by the $\mathcal{T}r$ operation $(T_1T_2) = \overset{T_1\ T_2}{\curlyvee}$. We give a bijection between potentially non-zero terms in the summands defining $\big\langle\gamma,\,(\tau_1\tau_2)\big\rangle$ and $\big\langle\,]\gamma[\,,\,\tau_1\otimes\tau_2\big\rangle$. In particular, we focus on those terms whose graph/tree pairing component may be non-zero.

Begin by fixing graph and tree representatives. Let

$$\gamma = [G\otimes\vec{w}] \in \big(\mathcal{G}r(n)\otimes W^{\otimes n}\big)_{\Sigma_n}$$

(where $\vec{w}\in W^{\otimes n}$) and $\tau_i = [T_i\otimes\vec{v}_i] \in \big(\mathcal{T}r(k_i)\otimes V^{\otimes k_i}\big)_{\Sigma_{k_i}}$ (where $\vec{v}_i\in V^{\otimes k_i}$, $k_1+k_2=n$). Also let $\gamma_i^{\hat{e}} = [G_i^{\hat{e}}\otimes\vec{w}_i^{\hat{e}}]$ (for $i=1,2$) be the graph coalgebra elements given by cutting $\gamma$ at the edge $e$. Recall that

$$\big\langle\gamma,\ (\tau_1\tau_2)\big\rangle = \sum_{\sigma\in\Sigma_n}\big\langle\sigma G,(T_1T_2)\big\rangle\,\big\langle\sigma^{-1}\vec{w},\,\vec{v}_1\otimes\vec{v}_2\big\rangle$$

$$\big\langle\gamma_1^{\hat{e}},\,\tau_1\big\rangle\,\big\langle\gamma_2^{\hat{e}},\,\tau_2\big\rangle = \sum_{\sigma_i\in\Sigma_{k_i}}\big\langle\sigma_1G_1^{\hat{e}},\,T_1\big\rangle\,\big\langle\sigma_2G_2^{\hat{e}},\,T_2\big\rangle\,\big\langle\sigma_1^{-1}\vec{w}_1^{\hat{e}},\,\vec{v}_1\big\rangle\,\big\langle\sigma_2^{-1}\vec{w}_2^{\hat{e}},\,\vec{v}_2\big\rangle.$$

Suppose that some $\big\langle\sigma_1G_1^{\hat{e}},\,T_1\big\rangle\big\langle\sigma_2G_2^{\hat{e}},\,T_2\big\rangle$ is non-zero. Since $G_1^{\hat{e}}$ and $G_2^{\hat{e}}$ are the graphs resulting from cutting $G$ at the edge $e$, there is a unique permuation $\sigma$ which (modulo arrow-reversing at $e$) displays $G$ as

$$(\sigma G) = \pm\ \begin{matrix}(\sigma_2G_2^{\hat{e}})+k_1\\ e\nearrow\\ (\sigma_1G_1^{\hat{e}})\end{matrix} \tag{1}$$

with sign $\pm$ coming from whether the arrow $e$ was reversed when giving $G$ this form (here $(\sigma_2B_e)+k_1$ denotes adding $k_1$ to each vertex label of $(\sigma_2B_e)$). Since the configuration pairing respects the arrow-reversing relation on graphs, it follows that

$$\big\langle\sigma G,\,(T_1T_2)\big\rangle = \pm\big\langle\sigma_1G_1^{\hat{e}},\,T_1\big\rangle\,\big\langle\sigma_2G_2^{\hat{e}},\,T_2\big\rangle$$

with the same sign as in Equation 1.

Conversely, if $\big\langle\sigma G,\,(T_1T_2)\big\rangle$ is non-zero then there is a corresponding non-zero $\big\langle\sigma_1G_1^{\hat{e}},\,T_1\big\rangle\,\big\langle\sigma_2G_2^{\hat{e}},\,T_2\big\rangle$. Given a subset $S\subset\{1,\dots,n\}$ let $G|_S$ denote the full subgraph of $G$ on the vertices with labels in $S$. It follows from Definition 2.12 that $\big\langle\sigma G,\,(T_1T_2)\big\rangle = 0$ unless there is exactly one edge in $\sigma G$ between the full subgraphs $(\sigma G)|_{\{1,\dots,k_1\}}$ and $(\sigma G)|_{\{k_1+1,\dots,n\}}$. Thus these graphs must be connected and (modulo arrow-reversing at $e$) the graph $\sigma G$ must be of the form

$$\sigma G = \pm\ \begin{matrix}(\sigma G)|_{\{k_1+1,\dots,n\}}\\ \sigma e\nearrow\\ (\sigma G)|_{\{1,\dots,k_1\}}\end{matrix} \tag{2}$$

with the sign $\pm$ reflecting whether the arrow $\sigma e$ was reversed when writing $\sigma G$ in this way. Since the configuration pairing respects the arrow-reversing relation on graphs, it follows that

$$\big\langle\sigma G,\ (T_1T_2)\big\rangle = \pm\big\langle(\sigma G)|_{\{1,\dots,k_1\}},\,T_1\big\rangle\,\big\langle(\sigma G)|_{\{k_1+1,\dots,n\}}-k_1,\,T_2\big\rangle$$

with the same sign as in Equation 2 (where by $G-k$ we mean to shift all labels of $G$ down by $k$). We may obtain a non-zero term of the form $\langle\sigma_1G_1^{\hat{e}},T_1\rangle\langle\sigma_2G_2^{\hat{e}},T_2\rangle$, by setting $\sigma_1$ and $\sigma_2$ so that $\sigma_1G_1^{\hat{e}} = (\sigma G)_{\{1,\dots,k_1\}}$ and $\sigma_2G_2^{\hat{e}} = (\sigma G)_{\{k_1+1,\dots,n\}} - k_1$.

The remainder of the proof is straightforward calculation, which relies on compatibility of Koszul signs. □

**Corollary 3.15.** *The graph coalgebra cobracket agrees with the Lie coalgebra cobracket through the quotient map from cofree graph coalgebras to cofree Lie coalgebras.*

In light of this proposition, an alternate approach to exhibiting the pairing between $\mathbb{L}V$ and $\mathbb{G}W$ for dual $V$ and $W$ would be to define the pairing inductively using the bracket and cobracket.

*Remark 3.16.* Corollary 3.11 and Proposition 3.14 give a method for constructing functionals on graded Lie algebras which are not free. Any finitely generated graded Lie algebra is the homology of some free finitely generated differential graded Lie algebra. That is, $L \cong H_*(\mathbb{L}V, d)$. The complex $(\mathbb{L}V, d)$ is dual to $(\mathbb{E}V^*, d^*)$, whose homology pairs with that of $H_*(\mathbb{L}V, d)$, namely $L$, through the configuration pairing. Using bar basis elements from $\mathbb{E}V^*$ one can recover the embedding of $L$ in its universal enveloping algebra, but using other spanning sets yields new possibilities.

### 3.3. Coenveloping graph coalgebras

There are four basic approaches to the free Lie algebra $\mathbb{L}(V)$ on a vector space $V$.

1. $\mathbb{L}$ is the left adjoint of the forgetful functor from Lie algebras to vector spaces.
2. $\mathbb{L}(V) \cong \bigoplus_n \mathcal{L}ie(n) \otimes_{\Sigma_n} V^{\otimes n}$, where $\sigma \in \Sigma_n$ acts on $\mathcal{L}ie(n) \otimes V^{\otimes n}$ as $\sigma \otimes \sigma^{-1}$, and the $\Sigma_n$ action on $V^{\otimes n}$ is governed by the Koszul sign convention.
3. $\mathbb{L}(V)$ is a quotient of the free non-associative algebra on $V$, $\bigoplus_n (\mathcal{T}r(n) \otimes V^{\otimes n})_{\Sigma_n}$ by the anti-symmetry and Jacobi relations on $\mathcal{T}r(n)$.
4. $\mathbb{L}(V)$ is the smallest subspace of the tensor algebra on $V$ which contains $V$ and is closed under commutators.

So far our development of Lie coalgebras has paralleled the second and third approaches, while the adjointness properties are immediate. To complete our picture, and connect with previous work, we now focus on developing the last approach. We give a representation of $\mathbb{E}(W)$ which is dual to the Poincaré-Birkhoff-Witt embedding of $\mathbb{L}(V)$ in the tensor algebra $TV$. We will exhibit $\mathbb{E}(W)$ as a quotient of the cotensor coalgebra. This representation is the starting point for the seminal work of Michaelis [**12**] on Lie coalgebras, so we in particular identify how our graph model for cofree Lie coalgebras encompasses that approach.

