Homology, Homotopy and Applications, vol. 13(1), 2011, pp.205–221

GENERALIZED DAVIS-JANUSZKIEWICZ SPACES, MULTICOMPLEXES AND MONOMIAL RINGS

ALVISE J. TREVISAN

(*communicated by Nigel Ray*)

Abstract

We show that every monomial ring can be realized topologically by a certain topological space. This space is called a generalized Davis-Januszkiewicz space and can be thought of as a colimit over a multicomplex, a combinatorial object generalizing a simplicial complex. Furthermore, we show that such a space is obtained as the homotopy fiber of a certain map with total space the classical Davis-Januszkiewicz space.

Introduction

A classical problem in algebraic topology is the realizability of certain rings or algebras, that is, finding a topological space whose cohomology ring is a given ring or algebra. A notable example is the realizability of graded polynomial rings, posed as a question by Steenrod in [**15**]. Complete answers were given only decades later, for example, by Notbohm in [**12**] in the case when the ground ring is $\mathbb{F}_p$ with p an odd prime and by Andersen and Grodal in [**1**] in the general case.

In this paper we focus on the realizability of monomial algebras, i.e., quotients of a polynomial algebra by an ideal generated by monomials. Let $R = \Bbbk[x_1, \dots, x_m]/I$ be such an algebra; if all the monomials in I are squarefree, then $R = SR(K)$ is the Stanley-Reisner algebra of some simplicial complex K. The algebra $SR(K)$ is an algebraic invariant of K, and it is one of the central objects of study in combinatorial commutative algebra. In [**7**] Davis and Januszkiewicz showed that every Stanley-Reisner algebra $R(K)$ is realized topologically by a space $DJ(K)$. Subsequent work of Buchstaber and Panov ([**3**]) has shown the existence of a homotopy equivalent model for $DJ(K)$ as a cellular subcomplex of $BT^m = (\mathbb{C}\mathrm{P}^\infty)^m$, and Notbohm and Ray ([**13**]) expressed it as a homotopy colimit of a certain diagram over the poset category of K.

In this paper we show that in fact every monomial algebra can be realized topologically by a generalized Davis-Januszkiewicz space. We emphasize the fact that a monomial ring can be thought of as the Stanley-Reisner ring of a combinatorial object generalizing a simplicial complex, called a multicomplex. This is the content of Theorem 3.6.

Received September 9, 2010, revised January 13, 2011; published on May 17, 2011.
2000 Mathematics Subject Classification: 13F55, 55U05, 55P99.
Key words and phrases: Davis-Januszkiewicz space, monomial ring, Stanley-Rainer ring, simplicial complex, polarization, homotopy fiber.
Article available at `http://intlpress.com/HHA/v13/n1/a7` and `doi:10.4310/HHA.2011.v13.n1.a7`

Any simplicial complex can be viewed as a multicomplex in a canonical way. Moreover, to any multicomplex K one can associate a certain classical simplicial complex K^{pol}. The corresponding Davis-Januszkiewicz spaces $DJ(K)$ and $DJ(K^{\mathrm{pol}})$ are related by a fibration that reflects topologically the algebraic process of polarization. Explicitly, the second main result of this paper is Theorem 4.3, in which we show that $DJ(K)$, the topological space realizing the Stanley-Reisner ideal of a multicomplex K, is the homotopy fiber of a map with total space $DJ(K^{\mathrm{pol}})$, a classical Davis-Januszkiewicz space, and base space the classifying space of a torus.

Organization of the paper

In Section 1 we introduce the notion of a multicomplex, study the basic facts about them and show how they generalize simplicial complexes. In Section 2 we review a classical construction, due to Fröberg, called polarization. It associates a squarefree monomial ring to any monomial ring, while at the same time preserving homological properties. In Section 3 we define the Davis-Januszkiewicz space of a multicomplex and describe its main properties. In particular, Theorem 3.6 describes its cohomology ring. In Section 4 we show that the polarization of a monomial ring is reflected topologically by a fibration involving the Davis-Januszkiewicz spaces of a multicomplex and a simplicial complex. This is the content of Theorem 4.3.

Acknowledgements

The author would like to thank the referee for helpful comments and suggestions that improved the contents and exposition of the paper and Dietrich Notbohm for many helpful discussions and careful proofreading.

1. Multicomplexes

We introduce the definition of a *multicomplex*. This is inspired by [**6**], but the treatment is substantially different. A simplicial complex K on the vertex set $[m] = \{1, \dots, m\}$ is a collection of subsets of $[m]$ such that whenever $\sigma \in K$ and $\tau \subset \sigma$, then also $\tau \in K$. We can also think of K as a collection of elements of $\{0,1\}^m$ such that whenever $\sigma \in K$ and $\tau \leqslant \sigma$, then also $\tau \in K$. The order on the two-element set $\{0,1\}$ is just $0 < 1$, and the order on $\{0,1\}^m$ is the standard Cartesian partial order. Suppose K is a simplicial complex in the first sense; then a simplex σ corresponds to the element $(\sigma_1, \dots, \sigma_m)$ given by

$$\sigma_i = \begin{cases} 1 & \text{if } i \in \sigma, \\ 0 & \text{if } i \notin \sigma. \end{cases}$$

In other words, $(\sigma_i, \dots, \sigma_m)$ consists of the values of the characteristic function of σ as a subset of $[m]$. In the opposite direction, by taking the support of an m-tuple $(\sigma_1, \dots, \sigma_m) \in \{0,1\}^m$ we obtain $\sigma = \{i \in [m] \mid \sigma_i \neq 0\}$. For example, the boundary $\partial\Delta^2$ of the 2-simplex is represented in the first sense by the set

$$\big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\big\}$$

and in the second sense by the set

$$\big\{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1)\big\}.$$

Using slightly different notation, we could replace 1 with the symbol ∞. We then represent a simplicial complex by a set of elements of $\{0,\infty\}$. We can define a generalization of this classical concept by considering collections of elements of $\mathbb{N}_\infty^m$ with the analogous property of a simplicial complex and satisfying an additional "boundedness" condition. Here $\mathbb{N}_\infty = \mathbb{N}\cup\{\infty\} = \{0,1,2,\dots,\infty\}$ with $i<\infty$ for each $i\in\mathbb{N}$ and $\mathbb{N}_\infty^m$ is endowed with the structure of a partially ordered set given by the standard Cartesian order. A *chain* in K is a totally ordered subset L of K. Since ∞ is larger than every natural number, every chain L has a least upper bound in $\mathbb{N}_\infty^m$, i.e., an element $\sup(L)\in\mathbb{N}_\infty^m$, possibly ∞, such that $\sup(L)\geqslant\tau$ for every $\tau\in L$.

Definition 1.1. A multicomplex K on $[m]$ is a subset of $\mathbb{N}_\infty^m$ such that

1. If $\sigma\in K$ and $\tau\leqslant\sigma$, then also $\tau\in K$;
2. Any chain L in K has an upper bound in K, i.e., $\sup(L)\in K$.

The elements of $\mathbb{N}_\infty^m$ are called multisimplices. A subcomplex of K is any subset of K which is also a multicomplex in its own right.

For any multisimplex σ, we denote by σ_i its i-th coordinate in $\mathbb{N}_\infty^m$.

Remark 1.2. The poset $\mathbb{N}_\infty^m$ can be endowed with the structure of a meet-semilattice via a coordinatewise minimum of multisimplices. More precisely, if S is a (not necessarily finite) subset of $\mathbb{N}_\infty^m$, then its *meet* $\wedge S$ is the element whose i-th coordinate is given by $\min\{\tau_i \mid \tau\in S\}$. If $S=\{\sigma_1,\dots,\sigma_n\}$ we write $\wedge S=\sigma_1\cap\cdots\cap\sigma_n$ and call it the *intersection* of $\sigma_1,\dots,\sigma_n$.

Since $\sigma_1\cap\cdots\cap\sigma_n\leqslant\sigma_i$ for $1\leqslant i\leqslant n$, condition 1 of Definition 1.1 says that a multicomplex is actually a sub-semilattice of $\mathbb{N}_\infty^m$.

