

COHOMOLOGY OF HECKE ALGEBRAS

DAVID BENSON, KARIN ERDMANN AND ARAM MIKAELIAN

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Abstract

We compute the cohomology $H^*(\mathcal{H}, k) = \text{Ext}_{\mathcal{H}}^*(k, k)$ where $\mathcal{H} = \mathcal{H}(n, q)$ is the Hecke algebra of the symmetric group $\mathfrak{S}_n$ at a primitive ℓ th root of unity q , and k is a field of characteristic zero. The answer is particularly interesting when $\ell = 2$, which is the only case where it is not graded commutative. We also carry out the corresponding computation for Hecke algebras of type B_n and D_n when ℓ is odd.

1. Introduction

Let $\mathcal{H} = \mathcal{H}(n, q)$ be the Hecke algebra of the symmetric group $\mathfrak{S}_n$ over a field k of characteristic zero and where q is a primitive ℓ th root of unity. This has generators $T_1, \dots, T_{n-1}$ satisfying braid relations together with the relations $(T_i + 1)(T_i - q) = 0$. We assume that $\ell \geq 2$. Write $n = \ell m + a$ where $0 \leq a < \ell$, and let $\mathcal{B} = \mathcal{H}(\lambda, q)$ where λ is the partition

$$\lambda = (\ell^m, 1^a).$$

That is, $\mathcal{B}$ is the subalgebra of $\mathcal{H}$ generated by all T_i , where $i < \ell m$ and ℓ does not divide i . Then $\mathcal{B}$ is isomorphic to the tensor product of m copies of $\mathcal{H}(\ell, q)$ and is a maximal ℓ -parabolic subalgebra of $\mathcal{H}$. It has been proved by Du [7] that every $\mathcal{H}$ -module is relatively $\mathcal{B}$ -projective. This suggests that $\mathcal{B}$ should play a role similar to that of the group algebra of a Sylow subgroup of a finite group.

The algebra $\mathcal{H}$ has a trivial module k , so it has cohomology $H^*(\mathcal{H}, k) = \text{Ext}_{\mathcal{H}}^*(k, k)$. Similarly we define $H^*(\mathcal{B}, k) = \text{Ext}_{\mathcal{B}}^*(k, k)$. Here we relate the cohomology of $\mathcal{H}$ to that of $\mathcal{B}$. We prove an analogue of the result for group algebras, which states that if G is a finite group and F is a field of characteristic p , then $H^*(G, F)$ is isomorphic to the stable part of $H^*(P, F)$ where P is a Sylow p -subgroup of G (Cartan and Eilenberg [2, Theorem XII.10.1]), and that if, furthermore, P is abelian, then the stable elements are the invariants of the action of $N_G(P)/P$ (Swan [13], corrected in [14]).

The symmetric group $\mathfrak{S}_m$ acts naturally on $\mathcal{B}$ and on $H^*(\mathcal{B}, k) = \text{Ext}_{\mathcal{B}}^*(k, k)$. Our main theorem is as follows.

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Theorem 1.1. *The restriction map in cohomology induces an isomorphism*

$$H^*(\mathcal{H}, k) \rightarrow H^*(\mathcal{B}, k)^{\mathfrak{S}_m}.$$

Case 1. If $\ell > 2$, then $H^(\mathcal{B}, k) = \Lambda(y_1, \dots, y_m) \otimes_k k[x_1, \dots, x_m]$ with $|y_i| = 2\ell - 3$ and $|x_i| = 2\ell - 2$. Defining a derivation d on $H^*(\mathcal{B}, k)$ via $d(x_i) = y_i$, $d(y_i) = 0$, we have*

$$H^*(\mathcal{H}, k) \cong H^*(\mathcal{B}, k)^{\mathfrak{S}_m} = \Lambda(d\sigma_1, \dots, d\sigma_m) \otimes_k k[\sigma_1, \dots, \sigma_m],$$

where σ_i is the i th elementary symmetric polynomial in $x_1, \dots, x_m$.

Case 2. If $\ell = 2$, then $H^(\mathcal{B}, k) = k\langle z_1, \dots, z_m \rangle / (z_i z_j + z_j z_i, i \neq j)$ with $|z_i| = 1$. This algebra is not graded commutative, because the degree one elements z_i do not square to zero.*

Let v_i be the i th elementary symmetric function in $z_1^2, \dots, z_m^2$, so that $|v_i| = 2i$. There are elements $u_i \in H^{2i-1}(\mathcal{B}, k)^{\mathfrak{S}_m}$ ($1 \leq i \leq m$) satisfying

$$u_i^2 = \sum_{l=0}^{i-1} (2l+1) v_{i-l-1} v_{i+l} \quad (1)$$

for $1 \leq i \leq m$ (for $i = 1$ this relation says that $u_1^2 = v_1$), and

$$u_i u_j + u_j u_i = 2 \sum_{l=0}^{j-1} (i-j+2l+1) v_{j-l-1} v_{i+l} \quad (2)$$

for $1 \leq j < i \leq m$, where v_i is taken to be zero if $i > m$ and $v_0 = 1$. We have

$$H^*(\mathcal{H}, k) \cong H^*(\mathcal{B}, k)^{\mathfrak{S}_m} = k\langle u_1, \dots, u_m, v_2, \dots, v_m \rangle / (R),$$

where (R) is the following set of relations:

1. $v_i v_j = v_j v_i$ ($1 \leq i, j \leq m$),
2. $u_i v_j = v_j u_i$ ($1 \leq i, j \leq m$),
3. relation (1) ($2 \leq i \leq m$),
4. relation (2) ($1 \leq j < i \leq m$),

where, in the right-hand side of relations (1) and (2), we take v_1 to be u_1^2 .

In both cases, $\ell > 2$ and $\ell = 2$, the following is the Poincaré series for the cohomology:

$$\sum_{i \geq 0} t^i \dim_k H^i(\mathcal{H}, k) = \frac{(1 + t^{2(\ell-1)-1})(1 + t^{4(\ell-1)-1}) \dots (1 + t^{2m(\ell-1)-1})}{(1 - t^{2(\ell-1)})(1 - t^{4(\ell-1)}) \dots (1 - t^{2m(\ell-1)})}.$$

In the final section of the paper we prove the analogous theorem for the Hecke algebras of types B_n and D_n when ℓ is odd.

2. Background on Hecke algebras

The standard approach to working with the representation theory of Hecke algebras of type A was developed by Dipper and James [5]. Given a standard Young subgroup

S_λ of $\mathfrak{S}_n$, they work with a set $\mathcal{D}_\lambda$ of right coset representatives of minimal length. Since we are working with left modules rather than right modules, we use the set $\mathcal{D}_\lambda^{-1}$ of inverses of these elements.

In detail, let λ be a partition of n . Let t^λ be the tableau of shape λ in which the numbers $1, 2, \dots, n$ appear in order along successive rows. Then we take for S_λ the standard Young subgroup, in which the rows of t^λ are the orbits of S_λ . Then the distinguished set $\mathcal{D}_\lambda$ of right coset representatives of S_λ in $\mathfrak{S}_n$ consists precisely of elements $g \in \mathfrak{S}_n$ such that the tableau $t^\lambda g$ is row standard. We write $\mathcal{D}_\lambda^{-1}$ for the set of g^{-1} with $g \in \mathcal{D}_\lambda$.

We recall from [5] the basic properties of the distinguished coset representatives.

Lemma 2.1. *Let $d \in \mathcal{D}_\lambda^{-1}$. Then*

- (i) *For each $w \in S_\lambda$, we have $l(dw) = l(w) + l(d)$.*
- (ii) *Write S for the set of transpositions of the form $s_i = (i, i+1)$ in $\mathfrak{S}_n$. If $v \in S$, then either $vd \in \mathcal{D}_\lambda^{-1}$, or $d^{-1}vd \in S_\lambda \cap S$ and $l(vd) = l(d) + 1$.*

We will also need to work with double cosets of Young subgroups, and we recall what we need from Lemma 1.6 of Dipper and James [5].

