

## THE $RO(G)$ -GRADED SERRE SPECTRAL SEQUENCE

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### *Abstract*

In this paper the Serre spectral sequence of Moerdijk and Svensson is extended from Bredon cohomology to  $RO(G)$ -graded cohomology for finite groups  $G$ . Special attention is paid to the case  $G = \mathbb{Z}/2$  where the spectral sequence is used to compute the cohomology of certain projective bundles and loop spaces.

### 1. Introduction

In [Bre67], Bredon created equivariant homology and cohomology theories of  $G$ -spaces, now called Bredon homology and Bredon cohomology, which yield the usual singular homology and cohomology theories when the group acting is taken to be the trivial group. In [LMM81], a cohomology theory for  $G$ -spaces is constructed that is graded on  $RO(G)$ , the Grothendieck ring of virtual representation of  $G$ . This  $RO(G)$ -graded theory extends Bredon cohomology in the sense that  $H^{\underline{n}}(X) = H_{B_r}^n(X)$  when  $\underline{n}$  is the trivial  $n$ -dimensional representation of  $G$ .

Many of the usual tools for computing cohomology have their counterparts in the  $RO(G)$ -graded setting. These include Mayer-Vietoris sequences, Künneth theorem, suspension isomorphisms, etc. Missing from the  $RO(G)$  computational tool box was an equivariant version of the Serre spectral sequence associated to a fibration  $F \rightarrow E \rightarrow B$ . Also, perhaps partially because of a lack of this spectral sequence, the theory of equivariant characteristic classes has not yet been developed.

The main result of this paper is to extend the spectral sequence of a  $G$ -fibration given in [MS93] from Bredon cohomology to the  $RO(G)$ -graded theory with special attention to the case  $G = \mathbb{Z}/2$ . The restriction to  $G = \mathbb{Z}/2$  is for two reasons. The first is that there is a map from Voevodsky's motivic cohomology and  $RO(\mathbb{Z}/2)$ -graded equivariant cohomology, and so one can try to answer questions in motivic cohomology by considering instead the relevant equivariant cohomology. The second reason is that the general algebra of Mackey functors is extremely complicated for arbitrary groups (even compact Lie), yet the  $G = \mathbb{Z}/2$  case is manageable.

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A  $p$ -dimensional real  $\mathbb{Z}/2$ -representation  $V$  decomposes as

$$V = (\mathbb{R}^{1,0})^{p-q} \oplus (\mathbb{R}^{1,1})^q = \mathbb{R}^{p,q}$$

where  $\mathbb{R}^{1,0}$  is the trivial representation and  $\mathbb{R}^{1,1}$  is the nontrivial 1-dimensional representation. Thus the  $RO(\mathbb{Z}/2)$ -graded theory is a bigraded theory, one grading measuring dimension and the other measuring the number of “twists”. In this case, we write  $H^V(X; M) = H^{p,q}(X; M)$  for the  $V^{\text{th}}$  graded component of the  $RO(\mathbb{Z}/2)$ -graded equivariant cohomology of  $X$  with coefficients in a Mackey functor  $M$ . Here is the spectral sequence:

**Theorem 1.1.** *If  $f: E \rightarrow X$  is a fibration of  $\mathbb{Z}/2$  spaces, then for every  $r \in \mathbb{Z}$  and every Mackey functor  $M$  there is a natural spectral sequence with*

$$E_2^{p,q} = H^{p,0}(X; \mathcal{H}^{q,r}(f; M)) \Rightarrow H^{p+q,r}(E; M).$$

This is a spectral sequence that takes as inputs the Bredon cohomology of the base space with coefficients in the local coefficient system  $\mathcal{H}^{q,r}(f; M)$  and converges to the  $RO(\mathbb{Z}/2)$ -graded cohomology of the total space.

This is really a family of spectral sequences, one for each integer  $r$ . If the Mackey functor  $M$  is a ring Mackey functor, then this family of spectral sequences is equipped with a tri-graded multiplication. If  $a \in H^{p,0}(X; \mathcal{H}^{q,r}(f; M))$  and

$$b \in H^{p',0}(X; \mathcal{H}^{q',r'}(f; M)),$$

then  $a \cdot b \in H^{p+p',0}(X; \mathcal{H}^{q+q',r+r'}(f; M))$ . There is also an action of  $H^{*,*}(pt; M)$  so that if  $\alpha \in H^{q',r'}(pt; M)$  and  $a \in H^{p,0}(X; \mathcal{H}^{q,r}(f; M))$ , then

$$\alpha \cdot a \in H^{p,0}(X; \mathcal{H}^{q+q',r+r'}(f; M)).$$

Under certain connectivity assumptions on the base space, the local coefficients  $\mathcal{H}^{q,r}(f; M)$  are constant, and the spectral sequence becomes the following. This result is restated and proved as Theorem 3.2.

**Theorem 1.2.** *If  $X$  is equivariantly 1-connected and  $f: E \rightarrow X$  is a fibration of  $\mathbb{Z}/2$  spaces with fiber  $F$ , then for every  $r \in \mathbb{Z}$  and every Mackey functor  $M$  there is a spectral sequence with*

$$E_2^{p,q} = H^{p,0}(X; \underline{H}^{q,r}(F; M)) \Rightarrow H^{p+q,r}(E; M).$$

The coefficient systems  $\mathcal{H}^{q,r}(f; M)$  and  $\underline{H}^{q,r}(F; M)$  that appear in the spectral sequence are explicitly defined in the next section. They are the equivariant versions of the usual local coefficient systems that arise when working with the usual Serre spectral sequence.

This spectral sequence is rich with information about the fibration involved, even in the case of the trivial fibration  $id: X \rightarrow X$ . In this case, the  $E_2$  page takes the form  $E_2^{p,q} = H^{p,0}(X; \underline{H}^{q,r}(pt; M)) \Rightarrow H^{p+q,r}(X; M)$ . Set  $M = \underline{\mathbb{Z}/2}$  and consider the case  $r = 1$ . Then  $H^{p,0}(X; \underline{H}^{q,r}(pt; \underline{\mathbb{Z}/2})) = 0$  if  $q \neq 0, 1$ . The case  $q = 0$  gives  $H^{p,0}(X; \underline{H}^{0,1}(pt; \underline{\mathbb{Z}/2})) = H^{p,0}(X; \underline{\mathbb{Z}/2})$ , and if  $q = 1$ ,

$$H^{p,0}(X; \underline{H}^{1,1}(pt; \underline{\mathbb{Z}/2})) = H_{non-eq}^p(X^G; \underline{\mathbb{Z}/2}).$$

The spectral sequence then has just two non-zero rows as shown in Figure 1 below.

0	0	0	
0	0	0	
0	0	0	
$H^0_{non-eq}(X^G)$	$H^1_{non-eq}(X^G)$	$H^2_{non-eq}(X^G)$	$\dots$
$H^{0,0}(X)$	$H^{1,0}(X)$	$H^{2,0}(X)$	$\dots$

Figure 1: The  $r = 1$  spectral sequence for  $id: X \rightarrow X$ .

As usual, the two row spectral sequence yields the following curious long exact sequence:

$$0 \rightarrow H^{0,0}(X) \rightarrow H^{0,1}(X) \rightarrow 0 \rightarrow H^{1,0}(X) \rightarrow H^{1,1}(X) \rightarrow H^0_{non-eq}(X^G) \rightarrow H^{2,0}(X) \rightarrow H^{2,1}(X) \rightarrow H^1_{non-eq}(X^G) \rightarrow \dots$$

Now, to any equivariant vector bundle  $f: E \rightarrow X$ , there is an associated equivariant projective bundle  $\mathbb{P}(f): \mathbb{P}(E) \rightarrow X$  whose fibers are lines in the fibers of the original bundle. Applying the above spectral sequence to this new bundle yields the following result, which appears later as Theorem 3.7.

**Theorem 1.3.** *If  $X$  is equivariantly 1-connected and  $f: E \rightarrow X$  is a vector bundle with fiber  $\mathbb{R}^{n,m}$  over the base point, then the spectral sequence of Theorem 3.2 for the bundle  $\mathbb{P}(f): \mathbb{P}(E) \rightarrow X$  with constant  $M = \underline{\mathbb{Z}/2}$  coefficients “collapses”.*

Here, when we say the spectral sequence collapses, we do not mean it collapses in the usual sense. Each fibration  $f: E \rightarrow X$  maps to the trivial fibration  $id: X \rightarrow X$  in an obvious way. Naturality then provides a map from the spectral sequence for  $id: X \rightarrow X$  to the spectral sequence for  $f: E \rightarrow X$ . In the above theorem, the spectral sequence “collapses” in the sense that the only nonzero differentials are those arising from the trivial fibration  $id: X \rightarrow X$ .