**Definition 3.17.** Define the graded vector space $\overline{\mathbb{G}}(W)/_{\ker\,]\cdot[}$ inductively, setting $\overline{\mathbb{G}}^1(W)/_{\ker\,]\cdot[} = W$ and letting $\overline{\mathbb{G}}^n(W)/_{\ker\,]\cdot[}$ be the quotient of $\overline{\mathbb{G}}^n(W)$ by the kernel of the map

$$]\cdot[\,: \overline{\mathbb{G}}^n(W) \longrightarrow \left(\overline{\mathbb{G}}^{<n}(W)/_{\ker\,]\cdot[}\right) \otimes \left(\overline{\mathbb{G}}^{<n}(W)/_{\ker\,]\cdot[}\right).$$

**Proposition 3.18.** $\overline{\mathbb{E}}(W) \cong \overline{\mathbb{G}}(W)/_{\ker\,]\cdot[}$.

*Proof.* By Proposition 3.7, $\ker(]\cdot[) \supset \mathrm{Arn}(W)$. It remains to show only that $\ker(]\cdot[) \subset \mathrm{Arn}(W)$. By Corollary 3.11 it is enough to show that the kernel of the pairing between $\mathbb{G}^n(W)$ and $\mathbb{L}^n(V)$ contains $\ker(]\cdot[)$. This follows by induction using Proposition 3.14. □

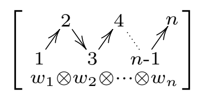
We encourage the interested reader to work through a direct proof of this proposition by explicitly showing the converse of Proposition 3.7.

Proposition 3.18 implies that $\mathbb{E}(W)$ is the quotient of $\mathbb{G}(W)$ by the largest coideal in the kernel of $\mathbb{G}(W) \to W$. This extends the definition of Lie coalgebras given by [**17**] as the quotient of the cotensor coalgebra $TW$ by the largest coideal in the kernel of $TW \to W$. In particular the construction of [**17**] follows as an immediate corollary using the injection of operads $\mathcal{A}s^\vee \to \mathcal{NAG}r$. We record this in a more computationally useful form as follows.

**Corollary 3.19.** *$\overline{\mathbb{E}}(W)$ is isomorphic to the quotient of the cotensor coalgebra $TW$ by the non-primitive kernel of the anti-cocommutative coproduct.*

*Proof.* There is a "graphification" map $g$ from $TW$ to $\mathbb{G}W$ which is injective:

$$g\colon w_1|w_2|\cdots|w_n \longmapsto$$



By abuse, call the anti-cocommutative coproduct on $TW$ the cobracket, and denote it $\,]\cdot[\, = \Delta - \tau\,\Delta$. As mentioned in Remark 3.8, $g$ sends cobrackets of cotensors in $TW$ to cobrackets of long graphs in $\mathbb{E}W$. Now apply Proposition 3.18. □

Proposition 3.18 suggests a simple algorithm for checking whether a Lie coalgebra element is trivial. Inductively define the iterated cobracket on graph coalgebras $]\cdot[^n\colon G \to G^{\otimes n}$ by

$$]g[^n = \sum_e \, ]g_1^{\hat{e}}[^{n-1} \otimes g_2^{\hat{e}},$$

where $]g[ = \sum_e g_1^{\hat{e}} \otimes g_2^{\hat{e}}$.

**Proposition 3.20.** *An element $g \in \mathbb{E}^n(W)$ is trivial if and only if $]g[^{n-1} = 0$.*

*Proof.* By Proposition 3.18, a necessary condition for a graph expression $g \in \mathbb{E}^n(W)$ to be trivial is for $]g[^{n-1} = 0$.

Conversely, applying Proposition 3.14,

$$\Big\langle g,\ \Big[\big[[v_1,v_2],v_3\big],\cdots v_n\Big]\Big\rangle \ = \ \Big\langle ]g[^{n-1}\,,\ v_1\otimes v_2\otimes\cdots\otimes v_n\Big\rangle.$$

Since bracket expressions of the form $\big[[[v_1,v_2],v_3]\cdots v_n\big]$ span $\mathbb{L}W^*$ and the configuration pairing is perfect between $\mathbb{E}W$ and $\mathbb{L}W^*$, $g=0$ if and only if $]g[^{n-1} = 0$. □

Our recovery of the approaches to cofree Lie coalgebras of Michaelis [**12**] and Schlessinger-Stasheff [**17**] allows us to highlight some advantages of the graph model. Working from $\mathbb{G}(W)$ the list of relations satisfied by Lie coalgebra elements is relatively simple, given by arrow-reversing and Arnold relations. Once we have restricted to the bar generators, however, the relations become harder to describe. For example, below are a two relations satisfied by bar generators of $\mathbb{E}^nW$ (neglecting Koszul signs).

$$\big(w_1|w_2|\cdots|w_n\big) - (-1)^{n-1}\big(w_n|\cdots|w_2|w_1\big) = 0 \tag{3}$$

$$\sum_{\substack{\sigma \text{ a cyclic} \\ \text{permuation of } (1,\ldots,n)}} \big(w_{\sigma(1)}|\cdots|w_{\sigma(n)}\big) = 0 \tag{4}$$

Relation (3) above comes from applying the arrow-reversing identity at every arrow of a long graph. Relation (4) is easily verified using Proposition 3.20. To complete the comparison to [**17**] we use our graph model to show that quotienting cotensor coalgebras by shuffle relations gives Lie coalgebras.

**Proposition 3.21.** *The Harrison shuffles give a spanning set of relations among bar generators of $\mathbb{E}^n W$. That is,*

$$\sum_{\substack{\sigma \text{ a shuffle of} \\ (1,2,\ldots k) \text{ into } (k+1,\ldots,n)}} \big(w_{\sigma(1)}|\cdots|w_{\sigma(n)}\big) = 0,$$

*and all other relations follow from these.*

*Proof.* Write $\mathrm{Sh}(W)$ for the vector subspace of $\mathbb{G}W$ generated by the Harrison shuffles of bar expressions. It is straightforward to show that $\mathrm{Sh}(W)$ is a coideal:

$$\big]\mathrm{Sh}(W)\big[ \;\subset\; \mathrm{Sh}(W) \otimes \mathbb{G}W + \mathbb{G}W \otimes \mathrm{Sh}(W).$$

On bar expressions of either 2 or 3 elements, the Harrison shuffles are merely the arrow-reversing and Arnold relations. Thus by Proposition 3.20, $\mathrm{Sh}(W) \subset \mathrm{Arn}(W)$.

That $\mathrm{Sh}(W)$ gives all relations among bar generators is now an immediate application of Proposition 3.18 and comments at the end of the first section of [**17**]. ☐

Recall that a Lyndon word in an ordered alphabet is a word which is lexicographically minimal among its cyclic reorderings. Classically, the Lyndon words form a basis for the shuffle ring [**16**]. Proposition 3.21 has the following corollary.

**Corollary 3.22.** *Let $W$ be a vector space with a basis. The cobar expressions whose entries form Lyndon words in those basis elements are a vector space basis for the Lie coalgebra $\mathbb{E}W$.*

For a further discussion of Lie algebra and Lie coalgebra bases using the configuration pairing along with a constructive proof of Corollary 3.22, see [**27**].

## 4. Bar constructions to and from the category of graph coalgebras

In parallel with our notation for Lie algebras, we use $\mathbb{C}V$ to denote the cofree graded-cocommutative coalgebra on a vector space $V$. If $V$ is reduced then $\mathbb{C}V$ is given by the symmetric invariants of the cotensor coalgebra $T^cV$ on $V$ (where the symmetric group acts with Koszul signs). Working rationally (with $V$ finitely generated), the norm map gives a vector space isomorphism with $\mathbb{A}V$, the free graded-commutative algebra generated by $V$, which is given by the symmetric coinvariants of the tensor algebra $TV$ on $V$ ($\mathbb{A}V$ is often called $\Lambda V$ or $SV$ elsewhere).

Note that $\mathbb{C}^0V = \langle 1\rangle = \mathbb{A}^0V$, while $\mathbb{L}^0V = 0 = \mathbb{E}^0V$. In various instances we will take augmentation ideals of algebras (denoted $\bar{A}$) or coaugmentation coideals of coalgebras (denoted $\bar{C}$).