Lemma 2.2. *Let $\mathcal{D}_{\lambda,\lambda} = \mathcal{D}_\lambda \cap \mathcal{D}_\lambda^{-1}$. Note that $\mathcal{D}_{\lambda,\lambda} = \mathcal{D}_{\lambda,\lambda}^{-1}$.*

- (a) *$\mathcal{D}_{\lambda,\lambda}$ is a system of (S_λ, S_λ) double coset representatives in $\mathfrak{S}_n$.*
- (b) *Each $d \in \mathcal{D}_{\lambda,\lambda}$ is the unique element of minimal length in its double coset.*
- (c) *If $d \in \mathcal{D}_{\lambda,\lambda}$, then $dS_\lambda d^{-1} \cap S_\lambda$ is a standard Young subgroup, which we denote by $S_{\lambda(d)}$ where $\lambda(d)$ is a composition of n .*
- (d) *If $v = dud^{-1} \in S_{\lambda(d)}$ for $u \in S_\lambda$, then $l(u) = l(v)$.*
- (e) *Every element $w \in \mathfrak{S}_n$ has a unique expression as $w = udv$ with $u \in \mathcal{D}_{\lambda(d)}^{-1} \cap S_\lambda$ and $d \in \mathcal{D}_{\lambda,\lambda}$ and $v \in S_\lambda$. Moreover, $l(w) = l(u) + l(d) + l(v)$ and*

$$\mathcal{D}_\lambda^{-1} = \bigcup_{d \in \mathcal{D}_{\lambda,\lambda}} (\mathcal{D}_{\lambda(d)}^{-1} \cap S_\lambda) d.$$

By part (e) of the lemma, any $w \in \mathcal{D}_\lambda^{-1}$ has a unique expression of the form $w = td$ for $d \in \mathcal{D}_{\lambda,\lambda}$ and $t \in \mathcal{D}_{\lambda(d)}^{-1} \cap S_\lambda$. We set $\mathcal{T}_d = \mathcal{D}_{\lambda(d)}^{-1} \cap S_\lambda$ for $d \in \mathcal{D}_{\lambda,\lambda}$, and then we have

$$\mathcal{D}_\lambda^{-1} = \bigcup_{d \in \mathcal{D}_{\lambda,\lambda}} \bigcup_{t \in \mathcal{T}_d} \{td\}. \quad (3)$$

The q -analogue of Lagrange's Theorem

Suppose that $S_\mu \subseteq S_\lambda \subseteq \mathfrak{S}_n$ are standard parabolic subgroups. Then S_λ can be written as a disjoint union of left cosets

$$S_\lambda = \bigcup_{d \in \mathcal{D}_\mu^{-1} \cap S_\lambda} dS_\mu.$$

Definition 2.3. We define the q -index of S_μ in S_λ to be the number

$$(S_\gamma : S_\mu)_q = \sum_{d \in \mathcal{D}_\mu^{-1} \cap S_\gamma} q^{l(d)}.$$

The following is an analogue for Hecke algebras of Lagrange's Theorem for groups:

Lemma 2.4. *We have*

$$(S_\lambda : S_\mu)_q (S_\mu : 1)_q = (S_\lambda : 1)_q.$$

Proof. If $w \in S_\gamma$, then w has a unique factorisation as $w = dv$ with $v \in S_\mu$ and d a distinguished coset representative, and $l(w) = l(d) + l(v)$. $\square$

Furthermore, we have a summation formula, corresponding to the double coset decomposition.

Lemma 2.5. *We have*

$$(\mathfrak{S}_n : S_\mu)_q = \sum_{d \in \mathcal{D}} (S_\mu : S_{\mu(d)})_q q^{l(d)}.$$

Proof. This follows from Lemma 2.4, using (3), since

$$\sum_{d \in \mathcal{D}_\lambda^{-1}} q^{l(d)} = \sum_{d \in \mathcal{D}_{\lambda, \lambda}} \left(\sum_{t \in \mathcal{T}_d} q^{l(t)} \right) q^{l(d)} = \sum_{d \in \mathcal{D}_{\lambda, \lambda}} (S_\mu : S_{\mu(d)})_q. \quad \square$$

A divisibility lemma

From now we take $\lambda = (\ell^m, a)$ where $n = \ell m + a$ and $0 \leq a < \ell$. We shall need to use the fact that the length of an element

$$g \in N(S_\lambda) \cap \mathcal{D}_\lambda = N(S_\lambda) \cap \mathcal{D}_\lambda^{-1} = N(S_\lambda) \cap \mathcal{D}_{\lambda, \lambda},$$

which fixes the fixed points of S_λ , is divisible by ℓ . This will play a role several times in simplifying identities.

Recall that the length of a permutation $x \in \mathfrak{S}_n$ is the size of its *inversion set*, which is defined to be

$$\text{inv}(x) = \{(i, j) : 1 \leq i < j \leq n \text{ and } ix > jx\}$$

(see Exercise 2 in Section 1.6 of Humphreys [10]). Take $x \in \mathcal{D}_\lambda$, and assume that x fixes each fixed point of S_λ . Then the tableau $t^\lambda x$ is row standard, and hence if $i < j$ and i, j are in the same row of t^λ , then (i, j) is not in $\text{inv}(x)$. Writing R_i for the i th row of t^λ , the inversion set is therefore the disjoint union of sets $\text{inv}(x)_{b,c} = \text{inv}(x) \cap (R_b \times R_c)$ where $1 \leq b < c \leq m$.

Lemma 2.6. *Let $x \in \mathcal{D}_\lambda$. Assume that one of $R_b x$ or $R_c x$ is equal to some row of t^λ where $b < c \leq m$. Then the size of $\text{inv}(x)_{b,c}$ is divisible by ℓ .*

Proof. Say $R_b x$ is some row of t^λ ; then $R_b x$ consists of consecutive numbers. Fix some $j \in R_c$; then jx is different from the numbers in $R_b x$, so either $ix > jx$ for all $i \in R_b$, or $ix < jx$ for all $i \in R_b$. So if M is the set of all $j \in R_c$ such that $(i, j) \in \text{inv}(x)$ for some $i \in R_b$, then $\text{inv}(x)_{b,c}$ has size $|R_b| \cdot |M| = \ell \cdot |M|$. $\square$

Corollary 2.7. *Suppose that $g \in N(S_\lambda) \cap \mathcal{D}_\lambda$, and g fixes each fixed point of S_λ . Then the length of g is divisible by ℓ .*

Proof. In this case, for any $1 \leq b \leq m$, the set $R_b g$ is a row of t^λ and hence the size of $\text{inv}(x)_{b,c}$ is divisible by ℓ . Furthermore, the fixed points of S_λ do not contribute to the inversion set of g . $\square$

Some explicit q -indices

Recall that we are assuming that $\lambda = (\ell^m, 1^a)$ where $n = \ell m + a$ and $0 \leq a < \ell$, so that S_λ is maximal ℓ -parabolic.

Lemma 2.8.

1. We have $(S_n : 1)_q = [n]_q!$. Furthermore, if $\mu = (\mu_i)$ is a composition of n , then $(S_\mu : 1)_q = \prod_i (S_{\mu_i} : 1)_q$.
2. We have $(S_n : S_\lambda)_q = m![a]_q!$.
3. If $S_{\lambda(d)}$ is a proper subgroup of S_λ , then $(S_\lambda : S_{\lambda(d)})_q = 0$.

Recall that the q -factorial is defined as $[a]_q! = [1]_q[2]_q \cdots [a]_q$, where $[a]_q$ denotes $1 + q + \cdots + q^{a-1}$.

Proof. Part (1) is well known, and part (2) is proved in Corollary 2.5 of [4]. To prove part (3), note that $\lambda(d)$ is a refinement of λ , so we can factorise the q -index, and it suffices to show that $(\mathfrak{S}_\ell : S_\mu)_q = 0$ if μ is a composition of ℓ and $\mu \neq (\ell)$. We have

$$(\mathfrak{S}_\ell : S_\mu)_q (S_\mu : 1)_q = (\mathfrak{S}_\ell : 1)_q = [\ell]_q! = 0,$$

and $(S_\mu : 1)_q$ is non-zero since all parts of μ are strictly less than ℓ . So it follows that $(\mathfrak{S}_\ell : S_\mu)_q = 0$. $\square$

The transfer map

Assume that M and N are $\mathcal{H}$ -modules, and $y \in \text{Hom}_{\mathcal{B}}(M, N)$. Define

$$\text{tr}_{\mathcal{B}, \mathcal{H}}(y)(x) = \sum_{d \in \mathcal{D}_\lambda^{-1}} q^{-l(d)} T_d y(T_{d^{-1}} x) \quad \text{for } x \in M.$$

Lemma 2.9. *The map $\text{tr}_{\mathcal{B}, \mathcal{H}}(y)$ is a $\mathcal{H}$ -module homomorphism.*

Proof. The argument for this is given in [4], but since the context and notation are slightly different we give the proof here for the convenience of the reader.