In non-equivariant topology, the Leray-Serre spectral sequence gives rise to a description of characteristic classes of vector bundles. Consider the universal bundle  $E_n \rightarrow G_n$  over the Grassmannian of  $n$ -planes in  $\mathbb{R}^\infty$ . Forming the associated projective bundle  $\mathbb{P}(E_n) \rightarrow G_n$  yields a fiber bundle with fiber  $\mathbb{R}P^\infty$ . Applying the Leray-Serre spectral sequence to this projective bundle yields characteristic classes of  $E_n$  as the image of the cohomology classes  $1, z, z^2, \dots \in H^*_{non-eq}(\mathbb{R}P^\infty; \mathbb{Z}/2)$  under the transgressive differentials. Since this universal bundle classifies vector bundles,

characteristic classes of arbitrary bundles can be defined as pullbacks of the characteristic classes,  $c_i \in H^i(G_n; \mathbb{Z}/2)$ , of the universal bundle. It would be nice to adapt this construction to the  $\mathbb{Z}/2$  equivariant setting. However, the equivariant space  $G_n((\mathbb{R}^{2,1})^\infty) = G_n(\mathcal{U}) = G_n$  is not 1-connected, and so the spectral sequence is not as easy to work with. It seems that there is no way to avoid using local coefficient systems in this setting.

Section 2 provides some of the definitions and basics that are needed for this paper. The main theorem is stated and proved in section 3, making use of some technical homotopical details that are provided in section 4. In section 5, the spectral sequence is then applied to compute the cohomology of a projective bundle  $\mathbb{P}(E)$  associated to a vector bundle  $E \rightarrow X$ . Section 6 provides an application of the  $RO(\mathbb{Z}/2)$ -graded Serre spectral sequence to loop spaces on certain spheres.

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## 2. Preliminaries

The section contains some of the basic machinery and notation that will be used throughout the paper. In this section, let  $G$  be any finite group.

A  $G$ -CW complex is a  $G$ -space  $X$  with a filtration  $X^{(n)}$  where  $X^{(0)}$  is a disjoint union of  $G$ -orbits and  $X^{(n)}$  is obtained from  $X^{(n-1)}$  by attaching cells of the form  $G/H_\alpha \times \Delta^n$  along maps  $f_\alpha: G/H_\alpha \times \partial\Delta^n \rightarrow X^{(n-1)}$ . The space  $X^{(n)}$  is referred to as the  $n$ -skeleton of  $X$ . Such a filtration on a space  $X$  is called a cell structure for  $X$ .

Given a  $G$ -representation  $V$ , let  $D(V)$  and  $S(V)$  denote the unit disk and unit sphere, respectively, in  $V$  with action induced by that on  $V$ . A  $\text{Rep}(G)$ -complex is a  $G$ -space  $X$  with a filtration  $X^{(n)}$  where  $X^{(0)}$  is a disjoint union of  $G$ -orbits and  $X^{(n)}$  is obtained from  $X^{(n-1)}$  by attaching cells of the form  $D(V_\alpha)$  along maps  $f_\alpha: S(V_\alpha) \rightarrow X^{(n-1)}$  where  $V_\alpha$  is an  $n$ -dimensional real representation of  $G$ . The space  $X^{(n)}$  is again referred to as the  $n$ -skeleton of  $X$ , and the filtration is referred to as a cell structure.

Let  $\Delta_G(X)$  be the category of equivariant simplices of the  $G$ -space  $X$ . Explicitly, the objects of  $\Delta_G(X)$  are maps  $\sigma: G/H \times \Delta^n \rightarrow X$ . A morphism from  $\sigma$  to  $\tau: G/K \times \Delta^m \rightarrow X$  is a pair  $(\varphi, \alpha)$  where  $\varphi: G/H \rightarrow G/K$  is a  $G$ -map and  $\alpha: \Delta^n \rightarrow \Delta^m$  is a simplicial operator such that  $\sigma = \tau \circ (\varphi \times \alpha)$ .

Let  $\Pi_G(X)$  be the fundamental groupoid of  $X$ . Explicitly, the objects of  $\Pi_G(X)$  are maps  $\sigma: G/H \rightarrow X$  and a morphism from  $\sigma$  to  $\tau: G/K \rightarrow X$  is a pair  $(\varphi, \alpha)$  where  $\varphi: G/H \rightarrow G/K$  is a  $G$ -map and  $\alpha$  is a  $G$ -homotopy class of paths from  $\sigma$  to  $\tau \circ \varphi$ .

There is a forgetful functor  $\pi: \Delta_G(X) \rightarrow \Pi_G(X)$  that sends  $\sigma: G/H \times \Delta^n \rightarrow X$  to  $\sigma: G/H \rightarrow X$  by restricting to the last vertex  $e^n$  of  $\Delta^n$ . A morphism  $(\varphi, \alpha)$  in  $\Delta_G(X)$  is restricted to  $(\varphi, \alpha)$  in  $\Pi_G(X)$  by restricting  $\alpha$  to the linear path from  $\alpha(e^n)$  to  $e^m$  in  $\Delta^m$ . There is a further forgetful functor to the orbit category  $\mathcal{O}(G)$ , which

will also be denoted by  $\pi$ , as shown below.

$$\Delta_G(X) \xrightarrow{\pi} \Pi_G(X) \xrightarrow{\pi} \mathcal{O}(G)$$

A coefficient system on  $X$  is a functor  $M: \Delta_G(X)^{op} \rightarrow \mathcal{A}b$ . We say that the coefficient system  $M$  is a local coefficient system if it factors through the forgetful functor to  $\Pi_G(X)^{op}$  (up to isomorphism). If  $M$  further factors through  $\mathcal{O}(G)^{op}$ , then we call  $M$  a constant coefficient system.

For the precise definition of a Mackey functor for  $G = \mathbb{Z}/2$ , the reader is referred to [May96] or [Dug05]. A summary of the important aspects of a  $\mathbb{Z}/2$  Mackey functor is given here. The data of a Mackey functor are encoded in a diagram of abelian groups like the one below.

$$\begin{array}{ccc} \downarrow (t^*) & \begin{array}{c} \xrightarrow{i_*} \\ \xleftarrow{i^*} \end{array} & \\ M(\mathbb{Z}/2) & & M(e) \end{array}$$

The maps must satisfy the following four conditions.

1.  $(t^*)^2 = id$
2.  $t^*i^* = i^*$
3.  $i_*(t^*)^{-1} = i_*$
4.  $i^*i_* = id + t^*$

According to [LMM81], each Mackey functor  $M$  uniquely determines an  $RO(G)$ -graded cohomology theory characterized by

1.  $H^n(G/H; M) = \begin{cases} M(G/H) & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$
2. The map  $H^0(G/K; M) \rightarrow H^0(G/H; M)$  induced by  $i: G/H \rightarrow G/K$  is the transfer map  $i^*$  in the Mackey functor.

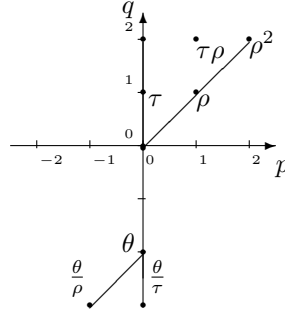
Given a Mackey functor  $M$ , a  $G$ -representation  $V$ , and a  $G$ -space  $X$ , we can form a coefficient system  $\underline{H}^V(X; M)$ . This coefficient system is determined on objects by  $\underline{H}^V(X; M)(G/H) = H^V(X \times G/H; M)$  with maps induced by those in  $\mathcal{O}(G)$ .

In this paper,  $G$  will usually be  $\mathbb{Z}/2$  and the Mackey functor will almost always be constant  $M = \underline{\mathbb{Z}/2}$  which has the following diagram.

$$\begin{array}{ccc} \downarrow (id) & \begin{array}{c} \xrightarrow{0} \\ \xleftarrow{id} \end{array} & \\ \mathbb{Z}/2 & & \mathbb{Z}/2 \end{array}$$

With these constant coefficients, the  $RO(\mathbb{Z}/2)$ -graded cohomology of a point is given by the picture in Figure 2.