### 4.1. The Quillen functors $\mathcal{L}$ and $\mathcal{C}$

Recall the standard definition of the Quillen adjoint pair of functors $\mathcal{L}\colon$ DGCC $\rightleftarrows$ DGLA $:\mathcal{C}$. The functor $\mathcal{L}$ can be viewed as the cobar construction followed by taking Hopf algebra primitives; $\mathcal{C}$ can be viewed as the bar construction on the universal enveloping algebra of a Lie algebra. Topologically these are identifying the rational homotopy of a space inside the cohomology of its loopspace via the Milnor-Moore theorem. In explicit algebra, given a differential graded-cocommutative coalgebra $(C,\ \Delta_C,\ d_C)$, the functor $\mathcal{L}$ produces the free graded Lie algebra on $s^{-1}\bar{C}$ with a differential consisting of the free extension of the differential $d_C$ plus a "twisting differential" freely induced by $\Delta_C$. Explicitly, we have the following.

**Definition 4.1.** Let $\mathcal{L}\colon$ DGCC $\to$ DGLA be the total complex of the bicomplex

$$\mathcal{L}(C,\ \Delta_C,\ d_C) = \big(\mathbb{L}(s^{-1}\bar{C}),\ d_{\mathbb{L}s^{-1}C},\ d_\Delta\big),$$

where $d_{\mathbb{L}s^{-1}C}$ is the differential inherited from the differential $d_C$ on $C$; and $d_\Delta$ is the free extension of the map given on the generators of $\mathbb{L}(s^{-1}\bar{C})$ by

$$d_\Delta(s^{-1}c) = \tfrac{1}{2}\sum_i (-1)^{|a_i|}[s^{-1}a_i, s^{-1}b_i], \qquad \text{where } \bar{\Delta}_C c = \sum_i a_i\otimes b_i.$$

The functor $\mathcal{C}$ is defined dually – given $(L,\ [\cdot,\cdot]_L,\ d_L)$ in DGLA, the functor $\mathcal{C}$ takes this to the cofree graded-cocommutative coalgebra primitively cogenerated by $sL$ with a differential consisting of the cofree extension of the differential $d_L$ plus a differential cofreely induced by the bracket $[\cdot,\cdot]_L$.

**Definition 4.2.** Let $\mathcal{C}\colon$ DGLA $\to$ DGCC be the total complex of the bicomplex

$$\mathcal{C}(L,\ [\cdot,\cdot]_L,\ d_L) = \big(\mathbb{C}(sL),\ d_{\mathbb{C}sL},\ d_{[\cdot,\cdot]}\big),$$

where $d_{\mathbb{C}sL}$ is the differential inherited from $d_L$ on $L$; and

$$d_{[\cdot,\cdot]}(sv_1\cdot sv_2\cdots sv_n) = \sum_{i<j}(-1)^{n_{ij}+|v_i|}s[v_i,v_j]\cdot sv_1\cdots\widehat{sv_i}\cdots\widehat{sv_j}\cdots sv_n,$$

where $(-1)^{n_{ij}}$ is the Koszul sign change incurred by moving $sv_i$, and $sv_j$ to the beginning of this expression.

An alternate way to view $d_{[\cdot,\cdot]}$, in parallel to Definition 4.1, is the following.

**Proposition 4.3.** *The differential $d_{[\cdot,\cdot]}$ is the cofree extension of the graded vector space map $[\mathbb{C}sL]_{\mathrm{G}} \to s[L]_{\mathrm{G}}$ given on $\mathbb{C}^{\neq 2}sL$ by the zero map and on $\mathbb{C}^2sL$ by $(sv_1\cdot sv_2)\mapsto(-1)^{|v_1|}s[v_1,v_2]$.*

Adjointness of $\mathcal{L}$ and $\mathcal{C}$ follows from that of the bar and cobar construction as well as that of the universal enveloping algebra and Lie primitives functors.

*Remark 4.4.* We will shortly construct functors $\mathcal{E}\colon$ DGCA $\leftrightarrows$ DGLC $:\mathcal{A}$ (dual to $\mathcal{L}$ and $\mathcal{C}$) as quotients of functors $\mathcal{G}\colon$ DGCA $\leftrightarrows$ DGGC $:\hat{\mathcal{A}}$ to and from graph coalgebras.

We may attempt to define a functor $\hat{\mathcal{L}}\colon$ DGCC $\to$ DGTA (where DGTA denotes dg-non-associative binary algebras and $\mathbb{T}$ denotes the free such algebra) by

$$\hat{\mathcal{L}}(C,\ \Delta_C,\ d_C) = \big(\mathbb{T}(s^{-1}\bar{C}),\ d_{\mathbb{T}s^{-1}\bar{C}},\ d_\Delta\big).$$

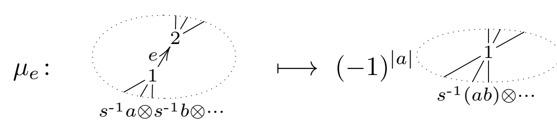
Unfortunately, $d_\Delta$ is not a differential on the non-associative algebra $\mathbb{T}(s^{-1}\bar{C})$ so $\hat{\mathcal{L}}$ isn't a differential complex. This is a striking difference between the non-associative algebra approach to Lie algebras and the graph coalgebra approach to Lie coalgebras we present in the next section. One can presumably replace non-associative algebras by graph algebras in the above construction and get a functor $\hat{\mathcal{L}}$ which maps to differential graded complexes and generalizes both Adams' bar construction and Quillen's $\mathcal{L}$ functor appropriately, an idea which we leave for furture work.

### 4.2. The functor $\mathcal{G}$

To define $\mathcal{G}$, we start with a differential graded (commutative, augmented, unital) algebra $(A,\ \mu_A,\ d_A)$, with augmentation ideal $\bar{A}$. The functor $\mathcal{G}$ produces the cofree graded anti-commutative graph coalgebra on $s^{-1}\bar{A}$ with differential consisting of the cofree extension of the differential $d_A$ along with another part coming from the multiplication $\mu_A$, defined by contracting edges. In order to make this precise, we must carefully define the sign associated to contracting an edge.

**Definition 4.5.** Let $[g]$ be a homogeneous element of $\mathbb{G}(s^{-1}\bar{A})$, namely an ordered directed graph with $n$ vertices along with a tensor of $n$ elements of $s^{-1}\bar{A}$ modulo the usual $\Sigma_n$-action. For every edge $e$ of $g$ we may construct a new ordered labeled graph $\mu_e(g)$ as follows.

Pick a representative of $[g]$ modulo $\Sigma_n$ in which edge $e$ goes from vertex number 1 to vertex number 2, with the first two entries of the associated tensor being $a$ and $b$. Contract the edge from 1 to 2 in this representative to a vertex which is then given the number 1 and first entry in the tensor of $(-1)^{|a|}s^{-1}(ab)$. In this operation, the ordering of all other vertices in the graph is shifted down by one to make up for the now missing 2 (associated elements in the tensor remain the same).




**Definition 4.6.** Let $\mathcal{G}\colon$ DGCA $\to$ DGGC be the total complex of the bicomplex

$$\mathcal{G}(A,\ \mu_A,\ d_A) = \big(\mathbb{G}(s^{-1}\bar{A}),\ d_{\mathbb{G}s^{-1}\bar{A}},\ d_\mu\big),$$

where $d_{\mathbb{G}s^{-1}\bar{A}}$ takes $d_{s^{-1}\bar{A}} = -s^{-1}d_A$ term-wise in the tensor associated to a graph coalgebra element and $d_\mu([g]) = \sum_e [\mu_e g]$.

This is a bicomplex by the same calculation which shows that Adams' classical bar construction is a bicomplex. Indeed, $\mathcal{G}$ extends Adams' bar construction to the category of graph coalgebras.

**Proposition 4.7.** *The map $d_\mu$ is compatible with the cobracket on $\mathbb{G}(s^{-1}\bar{A})$. That is, $d_\mu(\,]g[\,) = ]d_\mu g[$.*

*Moreover, $d_\mu$ is the cofree extension of the graded vector space map $\left[\mathbb{G}s^{-1}\bar{A}\right]_{\mathrm{G}} \to s^{-1}[\bar{A}]_{\mathrm{G}}$ given on $\mathbb{G}^{\neq 2}s^{-1}\bar{A}$ by the zero map and on $\mathbb{G}^2 s^{-1}\bar{A}$ by*

$$\overset{2}{\underset{1}{\nearrow}} \bigotimes s^{-1}a \otimes s^{-1}b \;\longmapsto\; (-1)^{|a|}s^{-1}(ab).$$

The Lie coalgebraic bar construction $\mathcal{E}$ is now given as a quotient of our graphical bar constuction $\mathcal{G}$.