Fix $s \in S$ where S is the set of basic transpositions. We must show that $\text{tr}_{\mathcal{B}, \mathcal{H}}(y)$ commutes with T_s . Let

$$\begin{aligned} \mathcal{D}_1 &= \{d \in \mathcal{D}_\lambda^{-1} : d^{-1}sd \in S_\lambda \cap S, l(sd) = l(d) + 1\}, \\ \mathcal{D}_2 &= \{d \in \mathcal{D}_\lambda^{-1} : sd \in \mathcal{D}_\lambda^{-1}, l(sd) = l(d) + 1\}, \\ \mathcal{D}_3 &= \{d \in \mathcal{D}_\lambda^{-1} : sd \in \mathcal{D}_\lambda^{-1}, l(sd) = l(d) - 1\} = s\mathcal{D}_2. \end{aligned}$$

By Lemma 2.1, these sets form a partition of $\mathcal{D}_\lambda^{-1}$. So we have

$$\text{tr}_{\mathcal{B}, \mathcal{H}}(y) = \sum_{d \in \mathcal{D}_1} q^{-l(d)} T_d y T_{d^{-1}} + \sum_{d \in \mathcal{D}_2} (q^{-l(d)} T_d y T_{d^{-1}} + q^{-l(d)-1} T_{sd} y T_{(sd)^{-1}}). \quad (4)$$

If $d \in \mathcal{D}_1$, write $s' = d^{-1}sd$. Then

$$T_{d^{-1}} T_s = T_{d^{-1}s} = T_{s'd^{-1}} = T_{s'} T_{d^{-1}},$$

and

$$T_d T_{s'} = T_{ds'} = T_{sd} = T_s T_d.$$

So

$$T_d y T_{d^{-1}} T_s = T_d y T_{s'} T_{d^{-1}} = T_d T_{s'} y T_{d^{-1}} = T_s T_d y T_{d^{-1}}.$$

This proves that T_s commutes with the first sum in (4).

Now let $d \in \mathcal{D}_2$; then

$$\begin{aligned} (q^{-l(d)} T_d y T_{d^{-1}} + q^{-l(d)-1} T_{(sd)} y T_{(sd)^{-1}}) T_s &= q^{-l(d)} T_d y T_{d^{-1} s} + q^{-l(d)-1} T_{sd} y T_{d^{-1}} (T_s)^2 \\ &= q^{-l(d)} T_d y T_{d^{-1} s} + q^{-l(d)-1} (q-1) T_{sd} y T_{d^{-1} s} + q^{-l(d)} T_{sd} y T_{d^{-1}} \\ &= T_s (q^{-l(d)-1} T_{(sd)} y T_{(sd)^{-1}} + q^{-l(d)} T_d y T_{d^{-1}}). \end{aligned}$$

This proves that T_s commutes with the second sum in (4). $\square$

We will only use this when $N = k$, the trivial module. Then the formula becomes

$$\mathrm{tr}_{\mathcal{B}, \mathcal{H}}(y)(x) = \sum_{d \in \mathcal{D}_\lambda^{-1}} y(T_{d^{-1}} x).$$

Next, still assuming $N = k$, we write $\mathrm{tr}_{\mathcal{B}, \mathcal{H}}(y) = \sum_{d \in \mathcal{D}_{\lambda, \lambda}} y_d$, where

$$y_d(x) = \sum_{t \in \mathcal{T}_d} y(T_{d^{-1}} T_{t^{-1}} x) \quad (x \in M) \quad (5)$$

and $\mathcal{T}_d = \mathcal{D}_{\lambda(d)}^{-1} \cap S_\lambda$.

Definition 2.10. Suppose that M is an $\mathcal{H}$ -module; then $y \in \mathrm{Hom}_{\mathcal{B}}(M, k)$ is *stable* provided for all $d \in \mathcal{D}_{\lambda, \lambda}$ we have $y[T_{d^{-1}}(-)] = q^{l(d)} y(-)$.

Remark 2.11. Any d in $N(S_\lambda) \cap \mathcal{D}_\lambda$ can be written as $d = d_1 d_2 = d_2 d_1$, where d_1 fixes the fixed points of S_λ and d_2 fixes the remaining points. Furthermore, d_1 and d_2 are also in $N(S_\lambda) \cap \mathcal{D}_\lambda$. For such a d , the stability condition of Definition 2.10 reduces to the condition for d_1 . This observation will be used in the proof of Proposition 3.2(ii).

Lemma 2.12. *If $y \in \mathrm{Hom}_{\mathcal{B}}(M, k)$ is an $\mathcal{H}$ -module homomorphism, then it is stable.*

Proof. In this case, for $x \in M$ we have

$$y[T_{d^{-1}}(x)] = T_{d^{-1}} y(x) = q^{l(d)} y(x)$$

since y maps into the trivial module. $\square$

Now assume that $y \in \mathrm{Hom}_{\mathcal{B}}(M, k)$ is stable, and consider the map y_d defined in (5). Since y is a $\mathcal{B}$ -module homomorphism, this can then be written as

$$y_d(x) = q^{l(d)} \sum_{t \in \mathcal{T}_d} T_{t^{-1}} y(x) = q^{l(d)} \left[\sum_t q^{l(t)} \right] y(x) = q^{l(d)} (S_\lambda : S_{\lambda(d)})_q y(x). \quad (6)$$

3. Relating cohomology of $\mathcal{H}$ and $\mathcal{B}$

In the following we write $H^*(\mathcal{H}, k)$ for $\mathrm{Ext}_{\mathcal{H}}^*(k, k)$ and $H^*(\mathcal{B}, k)$ for $\mathrm{Ext}_{\mathcal{B}}^*(k, k)$. The multiplicative structure on cohomology is given by Yoneda composition. Since $\mathcal{H}$ and $\mathcal{B}$ are not Hopf algebras, there is no *a priori* reason why these cohomology rings should be graded commutative, and indeed we shall see that for $\ell = 2$ it is not.

To compute $H^*(\mathcal{H}, k)$, we take a projective resolution of k as an $\mathcal{H}$ -module

$$\cdots \rightarrow P_i \rightarrow P_{i-1} \rightarrow \cdots \rightarrow P_0 \rightarrow 0 \quad (7)$$

and then take the cohomology of the complex

$$0 \rightarrow \text{Hom}_{\mathcal{H}}(P_0, k) \rightarrow \cdots \rightarrow \text{Hom}_{\mathcal{H}}(P_{i-1}, k) \rightarrow \text{Hom}_{\mathcal{H}}(P_i, k) \rightarrow \cdots.$$

We take for P_* a minimal resolution for $\mathcal{H}$, so that we have $H^i(\mathcal{H}, k) \cong \text{Hom}_{\mathcal{H}}(\Omega^i k, k)$, where $\Omega^i k$ is the kernel of $P_{i-1} \rightarrow P_{i-2}$ for $i \geq 2$, Ωk is the kernel of $P_0 \rightarrow k$, and $\Omega^0 k = k$.

Since $\mathcal{H}$ is a free $\mathcal{B}$ -module (see, for example, Lemma 2.4 of [5]), the restriction of a projective $\mathcal{H}$ -module to $\mathcal{B}$ is a projective $\mathcal{B}$ -module. So we may compute $H^*(\mathcal{B}, k)$ using the same resolution, but we need to note that for $\mathcal{B}$ it is not the minimal resolution. So $H^i(\mathcal{B}, k)$ is the quotient $\underline{\text{Hom}}_{\mathcal{B}}(\Omega^i k, k)$ of $\text{Hom}_{\mathcal{B}}(\Omega^i k, k)$ by the maps $P\text{Hom}_{\mathcal{B}}(\Omega^i k, k)$ that factor through a projective module.