Every lattice point in the picture that is inside the indicated cones represents a copy of the group  $\mathbb{Z}/2$ . The top cone is a polynomial algebra on the elements  $\rho \in H^{1,1}(pt; \underline{\mathbb{Z}/2})$  and  $\tau \in H^{0,1}(pt; \underline{\mathbb{Z}/2})$ . The element  $\theta$  in the bottom cone is infinitely

Figure 2:  $H^{*,*}(pt; \underline{\mathbb{Z}/2})$ 

divisible by both  $\rho$  and  $\tau$ . The cohomology of  $\mathbb{Z}/2$  is easier to describe:

$$H^{*,*}(\mathbb{Z}/2; \underline{\mathbb{Z}/2}) = \mathbb{Z}/2[t, t^{-1}]$$

where  $t \in H^{0,1}(\mathbb{Z}/2; \underline{\mathbb{Z}/2})$ . Details can be found in [Dug05] and [Car99].

Given a  $G$ -map  $f: \overline{E} \rightarrow X$  and a Mackey functor  $M$ , we can define a coefficient system  $\mathcal{H}^{q,r}(-, M): \Delta_G(X)^{op} \rightarrow \mathcal{A}b$  by taking cohomology of pullbacks:

$$\mathcal{H}^{q,r}(f, M)(\sigma) = H^{q,r}(\sigma^*(E), M).$$

In [MS93], it is shown that this is a local coefficient system when  $f$  is a  $G$ -fibration.

Given a  $G$ -fibration  $f: E \rightarrow X$ , we can define a functor  $\Gamma_f: \Delta_G(X) \rightarrow \mathcal{T}op$ . On objects,  $\Gamma(\sigma) = \sigma^*(E)$ . On morphisms,  $\Gamma(\varphi, \alpha) = \overline{\varphi \times \alpha}$ , where  $\varphi \times \alpha$  is the map of total spaces in the diagram

$$\begin{array}{ccc} \sigma^*(E) & \xrightarrow{\overline{\varphi \times \alpha}} & \tau^*(E) \\ \downarrow & & \downarrow \\ G/H \times \Delta^n & \xrightarrow{\varphi \times \alpha} & G/K \times \Delta^m. \end{array}$$

Let  $f: E \rightarrow X$  be a  $G$ -fibration over an equivariantly 1-connected  $G$ -space  $X$  with base point  $x \in X$  and let  $F = f^{-1}(x)$ . Define a constant coefficient system  $\underline{H}^{q,r}(F; M)$  as follows:  $\underline{H}^{q,r}(F; M)(G/H) = H^{q,r}((G/H) \times F; M)$  and if  $\varphi: G/H \rightarrow G/K$  is a  $G$ -map, then  $\underline{H}^{q,r}(F; M)(\varphi) = (\varphi \times id)^*$ . It is this coefficient system that appears in the spectral sequence of Theorem 3.7.

### 3. Construction of the Spectral Sequence

Unlike in ordinary topology, the equivariant Serre spectral sequence for a fibration  $f: E \rightarrow X$  will not be deduced from lifting a cellular filtration of  $X$  to one on  $E$ . Instead, the spectral sequence is a special case of the one for a homotopy colimit. Recall (from [Dug08] for example) that given a cohomology theory  $\mathcal{E}^*$  and a diagram of spaces  $D: I \rightarrow \mathcal{T}op_G$ , there is a natural spectral sequence as follows:

$$E_2^{p,q} = H^p(I^{op}; \mathcal{E}^q(D)) \implies \mathcal{E}^{p+q}(\text{hocolim } D). \quad (1)$$

For the case  $I = \Delta_G(X)$ , we know from [MS93] that the cohomology of  $\Delta_G(X)^{op}$  is the same as Bredon cohomology. For a  $G$ -fibration  $f: E \rightarrow B$ , we can consider the diagram  $\Gamma_f: \Delta_G(X) \rightarrow \mathcal{J}op_G$  that sends  $\sigma: G/H \times \Delta^n \rightarrow X$  to the pullback  $\sigma^*(E)$ . We then have the following technical lemma, whose proof is given in the next section where it appears as Lemma 4.4.

**Lemma 3.1.** *The composite map  $\text{hocolim}_{\Delta_G(X)} \Gamma_f \rightarrow \text{colim}_{\Delta_G(X)} \Gamma_f \rightarrow E$  is a weak equivalence.*

Here is the desired spectral sequence.

**Theorem 3.2.** *If  $f: E \rightarrow X$  is a fibration of  $G$  spaces, then for every  $V \in RO(G)$  and every Mackey Functor  $M$  there is a natural spectral sequence with*

$$E_2^{p,q}(M, V) = H^{p,0}(X; \mathcal{H}^{V+q}(f; M)) \Rightarrow H^{V+p+q}(E; M).$$

*Proof.* The homotopy colimit spectral sequence of (1) associated to  $\Gamma_f$  and the cohomology theory  $H^{V+*}(-; M)$  takes the form

$$E_2^{p,q}(M, V) = H^p(\Delta_G(X); \mathcal{H}^{V+q}(\Gamma_f; M)) \Rightarrow H^{V+p+q}(\text{hocolim}(\Gamma_f); M).$$

By [MS93, Theorem 3.2] and Lemma 4.4, this spectral sequence becomes

$$E_2^{p,q}(M, V) = H^{p,0}(X; \mathcal{H}^{V+q}(f; M)) \Rightarrow H^{V+p+q}(E; M).$$

Naturality of this spectral sequence follows from the naturality of the homotopy colimit spectral sequence.  $\square$

The standard multiplicative structure on the spectral sequence is given by the following theorem. Recall that the analogue of tensor product for Mackey functors is the box product, denoted by  $\square$ . See, for example, [FL04] for a full description of the box product.

**Theorem 3.3.** *Given a  $G$ -fibration  $f: E \rightarrow X$ , Mackey functors  $M$  and  $M'$  and  $V, V' \in RO(G)$ , there is a natural pairing of the spectral sequences of 3.2*

$$E_r^{p,q}(M, V) \otimes E_r^{p',q'}(M'; V') \rightarrow E_r^{p+p',q+q'}(M \square M'; V + V')$$

converging to the standard pairing

$$\cup: H^*(E; M) \otimes H^*(E; M') \rightarrow H^*(E; M \square M').$$

Furthermore, the pairing of  $E_2$  terms agrees, up to a sign  $(-1)^{p'q}$ , with the standard pairing

$$\begin{array}{c} H^{p,0}(X; \mathcal{H}^{V+q}(f; M)) \otimes H^{p',0}(X; \mathcal{H}^{V'+q'}(f; M')) \\ \cup \downarrow \\ H^{p+p',0}(X; \mathcal{H}^{V+V'+p+p'+q+q'}(f; M \square M')) \end{array}$$

*Proof.* This is a straightforward application of [MS93, Theorem 4.1].  $\square$

*Remark 3.4.* If  $M$  is a ring Mackey functor, then the product  $M \square M \rightarrow M$  gives a pairing of spectral sequences

$$E_r^{p,q}(M, V) \otimes E_r^{p',q'}(M, V') \rightarrow E_r^{p+p',q+q'}(M, V + V').$$

*Remark 3.5.* Since every  $G$ -fibration  $f: E \rightarrow X$  maps to the  $G$ -fibration  $id: X \rightarrow X$ , every spectral sequence of Theorem 3.2 admits a map from the spectral sequence for the identity of  $X$ .

**Lemma 3.6.** *If  $f: E \rightarrow X$  is a  $G$ -fibration over an equivariantly 1-connected based  $G$ -space  $X$ , then any local coefficient system  $\mathcal{A}$  on  $X$  is constant.*

*Proof.* Choose a base point  $x \in X$ . Then  $x$  can be considered as a map  $x: G/G \rightarrow X$ . Denote by  $x_H$  the point  $x$  thought of as a  $G/H$  point. That is  $x_H = x \circ \pi$  where  $\pi: G/H \rightarrow G/G$  is the projection. Notice that if  $\varphi: G/H \rightarrow G/K$ , then  $x_K = x_H \circ \varphi$ .

Define a constant coefficient system  $\bar{\mathcal{A}}: \mathcal{O}(G) \rightarrow \mathcal{A}b$  by  $\bar{\mathcal{A}}(G/H) = \mathcal{A}(x_H)$  and  $\bar{\mathcal{A}}(\varphi: G/H \rightarrow G/K) = \mathcal{A}(\varphi, c_x)$ , where  $c_x$  is the constant path from  $x_H$  to  $x_K$ . The claim is that  $\mathcal{A}$  factors through  $\bar{\mathcal{A}}$  up to isomorphism.