The maximal elements in a multicomplex K considered as a poset are called its *maximal multisimplices*. The following lemma states that every multicomplex is determined by its maximal multisimplices. We use the notation $\langle\mu_1,\dots,\mu_s\rangle$ to denote the multicomplex determined by the simplices $\mu_1,\dots,\mu_s$. In detail, we have that $\langle\mu_1,\dots,\mu_s\rangle=\{\sigma\in\mathbb{N}_\infty^m \mid \sigma\leqslant\mu_i \text{ for some } i, 1\leqslant i\leqslant s\}$. The multicomplex $\langle\sigma\rangle$ generated by a single multisimplex σ is denoted by $\Delta(\sigma)$ or just $\Delta\sigma$.

In principle a multicomplex could have an infinite number of maximal multisimplices. The next proposition shows that this cannot happen.

Proposition 1.3. *Every multicomplex has a finite number of maximal multisimplices.*

We prove Proposition 1.3 with the help of the following lemma.

Lemma 1.4. *Every infinite sequence of elements of $\mathbb{N}_\infty^m$, with $m\geqslant 1$, admits a non-decreasing infinite subsequence.*

Proof. We proceed by induction on the number of coordinates m. Suppose first that $m=1$. Let $S=(s_n)_{n\geqslant 1}$ be an infinite sequence in $\mathbb{N}_\infty$. If S is not bounded above we define a subsequence by setting $n_1=1$ and then, assuming n_i has been chosen,

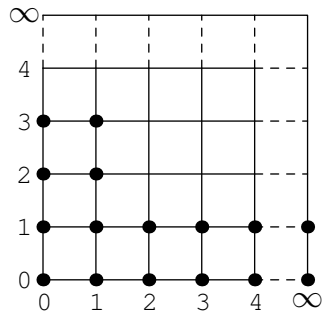


Figure 1: The multicomplex in $\mathbb{N}_\infty^2$ generated by $(\infty, 1)$ and $(1, 3)$.

set n_{i+1} to be the least index of S such that $n_i < n_{i+1}$ and $s_{n_i} < s_{n_{i+1}}$. Such an index can always be chosen; otherwise S would be bounded above. The subsequence $(s_{n_i})_{i \geqslant 1}$ is clearly strictly increasing as required.

Suppose now S has an upper bound different from ∞. It follows that only finitely many different elements of $\mathbb{N}_\infty$ appear in the sequence, and therefore that one of them must occur infinitely many times. This gives the required constant sequence.

If S has ∞ as an upper bound and infinitely many terms equal ∞, then the subsequence given by such terms is infinite and constant. Finally, if there are only finitely many terms equal to ∞ we can discard them to obtain a subsequence which falls in one of the first two cases. The resulting subsequence will also be a subsequence of the original one. This concludes the proof of the base step of the induction.

Suppose now that the proposition holds for $m - 1$ and let $S = (s_n)_{n \geqslant 1}$ be an infinite sequence of elements of $\mathbb{N}_\infty^m$. By considering only the first $m - 1$ coordinates of the s_n, we obtain, by applying the induction hypothesis, a non-decreasing infinite subsequence indexed by the integers $n_1 < n_2 < \cdots$. Consider the corresponding subsequence $S' = (s_{n_i})_{i \geqslant 1}$ of S and take the sequence given by the last coordinate of each s_{n_i}. By induction we can again find an infinite non-decreasing subsequence, indexed by integers $p_1 < p_2 < \cdots$, to obtain a subsequence $(s_{p_j})_{j \geqslant 1}$ of S' and hence of S which, by construction, is infinite and non-decreasing. □

Corollary 1.5. *Let $\mu_1, \ldots, \mu_s$ be the maximal multisimplices of a multicomplex K. Then $K = \langle \mu_1, \ldots, \mu_s \rangle$.*

Proof of Proposition 1.3. Any two maximal multisimplices have to be incomparable, but the previous lemma states that any infinite sequence of elements of $\mathbb{N}_\infty^m$ admits at least two (in fact infinitely many) comparable ones. □

Remark 1.6. Even though a regular simplicial complex is not directly a multicomplex, there is a canonical way to embed it into one. Let indeed K be an abstract simplicial complex, represented by a set of subsets of $[m] = \{1, \ldots, m\}$. Let $F \subset [m]$ be a face of K; then we assign a multisimplex $\sigma(F)$ given by

$$\sigma(F)_i = \begin{cases} \infty & \text{if } i \in F, \\ 0 & \text{otherwise.} \end{cases}$$

The multicomplex $\langle \sigma(F) \mid F \text{ maximal face of } K \rangle$ corresponds uniquely to K.

Conversely, let L be a multicomplex on m vertices whose maximal multisimplices have only ∞'s and 0's as coordinates. We assign to each maximal multisimplex μ of L the subset $F(\mu) = \{i \in m \mid \mu_i = \infty\}$ of $[m]$. In this way, the multicomplex L corresponds uniquely to the simplicial complex $\langle F(\mu) \mid \mu \text{ maximal face of } L\rangle$.

Definition 1.7. Let K be a multicomplex on m vertices. Its ν-vector $\nu(K)$ is the m-tuple with i-th coordinate $\nu(K)_i = \max\{\sigma_i \mid \sigma \in K, \sigma_i \neq \infty\}$, or $\nu(K)_i = 0$ if such a maximum does not exist.

Definition 1.8. An m-tuple τ in $\mathbb{N}_\infty^m$ is said to be a non-multisimplex of K if it does not belong to K. A non-multisimplex τ of K is said to be a missing face if all the $\sigma \in \mathbb{N}_\infty^m$ with $\sigma \leqslant \tau$ belong to K. In other words, a missing face is a minimal non-multisimplex.

Remark 1.9. By the second defining condition of a multicomplex, a multisimplex containing ∞ in one of its coordinates can never be a missing face.

If K and K' are multicomplexes, then we define their union and intersection to be the corresponding set-theoretic union and intersection. By condition 1 of Definition 1.1, the intersection $K \cap K'$ can be written as

$$K \cap K' = \{\sigma \cap \sigma' \mid \sigma \in K, \sigma \in K'\},$$

where the intersection is the meet operation of K as a semilattice, as described in Remark 1.2.

It is obvious that $K \cup K'$ is again a multicomplex and that for $K \cap K'$ the first defining condition of a multicomplex is clearly satisfied. For the boundedness condition, suppose L is a chain in $K \cap K'$; then L has an upper bound σ_K in K and an upper bound $\sigma_{K'}$ in K'. They might be different, but $\sigma_K \cap \sigma_{K'}$ belongs to both K and K' and is still an upper bound for L, so that $K \cap K'$ really is a multicomplex.

Suppose now that K is a multicomplex on $[m]$. The set of all possible intersections of its maximal multisimplices forms a partially ordered set with respect to the induced order. This is called the *intersection poset* of K and is denoted by $\mathcal{L}(K)$.

Remark 1.10. The intersection poset of a multicomplex is finite by Proposition 1.3. Moreover, it is a sub-semilattice of K, since it contains by construction all possible intersections of elements of K.

Lemma 1.11. *For any multisimplex σ there exists a unique element $i(\sigma)$ of $\mathcal{L}(K)$ with $i(\sigma) \geqslant \sigma$ which is minimal with respect to the induced partial order on $\mathcal{L}(K)$.*

Proof. As $\mathcal{L}(K)$ contains the maximal multisimplices of K, the required $i(\sigma)$ always exists. Moreover, if there were two incomparable elements $i(\sigma)$ and $i'(\sigma)$, with $i(\sigma) \geqslant \sigma$, $i'(\sigma) \geqslant \sigma$, and minimal in the set of elements with this property, then we would have that their intersection also satisfies $i(\sigma) \cap i'(\sigma) \geqslant \sigma$. This would contradict their minimality, as $i(\sigma) \cap i'(\sigma)$ is smaller than both $i(\sigma)$ and $i'(\sigma)$. □

The previous lemma has a very important consequence, since it implies that our main object of study, the Davis-Januszkiewicz space introduced in the next section, can be obtained as a colimit indexed over the finite poset $\mathcal{L}(K)$ instead of that over the possibly infinite face poset of K.