In particular, the composite of

$$\text{res}_{\mathcal{H}, \mathcal{B}}: \text{Hom}_{\mathcal{H}}(\Omega^i k, k) \rightarrow \text{Hom}_{\mathcal{B}}(\Omega^i k, k)$$

with the surjection $\text{Hom}_{\mathcal{B}}(\Omega^i k, k) \rightarrow \underline{\text{Hom}}_{\mathcal{B}}(\Omega^i k, k)$ gives the restriction map

$$\text{res}_{\mathcal{H}, \mathcal{B}}: H^i(\mathcal{H}, k) \rightarrow H^i(\mathcal{B}, k),$$

which is a ring homomorphism with respect to Yoneda composition.

We also have the transfer map

$$\text{tr}_{\mathcal{B}, \mathcal{H}}: \text{Hom}_{\mathcal{B}}(P_i, k) \rightarrow \text{Hom}_{\mathcal{H}}(P_i, k),$$

which commutes with the differentials and hence induces a transfer map in cohomology

$$\text{tr}_{\mathcal{B}, \mathcal{H}}: H^i(\mathcal{B}, k) \rightarrow H^i(\mathcal{H}, k).$$

As an intermediary, the transfer map

$$\text{tr}_{\mathcal{B}, \mathcal{H}}: \text{Hom}_{\mathcal{B}}(\Omega^i k, k) \rightarrow \text{Hom}_{\mathcal{H}}(\Omega^i k, k)$$

sends maps that factor through a projective module to zero, and this gives the map in cohomology.

Lemma 3.1. *Let $y \in \text{Hom}_{\mathcal{H}}(\Omega^i k, k)$. Then we have*

$$\text{tr}_{\mathcal{B}, \mathcal{H}} \text{res}_{\mathcal{H}, \mathcal{B}}(y) = (\mathfrak{S}_n : S_\lambda)_q y.$$

In particular, $(\mathfrak{S}_n : S_\lambda)_q \neq 0$, so that

$$\text{res}_{\mathcal{H}, \mathcal{B}}: H^*(\mathcal{H}, k) \rightarrow H^*(\mathcal{B}, k)$$

is injective.

Proof. Since y is an $\mathcal{H}$ -module homomorphism, we can write for $x \in \Omega^i k$

$$\text{tr}_{\mathcal{B}, \mathcal{H}} \text{res}_{\mathcal{H}, \mathcal{B}}(y)(x) = \sum_{d \in \mathcal{D}_\lambda^{-1}} y(T_{d^{-1}} x) = \sum_{d \in \mathcal{D}_\lambda^{-1}} T_{d^{-1}} y(x) = (\mathfrak{S}_n : S_\lambda)_q y(x),$$

and by Lemma 2.8 the q -index is non-zero. This shows that

$$\text{res}_{\mathcal{H}, \mathcal{B}}: \text{Hom}_{\mathcal{H}}(\Omega^i k, k) \rightarrow \text{Hom}_{\mathcal{B}}(\Omega^i k, k)$$

is injective. Since the transfer of an element of $P\text{Hom}_{\mathcal{B}}(\Omega^i k, k)$ is always zero, we deduce that $\text{res}_{\mathcal{H}, \mathcal{B}}: H^*(\mathcal{H}, k) \rightarrow H^*(\mathcal{B}, k)$ is injective as required. $\square$

This shows that $H^*(\mathcal{H}, k)$ is isomorphic to the image of the restriction. We will now give a description of this in terms of stable elements.

Proposition 3.2.

- (i) *The intersection of the stable elements of $\text{Hom}_{\mathcal{B}}(\Omega^i k, k)$ with the projective homomorphisms $P\text{Hom}_{\mathcal{B}}(\Omega^i k, k)$ is equal to $\{0\}$, so that it makes sense to talk of stable elements of $H^i(\mathcal{B}, k) \cong \underline{\text{Hom}}_{\mathcal{B}}(\Omega^i k, k)$. The image of the restriction map*

$$\text{res}_{\mathcal{H}, \mathcal{B}}: H^*(\mathcal{H}, k) \rightarrow H^*(\mathcal{B}, k)$$

consists precisely of the stable elements.

- (ii) *The inclusion $\mathcal{H}(\ell m, q) \subseteq \mathcal{H}$ induces an isomorphism*

$$H^*(\mathcal{H}, k) \rightarrow H^*(\mathcal{H}(\ell m, q), k).$$

Proof of (i). Suppose that $y \in H^i(\mathcal{H}, k)$. Then $\text{res}_{\mathcal{H}, \mathcal{B}}(y) \in \text{Hom}_{\mathcal{B}}(\Omega^i k, k)$ is stable; see Lemma 2.12.

Conversely, suppose that $y \in \text{Hom}_{\mathcal{B}}(\Omega^i k, k)$ and assume that y is stable. Using the Mackey formula, we have

$$\text{res}_{\mathcal{H}, \mathcal{B}} \text{tr}_{\mathcal{B}, \mathcal{H}}(y) = \sum_{d \in \mathcal{D}_{\lambda, \lambda}} y_d$$

with y_d as in (5). Since y is assumed to be stable, we can apply (6) to get

$$y(x) = \sum_{d \in \mathcal{D}_{\lambda, \lambda}} q^{l(d)} (S_{\lambda} : S_{\lambda(d)})_q y(x),$$

where $x \in \Omega^i(k)$ and $S_{\lambda(d)} = dS_{\lambda}d^{-1} \cap S_{\lambda}$. By Lemma 2.8 if $S_{\lambda(d)}$ is a proper subset of S_{λ} , then the q -index is zero. So we get

$$y(x) = \sum_{d \in N(S_{\lambda}) \cap \mathcal{D}_{\lambda}} q^{l(d)} y(x). \quad (8)$$

By Lemma 2.5 and parts 2 and 3 of Lemma 2.8, we have that

$$\sum_{d \in N(S_{\lambda}) \cap \mathcal{D}_{\lambda}} q^{l(d)} = m! [a]_q!$$

hence is non-zero. Therefore we have proved that

$$y = \text{res}_{\mathcal{H}, \mathcal{B}} [\text{tr}_{\mathcal{B}, \mathcal{H}}(y)] \cdot \frac{1}{m! [a]_q!},$$

so it is in the image of the restriction map.

Finally, the transfer of any projective map is zero, so that if a stable map is also projective, then it is equal to zero.

Proof of (ii). From Remark 2.11 and equation (8) it follows that an element of $\text{Hom}_{\mathcal{B}}(\Omega^i k, k)$ is stable with respect to $\mathcal{H}$ if and only if it is stable with respect to $\mathcal{H}(\ell m, q)$. $\square$

4. Action of $\mathfrak{S}_m$ on $\mathcal{B}$ and on $H^*(\mathcal{B}, k)$

The next aim is to characterise the image of the restriction map as fixed points of an action. According to Proposition 3.2(ii), we may assume without loss of generality that $n = \ell m$, i.e., that $a = 0$. We write λ for the partition (ℓ^m) .

The braid group

We begin by recalling that the braid group $\mathfrak{B}_n$ on n strings has a presentation with generators T_i ($1 \leq i \leq n-1$) and relations $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ and $T_i T_j = T_j T_i$ for $|i-j| \geq 2$. For any $w \in \mathfrak{S}_n$, let $w = s_{i_1} \cdots s_{i_l}$ be a shortest word in the transpositions $s_i = (i \ i+1)$ giving w , and set $T_w = T_{i_1} \cdots T_{i_l}$; this is independent of the choices.

There is a homomorphism from $\mathfrak{B}_n$ to $\mathcal{H}(n, q)$, taking T_i to the element of the same name. Note that in $\mathcal{H}(n, q)$, T_i is invertible with inverse $q^{-1}(T_i - q + 1)$.

For each i with $1 \leq i \leq m$, let $d_i \in N(S_\lambda)$ be the element swapping the i th block of ℓ elements with the $i+1$ st. Thus

$$jd_i = \begin{cases} j + \ell & \text{if } \ell i < j \leq \ell(i+1) \\ j - \ell & \text{if } \ell(i+1) < j \leq \ell(i+2) \\ j & \text{otherwise.} \end{cases}$$

Then d_i swaps the i th and $i+1$ st rows of the tableau t^λ and leaves the order of the elements within each row unchanged. Thus d_i is an involution and $t^\lambda d_i$ is row standard, so $d_i \in \mathcal{D}_{\lambda, \lambda}$.