For any object  $\sigma: G/H \rightarrow X$ , the connectivity assumptions ensure that there is one homotopy class of paths from  $\sigma$  to  $x_H$ . Let  $\beta_\sigma$  be a representative path.

For any morphism  $(\varphi, \alpha)$  in  $\Pi_G(X)$  from  $\sigma$  to  $\tau$ , one then has the following commutative diagram:

$$\begin{array}{ccc} \mathcal{A}(\tau) & \xrightarrow{\mathcal{A}(\varphi, \alpha)} & \mathcal{A}(\sigma) \\ \mathcal{A}(id, \beta_\tau) \uparrow & & \uparrow \mathcal{A}(id, \beta_\sigma) \\ \mathcal{A}(x_K) & \xrightarrow{\mathcal{A}(\varphi, c_x)} & \mathcal{A}(x_H). \end{array}$$

Now, each of the vertical maps is an isomorphism since, for example, the path  $\beta_\sigma$  has the inverse path  $\bar{\beta}_\sigma$  and each of the compositions  $\beta_\sigma * \bar{\beta}_\sigma$  and  $\bar{\beta}_\sigma * \beta_\sigma$  are homotopic to constant paths. The same is true for  $\tau$ .

Moreover,  $(\bar{\mathcal{A}} \circ \pi)(\sigma) = \bar{\mathcal{A}}(G/H) = \mathcal{A}(x_H)$ , and  $(\bar{\mathcal{A}} \circ \pi)(\varphi, \alpha) = \bar{\mathcal{A}}(\varphi) = \mathcal{A}(\varphi, c_x)$ .

This means that the above diagram exhibits an isomorphism from  $\bar{\mathcal{A}} \circ \pi \rightarrow \mathcal{A}$ .  $\square$

**Theorem 3.7.** *If  $X$  is equivariantly 1-connected and  $f: E \rightarrow X$  is a fibration of  $G$  spaces with fiber  $F$ , then for every  $V \in RO(G)$  and every Mackey Functor  $M$  there is a spectral sequence with  $E_2^{p,q} = H^{p,0}(X; \underline{H}^{V+q}(F; M)) \Rightarrow H^{V+p+q}(E; M)$ .*

*Proof.* By Theorem 3.2 and the above Lemma 3.6, it suffices to show that for the local coefficient system  $\mathcal{A} = \mathcal{H}^{V+q}(f; M)$ , the associated constant coefficient  $\bar{\mathcal{A}}$  is  $\underline{H}^{V+q}(F; M)$ .

Notice that  $x_H = x \circ \pi$  and so  $x_H^* E = (x \circ \pi)^* E = \pi^* x^* E = \pi^* F = (G/H) \times F$ .

We then have

$$\bar{\mathcal{A}}(G/H) = \mathcal{A}(x_H) = H^{V+q}(x_H^* E) = H^{V+q}((G/H) \times F) = \underline{H}^{V+q}(F; M)(G/H). \quad \square$$

## 4. Homotopical Considerations

What follows is an equivariant version of some of the statements about homotopical decompositions in [Dug08]. These are needed for the proof of Lemma 4.4, and may



also be of independent interest. Because of the technical nature of this material, the uninterested reader may skip ahead to the next section.

Consider the category  $G\text{-}sSet$  of equivariant simplicial sets. The objects are simplicial sets endowed with a  $G$ -action and all of the face and degeneracy maps respect the action. The morphisms are equivariant versions of the usual simplicial maps. An  $n$ -simplex of an equivariant simplicial set  $X$  is an element  $\sigma \in X_n$ . Alternatively, we can think of an equivariant  $n$ -simplex as an equivariant simplicial map  $\sigma: G/H \times \Delta^n \rightarrow X$ . Both points of view can be useful.

$G\text{-}sSets$  has a model category structure in which fibrations and weak equivalences are defined in terms of the fixed sets, that is  $f$  is a fibration if for all subgroups  $H$  the simplicial map  $f^H$  is a fibration, and similarly for weak equivalences. The cofibrations are then the maps with the appropriate lifting properties.

Let  $D: I \rightarrow G\text{-}sSet$  be a diagram of equivariant simplicial sets. Suppose there is a map  $\text{colim}_I D \rightarrow X$ . For each simplex  $\sigma \in X$  let  $F(D)_\sigma$  denote the category whose objects are pairs  $[i, \alpha \in (D_i)_n]$  such that the map  $D_i \rightarrow X$  sends  $\alpha$  to  $\sigma$ . A map in  $F(D)_\sigma$  from  $[i, \alpha \in (D_i)_n]$  to  $[j, \beta \in (D_j)_n]$  is a map  $i \rightarrow j$  such that  $D_i \rightarrow D_j$  sends  $\alpha$  to  $\beta$ . Then  $F(D)_\sigma$  is called the fiber category of  $D$  over  $\sigma$ .

**Proposition 4.1.** *Suppose that  $D: I \rightarrow G\text{-}sSet$  and  $X$  are as above, and assume that for every  $n \geq 0$  and every  $\sigma \in X_n$  the fiber category  $F(D)_\sigma$  is contractible. Then the map  $\text{hocolim } D \rightarrow X$  is a weak equivalence of equivariant simplicial sets.*

*Proof.* The proof is nearly identical to that of Proposition 19.9 in [Dug08]. The key facts are that for a bisimplicial set  $B$ , the geometric realization satisfies  $|B|^H = |B^H|$  and that weak equivalences are determined by their fixed sets.  $\square$

Now, suppose that  $D: I \rightarrow \mathcal{J}op_G$  is a diagram of  $G$ -spaces. Suppose we have a map  $p: \text{colim } D \rightarrow X$ . Then for each  $n \geq 0$ , each subgroup  $H \leq G$  and each  $\sigma: G/H \times \Delta^n \rightarrow X$  define the fiber category  $F(D)_\sigma$  of  $D$  over  $\sigma$  to be the category with objects pairs  $[i, \alpha: G/H \times \Delta^n \rightarrow D_i]$  such that  $p \circ \alpha = \sigma$ . A map from  $[i, \alpha: G/H \times \Delta^n \rightarrow D_i]$  to  $[j, \beta: G/H \times \Delta^n \rightarrow D_j]$  is a map  $i \rightarrow j$  making the obvious diagram commute.

**Proposition 4.2.** *In the above setting, suppose that for each  $n \geq 0$ ,  $H \leq G$ , and  $\sigma: G/H \times \Delta^n \rightarrow X$  the category  $F(D)_\sigma$  is contractible. Then the composite*

$$\text{hocolim } D \rightarrow \text{colim } D \rightarrow X$$

*is a weak equivalence.*

*Proof.* A map  $\sigma: G/H \times \Delta^n \rightarrow X$  is equivalent to a map  $\bar{\sigma}: \Delta^n \rightarrow X^H$ . Thus we can reduce to looking at the fixed sets. But, this is exactly Theorem 19.2 in [Dug08]. The condition that  $F(D)_\sigma$  is contractible is equivalent to the condition that  $F(D)_{\bar{\sigma}}$  is contractible. Thus the composite is a weak equivalence on fixed sets, and so is an equivariant weak equivalence.  $\square$

There is a related simplicial version of the above theorem. Assume that in addition there is a diagram  $\tilde{D}: I \rightarrow G\text{-}sSet$  and a natural isomorphism  $\phi_i: |\tilde{D}_i| \rightarrow D_i$ . For each  $\sigma: G/H \times \Delta^n \rightarrow X$  define the category  $\tilde{F}(D)_\sigma$  to have objects pairs  $[i, G/H \times \Delta^n \rightarrow \tilde{D}_i]$  such that the composite  $|G/H \times \Delta^n| \rightarrow |\tilde{D}_i| \rightarrow D_i \rightarrow X$  is  $\sigma$ .

The morphisms are as expected. Here,  $\Delta_s^n \in sSet$  is the  $n$ -simplex. The following is a refinement of the previous theorem.

**Proposition 4.3.** *In the above setting, suppose that for each  $n \geq 0$ ,  $H \leq G$ , and  $\sigma: G/H \times \Delta^n \rightarrow X$  the category  $\tilde{F}(D)_\sigma$  is contractible. Then the composite*

$$\text{hocolim } D \rightarrow \text{colim } D \rightarrow X$$

*is a weak equivalence.*

*Proof.* Again, we can reduce to looking at fixed sets, this time invoking Proposition 19.3 in [Dug08].  $\square$

For a  $G$ -fibration  $f: E \rightarrow X$ , we can consider the diagram  $\Gamma_f: \Delta_G(X) \rightarrow \mathcal{J}op$  that sends  $\sigma: G/H \times \Delta^n \rightarrow X$  to the pullback  $\sigma^*(E)$ . We then have the following technical lemma used in the construction of the spectral sequence.