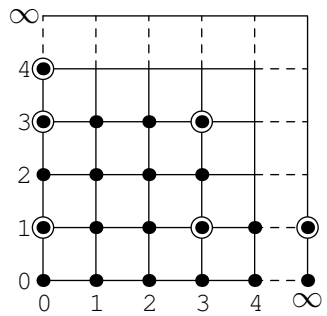


Figure 2: The multicomplex $\langle (0,4),(3,3),(\infty,1)\rangle$.
The elements of its intersection semilattice are circled.

Definition 1.12. Let K be a multicomplex on $[m]$ and k a commutative ring with unity. The Stanley-Reisner ring or face ring of K is the quotient

$$SR(K) = k[x_1, \ldots, x_m]/I_K$$

of the polynomial ring $k[x_1, \ldots, x_m]$ on m generators of degree 2 by the ideal

$$I_K = (x^\sigma \mid \sigma \text{ is a missing face of } K),$$

where $x^\sigma = x_1^{\sigma_1} x_2^{\sigma_2} \cdots x_m^{\sigma_m}$ for $\sigma = (\sigma_1, \ldots, \sigma_m)$. When $\sigma_i = 0$, the corresponding variable x_i does not appear. The ideal I_K is called the Stanley-Reisner ideal of K.

Note that the Stanley-Reisner ideal is well defined as a consequence of Remark 1.9.

Remark 1.13. If τ is a non-multisimplex of K, there exists a missing face σ such that $\sigma \leqslant \tau$. The corresponding monomial x^σ divides x^τ, and therefore x^τ belongs to the Stanley-Reisner ideal of K. This means that I_K contains all the monomials corresponding to non-multisimplices of K. In particular, there is a one-to-one correspondence between monomial ideals in the polynomial ring on m variables and multicomplexes on m vertices.

Remark 1.14. Let K and L be two multicomplexes on $[m]$. From the definition of the Stanley-Reisner ideal it follows immediately that

$$I_K + I_L = I_{K \cap L} \qquad \text{and} \qquad I_K \cap I_L = I_{K \cup L}.$$

2. Polarization

Unlike the Stanley-Reisner ring of a regular simplicial complex, the Stanley-Reisner ring of a multicomplex needs not be squarefree. Nonetheless, there is a canonical way to obtain a squarefree monomial ideal out of an arbitrary one, known as polarization. In this section we introduce a slight generalization of this concept, suitable for the treatment of generalized Davis-Januszkiewicz space. For a reference on Stanley-Reisner and monomial rings the reader can consult, among others, [**5**] and [**16**]. Throughout this section we assume that the base ring k is a field.

Suppose I is a monomial ideal in $R = k[x_1, \dots, k_n]$. For each variable x_i, denote by p_i the maximum exponent such that $x_i^{p_i}$ divides at least one of the minimal generators. Fix an m-tuple of integers $v = (v_1, \dots, v_m)$ such that $v_i \geqslant p_i, 1 \leqslant i \leqslant m$ and set $|v| = \sum_{i=1}^{n} v_i$. We define the *v-polarization* of R to be the polynomial ring

$$R_v^{\mathrm{pol}} = k[x_{11}, x_{12}, \dots, x_{1v_1}, \dots, x_{n1}, \dots, x_{nv_n}]$$

on v new variables. If some power x_i^s divides one of the minimal generators of I, then $s \leqslant p_i$, and we set its polarization to be the monomial $x_{i1}x_{i2}\cdots x_{is}$ of R^{pol}. Accordingly, the polarization of a generator $x_1^{s_1}\cdots x_n^{s_n}$ of I is defined to be the product of the polarization of the powers $x_i^{s_i}$ and is thus itself a monomial in R^{pol} which is obviously squarefree. The *v-polarization* I_v^{pol} of I is finally the ideal of R^{pol} generated by the polarization of the minimal generators of I.

Example 2.1. Let I be the ideal (x_1^2, x_1x_2) in $k[x_1, x_2]$; then $p_1 = 2, p_2 = 1$ so that $R_{(2,1)}^{\mathrm{pol}} = k[x_{11}, x_{12}, x_{21}]$. The polarized ideal $I_{(2,1)}^{\mathrm{pol}}$ is generated by the monomials $x_{11}x_{12}$ and $x_{11}x_{21}$.

There is an obvious map $R_v^{\mathrm{pol}} \to R$, sending a variable x_{ij} to x_i, which induces a homomorphism on the quotient rings $R_v^{\mathrm{pol}}/I_v^{\mathrm{pol}} \to R/I$. Its kernel is the ideal $(I_v^{\mathrm{pol}} + L)/I_v^{\mathrm{pol}}$, where L is generated by the differences

$$x_{11} - x_{12}, x_{12} - x_{13}, \dots, x_{1v_1-1} - x_{1v_1}, \dots, x_{n1} - x_{n2}, \dots, x_{nv_n-1} - x_{nv_n}. \qquad (1)$$

This gives at once an isomorphism between R/I and $R_v^{\mathrm{pol}}/(I_v^{\mathrm{pol}} + L)$. One of the most interesting features of polarization is that such differences form a regular sequence in $R_v^{\mathrm{pol}}/I_v^{\mathrm{pol}}$, and thus the homological properties of R/I are preserved. For example, R/I is a Cohen-Macaulay, respectively, Gorenstein, Golod, complete intersection ring if and only if the polarized quotient is (see [**8**]).

Definition 2.2. Let K be a multicomplex with Stanley-Reisner ideal I_K. The multicomplex K_v^{pol} defined by the polarized ideal $I_{K_v^{\mathrm{pol}}}$ is called the v-polarization of K.

Let $\nu(K) = (\nu_1, \dots, \nu_m)$ be the ν-vector of a multicomplex K on m vertices. By definition of ν-vector, the $(\nu_1 + 1, \dots, \nu_m + 1)$-polarization of K is well defined and we call it simply the *polarization* K^{pol} of K.

Remark 2.3. Since a polarized Stanley-Reisner ideal is squarefree, the multicomplex determined by it corresponds to a regular simplicial complex.

The following remark is elementary, but is a key step in studying the cohomology structure of Davis-Januszkiewicz spaces in Section 3.

Remark 2.4. Let K_1 and K_2 be two multicomplexes on m vertices and let $K = K_1 \cup K_2$ with ν-vector $\nu(K) = \nu$. Then for any m-tuple $v = (v_1, \dots, v_m)$ with $v_i \geqslant \nu_i + 1, 1 \leqslant i \leqslant m$ we have that $K_v^{\mathrm{pol}} = K_v^{\mathrm{pol}} = (K_1)_v^{\mathrm{pol}} \cup (K_2)_v^{\mathrm{pol}}$ and $(K_1 \cap K_2)_v^{\mathrm{pol}} = (K_1)_v^{\mathrm{pol}} \cap (K_2)_v^{\mathrm{pol}}$.

Even though we do not need it in the rest of this paper, it is interesting to note that we can describe the maximal multisimplices of K_v^{pol} directly in terms of the

maximal multisimplices of K, that is, we can give a direct combinatorial description of polarization.

We introduce an operation on sets of simplices. Applying this operation to the set of maximal multisimplices of a multicomplex, we obtain a set of generators for K_v^{pol}.