**Proposition 4.8.** *The differential $d_\mu$ preserves the vector subspace generated by arrow-reversing and Arnold expressions. Thus the arrow-reversing and Arnold coideal is a subcomplex of $\mathcal{G}$.*

*Proof.* This proposition follows immediately from the compatibility of $d_\mu$ and the cobracket and Proposition 3.7 once we show that $d_\mu$ vanishes on arrow-reversing expressions. Using the bar representation for graphs, this is shown by:

$$d_\mu\Big(s^{-1}a|s^{-1}b + (-1)^{(|a|+1)(|b|+1)}s^{-1}b|s^{-1}a\Big)$$
$$= (-1)^{|a|}s^{-1}(ab) + (-1)^{|a||b|+|a|+1}s^{-1}(ba) \;=\; 0. \qquad \square$$

**Definition 4.9.** Let $\mathcal{E}(A)$ be $\mathcal{G}(A)$ modulo the arrow-reversing and Arnold subcomplex.

In terms of the bar generators, the differentials in the definition of $\mathcal{E}$ coincide with the differentials used to define the usual algebraic (associative) bar construction. By Proposition 3.21, $\mathcal{E}(A)$ is isomorphic to the Harrison complex of the commutative algebra $A$ equipped with the Lie coalgebra structure from [**17**].

## 4.3. The functor $\hat{\mathcal{A}}$

The functor $\hat{\mathcal{A}}$ is given by Adams' cobar construction applied to a graph coalgebra. Explicitly it takes the differential graded graph coalgebra $(G,\ ]\cdot[_G,\ d_G)$ to the free graded-commutative algebra generated by $sG$ with a differential consisting of the free extension of $d_G$ along with another part coming from the graph cobracket.

**Definition 4.10.** Let $\hat{\mathcal{A}}$: DGGC $\to$ DGCA be the total complex of the bicomplex

$$\hat{\mathcal{A}}(G,\ ]\cdot[_G,\ d_G) = \big(\mathbb{A}sG,\ d_{\mathbb{A}sG},\ d_{]\cdot[_G}\big),$$

where $d_{]\cdot[}$ is the free extension of the map given on the generators of $\mathbb{A}sG$ by

$$d_{]\cdot[}(sg) = \tfrac{1}{2}\sum_e (-1)^{|g_1^{\hat{e}}|}\, sg_1^{\hat{e}} \cdot sg_2^{\hat{e}}, \qquad \text{for}\ \ ]g[ = \sum_e g_1^{\hat{e}} \otimes g_2^{\hat{e}}.$$

Unlike in Remark 4.4, this defines a differential graded complex.

**Theorem 4.11.** *$\hat{\mathcal{A}}(G)$ is a bicomplex.*

*Proof.* The non-trivial part of the argument is to show $d_{]\cdot[}^2 = 0$, for which it is enough to show that $d_{]\cdot[}^2 = 0$ on $sG \subset \mathbb{A}sG$. Furthermore it is enough to show $d_{]\cdot[}^2$ vanishes on graphs with only three vertices, since the general case is then solved by replacing vertices by graphs.

$$\begin{aligned} d^2_{]\cdot[}\left(s \underset{a\otimes b\otimes c}{\begin{smallmatrix} & 2 & \\ \nearrow & & \searrow \\ 1 & & 3 \end{smallmatrix}}\right) &= d_{]\cdot[}\left((-1)^{|a|+|b|} s\Big(\underset{a\otimes b}{\begin{smallmatrix} & 2 \\ \nearrow & \\ 1 & \end{smallmatrix}}\Big)\cdot sc + (-1)^{|a|} sa \cdot s\Big(\underset{b\otimes c}{\begin{smallmatrix} & 2 \\ \nearrow & \\ 1 & \end{smallmatrix}}\Big)\right) \\ &= (-1)^{|a|+|b|+|a|} sa\cdot sb\cdot sc + (-1)^{|a|+(|a|+1)+|b|} sa\cdot sb \cdot sc \;=\; 0 \end{aligned}$$

The computations for $\Big(\underset{a\otimes b\otimes c}{\begin{smallmatrix} & 2 & \\ \swarrow & & \searrow \\ 1 & & 3 \end{smallmatrix}}\Big)$ and $\Big(\underset{a\otimes b\otimes c}{\begin{smallmatrix} & 2 & \\ \nearrow & & \nwarrow \\ 1 & & 3 \end{smallmatrix}}\Big)$ are similar. $\square$

The following proposition is an immediate consequence of Proposition 3.7.

**Proposition 4.12.** *Let* Arn *be the arrow-reversing and Arnold vector subspace of $G$. Then $d_{]\cdot[}\big(s\mathrm{Arn}\big) \subset \big(s\mathrm{Arn}\big)\cdot\big(sG\big)$.*

Graded anti-commutativity of the graph cobracket in $G$ corresponds via $d_{]\cdot[}$ to graded commutativity of multiplication in $\mathbb{A}sG$.

$$\begin{aligned} ]g[ \quad &= \quad \sum_e g_1^{\hat{e}} \otimes g_2^{\hat{e}} \quad &=& \quad \sum_e -(-1)^{|g_1^{\hat{e}}||g_2^{\hat{e}}|}\, g_2^{\hat{e}} \otimes g_1^{\hat{e}} \\ d_{]\cdot[}\, sg \quad &= \quad \tfrac{1}{2}\sum_e (-1)^{|g_1^{\hat{e}}|}\, sg_1^{\hat{e}} \cdot sg_2^{\hat{e}} \quad &=& \quad \tfrac{1}{2}\sum_e -(-1)^{|g_1^{\hat{e}}||g_2^{\hat{e}}|+|g_2^{\hat{e}}|}\, sg_2^{\hat{e}}\cdot sg_1^{\hat{e}}. \end{aligned}$$

**Corollary 4.13.** *$\hat{A}$ descends to a well-defined map $\mathcal{A}\colon$ DGLC $\to$ DGCA by $\mathcal{A}([G]) = \hat{\mathcal{A}}(G)$.*

### 4.4. Adjointness of $\mathcal{G}$ and $\hat{\mathcal{A}}$

Let $G$ be a DGGC and $A$ be a DGCA, and use $[-]_{\mathrm{G}}$ to denote the forgetful functor to underlying graded vector spaces and $[-]_{\mathrm{GGC}}$, $[-]_{\mathrm{GCA}}$ to denote forgetting only differentials. It follows from the adjointness properties of $\mathbb{G}$ and $\mathbb{A}$ that the following spaces of homomorphisms are isomorphic:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{GGC}}\big([G]_{\mathrm{GGC}},\, \mathbb{G}s^{-1}[\bar{A}]_{\mathrm{G}}\big) \;&\cong\; \mathrm{Hom}_{\mathrm{G}}\big([G]_{\mathrm{G}},\, s^{-1}[\bar{A}]_{\mathrm{G}}\big) \\ &\cong\; \mathrm{Hom}_{\mathrm{G}}\big(s[G]_{\mathrm{G}},\, [\bar{A}]_{\mathrm{G}}\big) \;\cong\; \mathrm{Hom}_{\mathrm{GCA}}(\mathbb{A}s[G]_{\mathrm{G}},\, [A]_{\mathrm{GCA}})\,. \end{aligned} \tag{5}$$

This establishes adjointness of $\mathcal{G}$ and $\hat{\mathcal{A}}$ on the level of graded commutative algebras and graded graph coalgebras, forgetting differentials.

To display an adjointness which respects $d_\mu$ and $d_{]\cdot[}$, we translate the classical argument showing adjointness of bar and cobar constructions using twisting functions. We include the proof only to underline that the classical proof translates perfectly to this setting without any modification, even though we are now working with the much larger category of graph coalgebras.

**Theorem 4.14.** *The functors $\mathcal{G}$ and $\hat{\mathcal{A}}$ are an adjoint pair.*

*Proof.* Given $G$ and $A$, a DGGC and a DGCA, we will say that a degree $-1$ map $\tau\colon [G]_{\mathrm{G}} \to [A]_{\mathrm{G}}$ is a *twisting function* if it satisfies the requirement

$$d_A\,\tau + \tau\, d_E - \tfrac{1}{2}\Big(\mu_A \circ \big(((-1)^{|\cdot|}\tau)\otimes\tau\big) \circ\ ]\cdot[_G\Big) = 0.$$

We show that there are bijections between DGCA-maps $\hat{\mathcal{A}}G \to A$, DGGC-maps

$G \to \mathcal{G}A$, and twisting functions $G \to A$. In terms of Equation 5, we show that if $[f]_{\mathrm{GCA}} \in \mathrm{Hom}_{\mathrm{GCA}}(\mathbb{A}s[G]_{\mathrm{G}},\, [A]_{\mathrm{GCA}})$ comes from applying the forgetful functor to a map $f \in \mathrm{Hom}_{\mathrm{DGCA}}\left(\hat{\mathcal{A}}G,\, A\right)$, then the adjoint map $\tau \in \mathrm{Hom}_{\mathrm{G}}\big(s[G]_{\mathrm{G}},\, [\bar{A}]_{\mathrm{G}}\big)$ is in fact a twisting function. Furthermore any $\tau \in \mathrm{Hom}_{\mathrm{G}}\big(s[G]_{\mathrm{G}},\, [\bar{A}]_{\mathrm{G}}\big)$ which is also a twisting function will be adjoint to a map $f \in \mathrm{Hom}_{\mathrm{GCA}}(\mathbb{A}s[G]_{\mathrm{G}},\, [A]_{\mathrm{GCA}})$ in the image of the forgetful functor from $\mathrm{Hom}_{\mathrm{DGCA}}(\hat{\mathcal{A}}E,\, A)$. This will complete one half of the argument. The half of the argument for homomorphisms $G \to \mathcal{G}A$ is similar.