**Lemma 4.4.** *The map  $\text{hocolim}_{\Delta_G(X)} \Gamma_f \rightarrow \text{colim}_{\Delta_G(X)} \Gamma_f \rightarrow E$  is a weak equivalence.*

*Proof.* Consider the diagram  $D: \Delta_G(X) \rightarrow G\text{-}sSet$  sending  $([k], \alpha: G/H \times \Delta^k)$  to the simplicial set obtained as the pull back

$$G/H \times \Delta_s^n \xrightarrow{\sim} S(G/H \times \Delta^n) \rightarrow S(X) \leftarrow S(E),$$

where  $S(-)$  is the singular functor.

There is a map of diagrams  $|D| \rightarrow \Gamma_f$  which is an objectwise weak equivalence since  $f$  is a fibration. We are reduced to showing that  $\text{hocolim } |D| \rightarrow \text{colim } |D| \rightarrow E$  is a weak equivalence.

For each  $n \geq 0$ ,  $H \leq G$ , and  $\sigma: G/H \times \Delta^n \rightarrow E$ , the category  $\tilde{F}(D)_\sigma$  is contractible. This is due to the presence of an initial object associated to the map  $f \circ \sigma: G/H \times \Delta^n \rightarrow X$ . By Proposition 4.3, we are done.  $\square$

## 5. Cohomology of Projective Bundles

In this chapter, we specialize exclusively to the case where  $G = \mathbb{Z}/2$ .

To any equivariant vector bundle  $f: E \rightarrow X$ , there is an associated equivariant projective bundle  $\mathbb{P}(f): \mathbb{P}(E) \rightarrow X$  whose fibers are lines in the fibers of the original bundle. Applying the spectral sequence of Theorem 3.7 to this new bundle yields the following result:

**Theorem 5.1.** *If  $X$  is equivariantly 1-connected and  $f: E \rightarrow X$  is a vector bundle with fiber  $\mathbb{R}^{n,m}$  over the base point, then the spectral sequence of Theorem 3.7 for the bundle  $\mathbb{P}(f): \mathbb{P}(E) \rightarrow X$  with constant  $M = \underline{\mathbb{Z}/2}$  coefficients “collapses”.*

Here, the spectral sequence “collapses” in the sense that the only nonzero differentials are those arising from the trivial fibration  $id: X \rightarrow X$ . One can also view this collapsing by identifying the terms of the spectral sequence (at least after the  $E_2$  page) as a certain tensor product. Letting  $E_*^{*,*}$  be the spectral sequence associated to the fibration  $\mathbb{P}(f): \mathbb{P}(E) \rightarrow X$  above and  $F_*^{*,*}$  be the one for  $id: X \rightarrow X$ , then for  $n \geq 2$  there is an isomorphism  $E_n^{p,q} \cong (F_n^{*,*} \otimes H^{*,*}(\mathbb{P}(R^{n,m})))^{p,q}$ , where the tensor product is taken over  $H^{*,*}(pt)$ .

*Remark 5.2.* Despite the spectral sequence  $E_*^{*,*}$  behaving like  $F_*^{*,*} \otimes H_*^{*,*}(\mathbb{P}(R^{n,m}))$ , it need not be the case that  $H_*^{*,*}(\mathbb{P}(E))$  and  $H_*^{*,*}(X) \otimes H_*^{*,*}(\mathbb{P}(R^{n,m}))$  are isomorphic.

The projective spaces involved here have actions on them induced by the action in the fibers. Briefly, we denote by  $\mathbb{R}\mathbb{P}_{tw}^n = \mathbb{P}(\mathbb{R}^{n+1, \lfloor \frac{n+1}{2} \rfloor})$ , the equivariant space of lines in  $\mathbb{R}^{n+1, \lfloor \frac{n+1}{2} \rfloor}$ . For the other projective spaces, we simply denote the space of lines in  $\mathbb{R}^{n,m}$  by  $\mathbb{P}(\mathbb{R}^{n,m})$ . These projective spaces themselves are studied in more detail in [Kro09]. Some of the relevant results are restated here for convenience.

**Lemma 5.3.**  $\mathbb{P}(\mathbb{R}^{p,q}) \cong \mathbb{P}(\mathbb{R}^{p,p-q})$ .

**Lemma 5.4.** As a  $H_*^{*,*}(pt)$ -module,  $H_*^{*,*}(\mathbb{P}(\mathbb{R}^{p,q}))$  is free with a single generator in dimensions  $(0,0), (1,1), (2,1), (3,2), (4,2), \dots, (2q,q), (2q+1,q), \dots, (p-1,q)$ .

**Theorem 5.5.**  $H_*^{*,*}(\mathbb{R}\mathbb{P}_{tw}^\infty) = H_*^{*,*}(pt)[a,b]/(a^2 = \rho a + \tau b)$ , where  $\deg(a) = (1,1)$  and  $\deg(b) = (2,1)$ .

**Proposition 5.6.**  $H_*^{*,*}(\mathbb{P}(\mathbb{R}^{p,q}))$  is a truncated polynomial algebra over  $H_*^{*,*}(pt)$  on generators in dimensions  $(1,1), (2,1), (2q+1,q), (2q+2,q), \dots, (p-1,q)$ , subject to the relations determined by the restriction of  $H_*^{*,*}(\mathbb{R}\mathbb{P}_{tw}^\infty)$  to  $H_*^{*,*}(\mathbb{P}(\mathbb{R}^{p,q}))$ .

With these observations, we are ready to prove Theorem 5.1.

*Proof of Theorem 5.1.* By Lemma 5.3 we need only consider the case where  $n \geq m/2$ .

First, consider the case where the vector bundle has fiber  $\mathbb{R}^{n, \lfloor \frac{n}{2} \rfloor}$  over the base point.

If  $n$  is odd, consider the vector bundle  $E \oplus \mathbb{R}^{1,0} \rightarrow X$ , and if  $n$  is even consider  $E \oplus \mathbb{R}^{1,1} \rightarrow X$ . In either case, denote this new bundle by  $E \oplus L$ . Taking the associated projective bundles gives a diagram

$$\begin{array}{ccccc}
 \mathbb{R}\mathbb{P}_{tw}^{n-1} & \longrightarrow & \mathbb{R}\mathbb{P}_{tw}^n & \longrightarrow & pt \\
 \downarrow & & \downarrow & \xleftarrow{s} & \downarrow \\
 \mathbb{P}(E) & \longrightarrow & \mathbb{P}(E \oplus L) & \xrightarrow{\mathbb{P}(f)} & X \\
 \downarrow & & \mathbb{P}(f) \downarrow \uparrow s & & \downarrow id \\
 X & \xrightarrow{id} & X & \xrightarrow{id} & X
 \end{array}$$

Here and below  $pt$  is the one point set with trivial  $\mathbb{Z}/2$  action. The map  $s$  above is the canonical splitting that assigns to each point  $x$  in  $X$  the line given by the trivial factor in  $E \oplus L$ . It is important to note that this is indeed an equivariant splitting in both the case of  $\mathbb{R}^{1,0}$  and  $\mathbb{R}^{1,1}$ . This diagram yields maps between the spectral sequences associated to these three bundles over  $X$ . Let us consider the  $r = 1$  spectral sequence for  $\mathbb{P}(E \oplus L)$ . This is the sequence with  $E_2$ -page given by

$$E_2^{p,q} = H^{p,0}(X; \underline{H}^{q,1}(\mathbb{R}\mathbb{P}_{tw}^n)) \Rightarrow H^{p+q,1}(\mathbb{P}(E \oplus L)).$$

This spectral sequence is generated as an algebra over  $H_*^{*,*}(pt)$  by the classes  $a \in H^{0,0}(X; H^{1,1}(\mathbb{R}\mathbb{P}_{tw}^n))$  and  $b \in H^{0,0}(X; H^{2,1}(\mathbb{R}\mathbb{P}_{tw}^n))$ . To see that the spectral sequence collapses, we need only see that these classes  $a$  and  $b$  have trivial differentials.