Let $\sigma = (\sigma_1, \dots, \sigma_m)$ be a multisimplex of $[m]$. Let i, j be integers such that $1 \leqslant i \leqslant m, 0 \leqslant j \leqslant \sigma_i$. Let w be an integer such that $w \geqslant \sigma_i$ if σ_i is finite or $w \geqslant 0$ if $\sigma_i = \infty$. We define a multisimplex $s_{i,j,w}(\sigma)$ on $m + w$ vertices as follows:

$$\begin{array}{cccccccccccccc} (\sigma_1, & \dots, & \sigma_{i-1}, & \infty, & \dots, & \infty, & 0, & \infty, & \dots, & \infty, & \sigma_{i+1}, & \dots, & \sigma_m). \\ & & & & & & \uparrow & & & & \uparrow & & & \\ & & & & & & i+j & & & & i+w+1 & & & \end{array}$$

If i, w are as above, we also define $s_{i,w} = \{s_{i,j,w}(\sigma) \mid 0 \leqslant j \leqslant \sigma_i\}$. More generally, we define $s_{i,w}(\Sigma) = \bigcup_{\sigma \in \Sigma} s_{i,w}(\sigma)$ where Σ is a set of simplices such that for each $\sigma \in \Sigma$ then $w \geqslant \sigma_i$ whenever $\sigma_i \neq \infty$,.

Let now K be a multicomplex on m vertices with ν-vector $\nu(K) = (\nu_1, \dots, \nu_m)$ and let $v = (v_1, \dots, v_m)$ be an m-tuple such that $v_i \geqslant \nu_i, 1 \leqslant i \leqslant m$. We can apply to $\sigma \in K$ the s-operation described above m times to obtain

$$s(\sigma) = s_{m+v_1+\cdots+v_{m-1},v_m}(\cdots(s_{3+v_1+v_2,v_3}(s_{2+v_1,v_2}(s_{1,v_1}(\sigma)))\cdots). \tag{2}$$

Remark 2.5. If σ is a multisimplex on m vertices, then by construction the set $s(\sigma)$ contains simplices on $m + v_1 + \cdots + v_m$ vertices, so we can re-index the vertices of simplices of $s(\sigma)$ by assigning the labels $(1,1), \dots, (1, v_1)$ to the first $v_1 + 1$, the labels $(2,1), \dots, (2, v_2)$ to the following $v_2 + 1$ vertices and so on. With this labeling in mind, it is elementary but lengthy to check that K_v^{pol} is generated by all the simplices in $s(\mu)$, as μ varies among the maximal faces of K.

Example 2.6. Let K be the multicomplex on two vertices with maximal multisimplices $(1, 0)$ and $(0, \infty)$. We see that the missing faces are given by $(2, 0)$ and $(1, 1)$, so that the Stanley-Reisner ring of K is $k[x_1, x_2]/(x_1^2, x_1x_2)$, as in Example 2.1. The algebraic polarization is then the ring

$$k[x_{11}, x_{12}, x_{21}]/(x_{11}x_{12}, x_{11}x_{21}). \tag{3}$$

The multicomplex corresponding to the ring (3) has $\{11, 12, 21\}$ as the vertex set and the missing faces $(1, 1, 0)$ and $(1, 0, 1)$, so that $K^{\mathrm{pol}} = \langle (0, \infty, 0), (\infty, 0, 0), (0, \infty, \infty)\rangle)$. According to Remark 2.5, we can describe K^{pol} directly as $\langle s(1, 0), s(0, \infty)\rangle$. The ν-vector is $\nu(K) = (1, 0)$, so by (2) we have

$$s(1, 0) = s_{3,0}(s_{1,1}(1, 0)) = s_{3,0}(\{(0, \infty, 0), (\infty, 0, 0)\}) = \{(0, \infty, 0), (\infty, 0, 0)\}$$

and

$$s(0, \infty) = s_{3,0}(s_{1,1}(0, \infty)) = s_{3,0}(\{(0, \infty, \infty)\}) = \{(0, \infty, \infty)\}.$$

Remark 2.7. The v-polarization of a simplicial complex carries essentially the same information as the polarization. It is mainly a technical tool to deal with the polarization of sub-multicomplexes of larger multicomplexes, as in the proof of Theorem 4.3.

Let K be a multicomplex on m vertices with ν-vector $\nu(K) = \nu$. It is easy to check, either from the algebraic definition or from the construction described above, that

there is a bijection between the intersection posets of K^{pol} and of K_v^{pol}. The map sends a simplex $\sigma \in \mathcal{L}(K^{\mathrm{pol}})$, which has $|\nu| + m$ coordinates, to a simplex $\sigma^v \in \mathcal{L}(K_v^{\mathrm{pol}})$ obtained by filling in the remaining $|v| - |\nu| - m$ coordinates with ∞.

Let us illustrate this fact with a simple example. Set $K = \langle(1)\rangle = \{0, 1\}$, the multicomplex on 1 vertex generated by (1). It has $\nu(K) = (1)$, so its polarization is $K^{\mathrm{pol}} = \langle(\infty, 0), (0, \infty)\rangle$. Now, if $v = (4)$, for $n > 1$, the v-polarization is $K_v^{\mathrm{pol}} = \langle(\infty, 0, \infty, \infty), (0, \infty, \infty, \infty)$. Then $\mathcal{L}(K^{\mathrm{pol}}) = \{(\infty, 0), (0, \infty), (0, 0)\}$ and $\mathcal{L}(K_v^{\mathrm{pol}}) = \{(\infty, 0, \infty, \infty), (0, \infty, \infty, \infty), (0, 0, \infty, \infty)\}$.

Algebraically, we see that the Stanley-Reisner ring of K^{pol} is $k[x_{11}, x_{12}]/(x_{11}x_{12})$ and that of K_v^{pol} is $k[x_{11}, x_{12}, x_{13}, x_{14}]/(x_{11}x_{12})$.

3. Davis-Januszkiewicz spaces

In this paper we consider complex projective spaces $\mathbb{C}\mathrm{P}^n$ as pointed CW-complexes with the standard cellular structure consisting of one i-cell in each even dimension $2i$, for $i = 0, \ldots, n$, the 0-cell being a fixed base point $*$. When $m \leqslant n$, there is a natural cellular inclusion $\mathbb{C}\mathrm{P}^m \hookrightarrow \mathbb{C}\mathrm{P}^n$. We also allow n to be zero in which case $\mathbb{C}\mathrm{P}^0 = *$, or to be ∞, in which case $\mathbb{C}\mathrm{P}^\infty = \cup_{n=0}^{\infty}\mathbb{C}\mathrm{P}^n$ with the induced cellular structure.

As a subset of a partially ordered set, any multicomplex K inherits the structure of a partially ordered set himself. We denote by $\mathrm{CAT}(K)$ the corresponding category. Its objects are then simplices σ of K and there is exactly one morphism whenever $\sigma \leqslant \tau$. For a given multicomplex K on $[m]$, we define a functor X^K from $\mathrm{CAT}(K)$ to the category TOP of topological spaces as

$$X^K(\sigma) = \mathbb{C}\mathrm{P}^{\sigma_1} \times \cdots \times \mathbb{C}\mathrm{P}^{\sigma_m}. \tag{4}$$

For each morphism $\sigma \leqslant \tau$, there is a corresponding inclusion $X^K(\sigma) \hookrightarrow X^K(\tau)$, induced by componentwise cellular inclusion of the projective spaces.

Definition 3.1. Let K be a multicomplex. The Davis-Januszkiewicz space $DJ(K)$ of K is the colimit of the functor X^K, i.e., $DJ(K) = \operatorname{colim} X^K$.

Remark 3.2. The space $DJ(K)$ sits naturally in the m-fold product

$$BT^m = \mathbb{C}\mathrm{P}^\infty \times \cdots \times \mathbb{C}\mathrm{P}^\infty$$

by considering K as a subcomplex of the full multisimplex $\Delta^m = \langle(\infty, \ldots, \infty)\rangle$. Since all the maps in the colimit of X^K are the natural cellular inclusions, the colimit reduces to the union of topological spaces

$$DJ(K) = \bigcup_{\sigma \in K} X^K(\sigma)$$

and the Davis-Januszkiewicz space of K inherits a cellular structure with cells in one-to-one correspondence with finite (i.e., with no coordinate equal to ∞) simplices of K.