Let $f\colon \hat{\mathcal{A}}G \to A$ and write $\tau\colon s[G]_{\mathrm{G}} \to [\bar{A}]_{\mathrm{G}}$ for the adjoint of $[f]_{\mathrm{GCA}}$. Note that $\tau = [f]_{\mathrm{GCA}} \circ i$ where $i$ is the injection map $i\colon s[G]_{\mathrm{G}} \hookrightarrow \mathbb{A}s[G]_{\mathrm{G}}$. The requirement that $d_A\, f = f\, d_{\hat{\mathcal{A}}G}$ ensures that $\tau$ gives a twisting function. Explicitly, let $sg \in s[G]_{\mathrm{G}}$, then

$$
\begin{aligned}
0 &= d_A\, f\, i(sg) - f\, d_{\hat{\mathcal{A}}G}\, i(sg)\\
&= d_A\, f\, i(sg) - f\,(d_{\mathbb{A}sG} + d_{]\cdot[})\, i(sg)\\
&= d_A\, \tau(sg) - f\,(-s\, d_G g) - f\left(\tfrac{1}{2}\sum_e (-1)^{|g_1^{\hat{e}}|}\, sg_1^{\hat{e}} \cdot sg_2^{\hat{e}}\right) \quad \text{where} \quad ]g[\; = \sum_e g_1^{\hat{e}} \otimes g_2^{\hat{e}}\\
&= d_A\, \tau(sg) + \tau(s\, d_G g) - \tfrac{1}{2}\sum_e (-1)^{|a_i|}\, \tau(sg_1^{\hat{e}}) \cdot \tau(sg_2^{\hat{e}}).
\end{aligned}
$$

Conversely, let $\tau\colon s[G]_{\mathrm{G}} \to [\bar{A}]_{\mathrm{G}}$ give a twisting function $G \to A$ and let $f\colon \mathbb{A}s[G]_{\mathrm{G}} \to A$ be the adjoint of $\tau$ given by free extension. To show that $d_A\, f = f\,(d_{\mathbb{A}sG} + d_{]\cdot[})$ it is enough to check on generators $sg \in \mathbb{A}s[G]_{\mathrm{G}}$. On generators we have

$$
\begin{aligned}
d_A\, f(sg) &= d_A\, \tau(sg)\\
f\, d_{\mathbb{A}sG}(sg) &= -\tau(s\, d_G g)\\
f\, d_{]\cdot[}(sg) &= f\Big(\tfrac{1}{2}\sum_e (-1)^{|g_1^{\hat{e}}|} sg_1^{\hat{e}} \cdot sg_2^{\hat{e}}\Big), \quad \text{where} \quad ]g[\; = \sum_e g_1^{\hat{e}} \otimes g_2^{\hat{e}}\\
&= \tfrac{1}{2}\sum_e (-1)^{|g_1^{\hat{e}}|}\, \tau(sg_1^{\hat{e}}) \cdot \tau(sg_2^{\hat{e}}).
\end{aligned}
$$

However, since $\tau$ is a twisting function we know that

$$d_A\, \tau(sg) + \tau(s\, d_G g) - \tfrac{1}{2}\sum_e (-1)^{|g_1^{\hat{e}}|}\, \tau(sg_1^{\hat{e}}) \cdot \tau(sg_2^{\hat{e}}) = 0.$$

Substitution yields the desired equality.

We only sketch the bijection between $G \to \mathcal{G}A$ and twisting functions $G \to A$, since it is given similarly. Let $f\colon G \to \mathcal{G}A$ and write $\tau\colon [G]_{\mathrm{G}} \to s^{-1}[\bar{A}]_{\mathrm{G}}$ for the adjoint of $[f]_{\mathrm{GGC}}\colon [G]_{\mathrm{GGC}} \to \mathbb{G}s^{-1}[\bar{A}]_{\mathrm{G}}$. Note that $\tau = \pi \circ [f]_{\mathrm{GGC}}$ where $\pi$ is the projection map $\pi\colon \mathbb{G}s^{-1}[\bar{A}] \twoheadrightarrow s^{-1}[\bar{A}]$. By direct computation, the requirement that $\pi\, f\, d_G = \pi\, d_{\mathcal{G}A}\, f$ is equivalent to the condition that $\tau$ is a twisting function. $\square$

The adjointness of our duals of Quillen's functors $\mathcal{L}$ and $\mathcal{C}$ now follows.

**Corollary 4.15.** *The functors $\mathcal{E}$ and $\mathcal{A}$ are an adjoint pair.*

We summarize our results as follows.

**Theorem 4.16.** *The functor $\mathcal{E}\colon$ DGCA $\to$ DGLC factors through the category of differential graded anti-commutative graph coalgebras.*

This factorization is crucial in one of the main applications of our approach to Lie coalgebras through graph coalgebras. In the sequel to this paper [**20**], we associate geometric functionals on homotopy groups – that is, elements of $\mathrm{Hom}(\pi_*(X), \mathbb{Q})$ – to cycles in both the classical and Lie coalgebraic bar constructions on the (commutative) cochains on $X$. We say geometric because for example if $X = S^2$ and $\omega$ is a 2-cochain representing the top cohomology class, we associate to the graph cycle $\underset{\omega}{\overset{\omega}{\nearrow}}$ the Whitehead integral whose value on $f\colon S^3 \to S^2$ is $\int_{S^3} d^{-1} f^*\omega \wedge \omega$, where $d^{-1} f^*\omega$ is an anti-derivative of $f^*\omega$ [**28**]. Thus we call these functionals Hopf invariants or generalized linking invariants of homotopy. To show that we get a complete set of invariants, we induct over the rational Postnikov tower of the space $X$. In order to show that our Hopf invariants respect the long exact sequence of a fibration, we need to establish that the functor $\mathcal{E}$ applied to a twisted tensor product of some $A$ with a free commutative algebra is naturally weakly equivalent to $\mathcal{E}(A)$. We do so explicitly by constructing a "weight increasing chain homotopy" which is done in the category of graph coalgebras and cannot be done in the category of coassociative coalgebras.

We are starting to see that this factorization through graph coalgebras is even more crucial in the setting of spaces which are not simply connected and in particular to group theory. We are constructing "Hopf invariants" which determine when an element of $G$, presented as a word, lies in the $n$th commutator subgroup of $G$. Such invariants thus generalize the Magnus expansion and solve the word problem for residually nilpotent groups. For example, let $G$ be free on a generating set $a, b, \ldots$ and denote the dual cocycles to the corresponding homology generators by $a^*, b^*, \ldots$. The graph cycle $\underset{a^*}{\overset{b^*}{\nearrow}}$ defines a functional on words with trivial abelianization: the "signed count of occurrences of $b$ between some $a$-$a^{-1}$ pair." While such invariants can be generalized for arbitrary graphs, they do not satisfy the Arnold identity, so are best understood at the level of graph coalgebras rather than Lie coalgebras. We have been able to define such invariants for many groups which are not free, but are still searching for the appropriate bar construction – presumably a quotient of the graph coalgebraic bar construction – to govern our theory.

### 4.5. Pairings of Quillen functors

Our graphical approach to the Lie coalgebraic bar construction not only gives rise to the factorization of the previous section, but allows us to explicitly understand canonical linear dualities of Lie algebraic and coalgebraic Quillen functors.