The splitting  $s$  induces a map  $s^*$  from the  $r = 1$  spectral sequences associated to the bundle  $f : \mathbb{P}(E) \rightarrow X$  to the one for the trivial bundle  $id : X \rightarrow X$ . This map sends the class  $a \in H^{0,0}(X; H^{1,1}(\mathbb{R}\mathbb{P}_{tw}^n))$  to  $0 \in H^{0,0}(X; H^{1,1}(pt))$  since  $s^* : H^{*,*}(\mathbb{R}\mathbb{P}_{tw}^n) \rightarrow H^{*,*}(pt)$  is the projection. Thus  $s^*da = d(s^*a) = 0$ . However,  $s^*$  gives an isomorphism between coefficient systems  $\underline{H}^{0,1}(\mathbb{R}\mathbb{P}_{tw}^n) \cong_{s^*} \underline{H}^{0,1}(pt)$ . Observe:

$$\begin{aligned} (\underline{H}^{0,1}(\mathbb{R}\mathbb{P}_{tw}^n))(\mathbb{Z}/2) &= H^{0,1}(\mathbb{Z}/2 \times \mathbb{R}\mathbb{P}_{tw}^n) \\ &= [\mathbb{Z}/2 \times \mathbb{R}\mathbb{P}_{tw}^n, K(\mathbb{Z}/2(1), 0)]_{\mathbb{Z}/2} \\ &= [\mathbb{R}\mathbb{P}_{tw}^n, K(\mathbb{Z}/2, 0)]_e \\ &= H_{non-eg}^0(\mathbb{R}\mathbb{P}^n; \mathbb{Z}/2) \\ &\cong_{s^*} H_{non-eg}^0(pt; \mathbb{Z}/2) \\ &= H^{0,1}(pt) \\ &= (\underline{H}^{0,1}(pt))(\mathbb{Z}/2). \end{aligned}$$

Here,  $K(A(q), p)$  is the representing space for  $H^{p,q}(-; \underline{A})$ .

Also,  $\underline{H}^{0,1}(\mathbb{R}\mathbb{P}_{tw}^n)(G/G) = H^{0,1}(\mathbb{R}\mathbb{P}_{tw}^n) \cong_{s^*} H^{0,1}(pt)$ . It now follows that since  $s^*da = 0$ , it must be that  $da = 0$ .

Now, from the relation  $a^2 = \rho a + \tau b$  we get that  $0 = d(a^2) = \rho da + \tau db$ . Hence  $\tau db = 0$ . But,  $\tau : \underline{H}^{1,1}(\mathbb{R}\mathbb{P}_{tw}^n) \rightarrow \underline{H}^{1,2}(\mathbb{R}\mathbb{P}_{tw}^n)$  is an isomorphism. Thus  $db = 0$ .

Now,  $\mathbb{P}(i)^*(a) = a$  and  $\mathbb{P}(i)^*(b) = b$ , where  $\mathbb{P}(i)^* : \mathbb{P}(E \oplus L) \rightarrow \mathbb{P}(E)$ . Thus,  $d(a) = 0$  and  $d(b) = 0$  in the spectral sequence for  $\mathbb{P}(E)$  as well. Therefore the spectral sequence ‘‘collapses,’’ in the sense that all differentials are zero, except for the part of the spectral sequence corresponding to the trivial fibration  $id : X \rightarrow X$ .

For the other projective spaces, we can proceed inductively. Fix  $m$  and induct on  $n \geq m/2$ . The base case is exactly the argument above. For the inductive step, in going from  $\mathbb{P}(\mathbb{R}^{n,m})$  to  $\mathbb{P}(\mathbb{R}^{n+1,m})$ , a single new cohomology generator  $c_{n,m}$  appears in degree  $(n, m)$ , according to Lemma 5.4. Also, by Proposition 5.6, we have  $ac_{n-1,m} = \tau c_{n,m}$ , where  $c_{n-1,m}$  is the highest dimensional cohomology generator in

$$H^{*,*}(\mathbb{P}(\mathbb{R}^{n,m})).$$

Then in the spectral sequence we have  $d(ac_{n-1,m}) = \tau d(c_{n,m})$ . But, by induction,  $d(ac_{n-1,m}) = 0$ . Since  $\tau$  is still an injection in the range we are working in, it must be that  $d(c_{n,m}) = 0$ . This gives the desired collapsing of the spectral sequence.  $\square$

In fact, we can deduce even more about such a projective bundle.

**Corollary 5.7.** *If  $f : E \rightarrow X$  is a vector bundle with  $X$  equivariantly 1-connected with fiber  $\mathbb{R}^{n,m}$  over the base point, then  $\mathbb{P}(F)^* : H^{*,*}(X) \rightarrow H^{*,*}(\mathbb{P}(E))$  is an injection.*

*Proof.* By the preceding theorem, there is a natural injection of the spectral sequence for  $id_X$  into the one for  $\mathbb{P}(f)$ , thus an injection on the filtrations. We get an injection on the  $E_\infty$  terms, and thus, by the following lemma, an injection  $H^{*,*}(X) \rightarrow H^{*,*}(\mathbb{P}(E))$ .  $\square$

**Lemma 5.8.** *Let  $f : E_r^{p,q} \rightarrow F_r^{p,q}$  be a map of first quadrant spectral sequences, converging to  $A_{p+q}$  and  $B_{p+q}$  respectively, which is an injection for every  $p, q$ , and  $r$ . Then  $f$  induces an injection  $\tilde{f} : A_{p+q} \rightarrow B_{p+q}$ .*

*Proof.* Fix  $n$ . Then there is a filtration  $0 \subseteq A_0 \subseteq \cdots \subseteq A_n$  with  $A_i/A_{i-1} \cong E_\infty^{n-i,i}$ . Similarly, there is a filtration  $0 \subseteq B_0 \subseteq \cdots \subseteq B_n$  with  $B_i/B_{i-1} \cong F_\infty^{n-i,i}$ . Notice that  $A_0 = E_\infty^{n,0}$  and  $B_0 = F_\infty^{n,0}$ . Thus  $f_0: A_0 \rightarrow B_0$  is injective. Induction starts.

Suppose that  $\tilde{f}_i: A_i \rightarrow B_i$  is injective. We also know that

$$f_{i+1}: A_{i+1}/A_i \rightarrow B_{i+1}/B_i$$

is injective. We have a map  $\tilde{f}_{i+1}: A_{i+1} \rightarrow B_{i+1}$  that restricts to  $\tilde{f}_i$  and we want to see that  $\tilde{f}_{i+1}$  is injective. Suppose  $\tilde{f}_{i+1}(a) = 0$ . Then  $f_{i+1}([a]) = 0$ . But this map is injective, so  $a \in A_i$ . Since  $\tilde{f}_{i+1}$  restricts to  $\tilde{f}_i$  on  $A_i$ , we have that  $\tilde{f}_{i+1}(a) = \tilde{f}_i(a) = 0$ . As  $\tilde{f}_i$  is injective,  $a = 0$ . By induction,  $\tilde{f}_n = \tilde{f}$  is injective.  $\square$

## 6. Equivariant Adams-Hilton Construction

This section provides a  $G$ -representation complex structure to the space of Moore loops of a  $G$ -representation space  $Y$  under certain assumptions on the types of cells involved.

Let  $(Y, *)$  be a based  $G$ -space. Let  $\Omega^M(Y, *) \subseteq \text{Map}([0, \infty), Y) \times [0, \infty)$  denote the subspace of all pairs  $(\varphi, r)$  for which  $\varphi(0) = *$  and  $\varphi(t) = *$  for  $t \geq r$ . The space  $\Omega^M(Y, *)$  is the space of Moore loops of  $Y$ . It inherits a  $G$ -action given by  $g \cdot (\varphi, r) = (g \cdot \varphi, r)$ , where  $(g \cdot \varphi)(t) = g \cdot \varphi(t)$ . (The action of  $G$  on both  $\mathbb{R}$  and  $[0, \infty)$  are assumed to be trivial, so this is the usual diagonal action of  $G$  on a product restricted to the subspace of Moore loops.)

**Proposition 6.1.**  $\Omega(Y, *)$  is a  $G$ -deformation retract of  $\Omega^M(Y, *)$ .

*Proof.* The argument from nonequivariant topology adapts effortlessly to the equivariant setting. What follows is essentially the argument from Proposition 5.1.1 of [MS93].

First consider  $\tilde{\Omega}(Y, *) \subseteq \Omega^M(Y, *)$ , the subspace of all  $(\varphi, t)$  with  $t \geq 1$ . A deformation retraction,  $H$ , of  $\Omega^M(Y, *)$  onto  $\tilde{\Omega}(Y, *)$  is given by the following formulae:

$$H(s, (\varphi, r)) = \begin{cases} (\varphi, r+s) & \text{when } r+s \leq 1 \\ (\varphi, 1) & \text{when } r \leq 1 \text{ and } r+s \geq 1 \\ (\varphi, r) & \text{when } r \geq 1. \end{cases}$$

Now a deformation retraction  $K$  from  $\tilde{\Omega}(Y, *)$  to  $\Omega(Y, *)$  is given by the formula

$$K(s, (\varphi, r)) = (\varphi_s, (1-s)r+s),$$

where  $\varphi_s(t) = \varphi\left(\frac{r}{(1-s)r+s}t\right)$ .

Notice that  $H$  and  $K$  are both equivariant deformation retractions.  $\square$

Given any based  $G$ -space  $(X, *)$ , one can form the free  $G$ -monoid  $M(X, *)$  just as in the nonequivariant setting. As a space,  $M(X, *) = \coprod X^n / \sim$ . Here,  $\sim$  is the equivalence relation generated by all the relations of the form

$$(x_1, \dots, x_{i-1}, *, x_{i+1}, \dots, x_n) \sim (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n).$$

The  $G$ -action on  $M(X, *)$  is inherited from the diagonal action of  $G$  on each of the products  $X^n$ . Note that since the basepoint  $*$  is fixed by  $G$ , this action factors through the relation  $\sim$ .

This free  $G$ -monoid on  $(X, *)$  enjoys the universal property that any based  $G$ -map  $f: X \rightarrow M$ , where  $M$  is any topological  $G$ -monoid with  $f(*) = e$ , can be extended uniquely to a  $G$ -monoid map  $\tilde{f}: M(X, *) \rightarrow M$ .

$\Omega^M(Y, *)$  is a topological  $G$ -monoid. The loop concatenation product respects the  $G$  action in the sense that  $g \cdot ((\varphi, r) * (\psi, s)) = ((g \cdot \varphi) * (g \cdot \psi), r + s)$ . The point  $(*, 0)$ , where  $*$  denotes the constant loop at the base point of  $Y$ , is the identity element.

Let  $(X, *)$  be a based  $G$ -space. The equivariant James map is the  $G$ -map  $J: (X, *) \rightarrow (\Omega\Sigma X, *)$  given by  $J(x)(t) = [t, x] \in \Sigma X$ . Here, the  $G$ -action is given by  $(g \cdot J(x))(t) = [t, g \cdot x]$ . Compose this  $G$ -map with the inclusion of  $\Omega\Sigma X$  into  $\Omega^M\Sigma X$  to obtain a  $G$ -map  $J: (X, *) \rightarrow (\Omega^M\Sigma X, *)$  that does not carry the base point to the identity. Let  $\hat{X} = X \amalg [0, 1]/(1 \sim *)$  and define an extension  $\hat{J}$  of  $J$  to  $\hat{X}$  by  $\hat{J}(s) = (*, s)$  for  $s \in [0, 1]$  where  $*$  denotes the constant path at the basepoint. Note that  $\hat{X}$  and  $X$  are based  $G$ -homotopy equivalent if  $X$  is a  $G$ -CW complex. By now considering 0 to be the basepoint of  $\hat{X}$ ,  $\hat{J}$  is now a based  $G$ -map. This now extends uniquely to a  $G$ -map  $\bar{J}: M(\hat{X}, 0) \rightarrow \Omega^M\Sigma X$ . This is the map in James' theorem.

James' Theorem states that if  $X$  is a connected CW complex, the map  $\bar{J}: M(\hat{X}, 0) \rightarrow \Omega^M\Sigma X$  is a homotopy equivalence. See [CM95] for a proof of James' theorem. This can be easily extended to the equivariant setting in the case that  $X$  has connected fixed sets.

**Theorem 6.2** (Equivariant James Theorem). *If  $X$  is a connected  $G$ -CW complex with  $X^H$  connected for all  $H \leq G$ , the  $G$ -map  $\bar{J}: M(\hat{X}, 0) \rightarrow \Omega^M\Sigma X$  is a  $G$ -homotopy equivalence.*

*Proof.* Observe that  $M(\hat{X}, 0)^H = M(\hat{X}^H, 0)$  and  $(\Omega^M\Sigma X)^H = \Omega^M\Sigma(X^H)$ . Now,

$$\bar{J}^H: M(\hat{X}, 0)^H \rightarrow (\Omega^M\Sigma X)^H$$

is a homotopy equivalence by James' theorem since  $X^H$  is a connected CW complex by assumption. Thus  $\bar{J}$  is a  $G$ -homotopy equivalence.  $\square$

The space  $J(X) = M(\hat{X}, 0)$  is called the James construction.  $J(X)$  is a free, associative, unital  $G$ -monoid. If the basepoint  $*$  of  $X$  is a vertex, then  $J(X)$  has a natural  $G$ -CW complex structure coming from the decomposition of  $X^n$  as a product  $G$ -CW complex. Thus  $J(X)$  has the following properties:

1. every element  $v \in J(X)$  has a unique expression  $v = *$  or  $v = x_1x_2 \cdots x_n$ ,  $x_i \in X \setminus *$  for  $1 \leq i \leq n$ .
2.  $x_1 \cdots x_n$  is contained in a unique cell of  $J(X)$ , the cell  $C_1 \times \cdots \times C_n$  where  $x_i \in \text{Int}(C_i)$ ,  $1 \leq i \leq n$ , so that no indecomposable cell contains decomposable points, and
3. non-equivariantly, the cell complex has the form of a tensor algebra  $T(C_{\#}(X))$ , where the sub complex  $C_{\#}(X)$  is exactly the indecomposables, and the generating cells in dimension  $i$  are in bijective correspondence with the cells in dimension  $i + 1$  of  $\Sigma X$ .

*Remark 6.3.* The James construction given above uses the trivial one dimensional representation for loops and suspensions. This differs from the construction in [Ryb91] where the nontrivial one dimensional  $\mathbb{Z}/2$  representation is used.

Nonequivariantly, we have the Adams-Hilton construction as follows. Let  $Y$  be a CW complex with a single vertex  $*$  and no 1-cells. Then there is a model for  $\Omega^M(Y)$  which is a free associative monoid, with  $*$  the only vertex, the generating cells in dimension  $i$  are in 1-1 correspondence with the  $(i+1)$ -dimensional cells of  $Y$ , and it satisfies (2) above. This will generalize to the following equivariant version.

**Theorem 6.4** (Equivariant Adams-Hilton). *Let  $Y$  be a  $\text{Rep}(G)$ -complex with a single vertex  $*$ , no 1-cells, and the only cells in higher dimensions are  $V \oplus 1$ -cells where  $V$  is a real representation of  $G$  with all fixed sets of  $S^V$  connected. Then there is a model for  $\Omega^M(Y)$  which is a free associative monoid, with  $*$  the only vertex, the generating cells in dimension  $V$  are in 1-1 correspondence with the  $(V \oplus 1)$ -dimensional cells of  $Y$ , and it satisfies (2) above.*

For the case  $G = \mathbb{Z}/2$  the theorem becomes the following.

**Theorem 6.5** ( $\mathbb{Z}/2$ -Equivariant Adams-Hilton). *Let  $Y$  be a  $\text{Rep}(\mathbb{Z}/2)$ -complex with a single vertex  $*$  and no  $(n, n)$ -cells or  $(n, n-1)$ -cells for  $n \geq 1$ . Then there is a model for  $\Omega^M(Y)$  which is a free associative monoid, with  $*$  the only vertex, the generating cells in dimension  $(p, q)$  are in 1-1 correspondence with the  $(p+1, q)$ -dimensional cells of  $Y$ , and it satisfies (2) above.*

*Proof.* With these restrictions on the types of cells in our  $\text{Rep}(G)$ -complex, the proof of the Adams-Hilton theorem in [CM95] adapts to the equivariant case. For example, in the base case of the inductive argument, one has that the 2-skeleton  $Y^{(2)} = \bigvee_{V_\alpha} S^{V_\alpha \oplus 1} = \Sigma^1(\bigvee_{V_\alpha} S^{V_\alpha})$ . Since each  $S^{V_\alpha}$  has connected fixed sets, the equivariant James construction applies and the result is immediate.

For the inductive step, the prolongation construction is equivariant and so by [Wan80] the quasifibration arguments extend to the equivariant setting. This allows the remainder of the argument to adapt to the equivariant setting.  $\square$

One application of this model is the computation of  $H^{*,*}(\Omega S^{p,q}; \underline{\mathbb{Z}/2})$  when  $S^{p,q}$  has a connected fixed set and  $p \geq 2$ .

**Proposition 6.6.** *If  $S^{p,q}$  is equivariantly 1-connected, then  $H^{*,*}(\Omega S^{p,q}; \underline{\mathbb{Z}/2})$  is an exterior algebra over  $H^{*,*}(pt; \underline{\mathbb{Z}/2})$  on generators  $a_1, a_2, \dots$ , where  $a_i$  has bidegree  $((p-1) \cdot 2^{i-1}, q \cdot 2^{i-1})$ .*

*Sketch of proof.* For each value of  $p$  and  $q$ , the argument is similar, so let's focus on the case  $p = 4$  and  $q = 2$  to compute  $H^{*,*}(\Omega S^{4,2})$ .

Now, since the fixed set of  $S^{4,2}$  is connected, by the Adams-Hilton construction we have an upper bound for the reduced cohomology of the loop space given in Figure 3.

In the spectral sequence of the filtration, it is clear that all differentials must be zero, and so Figure 3 reveals the structure of  $H^{*,*}(\Omega S^{4,2})$  as a free  $H^{*,*}(pt)$ -module. Denote the generators of  $H^{3 \cdot 2^{i-1}, 2 \cdot 2^{i-1}}(\Omega S^{p,q}; \underline{\mathbb{Z}/2})$  by  $a_i$ .

Consider the path-loop fibration  $\Omega S^{4,2} \rightarrow PS^{4,2} \rightarrow S^{4,2}$ . The base is 1-connected, so we can apply the spectral sequence of Theorem 3.7, which will converge to the cohomology of a point since the total space  $PS^{4,2} \simeq pt$ . Consider first the  $r = 2$  portion of the spectral sequence.

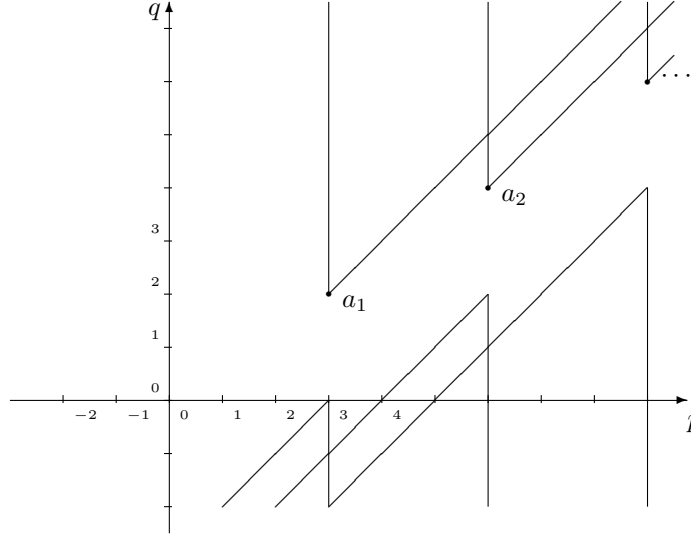


Figure 3: The  $E_1$  page of the cellular spectral sequence for  $\Omega S^{4,2}$ .

To fill in the entries in the spectral sequence, the Mackey functors  $\underline{H}^{q,2}(\Omega S^{4,2})$  need to be computed for various values of  $q$ . These can be obtained from the module structure above. The calculations yield that  $\underline{H}^{0,2}(\Omega S^{4,2}) = \underline{H}^{3,2}(\Omega S^{4,2}) = \underline{\mathbb{Z}/2}$ ,  $\underline{H}^{1,2}(\Omega S^{4,2}) = \underline{H}^{2,2}(\Omega S^{4,2}) = \langle \mathbb{Z}/2 \rangle$ , and  $\underline{H}^{4,2}(\Omega S^{4,2}) = \underline{H}^{5,2}(\Omega S^{4,2}) = 0$ . The Mackey functor  $\underline{H}^{6,2}(\Omega S^{4,2})$  is dual to  $\underline{\mathbb{Z}/2}$ , though this information will not be needed.

Given the above Mackey functors, we have that the  $q = 0$  and  $q = 3$  rows are  $H^{*,0}(S^{4,2}; \mathbb{Z}/2)$ , the  $q = 1$  and  $q = 2$  rows are  $H_{non-eq}^*(S^2; \mathbb{Z}/2)$ , and the  $q = 4$  and  $q = 5$  rows are entirely zeroes. Thus the spectral sequence is as shown in Figure 4.

Since the total space of the fibration is contractible, the spectral sequence converges to  $H^{p+q,2}(pt)$ . Since  $H^{4,2}(pt) = 0$ , there must be a nontrivial differential  $d_2: E^{0,3} \rightarrow E^{2,2}$  sending the generator  $a_1 \in H^{0,0}(S^{4,2}; \underline{H}^{3,2}(\Omega S^{4,2}))$  to the generator

$$z \in H^{2,0}(S^{4,2}; \underline{H}^{2,2}(\Omega S^{4,2})).$$

Now, the products  $a_1^2$  and  $a \cdot z$  live in the  $r = 4$  spectral sequence and so to determine the differentials on  $a_1^2$ , we need the picture of that spectral sequence. This is shown in Figure 5.

Since  $H^{7,4}(pt) = 0$ , there must be a nontrivial differential  $d_2: E^{0,6} \rightarrow E^{2,5}$  sending the generator  $a_2$  isomorphically to  $a \cdot z$ . Since  $d_2(a_1^2) = 0$ , it must be that  $a_1^2 = 0$ . An inductive argument will show that the ring structure is indeed that of an exterior algebra with the specified generators.  $\square$



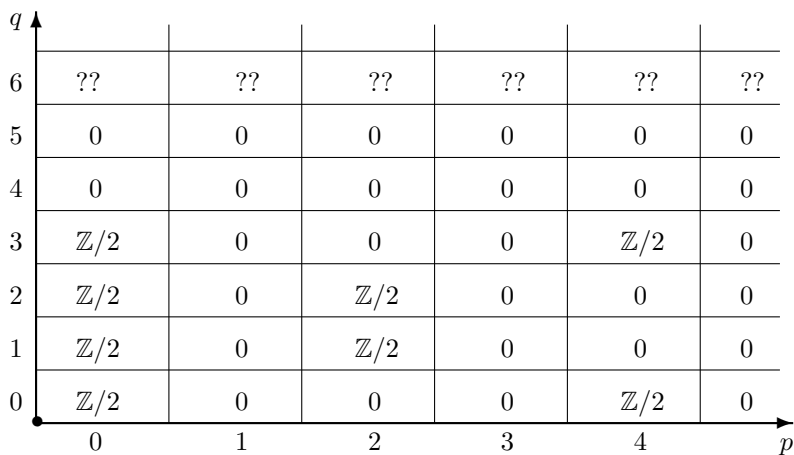


Figure 4: The  $r = 2$  spectral sequence for  $\Omega S^{4,2} \rightarrow PS^{4,2} \rightarrow S^{4,2}$ .

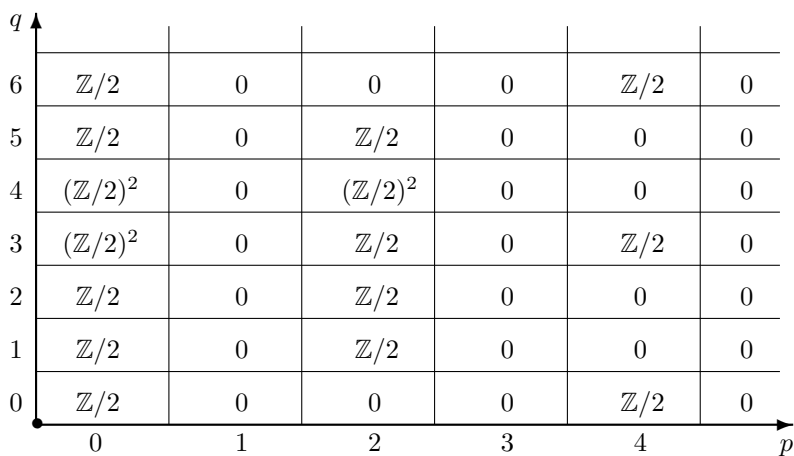


Figure 5: The  $r = 4$  spectral sequence for  $\Omega S^{4,2} \rightarrow PS^{4,2} \rightarrow S^{4,2}$ .

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