As soon as a multicomplex K contains a non-finite multisimplex, i.e., one having ∞ as one of the coordinates, the category $\mathrm{CAT}(K)$ becomes infinite. We would like to work over a finite category instead, mainly to be able to use induction arguments, as in the proof of Theorem 3.6. Luckily enough it is possible to do so by restricting

our attention to the full subcategory $\mathrm{CAT}(\mathcal{L}(K))$ whose objects are the simplices in the intersection poset of K. In the following lemma we describe what happens if we restrict the functor X^K to $\mathrm{CAT}(\mathcal{L}(K))$. Denote such a restriction $X^K|_{\mathrm{CAT}(\mathcal{L}(K))}$ by $X^{\mathcal{L}(K)}$.

Proposition 3.3. *The Davis-Januszkiewicz space of K is determined by the intersection poset $\mathcal{L}(K)$, i.e.,* $\operatorname{colim} X^K \approx \operatorname{colim} X^{\mathcal{L}(K)}$.

Proof. We show that the topological space on the right-hand side of the above equality satisfies the universal property of the colimit on the left-hand side. This automatically gives the required homeomorphism by uniqueness of colimits.

We start by indicating the maps from the spaces $X^K(\sigma)$ to the colimit over the intersection poset $\operatorname{colim} X^{\mathcal{L}(K)}$. By Lemma 1.11, for each multisimplex σ of K there is a unique element $i(\sigma)$ of $\mathcal{L}(K)$ satisfying $\sigma \leqslant i(\sigma)$ which is minimal with respect to such inequality. On one hand, the existence of $i(\sigma)$ gives by functoriality a morphism $X^K(\sigma) \hookrightarrow X^K(i(\sigma)) = X^{\mathcal{L}(K)}(i(\sigma))$ which, composed with the natural morphism $X^{\mathcal{L}(K)}(i(\sigma)) \to \operatorname{colim} X^{\mathcal{L}(K)}$, produces the required morphism

$$X^K(\sigma) \xrightarrow{f_\sigma} \operatorname{colim} X^{\mathcal{L}(K)}.$$

On the other hand, the minimality condition implies that every map from $X^K(\sigma)$ to some other space Z, compatible with the colimit maps $X^K(\sigma) \to \operatorname{colim} X^K$, factors through $\operatorname{colim} X^{\mathcal{L}(K)}$. Finally, the uniqueness of such $i(\sigma)$ guarantees that the factorization is unique, thus concluding the proof. □

Remark 3.4. In the case when K corresponds to a regular simplicial complex (see Remark 1.6), the maximal multisimplices of K contain only 0's and ∞'s and therefore so do all of their intersections. This means that only points $*$ and $\mathbb{C}\mathrm{P}^\infty$'s appear as factors in the spaces $X^{\mathcal{L}(K)}(\sigma), \sigma \in \mathcal{L}(K)$. In such case $DJ(K)$ is the classical Davis-Januszkiewicz space of the (unique) regular simplicial complex corresponding to K.

Suppose that K is a multicomplex admitting a vector $j = (j_1, \dots, j_m) \in \mathbb{N}^m$ of "weights" such that each maximal multisimplex σ has for each i either $\sigma_i = j_i$ or $\sigma_i = \infty$. We can associate to K a regular simplicial complex L whose simplices are in one to one correspondence to those of $\mathcal{L}(K)$. The maximal faces of L are given by $F(\mu)$, as μ varies among the maximal faces of K, with $F(\mu) = \{i \in m \mid \mu_i = \infty\}$. The Davis-Januszkiewicz space $DJ(K)$ associated to the multicomplex K is then the generalized moment-angle complex $Z(L; \underline{\mathrm{X}})$ of $[\mathbf{2}]$ for the m-tuple

$$\underline{\mathrm{X}} = ((\mathbb{C}\mathrm{P}^\infty, \mathbb{C}\mathrm{P}^{j_1}), \dots, (\mathbb{C}\mathrm{P}^\infty, \mathbb{C}\mathrm{P}^{j_m})).$$

We now move to description of the cohomological properties of Davis-Januszkiewicz spaces of multicomplexes. We noted in Remark 3.2 that the space $DJ(K)$ sits naturally in the m-fold product $BT^m = \mathbb{C}\mathrm{P}^\infty \times \cdots \times \mathbb{C}\mathrm{P}^\infty$. The cohomology ring of BT^m with coefficients in a commutative ring with unity k is a polynomial ring $k[v_1, \dots, v_m]$ on m variables of degree 2.

In analogy with the case of regular simplicial complexes, the following lemma expresses the Davis-Januszkiewicz space of a union of two multicomplexes as a certain pushout. This makes it possible to use Mayer-Vietoris arguments for the cohomology of those spaces.

Lemma 3.5. *If a multicomplex K can be written as a union of two subcomplexes K' and K'', then $DJ(K)$ sits in the following pushout diagram:*

$$\begin{array}{ccc} DJ(K' \cap K'') & \longrightarrow & DJ(K'') \\ \downarrow & & \downarrow \\ DJ(K') & \longrightarrow & DJ(K) \end{array}$$

where the maps originating from the top left corner are the natural inclusions. Moreover, the above diagram is also a homotopy pushout.

Proof. As $DJ(K')$ and $DJ(K'')$ are disjoint outside $DJ(K' \cap K'')$ and the maps are the natural inclusions, the pushout is given by the union, but then

$$DJ(K') \cup DJ(K'') = \left(\bigcup_{\sigma \in K'} X^{K'}(\sigma) \right) \cup \left(\bigcup_{\sigma \in K''} X^{K''}(\sigma) \right) = \bigcup_{\sigma \in K} X^{K}(\sigma)$$

because for $K = K' \cup K''$ the functor X^K satisfies $X^K(\sigma) = X^{K'}(\sigma)$ for any $\sigma \in K'$ and $X^K(\sigma) = X^{K''}(\sigma)$ for any $\sigma \in K''$. This concludes the proof of the first statement, since the last term in the previous equation is exactly $DJ(K)$. As all maps are inclusions of CW-complexes, they are cofibrations and therefore the pushout is also a homotopy pushout. □

Theorem 3.6. *Le k be a commutative ring with unity k and K be an arbitrary multicomplex on $[m]$. The cohomology ring $H^*(DJ(K); k)$ with coefficients in k is isomorphic, as a ring, to the Stanley-Reisner ring of K and the inclusion $i\colon DJ(K) \hookrightarrow BT^m$ induces the canonical projection $k[x_1, \ldots, x_m] \twoheadrightarrow k[x_1, \ldots, x_m]/I_K$.*

Proof. As noted in Remark 3.2, the cells of $DJ(K)$ are in one-to-one correspondence with the finite simplices of K. As there are no odd-dimensional cells, it is clear that additively $H^*(DJ(K); k) \cong k[x_1, \ldots, x_m]/I_K$, where x_i is the cohomology class corresponding to the 2-cell of the i-th coordinate $\mathbb{C}\mathrm{P}^\infty$ in BT^m and a finite multisimplex $\sigma = (\sigma_1, \ldots, \sigma_m)$ corresponds to the cohomology class given by $x^\sigma = x^{\sigma_1} \cdots x^{\sigma_m}$.

We now show that the inclusion $i\colon DJ(K) \hookrightarrow BT^m$ induces the canonical projection. By Proposition 3.3, we can take $DJ(K)$ to be $\operatorname{colim} X^{\mathcal{L}(K)}$ and proceed by induction on the number of elements of $\mathcal{L}(K)$.

The "smallest" possible $\mathcal{L}(K)$ consists only of one multisimplex $\sigma = (\sigma_1, \ldots, \sigma_m)$. The colimit $DJ(K)$ is given by $X^K((\sigma_1, \ldots, \sigma_m))$, which is just the product $\mathbb{C}\mathrm{P}^{\sigma_1} \times \cdots \times \mathbb{C}\mathrm{P}^{\sigma_m}$ of complex projective spaces. In this case the conclusion is clear.