**Theorem 4.17.** *The diagram*

$$\begin{tikzcd} \mathrm{DGCC} \arrow[r, shift left, "\mathcal{L}"] \arrow[d, shift right, "*"'] & \mathrm{DGLA} \arrow[l, shift left, "\mathcal{C}"] \arrow[d, shift left, "*"] \\ \mathrm{DGCA} \arrow[u, shift right, "*"'] \arrow[r, shift right, "\mathcal{E}"'] & \mathrm{DGLC} \arrow[l, shift right, "\mathcal{A}"'] \arrow[u, shift left, "*"] \end{tikzcd} \tag{6}$$

*displays a duality of adjoint pairs of functors. In particular, the square sub-diagrams obtained by starting at any corner and mapping to the opposite are commutative up to canonical isomorphism. In particular, if $C$ is a differential graded-cocommutative* *coalgebra which is linearly dual to a differential graded-commutative algebra $A$, then $\mathcal{E}(A)$ is linearly dual to $\mathcal{L}(C)$ through the configuration pairing.*

This result refines the work of Schlessinger-Stasheff by identifying the configuration pairing as giving rise to the canonical duality between the Lie algebraic and coalgebraic bar constructions.

*Proof.* We treat separately the commutativity of the squares which constitute the theorem. The first two are restated as follows.

*If $L$ is a differential graded Lie algebra which is linearly dual to a differential graded Lie coalgebra $E$, then $\mathcal{C}(L)$ is linearly dual to $\mathcal{A}(E)$.*

Write $L = (L_\bullet,\, d_L,\, [\cdot,\cdot])$ and $E = (E^\bullet,\, d_E,\, \;]\cdot[\,)$. By definition, we need to establish the duality of the bicomplexes

$$\begin{aligned} \mathcal{C}(L) &= \big(\mathbb{C}sL,\; d_{\mathbb{C}sL},\; d_{[\cdot,\cdot]}\big) \qquad \text{and} \\ \mathcal{A}(E) &= \big(\mathbb{A}sE,\; d_{\mathbb{A}sE},\; d_{]\cdot[}\big). \end{aligned}$$

The duality already developed between between $L_\bullet$ and $E^\bullet$ induces an algebra/coalgebra duality between $\mathbb{C}sL$ and $\mathbb{A}sE$. Furthermore, since $d_L$ and $d_E$ are linearly dual, their cofree/free extensions $d_{\mathbb{C}sL}$ and $d_{\mathbb{A}sE}$ will be as well. It remains to show that the maps $d_{[\cdot,\cdot]}$ and $d_{]\cdot[}$ are dual. However, these are also cofree/free extensions, namely of the maps

$$\begin{aligned} &\mathbb{C}sL \longrightarrow sL \quad \text{by} \quad sa\cdot sb \longmapsto (-1)^{|a|} s[a,\,b] \\ &sE \longrightarrow \mathbb{A}sE \quad \text{by} \quad s\gamma \longmapsto \tfrac{1}{2}\sum_e (-1)^{|\gamma_1^{\hat e}|}\, s\gamma_1^{\hat e}\cdot s\gamma_2^{\hat e}, \qquad \text{where} \quad ]\gamma[\; = \sum_e \gamma_1^{\hat e}\otimes\gamma_2^{\hat e}. \end{aligned}$$

We verify the duality of these restrictions explicitly, using compatibility of pairings with our assorted multiplications and comultiplications.

$$\begin{aligned} \big\langle s\gamma,\; (-1)^{|a|} s[a,\,b]\big\rangle &= \big\langle \gamma,\; (-1)^{|a|}[a,\,b]\big\rangle \\ &= \big\langle ]\gamma[\,,\; (-1)^{|a|} a\otimes b\big\rangle \\ &= (-1)^{|a|}\sum_e \big\langle \gamma_1^{\hat e},\, a\big\rangle\, \big\langle \gamma_2^{\hat e},\, b\big\rangle \\ \Big\langle \tfrac{1}{2}\sum_e (-1)^{|\gamma_1^{\hat e}|}\, s\gamma_1^{\hat e}\cdot s\gamma_2^{\hat e},\; sa\cdot sb\Big\rangle &= \tfrac{1}{2}\sum_e (-1)^{|\gamma_1^{\hat e}|}\big\langle s\gamma_1^{\hat e}\otimes s\gamma_2^{\hat e},\; \Delta(sa\cdot sb)\big\rangle \\ &= \tfrac{1}{2}\sum_e (-1)^{|\gamma_1^{\hat e}|}\Big(\langle s\gamma_1^{\hat e},\; sa\rangle\, \langle s\gamma_2^{\hat e},\; sb\rangle \\ &\qquad\qquad + (-1)^{(|a|+1)(|b|+1)}\langle s\gamma_1^{\hat e},\; sb\rangle\, \langle s\gamma_2^{\hat e},\; sa\rangle\Big) \\ &= \sum_e (-1)^{|\gamma_1^{\hat e}|}\langle s\gamma_1^{\hat e},\; sa\rangle\, \langle s\gamma_2^{\hat e},\; sb\rangle \end{aligned}$$

The equality of the last two lines above uses anti-cocommutativity of the cobracket $]\gamma[$ as well as the fact that, for the pairings to be nonzero, the degrees of $\gamma_1^{\hat e}$ and $a$ must match, as must the degrees of $\gamma_2^{\hat e}$ and $b$.

Since each of the above pairings are 0 unless $|\gamma_1^{\hat e}| = |a|$, we have equality, establishing the first half of the theorem.

The proof of the second half of the theorem proceeds in the same manner as that of the first half. Briefly, if we write $A = (A^\bullet,\, d_A,\, \mu)$ and $C = (C_\bullet,\, d_C,\, \Delta)$, then the duality of the bicomplexes defining $\mathcal{E}(A)$ and $\mathcal{L}(C)$ is immediate, given by the configuration pairing as stated, except for that of the differentials $d_\mu$ and $d_\Delta$. But $d_\mu$ and $d_\Delta$ are also cofree/free extensions, namely of the maps

$$\mathbb{E}s^{-1}\bar{A} \longrightarrow s^{-1}\bar{A} \quad \text{by} \quad {}_{1}\!\nearrow^{2} \bigotimes s^{-1}a \otimes s^{-1}b \longmapsto (-1)^{|a|}\, s^{-1}(ab)$$
$$s^{-1}\bar{C} \longrightarrow \mathbb{L}s^{-1}\bar{C} \quad \text{by} \quad s^{-1}\gamma \longmapsto \sum_i (-1)^{|\alpha_i|}[s^{-1}\alpha_i,\, s^{-1}\beta_i], \quad \text{where} \quad \bar{\Delta}\gamma = \sum_i \alpha_i \otimes \beta_i.$$

The duality of these restrictions follows from direct calculation, as before. □

The statements given in the previous proof do not require our underlying finiteness hypotheses. If we start with a linearly dual pair of an algebra and coalgebra, the functors $\mathcal{L}$ and $\mathcal{E}$ will produce a linearly dual Lie algebra and coalgebra. The finite generation hypotheses only ensure that our vertical linear duality maps are isomorphisms.

## Appendix A. Application to computing rational homotopy groups

We collect a number of facts and constructions that were either in the literature (Schlessinger-Stasheff [**17**], Bausfield-Gugenheim [**3**]) or were "in the air" during the formative years of rational homotopy theory. We are starting to see that a significant pay-off will be obtained when moving to the non-simply-connected case, where our graph coalgebra approach can give rise to additional understanding of fundamental groups themselves, rather than having the fundamental group act on a (minimal) model. For the sake of reference, we collect first results in the simply-connected setting here.

Let $A^*(X)$ be the $PL$ cochains functor [**4**], a commutative model for the rational cochains on $X$. Recall that finding good models for rational chains on a space with commutative coproduct is problematic, since one must pass to a finite model for $A^*(X)$ before applying linear duality to get some model $M_*(X)$ for chains which is commutative, but which is then not functorial. By Quillen's theorem, we know that $H_*(\mathcal{L}(M_*(X)))$ is isomorphic to $\pi_*(X) \otimes \mathbb{Q}$. Staying in the world of cochains we can proceed more directly. Let $\pi^*_{\mathbb{Q}}(X)$ denote $\mathrm{Hom}(\pi_*(X), \mathbb{Q})$.

**Corollary A.1.** *The homology of $\mathcal{E}(A^*(X))$ is isomorphic to $\pi^*_{\mathbb{Q}}(X)$.*

The standard way to recover homotopy data from cochains to this point has been essentially to replace $A^*(X)$ with a quasi-isomorphic $\mathcal{A}(E)$ for some Lie coalgebra $E$, from which it follows by Quillen's theorem that $\pi^*(X) \cong E$ (see also Corollary C.2 below). Our approach on the other hand is functorial and computational at the same time.

In the sequel to this paper [**20**], we develop geometry underlying Corollary A.1, defining homotopy periods for any cycles in $\mathcal{E}(A^*_{PL}(X))$. This geometry unifies and generalizes approaches of Hopf, Whitehead, Boardman-Steer, Sullivan, Novikov, Chen and Hain, and can yield $\mathbb{Z}$ and $\mathbb{Z}/p$-valued homotopy periods.

Finally, we may employ the spectral sequence of a bicomplex, which yields the following.

**Corollary A.2.** *If $X$ is a finite complex, there is a spectral sequence converging to $\pi^*(X)$ with $E^1$ given by $\mathcal{E}(H^*(X))$. This spectral sequence collapses at $E^2$ if $X$ is formal.*

After Corollary C.2 we show that this spectral sequence is isomorphic to one constructed by Halperin and Stasheff [**8**].

## Appendix B. Model structures

The adjointness results of Section 4.4 preserve model structures, so that $\mathcal{E}$, $\mathcal{A}$ and also $\mathcal{G}$, $\hat{\mathcal{A}}$ form Quillen adjoint pairs. Because we are in the finitely generated setting, we get only model structures, not closed model structures. All categories in this section are reduced appropriately.

**Theorem B.1** (Quillen [**15**]). *A model category structure on* DGLA *is given by the following:*

- *Weak equivalences are the quasi-isomorphisms.*
- *Fibrations are the level-wise surjections above the bottom degree.*
- *Cofibrations are determined by left lifting; they are the free* GLA*-maps.*

*A model category structure on* DGCC *is given by the following:*

- *Weak equivalences are the quasi-isomorphisms.*
- *Cofibrations are the levelwise injections.*
- *Fibrations are determined by right lifting.*

Recall that $\otimes$ gives finite products in GCC, since our coalgebras are counital, coaugmented. Note that all DGCC's are cofibrant and all DGLA's are fibrant.

*Remark B.2.* By the results of Quillen [**15**], these give model category structures even with the finiteness assumptions removed. Though Quillen did not show that these model categories are closed when finiteness hypotheses are removed, in particular that infinite limits exist in the coalgebra setting, there are now a number of proofs in the literature.

In the course of developing algebraic models for rational homotopy theory, Quillen established the following (see [**15**, Thm 5.3]).

**Theorem B.3** (Quillen). *The functors $\mathcal{L}\colon$ DGCC $\leftrightarrows$ DGLA $:\mathcal{C}$ are a Quillen adjoint pair. That is, $\mathcal{L}$ preserves cofibrations and trivial cofibrations; $\mathcal{C}$ preserves fibrations and trivial fibrations.*

*Furthermore, $\mathcal{L}$ and $\mathcal{C}$ give a Quillen equivalence. That is, if $C$ is a cofibrant* DGCC *and $L$ is a fibrant* DGLA*, then a map $\mathcal{L}(C) \to L$ is a weak equivalence if and only if the adjoint map $C \to \mathcal{C}(L)$ is a weak equivalence.*

We now give parallels to these results in our algebra–Lie coalgebra setting. In the following, we continue to restrict to finitely generated, reduced objects.

**Definition B.4.** We will say that a DGCA-map $f\colon A \to B$ is a free GCA-map if as a GCA-map, it is an inclusion of a graded algebra with free cokernel, as displayed in the diagram:

$$\begin{tikzcd}[A]_{\mathrm{GLA}} \arrow[r, "{[f]_{\mathrm{GLA}}}"] \arrow[dr, hook] & {[B]_{\mathrm{GLA}}} \\ & {[A]_{\mathrm{GLA}} \otimes \mathbb{A}W.} \arrow[u, phantom, "\cong"]\end{tikzcd}$$

In GCA, $\otimes$ is the categorical coproduct, since our algebras are unital.

We will say that a DGGC-map $f\colon D \to E$ is a cofree GGC-map if as a GGC-map, it is a projection of graded coalgebras with cofree kernel, as displayed in the diagram:

$$\begin{tikzcd}{[D]_{\mathrm{GGC}}} \arrow[r, "{[f]_{\mathrm{GGC}}}"] & {[E]_{\mathrm{GGC}}} \\ {[E]_{\mathrm{GGC}} \circledast \mathbb{G}W.} \arrow[u, phantom, "\cong"] \arrow[ur, two heads] & \end{tikzcd}$$

By $\circledast$ we mean the "cofree product" – the categorical product of graph coalgebras – given by the categorical equalizer of the pair of maps

$$G \circledast K := \mathrm{Eq}\Big(\mathbb{G}(G \oplus K) \rightrightarrows \mathbb{G}(\mathbb{G}G \oplus \mathbb{G}K)\Big)$$

coming from $\mathbb{G}$ being a cotriple and from $G$, $K$ being graph coalgebras.

**Theorem B.5.** *A model category structure on* DGCA *is given by the following:*

- *Weak equivalences are the quasi-isomorphisms.*
- *Fibrations are the levelwise surjections.*
- *Cofibrations are determined by left lifting; they are the free* GCA *maps.*

*A model category structure on* DGGC *is given by the following:*

- *Weak equivalences are the quasi-isomorphisms.*
- *Cofibrations are injections above degree one.*
- *Fibrations are determined by right lifting; they are the cofree* GGC *maps.*

*A model category structure on* DGLC *is given similarly.*

While it is possible to merely mimic the original proof of Quillen from [**15**], we may instead infer this from the literature on model categories.

*Proof Sketch.* The stated model category structure on DGCA is standard in the literature – it is given by lifting the projective model structure on (reduced) cochains. See [**9**] and [**18**, 4.1].

To see that the cofibrations are indeed the free maps may be done in the same way as Quillen shows the corresponding fact in DGLA (see [**15**, Prop 5.5, p256]) by attatching cells using pushouts of cofibrations. In this manner one may show that all cofibrations are retracts of free maps. However, subalgebras of free algebras are again free; so such maps must themselves be free.

The listed model category structure on DGGC is implied by general operad theory work of [**1**, Thm 3.2.3]. That fibrations are indeed the cofree maps follows in the finitely generated case from the dual of the corresponding statement about cofibrations in DGLA. □

*Remark B.6.* As in the DGCC and DGLA settings, the structures given in Theorem B.5 (minus the description of fibrations in DGGC) give closed model category structures when finiteness assumptions are removed. There is a discrepancy between this situation and that of [**1**], which defines cooperads using direct sums and orbits instead of products and fixed points.

**Lemma B.7.** *The model structures of* DGLA *and* DGLC *and of* DGCA *and* DGCC *given in Theorem B.1 and Theorem B.5 are linearly dual. That is, each vertical linear duality isomorphism sends fibrations to cofibrations, cofibrations to fibrations, and weak equivalences to weak equivalences.*

While we have generally chosen to give self-contained arguments, for showing that $\mathcal{E}$ and $\mathcal{A}$ give a Quillen equivalence we stray from this choice for the sake of brevity. We may deduce the following result from Lemma B.7, our main Theorem 4.17, and Quillen's Theorem as stated in Theorem B.3.

**Theorem B.8.** *The functors $\mathcal{G}$ and $\hat{\mathbb{A}}$ are a Quillen adjoint pair, as are the functors $\mathcal{E}$ and $\mathcal{A}$. Furthermore, $\mathcal{E}$ and $\mathcal{A}$ are a Quillen equivalence.*

## Appendix C. Minimal models

We end with some brief notes about minimal models, originally due to Sullivan [**23, 4**]. In our language, a minimal model in DGCA is an object of the form $(\mathbb{A}W,\, d)$ where $dW \subset \mathbb{A}^{\geqslant 2}W$. Sullivan's theorem [**23**] is that every DGCA supports a quasi-isomorphism from a minimal model $(\mathbb{A}W,\ d) \xrightarrow{\simeq} A$, and furthermore the minimal model $(\mathbb{A}W,\ d)$ is unique up to isomorphism. Minimal models in DGCA are useful because the Postnikov tower of a rational one-reduced space is encoded transparently in its minimal model as the increasing filtration by free sub-algebras.

Baues and Lemaire [**2**] note that the property satisfied by the differential of a minimal model may be more concisely stated as $(-)^{\mathrm{ind}} \circ d = 0$. Further, they show that making the analogous definition in DGLA also agrees with the naive definition, namely $(\mathbb{L}V,\, d)$ with $dV \subset \mathbb{L}^{\geqslant 2}V$. These minimal models have existence and uniqueness properties similar to those of Sullivan's minimal models in DGCA, but because of the switch from cochains to chains their construction is more difficult – see [**2**]. From the point of view of topology, minimal models in DGLA encode the Eckmann-Hilton homology decomposition of a rational space.

One lemma in the proof of the uniqueness of minimal models of algebras is interesting in its own right. We say that a DGCA is a "differential free graded algebra" if it has the form $(\mathbb{A}V,\, d)$, and similarly for a "differential free graded Lie algebra". Then we have the following [**2**, Prop 1.5].

**Proposition C.1** (Sullivan, Baues-Lemaire)**.** *A map $f$ of differential free graded (Lie) algebras is a quasi-isomorphism if and only if the induced* DG*-map $(f)^{ind}$ on indecomposables is a quasi-isomorphism.*

We apply this proposition to the units of the adjunctions $\mathcal{A}\,\mathcal{E} \to \mathbb{1}_{\mathrm{DGCA}}$ and $\mathcal{L}\,\mathcal{C} \to \mathbb{1}_{\mathrm{DGLA}}$.

**Corollary C.2.** *If $A$ is a differential free graded algebra, then $[\mathcal{E}A]_{\mathrm{DG}} \simeq s(A)^{ind}$. Similarly, if $L$ is a differential free graded Lie algebra, then $[\mathcal{C}L]_{\mathrm{DG}} \simeq s(L)^{ind}$.*

*In particular if $A$ is a* DGCA *minimal model, then $H^*\mathcal{E}A \cong s(A)^{ind}$ as a graded vector space. Similarly, if $L$ is a* DGLA *minimal model then $H_*\mathcal{C}L \cong s(L)^{ind}$.*

We use this corollary to recover the Halperin-Stasheff spectral sequence for calculating the linear dual of homotopy groups of a finite complex, as described in 4.14 of [**8**], from our Corollary A.2. The main construction of [**8**] is that of a filtered model for $(A,\, d_A)$ as a deformation of a minimal model for $(H_*(A),\, 0)$, which in our notation would be called $(\mathbb{A}Z,\, D)$ and $(\mathbb{A}Z,\, d)$ respectively. When $A = A^*(X)$, the results of Section 8 of [**23**] imply that $H_*(Z,\, D) \cong \pi^*(X)$. Because $D$ and $d$ differ by terms of lower filtration, there is a spectral sequence starting with $H_*(Z,\, d)$ and converging to $H_*(Z,\, D) \cong \pi^*(X)$.

By Corollary C.2, we have $H_*(Z,\, d) \cong H_*\left(\mathcal{E}(H^*(X)\right)$, so this spectral sequence has the same $E^2$ term as that of Corollary A.2. Indeed, we may relate these two spectral sequences by comparing them both to equivalent spectral sequences for $\mathcal{E}(\mathbb{A}Z,\, D)$, which on one hand is quasi-isomorphic to $\mathcal{E}(A,\, d_A)$ simply because $\mathcal{E}$ is quasi-isomorphism invariant; and on the other hand is quasi-isomorphic to $(Z,\, D)$ by Corollary C.2. Our approach through $\mathcal{E}(A,\, d_A)$ seems to have better functoriality properties, a more transparent cobracket structure, and greater flexibility in addition to the relationship with Hopf invariants.

Natural notions of minimal models in coalgebras are obtained by duality. Explicitly we require them to be cofree with differentials satisfying $d \circ (-)^{\mathrm{pr}} = 0$.

**Definition C.3.** A minimal model in DGCC is a coalgebra of the form $(\mathbb{C}V,\ d)$ where $dV = 0$.

A minimal model in DGLC is a coalgebra of the form $(\mathbb{E}W,\ d)$ where $dW = 0$.

We may speak of "differential cofree graded (Lie) coalgebras" similarly to obtain duals to Proposition C.1 and Corollary C.2.

**Proposition C.4.** *A map $f$ of differential cofree graded (Lie) coalgebras is a quasi-isomorphism if and only if the induced* DG*-map $(f)^{pr}$ on primitives is a quasi-isomorphism.*

**Corollary C.5.** *If $C$ is a differential cofree graded coalgebra, then $[\mathcal{L}C]_{\mathrm{DG}} \simeq s^{-1}(C)^{pr}$. Similarly, if $E$ is a differential cofree graded $\mathcal{E}il$ coalgebra, then $[\mathcal{A}E]_{\mathrm{DG}} \simeq s^{-1}(E)^{pr}$.*

*In particular if $C$ is a* DGCC *minimal model, then $H_*\mathcal{L}C \cong s^{-1}(C)^{pr}$ as a graded vector space. Similarly, if $E$ is a* DGLC *minimal model, then $H^*\mathcal{A}E \cong s^{-1}(E)^{pr}$.*

Minimal models in all cases are unique up to isomorphism for each object, and Bousfield-Gugenheim even give a functorial construction of them [**3**]. Minimal models of algebras are cofibrant replacements, and minimal models of coalgebras are fibrant replacements. There are other standard functorial fibrant and cofibrant replacements, namely in each setting by applying the appropriate pair of adjoint horizontal arrows from the diagram of Theorem 4.17. These generally differ from minimal models, and as indicated by our discussion of the Halperin-Stasheff spectral sequence the interplay between the two approaches can be enlightening.

## References


**[1]** M. Aubry and D. Chataur. Cooperads and coalgebras as closed model categories. *J. Pure Appl. Algebra*, 180(1-2):1–23, 2003.

**[2]** H. J. Baues and J.-M. Lemaire. Minimal models in homotopy theory. *Math. Ann.*, 225(3):219–242, 1977.

**[3]** A. Bousfield and V. Gugenheim. On PL de Rham theory and rational homotopy type. *Mem. Amer. Math. Soc.*, 179(8):ix+94, 1976.

**[4]** Y. Félix, S. Halperin, and J.-C. Thomas. *Rational homotopy theory*, volume 205 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.

**[5]** B. Fresse. Koszul duality of operads and homology of partition posets. In *Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory*, volume 346 of *Contemp. Math.*, pages 115–215. Amer. Math. Soc., Providence, RI, 2004.

**[6]** E. Getzler and J. D. S. Jones. Operads, homotopy algebra and iterated integrals for double loop spaces, arXiv:hep-th/9403055.

**[7]** V. Ginzburg and M. Kapranov. Koszul duality for operads. *Duke Math. J.*, 76(1):203–272, 1994.

**[8]** S. Halperin and J. Stasheff. Obstructions to homotopy equivalences. *Adv. in Math.*, 32(3):233–279, 1979.

**[9]** P. S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.

**[10]** M. Markl, S. Shnider, and J. Stasheff. *Operads in algebra, topology and physics*, vol 96 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.

**[11]** G. Melançon and C. Reutenauer. Free Lie superalgebras, trees and chains of partitions. *J. Algebraic Combin.*, 5(4):337–351, 1996.

**[12]** W. Michaelis. Lie coalgebras. *Adv. in Math.*, 38(1):1–54, 1980.

**[13]** J.C. Moore. Differential Homological Algebra. *Actes du Congr. Intern. des Mathématiciens*, (1970): 335–339.

**[14]** E. O'Neill Higher order massey products and links. *Trans. Amer. Math Soc.*, 248(1):37–66, 1979.

**[15]** D. Quillen. Rational homotopy theory. *Ann. of Math. (2)*, 90:205–295, 1969.

**[16]** D.E. Radford. A natural ring basis for the shuffle algebra. *J. Algebra*, 58:432–453, 1979.

**[17]** M. Schlessinger and J. Stasheff. The Lie algebra structure of tangent cohomology and deformation theory. *J. Pure Appl. Algebra*, 38(2):313–322, 1985.

**[18]** S. Schwede and B. E. Shipley. Algebras and modules in monoidal model categories. *Proc. London Math. Soc. (3)*, 80(2):491–511, 2000.

**[19]** D. P. Sinha. A pairing between graphs and trees. 2006, math.QA/0502547.

**[20]** D. P. Sinha and B. Walter. Lie coalgebras and rational homotopy theory, II: Hopf invariants. arXiv:0809.5084. To appear in *Trans. Amer. Math Soc.*.

**[21]** J. R. Smith. Cofree coalgebras over operads. *Topology Appl.*, 133(2):105–138, 2003.

**[22]** J. R. Smith. Homotopy theory of coalgebras over operads. 2003, arXiv:math.CT/0305317.

**[23]** D. Sullivan. Infinitesimal computations in topology. *Inst. Hautes Études Sci. Publ. Math.*, (47):269–331 (1978), 1977.

**[24]** V. Tourtchine. On the other side of the bialgebra of chord diagrams. 2004, math.QA/0411436.

**[25]** B. Walter. Cofree coalgebras over cooperads. In preparation.

**[26]** B. Walter. The configuration basis of a Lie algebra and its dual. 2010, arXiv:1010.4765v1 [math.RA]

**[27]** B. Walter. Lie algebra configuration pairing. 2010, arXiv:1010.4732v1 [math.RA]

**[28]** J. H. C. Whitehead, An expression of Hopf's invariant as an integral. *Proc. Nat. Acad. Sci. U. S. A.*, (33):117–123, 1947.



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