It is not hard to verify directly that the elements T_{d_i} ($1 \leq i \leq m$) satisfy the braid relations. So inside $\mathfrak{B}_n$ we have a wreath product $\mathfrak{B}_\ell \wr \mathfrak{B}_m$ generated by the elements T_i ($1 \leq i < n$, $\ell \nmid i$) and the elements T_{d_i} ($1 \leq i \leq m$). This is the same as the group generated by the T_w for $w \in N(S_\lambda)$.

Conjugation by d_i interchanges the generators in the i th factor of S_λ with the generators of the $i+1$ st factor and fixes all other generators. So for $s_j \in S_\lambda \cap S$ (i.e., $\ell \nmid j$) we have

$$d_i s_j = \begin{cases} s_{j+\ell} d_i & \ell i < j < \ell(i+1) \\ s_{j-\ell} d_i & \ell(i+1) < j < \ell(i+2) \\ s_j d_i & \text{otherwise.} \end{cases}$$

Lemma 4.1. *We have*

$$T_{d_i} T_j = \begin{cases} T_{j+\ell} T_{d_i} & \ell i < j < \ell(i+1) \\ T_{j-\ell} T_{d_i} & \ell(i+1) < j < \ell(i+2) \\ T_j T_{d_i} & \text{otherwise.} \end{cases}$$

Proof. Since d_i is a distinguished coset representative, we have for any s_j in S_λ that $l(d_i s_j) = l(d_i) + 1 = l(s_j d_i)$ and therefore if, for example, s_j is in the i th factor, then

$$T_{d_i} T_j = T_{d_i s_j} = T_{s_{j+\ell} d_i} = T_{j+\ell} T_{d_i}.$$

Similarly one gets the other identities. $\square$

The action of $\mathfrak{S}_m$ on $\mathcal{B}$

The algebra $\mathcal{B}$ is the tensor product of m copies of $\mathcal{H}(\ell, q)$, say $\mathcal{B} = \bigotimes_{i=1}^m \mathcal{B}_i$, where $\mathcal{B}_i$ is supported on the numbers in the i th row of t^λ . Therefore the symmetric group $\mathfrak{S}_m$ acts by permuting the factors. This will induce an action on the cohomology of $\mathcal{B}$, and we want to show that the fixed points under this action are precisely the stable elements. We shall realise this action through conjugation by elements of $\mathfrak{B}_\ell \wr \mathfrak{B}_m$. Namely, if $y \in \text{Hom}_{\mathcal{B}}(\Omega^i k, k)$ and $w \in N(S_\lambda)$, then we define $T_w \cdot y$ via

$$(T_w \cdot y)(x) = T_w(y(T_w^{-1}x)) = q^{l(w)}y(T_w^{-1}x) \quad (9)$$

for $x \in \Omega^i k$.

This is the usual formula for group actions on homomorphisms, but it is not the same as the formula for the action of the Hecke algebra on a dual space. Recall that for an $\mathcal{H}$ -module M , the dual space $\text{Hom}_k(M, k)$ is an $\mathcal{H}$ -module, using the anti-involution on $\mathcal{H}$ defined by

$$(T_g)^* = T_{g^{-1}}$$

and linear extension; see §4 in [5]. We write the action of $T_g \in \mathcal{H}$ on $\text{Hom}_k(M, k)$ as

$$T_g f(m) = f(T_{g^{-1}}m). \quad (10)$$

We view $\text{Hom}_{\mathcal{B}}(M, k)$ and $\text{Hom}_{\mathcal{H}}(M, k)$ as subspaces of M^* .

The advantage of (9) over (10) is that it gives a well-defined action of the group $\mathfrak{B}_\ell \wr \mathfrak{B}_m$ on $\text{Hom}_{\mathcal{B}}(\Omega^i k, k)$. Our strategy is as follows: The next proposition shows that the normal subgroup $\mathfrak{B}_\ell^{\times m}$ acts trivially, so that we are reduced to an action of $\mathfrak{B}_m$. Then Proposition 4.5 will show that the pure braid group acts trivially up to projective homomorphisms, so that we are reduced to an action of $\mathfrak{S}_m$ on $H^*(\mathcal{B}, k)$.

Proposition 4.2. *Let M be an $\mathcal{H}$ -module, $y \in \text{Hom}_{\mathcal{B}}(M, k)$ and $g \in S_\lambda$. Then $T_g \cdot y = y$.*

Proof. Using $T_i^{-1} = q^{-1}(T_i - q + 1)$ we see that T_g^{-1} belongs to $\mathcal{B}$, and then the statement is clear from the definition. $\square$

Lemma 4.3. *If $g \in N(S_\lambda) \cap \mathcal{D}_\lambda$, then the element $T_g T_{g^{-1}} \in \mathcal{H}(n, q)$ centralises $\mathcal{B}$. In particular, $T_{d_i}^2$ centralises $\mathcal{B}$.*

Proof. The algebra $\mathcal{B}$ is generated by the T_j for $s_j \in S_\lambda \cap S$. If $s_j \in S_\lambda \cap S$, then for some j' we have $s_j g = g s_{j'}$. Thus

$$\begin{aligned} T_g T_{g^{-1}} T_j &= T_g T_{g^{-1} s_j} = T_g T_{s_{j'} g^{-1}} = T_g T_{j'} T_{g^{-1}} \\ &= T_{g s_{j'}} T_{g^{-1}} = T_{s_j g} T_{g^{-1}} = T_j T_g T_{g^{-1}}. \end{aligned} \quad \square$$

The following lemma is copied from a standard argument in group cohomology. We shall need to use it not only when $\mathcal{Q} = \mathcal{B}$, but also when $\mathcal{Q}$ is a proper parabolic subalgebra of $\mathcal{B}$.

Lemma 4.4. *Let $\mathcal{Q}$ be a parabolic subalgebra of $\mathcal{H}$ and $\gamma \in \mathcal{H}$ be an element centralising $\mathcal{Q}$ and acting as the identity on k . Then $\gamma: \Omega^i k \rightarrow \Omega^i k$ is a $\mathcal{Q}$ -module map, and $\gamma - 1$ factors through a projective $\mathcal{Q}$ -module.*

Thus for $y \in \text{Hom}_{\mathcal{Q}}(\Omega^i k, k)$, the element $y \circ \gamma: x \mapsto y(\gamma x)$ of $\text{Hom}_{\mathcal{Q}}(\Omega^i k, k)$ differs from y by an element of $\text{PHom}_{\mathcal{Q}}(\Omega^i k, k)$, so that y and $y \circ \gamma$ represent the same element of $H^i(\mathcal{Q}, k)$.

Proof. Multiplication by γ commutes with the action of $\mathcal{Q}$ and therefore induces a map of chain complexes on the resolution (7), lifting the identity on k . Thus there is a chain homotopy to the identity, consisting of maps $h_i: P_i \rightarrow P_{i-1}$:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & P_i & \xrightarrow{\delta} & P_{i-1} & \xrightarrow{\delta} & P_{i-2} & \xrightarrow{\delta} & \cdots & \xrightarrow{\delta} & P_0 \\ & & \downarrow \gamma & \nearrow h_{i-1} & \downarrow \gamma & \nearrow h_{i-2} & \downarrow \gamma & & & & \downarrow \gamma \\ \cdots & \longrightarrow & P_i & \xrightarrow{\delta} & P_{i-1} & \xrightarrow{\delta} & P_{i-2} & \xrightarrow{\delta} & \cdots & \xrightarrow{\delta} & P_0 \end{array}$$

satisfying $\delta h_{i-1} + h_{i-2} \delta = \gamma - 1$. On $\Omega^i k = \ker(\delta: P_{i-1} \rightarrow P_{i-2})$, we have $\delta = 0$ and so $\gamma - 1: \Omega^i k \rightarrow \Omega^i k$ factors as $\Omega^i k \xrightarrow{h_{i-1}} P_i \xrightarrow{\delta} \Omega^i k$. $\square$

Proposition 4.5.

- (i) If $y \in \text{Hom}_{\mathcal{B}}(\Omega^i k, k)$, then $T_{d_i} \cdot y$ and $T_{d_i} y$ differ by an element of $\text{PHom}_{\mathcal{B}}(\Omega^i k, k)$

and hence represent the same element of $H^i(\mathcal{B}, k)$.

- (ii) The elements $T_{d_i}^2 \in \mathfrak{B}_m$ act trivially on $H^*(\mathcal{B}, k)$.

Proof. The elements T_{d_i} act as multiplication by $q^{l(d_i)}$ on k , and by Lemma 2.7 we have $q^{l(d_i)} = 1$. Thus $T_{d_i}^2$ also acts trivially on k . By Lemma 4.3, $T_{d_i}^2$ also centralises $\mathcal{B}$. From (9) and (10) we have $T_{d_i} \cdot y = y \circ T_{d_i}^{-1}$, while $T_{d_i} y = y \circ T_{d_i}$. Thus $T_{d_i} y = (T_{d_i} \cdot y) \circ T_{d_i}^2$, and (i) now follows from Lemma 4.4. Similarly for part (ii) we have $T_{d_i}^2 \cdot y = y \circ T_{d_i}^{-2}$, and we can again apply Lemma 4.4. $\square$

Theorem 4.6. The action of $\mathfrak{B}_\ell \wr \mathfrak{B}_m$ on $H^*(\mathcal{B}, k)$ factors through the surjection $\mathfrak{B}_\ell \wr \mathfrak{B}_m \rightarrow \mathfrak{S}_m$. Formulas (9) and (10) both describe the resulting action of $\mathfrak{S}_m$ on $H^*(\mathcal{B}, k)$.

Proof. This follows from Propositions 4.2 and 4.5. $\square$

Theorem 4.7. The image of the restriction map $H^*(\mathcal{H}, k) \rightarrow H^*(\mathcal{B}, k)$ consists precisely of the fixed points in $H^*(\mathcal{B}, k)$ under the action of the symmetric group $\mathfrak{S}_m$.

Proof. By Proposition 3.2, $H^*(\mathcal{H}, k)$ consists of the stable elements of $H^*(\mathcal{B}, k)$. If $y \in H^i(\mathcal{B}, k)$ is stable, then y is in particular a fixed point. For the converse, let $y \in H^i(\mathcal{B}, k)$ be a fixed point under $\mathfrak{S}_m$. We must show that then y is stable.

Take $d \in \mathcal{D}_\lambda^{-1}$, and let $\mathcal{Q}$ be the parabolic subalgebra of $\mathcal{H}$ corresponding to $S_{\lambda(d)}$. We must show that on restriction to $\mathcal{Q}$ we have $T_d y = q^{l(d)} y$ as an element in $H^i(\mathcal{Q}, k)$.

By the Lemma 4.8 below, there is $w \in \mathcal{D}_\lambda^{-1}$ such that $d S_\lambda d^{-1} = w S_\lambda w^{-1}$ and w centralises $S_{\lambda(d)}$, and such that, moreover, if $g = w^{-1} d$, then $q^{l(g)+l(w)} = q^{l(d)}$.

Since w centralises $\mathcal{Q}$, we know by applying Lemma 4.4 to $q^{-l(w)} T_{w^{-1}}$ that $T_w y = q^{l(w)} y$, as elements in $H^i(\mathcal{Q}, k)$. We must relate $T_d y$ and $T_w y$.

We have $d = wg$; therefore $(T_d)^* = (T_g)^*(T_w)^*$ and

$$T_dy(x) = y((T_d)^*x) = y((T_g)^*(T_w)^*x) = (T_gy)((T_w)^*x).$$

Since $g \in N(S_\lambda)$ and y is a fixed point, we know $T_gy = q^{l(g)}y$, so

$$T_dy(x) = q^{l(g)}(T_wy)(x) = q^{l(g)+l(w)}y(x).$$

By Lemma 4.8 this is precisely $q^{l(d)}y(x)$ as required. $\square$

Lemma 4.8. *Let $d \in \mathcal{D}_{\lambda,\lambda}$ and $Q = dS_\lambda d^{-1} \cap S_\lambda$. Then there is some $w \in C(Q) \cap \mathcal{D}_{\lambda,\lambda}$ such that $wS_\lambda w^{-1} = dS_\lambda d^{-1}$, and if $g = w^{-1}d$ then ℓ divides $l(g)$, and*

$$q^{l(d)} = q^{l(w)+l(g)} = q^{l(w)}.$$

Proof. Since the rows of t^λ are the support sets of S_λ , we see that the rows of $t^\lambda d^{-1}$ are the support sets of $dS_\lambda d^{-1}$. Hence the rows which are common to t^λ and $t^\lambda d^{-1}$ (not necessarily in the same places) are then precisely the support sets of Q . Note that the natural order on numbers induces a linear order on the rows of t^λ .

Let $R_{i_1} < R_{i_2} < \dots < R_{i_t}$ be the rows of t^λ such that if $R_{j_m} = R_{i_m} d^{-1}$, then the R_{j_m} are the support sets of the intersection Q . Then let $A_1 < A_2 < \dots < A_{r-t}$ be the rows of t^λ other than the R_{i_m} , and $B_1 < B_2 < \dots < B_{r-t}$ be the rows of t^λ other than the rows R_{j_m} .

We define now $g \in \mathfrak{S}_n$ by

$$B_i g = A_i, \quad R_{j_r} g = R_{i_r}.$$

Then g induces a permutation of the rows of t^λ , keeping each row in order. Hence $g \in N(S_\lambda) \cap \mathcal{D}_\lambda^{-1}$ (and therefore $g \in \mathcal{D}_{\lambda,\lambda}$). Define $w^{-1} = gd^{-1}$. Then

$$R_{j_r} w^{-1} = R_{j_r}, \quad B_i w^{-1} = A_i d^{-1}.$$

Hence $w^{-1} \in \mathcal{D}_\lambda$ and it fixes the support of Q pointwise and therefore it centralises Q . We will show that this satisfies the condition on the length. By Corollary 2.6, we know that $l(g)$ is divisible by ℓ .

So we must show that $l(d^{-1}) \equiv l(w^{-1}) \pmod{\ell}$. Let $b < c$. If one of $R_b d^{-1}$ or $R_c d^{-1}$ is a row of t^λ , then $\text{inv}(d^{-1})_{b,c}$ is divisible by ℓ , by Corollary 2.6. So we only need to consider such sets where neither $R_b d^{-1}$ nor $R_c d^{-1}$ is a row of t^λ , and the same reduction holds for the inversion set of w^{-1} .

Suppose that $b < c$ are such that neither $R_b d^{-1}$ nor $R_c d^{-1}$ are rows of t^λ . We show that there are $u < v$ such that $|\text{inv}(d^{-1})_{b,c}| = |\text{inv}(w^{-1})_{u,v}|$, where neither $R_u w^{-1}$ nor $R_v w^{-1}$ are rows of t^λ and where this produces a bijection between the relevant parts in the partitions of $\text{inv}(d^{-1})$ and $\text{inv}(w^{-1})$. Since the remaining parts of $\text{inv}(d^{-1})$ and $\text{inv}(w^{-1})$ have size divisible by ℓ , the lemma will follow.

By assumption, $R_b = A_{b'}$ and $R_c = A_{c'}$ for $b' < c'$. Now, $A_{b'} d^{-1} = B_{b'} w^{-1}$ and $A_{c'} d^{-1} = B_{c'} w^{-1}$. There are unique u, v such that $R_u = B_{b'}$ and $R_v = B_{c'}$, and then $u < v$. Let $(i, j) \in \text{inv}(d^{-1})_{b,c}$. Then $id^{-1} > jd^{-1}$, so

$$id^{-1} = i' w^{-1}, \quad jd^{-1} = j' w^{-1} \quad (i' \in B_{b'}, j' \in B_{c'}),$$

and then $(i', j') \in \text{inv}(w^{-1})_{u,v}$. The converse also holds, and this gives the required bijection between $\text{inv}(d^{-1})_{b,c}$ and $\text{inv}(w^{-1})_{u,v}$. $\square$

5. Invariant theory

Let k be a field of characteristic zero, and let $\mathcal{H} = \mathcal{H}(n, q)$ be the Hecke algebra of degree n with parameter q a primitive ℓ th root of unity, $\ell \geq 2$. Let $\mathcal{H}_1 = \mathcal{H}(\ell, q)$, let $n = \ell m + a$ with $0 \leq a < \ell$, and let $\mathcal{B} = \mathcal{H}_1^{\otimes m} \subseteq \mathcal{H}$. From the stable element computation in the last section, we have

$$H^*(\mathcal{H}, k) = H^*(\mathcal{B}, k)^{\mathfrak{S}_m}. \quad (11)$$

In this section we compute the right-hand side using invariant theory.