Suppose now that $\mathcal{L}(K)$ has a number of simplices greater than zero. Let μ be any maximal multisimplex of K and consider the two subcomplexes $\Delta\mu$ and $K \setminus \mu$ of K generated respectively by μ and by all the other maximal multisimplices. Denote by $\mathcal{L}'$ and $\mathcal{L}''$ the sub-posets of $\mathcal{L}(K)$ corresponding respectively to the intersection posets of $\Delta\mu$ and $K \setminus \mu$. As they are generated by a strictly smaller number of maximal multisimplices of K, the induction hypothesis can be applied. Note that $\mathcal{L}'$ really consists of just one multisimplex, that is, μ.

For the sake of simplicity, write $H^*(Y)$ for the cohomology with k-coefficients of any space Y and R for the polynomial ring $k[v_1, \ldots, v_m]$. Using the Mayer-Vietoris sequence in cohomology associated to the pushout diagram of Lemma 3.5 and noting

that $K \setminus \mu \cap \Delta\mu = \partial\mu$ and $K \setminus \mu \cup \Delta\mu = K$, we get a short exact sequence (of graded k-modules)

$$0 \longrightarrow H^*(DJ(K)) \longrightarrow H^*(DJ(K \setminus \mu)) \oplus H^*(DJ(\Delta\mu)) \longrightarrow H^*(DJ(\partial\mu)) \longrightarrow 0.$$

Let i_K, resp. $i_{K\setminus\mu}, i_{\Delta\mu}, i_{\partial\mu}$ be the inclusion of $DJ(K)$, resp.

$$DJ(K \setminus \mu), DJ(\Delta\mu), DJ(\partial\mu),$$

into BT^m. Applying the induction hypothesis we get a commutative diagram of k-modules

$$\begin{array}{ccccc}
H^*(DJ(K)) & \longrightarrow & H^*(DJ(K \setminus \mu)) \oplus H^*(DJ(\Delta\mu)) & \longrightarrow & H^*(DJ(\partial\mu)) \\
\uparrow{\scriptstyle i_K^*} & & \uparrow{\scriptstyle i^*_{K\setminus\mu} \oplus i^*_{\Delta\mu}} & & \uparrow{\scriptstyle i^*_{\partial\mu}} \\
R & \longrightarrow & R \oplus R & \longrightarrow & R \\
\uparrow & & \uparrow & & \uparrow \\
\ker i_K^* & \longrightarrow & I_K \oplus I_{\Delta\mu} & \longrightarrow & I_{\partial\mu},
\end{array}$$

where the first map in the second row is the diagonal sending a polynomial $h \in R$ to $(h, h) \in R \oplus R$ and the second map is the difference sending (f, g) to $f - g$.

Since all the rows are exact and the last two columns are exact by the inductive hypothesis, by the Nine Lemma, the first column is also exact. Moreover, the last row in the diagram identifies $\ker i_K^*$ with $I_K \cap I_{\Delta\mu}$. Since $I_K \cap I_{\Delta\mu} = I_K$, we have identified additively the cohomology of $DJ(K)$ with the quotient of R by $I(K)$, i.e., Stanley-Reisner ring of K.

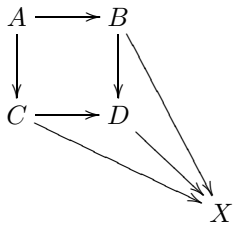
The only thing left to describe is the multiplicative structure. To this extent, let x^τ and x^σ be the respective cohomology classes corresponding to the finite simplices τ and σ. Since the induced map $i^*\colon H^*(BT^m) \to H^*(DJ(K))$ is just the canonical projection from the polynomial ring R to its quotient ring R/I_K, naturality of the cup product reduces the computation of $x^\tau \smile x^\sigma$ to the product of the cohomology classes in $H^*(BT^m)$ corresponding to the same simplices. It follows that $x^\tau \smile x^\sigma = x^{\tau+\sigma}$, thus showing that the additive isomorphism found above is also multiplicative. □

4. Davis-Januszkiewicz spaces as homotopy fibers

In this section we investigate the relation between the Davis-Januszkiewicz space of a multicomplex and that of its polarization.

In the proof of Theorem 4.3 we make use of the following lemma, which is an easy consequence of Mather's Cube Lemma ([**10**]).

Lemma 4.1. *For any commutative diagram of the form*



with the square A-B-C-D a homotopy pushout, the homotopy fibers fit in a homotopy pushout diagram

$$\begin{array}{ccc} F_A & \longrightarrow & F_B \\ \downarrow & & \downarrow \\ F_C & \longrightarrow & F_D, \end{array}$$

where F_A, F_B, F_C, F_D denote the homotopy fibers of the maps respectively from A, B, C, D to X.

We move on to the description of the Davis-Januszkiewicz space as a certain homotopy fiber. To do so, we need to consider an H-space structure on $\mathbb{C}\mathrm{P}^\infty$. A good reference for the material in the following paragraph is, for example, [**9**], in particular Part II.

Since the infinite complex projective space $\mathbb{C}\mathrm{P}^\infty$ is a $K(\mathbb{Z},2)$ space, for any CW-complex X there is a natural isomorphism $\eta\colon H^2(X;\mathbb{Z}) \to [X, \mathbb{C}\mathrm{P}^\infty]$ from the second cohomology group of X to the group of homotopy classes of maps $X \to \mathbb{C}\mathrm{P}^\infty$. Identifying $K(\mathbb{Z}\times\mathbb{Z},2)$ with $\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty$, there is a multiplication map $\mu\colon \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty$ which makes $\mathbb{C}\mathrm{P}^\infty$ into an H-group in such a way that addition in $H^2(X;\mathbb{Z})$ corresponds to the multiplication μ under the isomorphism η. This multiplication is unique up to homotopy, but it is not needed for what follows. By H-group we mean an H-space with a homotopy associative and homotopy commutative multiplication and a homotopy inverse. We write $\mu(t,u) = t * u$ for the product of two elements under this multiplication of $\mathbb{C}\mathrm{P}^\infty$ and t^{-1} for the image of an element t under the homotopy inverse.

Let K be a fixed multicomplex on m vertices with ν-vector $\nu(K) = (\nu_1, \ldots, \nu_m)$. If we fix an arbitrary vector $v = (v_1, \ldots, v_m)$ with $v_i \geqslant \nu_i + 1$ for $1 \leqslant i \leqslant m$, then the v-polarization K_v^{pol} is a well-defined multicomplex on $|v|$ vertices. Let $x_{11} - x_{12}, x_{12} - x_{13}, \ldots, x_{1v_1-1} - x_{1v_1}, \ldots, x_{n1} - x_{n2}, \ldots, x_{nv_n-1} - x_{nv_n}$ be the regular sequence generating the ideal L of Section 2. The number of linear forms in the sequence is given by $\sum_{i=1}^m (v_i - 1) = |v| - m$.

Think of the $|v|$-fold product $(\mathbb{C}\mathrm{P}^\infty)^{|v|}$ as indexed by pairs (i,j), where $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant i$ and the $(|v|-m)$-fold product $(\mathbb{C}\mathrm{P}^\infty)^{|v|-m}$ as indexed by pairs (r,s), where $1 \leqslant r \leqslant m$ and $1 \leqslant s \leqslant v_r - 1$. Define a map φ_v from $(\mathbb{C}\mathrm{P}^\infty)^{|v|} \to (\mathbb{C}\mathrm{P}^\infty)^{|v|-m}$ coordinatewise as $\varphi_v(t)_{(r,s)} = t_{(r,s)} * t_{(r,s+1)}^{-1}$, where the operation $*$ is the multiplication described above. The Davis-Januszkiewicz space $DJ(K_v^{\mathrm{pol}})$ of the v-polarization of K sits naturally inside $(\mathbb{C}\mathrm{P}^\infty)^{|v|}$, so by composition we have a map

$$\psi_v\colon DJ(K_v^{\mathrm{pol}}) \overset{\iota}{\hookrightarrow} (\mathbb{C}\mathrm{P}^\infty)^{|v|} \xrightarrow{\varphi_v} (\mathbb{C}\mathrm{P}^\infty)^{|v|-m}. \tag{5}$$

When $v = \nu + 1 = (\nu_1 + 1, \ldots, \nu_m + 1)$, that is, when $K_v^{\mathrm{pol}} = K^{\mathrm{pol}}$ is the polarization of K, we denote the corresponding map simply by $\psi\colon DJ(K^{\mathrm{pol}}) \overset{\iota}{\hookrightarrow} (\mathbb{C}\mathrm{P}^\infty)^{|\nu|+m} \xrightarrow{\varphi_{\nu+1}} (\mathbb{C}\mathrm{P}^\infty)^{|\nu|}$.