The structure of the algebra $H^*(\mathcal{H}_1, k)$ depends on whether $\ell = 2$ or $\ell > 2$. If $\ell > 2$, then

$$H^*(\mathcal{H}_1, k) = \Lambda(y) \otimes_k k[x], \quad |y| = 2\ell - 3, \quad |x| = 2\ell - 2, \quad (12)$$

while if $\ell = 2$, then

$$H^*(\mathcal{H}_1, k) = k[z], \quad |z| = 1. \quad (13)$$

Here $\Lambda(y)$ denotes an exterior algebra in one variable. Thus we have $y^2 = 0$.

To obtain $H^*(\mathcal{B}, k)$, we apply the version of the Künneth theorem given in Yoneda [15], which describes the Ext algebra of a tensor product of algebras as the graded tensor product of the Ext algebras, with the usual sign conventions.

The case $\ell > 2$.

We deal with the easy case $\ell > 2$ first. In this case, the Künneth theorem and (12) give

$$H^*(\mathcal{B}, k) = \Lambda(y_1, \dots, y_m) \otimes_k k[x_1, \dots, x_m], \quad |y_i| = 2\ell - 3, \quad |x_i| = 2\ell - 2. \quad (14)$$

The action of the symmetric group $\mathfrak{S}_m$ is by permutation of the variables $y_1, \dots, y_m$ and simultaneously the variables $x_1, \dots, x_m$. The invariants are given by a theorem of Solomon [12] as follows: Define a derivation d on $H^*(\mathcal{B}, k)$ via $d(x_i) = y_i$, $d(y_i) = 0$. This differential commutes with the action of $\mathfrak{S}_m$, so it sends invariants to invariants. Let $\sigma_1, \dots, \sigma_m$ be the elementary symmetric polynomials in $x_1, \dots, x_m$, so that

$$k[x_1, \dots, x_m]^{\mathfrak{S}_m} = k[\sigma_1, \dots, \sigma_m].$$

The main theorem of [12] shows in this case that

$$(\Lambda(y_1, \dots, y_m) \otimes_k k[x_1, \dots, x_m])^{\mathfrak{S}_m} = \Lambda(d\sigma_1, \dots, d\sigma_m) \otimes_k k[\sigma_1, \dots, \sigma_m]. \quad (15)$$

Combining (11), (14) and (15), we obtain

$$H^*(\mathcal{H}, k) = \Lambda(d\sigma_1, \dots, d\sigma_m) \otimes_k k[\sigma_1, \dots, \sigma_m].$$

A standard computation now shows that the Poincaré series for $H^*(\mathcal{H}, k)$ is as given in Theorem 1.1.

The case $\ell = 2$.

What is different in the case $\ell = 2$ is that the algebra (13) is not graded commutative, since there is an element of degree one that does not square to zero. So taking

into account the signs given by the Künneth theorem, we get

$$H^*(\mathcal{B}, k) = k\langle z_1, \dots, z_m \rangle / (z_i z_j + z_j z_i) \quad (i \neq j).$$

Here, the relations say that the variables anticommute, but do not square to zero.

In this case, we define a finite filtration on $H^*(\mathcal{B}, k)$ and pass to the associated graded. The filtration is given by setting $\mathcal{F}_i$ equal to the linear span of the monomials in the generating variables in which at most i of the variables appear with odd exponent. We have

$$k[z_1^2, \dots, z_m^2] = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_m = H^*(\mathcal{B}, k).$$

This is a multiplicative filtration, in the sense that $\mathcal{F}_i \mathcal{F}_j \subseteq \mathcal{F}_{i+j}$, so the associated graded

$$\mathrm{Gr} H^*(\mathcal{B}, k) = \bigoplus_{i=0}^m \mathcal{F}_i / \mathcal{F}_{i-1}$$

has a ring structure, where $\mathcal{F}_{-1}$ is interpreted as zero. In this ring, we write y_i for the image of $z_i \in \mathcal{F}_0$, so that $y_i^2 = 0$, and we write x_i for the image of $z_i^2 \in \mathcal{F}_2$. It is not hard to check that

$$\mathrm{Gr} H^*(\mathcal{B}, k) = \Lambda(y_1, \dots, y_m) \otimes_k k[x_1, \dots, x_m].$$

Since the filtration is invariant under the action of $\mathfrak{S}_m$, we have

$$\mathcal{F}_i^{\mathfrak{S}_m} = \mathcal{F}_i \cap H^*(\mathcal{B}, k)^{\mathfrak{S}_m},$$

and by Maschke's theorem we have

$$(\mathcal{F}_i / \mathcal{F}_{i-1})^{\mathfrak{S}_m} = \mathcal{F}_i^{\mathfrak{S}_m} / \mathcal{F}_{i-1}^{\mathfrak{S}_m}.$$

Setting

$$\mathrm{Gr} H^*(\mathcal{B}, k)^{\mathfrak{S}_m} = \bigoplus_{i=0}^m \mathcal{F}_i^{\mathfrak{S}_m} / \mathcal{F}_{i-1}^{\mathfrak{S}_m},$$

we again use Solomon's formula (15) to obtain

$$\mathrm{Gr} H^*(\mathcal{B}, k)^{\mathfrak{S}_m} = \Lambda(d\sigma_1, \dots, d\sigma_m) \otimes_k k[\sigma_1, \dots, \sigma_m]. \quad (16)$$

At this stage, it follows that the Poincaré series for $H^*(\mathcal{H}, k)$ with $\ell = 2$ is as given in Theorem 1.1, and it remains to ungrade the relations $(d\sigma_i)^2 = 0$ in the presentation for $\mathrm{Gr} H^*(\mathcal{B}, k)^{\mathfrak{S}_m}$ to give a presentation for $H^*(\mathcal{H}, k)$.

We define elements $v_i \in H^*(\mathcal{B}, k)$ ($1 \leq i \leq m$) to be the elementary symmetric functions in $z_1^2, \dots, z_r^2$, with image σ_i in $\mathrm{Gr} H^*(\mathcal{B}, k)$, so that $|v_i| = 2i$, and we define elements u_i ($1 \leq i \leq m$) by lifting $d\sigma_i$ in the obvious way to $H^*(\mathcal{B}, k)$. Namely,

$$u_i = \sum z_\alpha z_{\beta_1}^2 \cdots z_{\beta_{i-1}}^2,$$

where the sum is over indices that are all different and satisfy $\beta_1 < \dots < \beta_{i-1}$. Then $u_1^2 = v_1$, so v_1 is a redundant generator and may be removed from the list. More generally, for $i \geq 1$, u_i^2 and $u_i u_j + u_j u_i$ are polynomials of degree $2i - 1$, respectively $i + j - 1$, in $v_1, \dots, v_m$. Our next task is to determine these polynomials explicitly.

Lemma 5.1. *Let $\nu \geq \mu \geq 0$. Then*

$$(\nu - 2\mu) \binom{\nu}{\mu} + (\nu - 2\mu + 2) \binom{\nu}{\mu - 1} + \cdots + (\nu - 2) \binom{\nu}{1} + \nu \binom{\nu}{0} = (\nu - \mu) \binom{\nu}{\mu}.$$

Proof. This is an easy induction on μ using the fact that

$$(\nu - \mu) \binom{\nu}{\mu} = (\mu + 1) \binom{\nu}{\mu + 1}. \quad \square$$

Proposition 5.2. *Set $v_0 = 1$. Then for $1 \leq i \leq m$ we have*

$$\begin{aligned} u_i^2 &= \sum_{l=0}^{i-1} (2l+1) v_{i-l-1} v_{i+l} \\ &= v_{i-1} v_i + 3v_{i-2} v_{i+1} + 5v_{i-3} v_{i+2} + \cdots + (2i-1) v_{2i-1}, \end{aligned}$$

and for $1 \leq j < i \leq m$ we have

$$\begin{aligned} u_i u_j + u_j u_i &= 2 \sum_{l=0}^{j-1} (i-j+2l+1) v_{j-l-1} v_{i+l} \\ &= 2(i-j+1) v_{j-1} v_i + 2(i-j+3) v_{j-2} v_{i+1} + \cdots + 2(i+j-1) v_{i+j-1}. \end{aligned}$$

Proof. The formula for u_i^2 is really just the case $i = j$ of the second formula, after dividing both sides by 2, so we shall concentrate on the second formula assuming $1 \leq j \leq i \leq m$.

Recall that $u_i = \sum z_\alpha z_{\beta_1}^2 \cdots z_{\beta_{i-1}}^2$, where the sum is over indices that are all different and satisfy $\beta_1 < \cdots < \beta_{i-1}$. Since $z_1, \dots, z_m$ anticommute, we only get non-zero contributions to $u_i u_j + u_j u_i$ if the z_α terms have the same index, and then we get a linear combination of monomials of the form $z_{\alpha_1}^2 \cdots z_{\alpha_s}^2 z_{\beta_1}^4 \cdots z_{\beta_t}^4$ where $s = i + j - 2t - 1$. The coefficient of such a monomial in $u_i u_j + u_j u_i$ is

$$2 \frac{s!}{(i-t-1)!(j-t-1)!} = 2(i-t) \binom{i+j-2t-1}{j-t-1},$$

while the coefficient in $v_{j-1-l} v_{i+l}$ is

$$\frac{s!}{(i+l-t)!(j-l-t-1)!} = \binom{i+j-2t-1}{j-l-t-1}.$$

The proposition then follows from the identity

$$\sum_{l \geq 0} (i-j+2l+1) \binom{i+j-2t-1}{j-l-t-1} = (i-t) \binom{i+j-2t-1}{j-t-1}.$$

This identity is obtained from Lemma 5.1 by setting

$$\nu = i + j - 2t - 1 \quad \text{and} \quad \mu = j - t - 1. \quad \square$$

It follows from Proposition 5.2 that after ungrading the invariants given in (16), we obtain

$$H^*(\mathcal{H}, k) \cong H^*(\mathcal{B}, k)^{\mathfrak{S}_m} \cong k \langle u_1, \dots, u_m, v_2, \dots, v_m \rangle / (R),$$

where (R) is the set of relations given in Theorem 1.1.

6. Some Hecke algebras for other types

We consider the other infinite families of finite Coxeter groups of type B_n ($n \geq 2$) and D_n ($n \geq 4$). These groups are given by $W(B_n) = \mathfrak{S}_2 \wr \mathfrak{S}_n$ and $W(D_n) = W(B_n) \cap \mathcal{A}_{2n}$, where $\mathcal{A}_{2n}$ is the alternating group of degree $2n$. There is a Hecke algebra defined for any finite Coxeter system; see, for example, §68 of Curtis and Reiner [3] or §8.5 of Geck and Pfeiffer [8]. The Hecke algebra of type A_{n-1} is the algebra $\mathcal{H}(n, q)$ considered in previous sections. The Hecke algebra of a decomposable Coxeter system is isomorphic to the tensor product of the Hecke algebras for the indecomposable factors.

For type B_n the Hecke algebra involves two parameters q and Q , and we write $\mathcal{H}(B_n, Q, q)$. An explicit presentation can be found in §3 of Dipper and James [5]. There is a natural inclusion of $\mathcal{H}(n, q)$ in $\mathcal{H}(B_n, Q, q)$ which takes the elements T_i ($1 \leq i \leq n-1$) to the elements T_{s_i} of [5], ignoring the element T_t .

For type D_n there is just one parameter, and we write $\mathcal{H}(D_n, q)$. An explicit presentation in this case can be found in the introduction to Hu [9]. Again there is a natural inclusion of $\mathcal{H}(n, q)$ in $\mathcal{H}(D_n, q)$, which takes the elements T_i ($1 \leq i \leq n-1$) to the elements of the same name in [9], ignoring T_0 .

First we treat type B_n for $n \geq 2$. We set $f_n(Q, q) = \prod_{i=1}^{n-1} (Q + q^i)$.

Theorem 6.1 (Dipper-James, [5, Theorem 4.17]¹). *If $f_n(Q, q)$ is invertible in k , then the Hecke algebra $\mathcal{H}(B_n, Q, q)$ is Morita equivalent to the algebra*

$$\prod_{j=0}^n \mathcal{H}(j, q) \otimes_k \mathcal{H}(n-j, q).$$

It may be deduced from the character theory of the Hecke algebra (see, for example, §§5.5 and 10.3 of Geck and Pfeiffer [8], and especially the remark at the bottom of page 165) that the trivial module corresponds to the pair of partitions $([n], \emptyset)$, and is therefore a representation of the factor Morita equivalent to $\mathcal{H}(n, q)$ corresponding to the term $j = n$ in the above decomposition. Therefore we have the following.

Theorem 6.2. *If $f_n(Q, q)$ is invertible in k , then the natural inclusion $\mathcal{H}(n, q) \rightarrow \mathcal{H}(B_n, Q, q)$ induces an isomorphism*

$$H^*(\mathcal{H}(B_n, Q, q), k) \cong H^*(\mathcal{H}(n, q), k).$$

Remark 6.3. If $Q = q$ is an ℓ th root of unity and k is a field of characteristic zero, then the invertibility condition for $f_n(Q, q)$ is equivalent to the statement that ℓ is odd, so that Case 1 of Theorem 1.1 gives the structure of $H^*(\mathcal{H}(B_n, Q, q), k)$ in this case.

Next, we treat type D_n for $n \geq 4$. We set $f_n(q) = 2 \prod_{i=1}^{n-1} (1 + q^i)$. The next theorem is implicit in Theorems 3.6 and 3.7 of Pallikaros [11] and is made explicit in Hu [9].

¹There is a misprint in the statement of this theorem in [5]: they write $(n, n-a)$ but they mean $(a, n-a)$, where their a is our j .

Theorem 6.4. *If $f_n(q)$ is invertible in k and n is odd, then $\mathcal{H}(D_n, q)$ is Morita equivalent to the algebra*

$$\prod_{j=(n+1)/2}^n \mathcal{H}(j, q) \otimes_k \mathcal{H}(n-j, q).$$

The corresponding theorem for n even was proved by Hu [9]:

Theorem 6.5. *If $f_n(q)$ is invertible in k and n is even, then $\mathcal{H}(D_n, q)$ is Morita equivalent to the algebra*

$$A(n/2) \times \prod_{j=(n+1)/2}^n \mathcal{H}(j, q) \otimes_k \mathcal{H}(n-j, q),$$

where $A(n/2)$ is an explicitly described algebra.

In both cases, the trivial module again corresponds to the pair of partitions $([n], \emptyset)$ (see §§5.6 and 10.4 of [8]). It is therefore a representation of the factor Morita equivalent to $\mathcal{H}(n, q)$ corresponding to the term $j = n$ in the decomposition. Therefore we have the following.

Theorem 6.6. *If $f_n(q)$ is invertible in k , then the natural inclusion*

$$\mathcal{H}(n, q) \rightarrow \mathcal{H}(D_n, q)$$

induces an isomorphism

$$H^*(\mathcal{H}(D_n, q), k) \cong H^*(\mathcal{H}(n, q), k).$$

Remark 6.7. Again, if q is an ℓ th root of unity and k is a field of characteristic zero, then the invertibility condition for $f_n(q)$ is equivalent to the statement that ℓ is odd, so that Case 1 of Theorem 1.1 gives the structure of $H^*(\mathcal{H}(D_n, q), k)$ in this case. We do not know the answers for type B_n and D_n for ℓ even, or for the Hecke algebras of exceptional type.

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David Benson `bensondj@maths.abdn.ac.uk`

Institute of Mathematics, University of Aberdeen, Aberdeen AB24 3UE, United Kingdom

Karin Erdmann `erdmann@maths.ox.ac.uk`

Mathematical Institute, 24–29 St. Giles’, Oxford OX1 3LB, United Kingdom

Aram Mikaelian `Aram.Mikaelian@maths.ox.ac.uk`

Mathematical Institute, 24–29 St. Giles’, Oxford OX1 3LB, United Kingdom