Lemma 4.2. *Let K a multicomplex with ν-vector $\nu(K) = \nu$ and let v be a vector with $v_i \geqslant \nu_i + 1$ for $1 \leqslant i \leqslant m$. The following statements hold:*

1. *There is a homeomorphism $DJ(K_v^{\mathrm{pol}}) \approx DJ(K^{\mathrm{pol}}) \times (\mathbb{C}P^\infty)^{|v|-|\nu|-m}$.*
2. *There is a homotopy equivalence of homotopy fibers* $\mathrm{hofib}(\psi_v) \simeq \mathrm{hofib}(\psi)$.

Proof. By Remark 2.7 each simplex σ^v in the intersection poset $\mathcal{L}(K_v^{\mathrm{pol}})$ is obtained from a simplex $\sigma \in \mathcal{L}(K^{\mathrm{pol}})$ by adding $|v| - |\nu| - m$ coordinates all equal to ∞. This means that for a simplex σ of K_v^{pol} we have

$$X^{\mathcal{L}(K_v^{\mathrm{pol}})}(\sigma^v) = X^{\mathcal{L}(K^{\mathrm{pol}})}(\sigma) \times (\mathbb{C}\mathrm{P}^\infty)^{|v|-|\nu|-m}.$$

Since colimits of CW-complexes commute with products and

$$DJ(K_v^{\mathrm{pol}}) = \operatorname{colim} X^{\mathcal{L}(K_v^{\mathrm{pol}})},$$

the first statement holds.

For the second statement, let $p_1 \colon (\mathbb{C}\mathrm{P}^\infty)^{|v|} \to (\mathbb{C}\mathrm{P}^\infty)^{|\nu|+m}$ the projection which is the identity on the coordinates (i,j) with $j \leqslant \nu_i$ and maps everything else to the base point and similarly let $p_2 \colon (\mathbb{C}\mathrm{P}^\infty)^{|v|-m} \to (\mathbb{C}\mathrm{P}^\infty)^{|\nu|}$ be the projection which is the identity on the coordinates (i,j) with $j \leqslant \nu_i - 1$ and maps everything else to the base point. By the first statement in the lemma, the restriction $\widehat{p_1}$ of p_1 to $DJ(K_v^{\mathrm{pol}}) \approx DJ(K^{\mathrm{pol}}) \times (\mathbb{C}\mathrm{P}^\infty)^{|v|-|\nu|-m}$ is the identity on $DJ(K^{\mathrm{pol}})$ and maps everything else to the base point; hence it is a trivial fibration with fiber $(\mathbb{C}\mathrm{P}^\infty)^{|v|-|\nu|-m}$. Similarly, p_2 is a trivial fibration again with fiber $(\mathbb{C}\mathrm{P}^\infty)^{|v|-|\nu|-m}$. By definition of the map ψ_v, the following diagram commutes on the nose:

$$\begin{array}{ccc} DJ(K_v^{\mathrm{pol}}) & \xrightarrow{\psi_v} & (\mathbb{C}\mathrm{P}^\infty)^{|v|-m} \\ \downarrow{\scriptstyle \widehat{p_1}} & & \downarrow{\scriptstyle p_2} \\ DJ(K^{\mathrm{pol}}) & \xrightarrow{\psi} & (\mathbb{C}\mathrm{P}^\infty)^{|\nu|}. \end{array}$$

Since the induced map between the fibers is just the identity of $(\mathbb{C}\mathrm{P}^\infty)^{|v|-|\nu|-m}$, the homotopy fibers $\operatorname{hofib}(\psi)$ and $\operatorname{hofib}(\psi_v)$ are homotopy equivalent by [**14**, Proposition 7.6.1]. □

We are now ready to prove the second main result of this paper.

Theorem 4.3. *Let K be a multicomplex on m vertices with ν-vector $\nu(K)$. With notation as above, if $v = (v_1 \dots, v_m)$ is such that $v_i \geqslant \nu(K)_i + 1$ for $1 \leqslant i \leqslant m$, then $DJ(K)$ is the homotopy fiber of $\psi_v \colon DJ(K_v^{\mathrm{pol}}) \to (\mathbb{C}P^\infty)^{|v|-m}$.*

Proof. As in the proof of Theorem 3.6, by applying Proposition 3.3 we can use induction on the number of simplices of the intersection poset $\mathcal{L}(K)$. Moreover, by Lemma 4.2 we can restrict without loss of generality to the case when $v = \nu + 1 = (\nu_1 + 1, \dots, \nu_m + 1)$, that is, when $K_v^{\mathrm{pol}} = K^{\mathrm{pol}}$ is the polarization of K.

Suppose that $\mathcal{L}(K)$ has only one element, i.e., that K has only one maximal multisimplex $\sigma = (\sigma_1, \dots, \sigma_m)$, so that $K = \Delta\sigma$. Then $DJ(K) = X^K(\sigma) = \mathbb{C}\mathrm{P}^{\sigma_1} \times \cdots \cdots \times \mathbb{C}\mathrm{P}^{\sigma_m}$. The polarized multicomplex K^{pol} corresponds by Remark 2.3 to a regular simplicial complex L. If σ is finite, L is dual to the boundary of the product of simplices $\Delta^{\sigma_1} \times \cdots \times \Delta^{\sigma_m}$, viewed as a polytope. In this case, the map φ of (5) is induced by the canonical characteristic function of $\mathbb{C}\mathrm{P}^{\sigma_1} \times \cdots \times \mathbb{C}\mathrm{P}^{\sigma_m}$ as a quasitoric manifold (see [**4**]) over the above product of simplices. This expresses at once $DJ(K)$ as the homotopy fiber of ψ, because for any quasitoric $2n$-manifold M_P over an n-polytope P, the Borel construction gives a fibration $M_P \to M_P \times_{T^n} ET^n \to BT^n$, where the

total space is homotopy equivalent to the Davis-Januszkiewicz space $DJ(K_{\partial P})$, for $K_{\partial P}$ the regular simplicial complex dual to the boundary of P.

If σ is not finite, then the map ψ is just a product of the homotopy fibration of the previous case with a trivial fibration (everything maps to the base point). This takes care of the base step of our induction.

Suppose now that K is an arbitrary multicomplex. As in the proof of 3.6, we express it as a union $K = K' \cup K''$, where $K' = \Delta\mu$ and $K'' = K \setminus \mu$ for some maximal multisimplex μ. The intersection posets of K' and K'' (and consequently of $K' \cap K''$) contain strictly less elements than that of K, so we can apply the induction hypothesis to $DJ(K')$, $DJ(K'')$ and $DJ(K' \cap K'')$. Let $\nu = \nu(K)$ be the ν-vector of K. According to Remark 2.4, $K^{\mathrm{pol}} = (K')^{\mathrm{pol}}_{\nu+1} \cup (K'')^{\mathrm{pol}}_{\nu+1}$ and $(K' \cap K'')^{\mathrm{pol}}_{\nu+1} = (K')^{\mathrm{pol}}_{\nu+1} \cap (K'')^{\mathrm{pol}}_{\nu+1}$, so by Lemma 3.5, the following square:

$$\begin{array}{ccc} DJ((K' \cap K'')^{\mathrm{pol}}) & \longrightarrow & DJ((K'')^{\mathrm{pol}}_{\nu+1}u) \\ \downarrow & & \downarrow \\ DJ((K')^{\mathrm{pol}}_{\nu+1}) & \longrightarrow & DJ((K)^{\mathrm{pol}}_{\nu+1}), \end{array}$$

where all the maps are the natural inclusions, is a homotopy pushout. By definition of the map ψ_v of (5), the restriction of $\psi\colon DJ(K^{\mathrm{pol}}) \to (\mathbb{C}\mathrm{P}^\infty)^{|\nu|}$ to each of the spaces in the above diagram is the respective $\psi_{\nu+1}$. All these spaces then fit in a commutative diagram as in Lemma 4.1, where X is in this case $(\mathbb{C}\mathrm{P}^\infty)^{|\nu|}$ and the maps are the corresponding restrictions of of ψ.

It follows that the homotopy fiber of $DJ(K^{\mathrm{pol}}) \to (\mathbb{C}\mathrm{P}^\infty)^{|\nu|}$ is the homotopy pushout of the fibers, i.e., the homotopy pushout of

$$\begin{array}{ccc} DJ(K' \cap K'') & \longrightarrow & DJ(K'') \\ \downarrow & & \\ DJ(K') & & \end{array}$$

which is nothing but $DJ(K)$, as desired. □

Since the maximal simplices of the polarized multicomplex K^{pol} contain only ∞'s and 0's in their coordinates, by Remark 3.4 the corresponding Davis-Januszkiewicz space $DJ(K^{\mathrm{pol}})$ is a classical Davis-Januszkiewicz space. In light of this fact and with Theorem 4.3 in hand, we can provide an alternative viewpoint on Theorem 3.6. For notation and basic facts about spectral sequences the reader may wish to consult, for example, [**11**].

We consider the Eilenberg-Moore spectral sequence of the homotopy fibration

$$DJ(K) \to DJ(K^{\mathrm{pol}}) \xrightarrow{\psi} (\mathbb{C}\mathrm{P}^\infty)^{|\nu|} \tag{6}$$

given by Theorem 4.3. Let $H^*(X)$ denote the cohomology algebra of a topological space X with coefficients in a fixed field k. The Eilenberg-Moore spectral sequence associated to a fibration $F \to E \to B$ is a spectral sequence of commutative algebras, converging to the cohomology algebra $H^*(F)$ of the fiber, with E_2 term given by the Tor-algebra $\mathrm{Tor}_{H^*(B)}(H^*(E); k)$.

By construction, the map $\psi^*\colon H^*((\mathbb{C}\mathrm{P}^\infty)^{|\nu|}) \to H^*(DJ(K^{\mathrm{pol}}))$ sends the $|\nu|$ generators of $H^*((\mathbb{C}\mathrm{P}^\infty)^{|\nu|})$ to the linear forms of (1). Since those linear forms are a regular sequence, the algebra $H^*(DJ(K^{\mathrm{pol}}))$ is free as a module over $H^*((\mathbb{C}\mathrm{P}^\infty)^{|\nu|})$. The Tor-algebra then reduces to a tensor product, so we have

$$\begin{aligned}\mathrm{Tor}_{H^*((\mathbb{C}\mathrm{P}^\infty)^{|\nu|})}(H^*(DJ(K^{\mathrm{pol}})); k) &\cong H^*(DJ(K^{\mathrm{pol}})) \otimes_{H^*((\mathbb{C}\mathrm{P}^\infty)^{|\nu|})} k \\ &\cong H^*(DJ(K^{\mathrm{pol}}))/\psi^*(H^{>0}((\mathbb{C}\mathrm{P}^\infty)^{|\nu|})).\end{aligned}$$

The Eilenberg-Moore spectral sequence collapses then at the E_2 page, and therefore the cohomology of the fiber is given by the above expression

$$H^*(DJ(K)) \cong H^*(DJ(K^{\mathrm{pol}}))/\psi^*(H^{>0}((\mathbb{C}\mathrm{P}^\infty)^{|\nu|})),$$

i.e., by the quotient of the cohomology algebra of $DJ(K^{\mathrm{pol}})$ by the ideal

$$L = \psi^*(H^{>0}((\mathbb{C}\mathrm{P}^\infty)^{|\nu|}))$$

generated by the regular sequence (1).

Now, $DJ(K^{\mathrm{pol}})$ is a classical Davis-Januszkiewicz space and its cohomology algebra is, in the notation of Section 2 the (squarefree) Stanley-Reisner ring $SR(K^{\mathrm{pol}}) = R^{\mathrm{pol}}/I_K^{\mathrm{pol}}$, given by the quotient of the polarized ring R^{pol} by the the polarization I_K^{pol} of the Stanley-Reisner ideal of K. Putting everything back together, we have that

$$H^*(DJ(K)) \cong R^{\mathrm{pol}}/(I_K^{\mathrm{pol}} + L) \cong SR(K),$$

which recovers Theorem 3.6.

References

[1] K.K. Andersen and J. Grodal, The Steenrod problem of realizing polynomial cohomology rings, *J. Topol.* **1** (2008), no. 4, 747–760. MR2461854 (2009i:55022)

[2] A. Bahri, M. Bendersky, F.R. Cohen and S. Gitler, The polyhedral product functor: A method of decomposition for moment-angle complexes, arrangements and related spaces, *Adv. Math.* **225** (2010), no. 3, 1634–1668. MR2673742

[3] V.M. Bukhshtaber and T.E. Panov, Torus actions and the combinaitorics of polytopes, *Tr. Mat. Inst. Steklova* **225** (1999), 96–131. MR1725935 (2000m:52021)

[4] V.M. Buchstaber and T.E. Panov, *Torus actions and their applications in topology and combinatorics*, University Lecture Series **24**, Amer. Math. Soc., Providence, RI (2002). MR1897064 (2003e:57039),

[5] E. Miller and B. Sturmfels, *Combinatorial commutative algebra*, Grad. Texts in Math. **227**, Springer-Verlag, New York (2005). MR2110098 (2006d:13001)

[6] M. Cimpoeaş, Finite multicomplexes, *An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat.* **14** (2006), no. 2, 9–30. MR2338712 (2008f:13037)

[7] M.W. Davis and T. Januszkiewicz, Convex polytopes, Coxeter orbifolds and torus actions, *Duke Math. J.* **62** (1991), no. 2, 417–451. MR1104531 (92i:52012)

[8] R. Fröberg, A study of graded extremal rings and of monomial rings, *Math. Scand.* **52** (1982), no. 1, 22–34. MR0681256 (84j:13019)

[**9**] R.M. Kane, *The homology of Hopf spaces*, North-Holland Mathematical Library **40**, North-Holland Publishing Co., Amsterdam (1998). MR0961257 (90f:55018)

[**10**] M. Mather, Pull-backs in homotopy theory, *Canad. J. Math.* **28** (1976), no. 2, 225–263. MR0402694 (53 #6510)

[**11**] J. McCleary, *A user's guide to spectral sequences*, Cambridge Studies in Advanced Mathematics **58**, Cambridge University Press, Cambridge (2001). MR1793722 (2002c:55027).

[**12**] D. Notbohm, Spaces with polynomial mod-p cohomology, *Math. Proc. Cambridge Philos. Soc.* **126** (1999), no. 2, 277–292. MR1670237 (2000e:55013)

[**13**] D. Notbohm and N. Ray, On Davis-Januszkiewicz homotopy types. I. Formality and rationalisation, *Algebr. Geom. Topol.* **5** (2005), 31–51. MR2135544 (2006a:55016)

[**14**] P. Selick, *Introduction to homotopy theory*, Fields Institute Monographs **9**, Amer. Math. Soc., Providence, RI, 1997. MR1450595 (98h:55001)

[**15**] N.E. Steenrod, The cohomology algebra of a space, *Enseignement Math. (2)*, **7** (1961), 153–178 (1962). MR0160208 (28 #3422),

[**16**] R.H. Villarreal, *Monomial algebras*, Monographs and Textbooks in Pure and Applied Mathematics **238**, Marcel Dekker Inc., New York (2001). MR1800904 (2002c:13001)

Alvise J. Trevisan `a.trevisan@vu.nl`

VU University Amsterdam, Faculteit der Exacte Wetenschappen, De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands