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THE GLUING PROBLEM DOES NOT FOLLOW FROM HOMOLOGICAL PROPERTIES OF $\Delta_p(G)$

ASSAF LIBMAN

(*communicated by J. P. C. Greenlees*)

Abstract

Given a block b in kG where k is an algebraically closed field of characteristic p, there are classes $\alpha_Q \in H^2(\mathrm{Aut}_{\mathcal{F}}(Q); k^\times)$, constructed by Külshammer and Puig, where $\mathcal{F}$ is the fusion system associated to b and Q is an $\mathcal{F}$-centric subgroup. The gluing problem in $\mathcal{F}$ has a solution if these classes are the restriction of a class $\alpha \in H^2(\mathcal{F}^c; k^\times)$. Linckelmann showed that a solution to the gluing problem gives rise to a reformulation of Alperin's weight conjecture. He then showed that the gluing problem has a solution if for every finite group G, the equivariant Bredon cohomology group $H^1_G(|\Delta_p(G)|; \mathcal{A}^1)$ vanishes, where $|\Delta_p(G)|$ is the simplicial complex of the non-trivial p-subgroups of G and $\mathcal{A}^1$ is the coefficient functor $G/H \mapsto \mathrm{Hom}(H, k^\times)$. The purpose of this note is to show that this group does not vanish if $G = \Sigma_{p^2}$ where $p \geqslant 5$.

1. Introduction

Given a functor $M : \mathcal{C} \to \mathbf{Ab}$, where $\mathcal{C}$ is a small category, we will write $H^*(\mathcal{C}; M)$ for the groups $\varprojlim^*_{\mathcal{C}} M$. When $\mathcal{C}$ has one object with a group G of automorphisms, a functor $M : \mathcal{C} \to \mathbf{Ab}$ is the same thing as a G-module and $H^*(G; M) \cong \varprojlim^*_{\mathcal{C}} M$.

Let us now fix a prime p and let $\mathcal{F}$ be the fusion system of a block b of a finite group G. As usual, we will write $\mathcal{F}^c$ for the full subcategory generated by the $\mathcal{F}$-centric subgroups in $\mathcal{F}$. Let k be an algebraically closed field of characteristic p. In [**8**] Külshammer and Puig show that for every $\mathcal{F}$-centric subgroup Q there is a canonically chosen class $\alpha_Q \in H^2(\mathrm{Aut}_{\mathcal{F}}(Q); k^\times)$. We view $\mathrm{Aut}_{\mathcal{F}}(Q)$ as a full subcategory of $\mathcal{F}^c$ and say that the gluing problem has a solution in $\mathcal{F}$ if there exists a class $\alpha \in H^2(\mathcal{F}^c; k^\times)$, where $k^\times$ is the constant functor, such that the restriction $\alpha|_{\mathrm{Aut}_{\mathcal{F}}(Q)}$ is equal to α_Q for all $Q \in \mathcal{F}^c$.

Linckelmann showed in [**10**] that if the gluing problem has a solution in the fusion systems of all blocks then Alperin's weight conjecture is logically equivalent to a relation between the number $\mathbf{k}(b)$ of complex representations of G associated to b by

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Knörr and Robinson [**7**] and the Euler characteristic of a certain cochain complex built from the fusion system of b and the cohomology class α.

Let G be a finite group and $\mathcal{C}$ a finite G-poset. The (combinatorial) simplicial complex associated to $\mathcal{C}$, see [**13**, Chap. 3], is denoted $S(\mathcal{C})$. The n-simplices are sequences $c_0 \precneqq \cdots \precneqq c_n$ in $\mathcal{C}$ which we denote $\mathbf{c}$. Face maps are inclusion of simplices. We view $S(\mathcal{C})$ as a topological space via the geometric realization. Clearly G acts on $S(\mathcal{C})$ whose orbit space is denoted $[S(\mathcal{C})]$. It is a CW-complex obtained as the geometric realization of the simplicial set $\mathrm{Nr}(\mathcal{C})/G$ where $\mathrm{Nr}(-)$ is the nerve construction [**3**, XI.2.1]. By abuse of notation, $[S(\mathcal{C})]$ will also denote the poset of the cells of $[S(\mathcal{C})]$ ordered by inclusion.

As a special case consider the poset $\Delta_p(G)$ of the non-trivial p-subgroups of a finite group G. Note that the isotropy group of an n-simplex $\mathbf{P} = (P_0 < \cdots < P_n)$ in $S(\Delta_p(G))$ is

$$N_G(\mathbf{P}) = \cap_{i=0}^n N_G(P_i).$$

The objects of the poset $[S(\Delta_p(G))]$, viewed as a small category, are the G-conjugacy classes $[\mathbf{P}]$ of the simplices of $S(\Delta_p(G))$ and there is a unique morphism $[\mathbf{Q}] \to [\mathbf{P}]$ if the simplex $\mathbf{Q}$ is conjugate in G to a face of $\mathbf{P}$. There is a functor $\mathcal{N}_G \colon [S(\Delta_p(G))] \to \mathbf{Ab}$ defined by Linckelmann in [**9**]

$$\mathcal{N}_G([\mathbf{P}]) = \mathrm{Hom}(N_G(\mathbf{P}), k^\times) = \mathrm{Hom}(N_G(\mathbf{P})_{\mathrm{ab}}, k^\times).$$

Theorem 1.2 of [**9**] implies that the gluing problem in $\mathcal{F}$ has a solution if we can prove that $H^1([S(\Delta_p(K))]; \mathcal{N}_K) = 0$ for all $K = \mathrm{Aut}_{\mathcal{F}}(Q)/\mathrm{Inn}(Q)$ where Q is an $\mathcal{F}$-centric subgroup. Thus, if we can prove that $H^1([S(\Delta_p(G))]; \mathcal{N}_G) = 0$ for all finite groups G, then the gluing problem has a solution for all fusion systems. The purpose of this note is to show that this programme, suggested by Linckelmann, is not feasible.

Theorem 1.1. *Set $G = \Sigma_{p^2}$. If $p \geqslant 5$ then $H^1([S(\Delta_p(G))]; \mathcal{N}_G) \neq 0$.*

We remark that Σ_{p^2} appears as an outer $\mathcal{F}$-automorphism group of $Q = (C_p)^{p^2}$ in the fusion system of the principal block of $C_p \wr \Sigma_{p^2}$. We also remark, without proof, that Theorem 1.1 is valid for $p = 3$ but it fails if $p = 2$. For $p = 2$ one observes that $H^*_G(|\mathcal{B}_p(G)|; \mathcal{H}^1) = 0$, see equation (1), because $\mathcal{H}^1$ vanishes on all the orbits of $|\mathcal{B}_p(G)|$. For $p = 3$ one has to examine the exact sequence (3) more carefully than we do in Propositions 4.2–4.4.

2. Subdivision categories and higher limits

Let G be a finite group. As in the introduction, if $\mathcal{C}$ is a finite G-poset, let $S(\mathcal{C})$ denote the associated G-simplicial complex and let $[S(\mathcal{C})]$ denote its orbit space. We will denote the set of n-simplices of $S(\mathcal{C})$ by $S(\mathcal{C})_n$. It is the set of the non-degenerate n-simplices of $\mathrm{Nr}(\mathcal{C})$. Thus, the n-simplices of $S(\mathcal{C})$ are sequences $\mathbf{c}$ of the form $c_0 \precneqq \cdots \precneqq c_n$ in $\mathcal{C}$. The faces of $\mathbf{c}$ are its non-empty subsequences.

The space $[S(\mathcal{C})]$ is the geometric realization of the simplicial set $\mathrm{Nr}(\mathcal{C})/G$ whose set of non-degenerate simplices is $[S(\mathcal{C})]_n = S(\mathcal{C})_n/G$ which in turn, corresponds to the set of n-cells of $[S(\mathcal{C})]$. We obtain a poset, abusively denoted $[S(\mathcal{C})]$, whose objects are the G-orbits of the simplices of $S(\mathcal{C})$ with an arrow $[\mathbf{c}'] \to [\mathbf{c}]$ if $\mathbf{c}'$ is in the orbit of a face of $\mathbf{c}$. The objects of $[S(\mathcal{C})]$ will be referred to as "simplices".

Given an n-simplex $c_0 \precneqq \cdots \precneqq c_n$ in $S(\mathcal{C})$ where $n \geqslant 1$, we will write $\partial_i \mathbf{c}$ for the $(n-1)$-simplex obtained by removing c_i where $0 \leqslant i \leqslant n$. We obtain face maps

$$\partial_i \colon S(\mathcal{C})_n \to S(\mathcal{C})_{n-1} \qquad \text{and} \qquad [\partial_i] \colon [S(\mathcal{C})]_n \to [S(\mathcal{C})]_{n-1}, \qquad (0 \leqslant i \leqslant n)$$

where ∂_i is G-equivariant and an n-simplex $[\mathbf{c}]$ in $[S(\mathcal{C})]$ gives rise to a map of transitive G-sets $[\mathbf{c}] \xrightarrow{[\partial_i]} [\partial_i \mathbf{c}]$.

Definition 2.1. Fix a commutative ring R. Let $\mathcal{C}$ be a finite G-poset and consider a functor $\mathcal{A} \colon [S(\mathcal{C})] \to R\text{-mod}$. Define a cochain complex $C^*(\mathcal{A})$ as follows.

$$C^n(\mathcal{A}) = \prod_{[\mathbf{c}] \in [S(\mathcal{C})]_n} \mathcal{A}([\mathbf{c}]), \qquad \text{and} \qquad d \colon C^{n-1}(\mathcal{A}) \xrightarrow{\sum_{j=0}^n (-1)^j d^j} C^n(\mathcal{A}).$$

The homomorphisms $d^j \colon C^{n-1}(\mathcal{A}) \to C^n(\mathcal{A})$ are defined on the $[\mathbf{c}]$-th component of $C^n(\mathcal{A})$, where $[\mathbf{c}] \in [S(\mathcal{C})]_n$, by the composition

$$C^{n-1}(\mathcal{A}) \xrightarrow{\text{proj}} \mathcal{A}([\partial_j \mathbf{c}]) \xrightarrow{\mathcal{A}([\partial_j \mathbf{c}] \preceq [\mathbf{c}])} \mathcal{A}([\mathbf{c}]).$$

Lemma 2.2 (cf. [**10**, Proposition 3.2])**.** *With the notation of Definition 2.1, the cohomology groups of $C^*(\mathcal{A})$ are isomorphic to $H^*([S(\mathcal{C})]; \mathcal{A})$.*

Proof. For every $n \geqslant 0$ consider the projective functors $P_n \colon [S(\mathcal{C})] \to \mathbf{Ab}$ defined by

$$P_n = \bigoplus_{[\mathbf{c}] \in [S(\mathcal{C})]_n} \mathbb{Z} \otimes \mathrm{Mor}_{[S(\mathcal{C})]}([\mathbf{c}], -).$$

For every $0 \leqslant j \leqslant n$ there are morphisms $d^j_{n-1} \colon P_n \to P_{n-1}$ which are induced by Yoneda's lemma via the morphisms $[\partial_j \mathbf{c}] \to [\mathbf{c}]$ for every $[\mathbf{c}] \in [S(\mathcal{C})]_n$. Define morphisms $d_{n-1} \colon P_n \to P_{n-1}$ by $d_{n-1} = \sum_{j=0}^n (-1)^j d^j_{n-1}$. We claim that the resulting

$$\cdots \to P_n \xrightarrow{d_{n-1}} P_{n-1} \to \ldots \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \to \underline{\mathbb{Z}} \qquad (\text{denoted } P_\bullet \to \underline{\mathbb{Z}})$$

is a projective resolution of the constant functor $\underline{\mathbb{Z}}$. Indeed, the evaluation of $P_\bullet$ at every object $[\mathbf{x}] \in [S(\mathcal{C})]_n$ yields the chain complex $C_*(\Delta^n; \mathbb{Z})$ because the faces of $[\mathbf{x}]$ in $[S(\mathcal{C})]$ generate the standard simplex Δ^n. Finally, by Yoneda's Lemma $\mathrm{Hom}(P_\bullet, \mathcal{A}) = C^*(\mathcal{A})$ and its homology groups are isomorphic to $\varprojlim{}^* \mathcal{A}$. □

3. Bredon cohomology

Throughout this section a space means a simplicial set. Let G be a finite group. A *coefficient functor* $\mathcal{M}$ for G is a contravariant functor $\{G\text{-sets}\} \to \mathbf{Ab}$ which turns coproducts of G-sets into products of abelian groups. By applying $\mathcal{M}$ to the sets of simplices of a G-space X, one obtains a cosimplicial abelian group $\mathcal{M}(X)$. The cochain complex associated to $\mathcal{M}(X)$ is denoted $C^*(X; \mathcal{M})$, see [**15**, 8.2]. Its homology groups are called the Bredon cohomology groups $H^*_G(X; \mathcal{M})$, see e.g., [**5**, §4]. Note that $C^n(X; \mathcal{M}) = \prod_{[\mathbf{x}] \subseteq X} \mathcal{M}([\mathbf{x}])$ where the product runs through the orbits of the n-simplices of X.

If Y is a G-subspace of X then there is a canonical short exact sequence of cochain complexes

$$0 \to C^*_G(X, Y; \mathcal{M}) \to C^*_G(X; \mathcal{M}) \to C^*_G(Y; \mathcal{M}) \to 0$$

which defines the relative cohomology groups $H^*_G(X, Y; \mathcal{M})$ together with the usual long exact sequences. Bredon cohomology is an equivariant cohomology theory, cf. [**4**]. In particular it turns G-homotopy equivalences into isomorphisms and if X is a union of subspaces $Y_1 \cup Y_2$, one has the usual Mayer Vietoris sequence.

The normalized cochain complex $NC^*(X; \mathcal{M})$ is a sub-complex of $C^*(X; \mathcal{M})$ defined by

$$NC^n(X; M) = \bigcap_{i=0}^{n-1} \Big(\mathrm{Ker}(C^n(X; \mathcal{M}) \xrightarrow{s^i} C^{n-1}(X; \mathcal{M})) \Big),$$

where s^i are the codegeneracy maps of the cosimplicial group $\mathcal{M}(X)$. If $[\mathbf{x}]$ is the orbit of a simplex in X and s_i is a degeneracy operator of X, it is easy to check that $s_i \colon [\mathbf{x}] \to [s_i \mathbf{x}]$ is an isomorphism of transitive G-sets and in particular $\mathcal{M}([\mathbf{x}]) = \mathcal{M}([s_i \mathbf{x}])$. It easily follows that $NC^n(X; \mathcal{M}) = \prod_{[\mathbf{x}] \subseteq X} \mathcal{M}([\mathbf{x}])$ where $[\mathbf{x}]$ runs through the orbits of the *non-degenerate* n-simplices of X.

It is well known that the inclusion of $NC^*(X; \mathcal{M})$ in $C^*(X; \mathcal{M})$ is a homology equivalence. See [**15**, 8.3].

Recall that the Borel construction of a G-space U is $U \times_G EG$ where EG is a contractible space on which G acts freely. If $U = G/K$ then $U \times_G EG = BK$ is the classifying space of K.

Definition 3.1. Fix a finite group G, an abelian group A and an integer $n \geqslant 0$. Define a coefficient functor $\mathcal{H}^n$ for G by $\mathcal{H}^n(U) = H^n(U \times_G EG; A)$. Observe that $\mathcal{H}^n(G/K) = H^n(K; A)$ where A has the trivial action of K.

Definition 3.2. Let $\mathcal{C}$ be a finite G-poset and let $\mathcal{M}$ be a coefficient system. The underlying set of every object $[\mathbf{c}]$ of $[S(\mathcal{C})]$ is a transitive G-set and we define a functor $\mathcal{A}_{\mathcal{M}} \colon [S(\mathcal{C})] \to \mathbf{Ab}$ by

$$\mathcal{A}_{\mathcal{M}}([\mathbf{c}]) = \mathcal{M}([\mathbf{c}]).$$

If $[\mathbf{c}']$ is a face of $[\mathbf{c}]$, we define $\mathcal{A}_{\mathcal{M}}([\mathbf{c}'] \to [\mathbf{c}])$ by applying $\mathcal{M}$ to the map $[\mathbf{c}] \to [\mathbf{c}']$ of transitive G-sets.

By inspection of Definition 2.1, $C^*(\mathcal{A}_{\mathcal{M}}) \cong NC^*_G(|\mathcal{C}|; \mathcal{M})$ and the next result follows from Lemma 2.2. It has been observed by Słominska [**12**, p. 116] and by others e.g., Grodal in [**6**, Theorem 7.3], Linckelmann [**10**, Proposition 3.5] and Dwyer in [**5**].

Lemma 3.3. *Let $\mathcal{C}$ be a finite G-poset and let $\mathcal{M}$ be a coefficient functor for G. With the notation of Definition 3.2, $H^*([S(\mathcal{C})]; \mathcal{A}_{\mathcal{M}}) \cong H^*_G(|\mathcal{C}|; \mathcal{M})$.*

4. Proof of Theorem 1.1

Set $G = \Sigma_{p^2}$ and let $\mathcal{C}$ denote the poset $\Delta_p(G)$ of the non-trivial p-subgroups of G. First we observe that $\mathrm{Hom}(K, A) = H^1(K; A)$ for any finite group K and any abelian group A. Thus, the functor $\mathcal{N}_G \colon [S(\mathcal{C})] \to \mathbf{Ab}$ defined in the introduction is canonically isomorphic to $\mathcal{A}_{\mathcal{H}^1}$ as defined in 3.2 and in 3.1 with $A = k^\times$ where k is an algebraically closed field of characteristic p. In light of Lemma 3.3 we need to prove that $H^1_G(|\Delta_p(G)|; \mathcal{H}^1) \neq 0$. Consider the G-subposet $\mathcal{B}_p(G)$ of the non-trivial radical p-subgroups of G, namely the non-trivial p-subgroup $P \leqslant G$ such that $N_G(P)/P$ contains no non-trivial normal p-subgroup. It is well known that the inclusion $|\mathcal{B}_p(G)| \subseteq |\Delta_p(G)|$ is a G-homotopy equivalence, see e.g., [**2**, Proposition 6.6.5]. Therefore, it remains to prove that

$$H^1_G(|\mathcal{B}_p(G)|; \mathcal{H}^1) \neq 0. \tag{1}$$

The radical p-subgroups of the symmetric groups were classified by Alperin and Fong in [**1**]. In $G = \Sigma_{p^2}$ they form the following conjugacy classes:

(**R1**) The conjugacy class of the Sylow p-subgroup $V_{1,1} \overset{\text{def}}{=} C_p \wr C_p \leqslant \Sigma_{p^2}$. Its normalizer is $V_{1,1} \rtimes (\mathrm{GL}_1(p) \times \mathrm{GL}_1(p))$ with the diagonal action of $\mathrm{GL}_1(p)$ on the base group $(C_p)^p$ and the usual action of the second $\mathrm{GL}_1(p)$ on C_p at the top.

(**R2**) The conjugacy class of the subgroup $V_2 = C_p \times C_p$ embedded in Σ_{p^2} via its action on itself by translation. Its normalizer is $V_2 \rtimes \mathrm{GL}_2(p)$.

(**R3**) For every $k = 1, \dots, p$ the conjugacy class of the subgroup ${V_1}^{\times k}$ which is isomorphic to ${C_p}^{\times k}$ as a subgroup of ${\Sigma_p}^{\times k} \leqslant \Sigma_{p^2}$. The normalizer of ${V_1}^{\times k}$ is

$$\Big(\big(V_1 \rtimes \mathrm{GL}_1(p)\big) \wr \Sigma_k \Big) \times \Sigma_{p(p-k)}.$$

Definition 4.1. Consider the following subposets of $\mathcal{B}_p(G)$.

1. Let $\mathcal{D}_1$ be the subposet consisting of the conjugacy class of $V_{1,1}$ and the conjugacy classes of $V_1, {V_1}^{\times 2}, \dots, {V_1}^{\times p}$.
2. Let $\mathcal{V}_1$ be the subposet consisting of the conjugacy classes of $V_1, {V_1}^{\times 2}, \dots, {V_1}^{\times p}$.
3. Let $\mathcal{D}_2$ be the subposet consisting of the conjugacy classes of $V_{1,1}$ and V_2.
4. Let $\mathcal{D}_3$ be the subposet consisting of the conjugacy class of $V_{1,1}$.

Observe that V_2 is a transitive subgroup of Σ_{p^2} so it cannot be conjugate to a subgroup of $V_1^{\times k}$ whose orbits have cardinality p. Also, V_2 acts freely so it cannot contain a conjugate of $V_1^{\times k}$ since the latter do not act freely on the underlying set of p^2 elements. We see that up to conjugacy $\mathcal{B}_p(G)$ has the form

$$[V_1] \;<\; [V_1^{\times 2}] \;<\; \dots\; [V_1^{\times p}] \;<\; [V_{1,1}] \;>\; [V_2]$$

and it follows that

$$|\mathcal{B}_p(G)| = |\mathcal{D}_1| \cup |\mathcal{D}_2|, \qquad \text{and} \qquad |\mathcal{D}_3| = |\mathcal{D}_1| \cap |\mathcal{D}_2|. \tag{2}$$

The Mayer Vietoris sequence gives an exact sequence

$$\dots \to H^0_G(|\mathcal{D}_1|; \mathcal{H}^1) \oplus H^0_G(|\mathcal{D}_2|; \mathcal{H}^1) \to H^0_G(|\mathcal{D}_3|; \mathcal{H}^1) \to H^1_G(|\mathcal{B}_p(G)|; \mathcal{H}^1) \to \dots \tag{3}$$

For what follows, it will be convenient to denote

$$L = \mathrm{Hom}(\mathrm{GL}_1(p), k^\times) \cong \mathbb{F}_p^\times.$$

Proposition 4.2. *$H^0_G(|\mathcal{D}_3|; \mathcal{H}^1) \cong L \times L$ and $H^{*\geqslant 1}_G(|\mathcal{D}_3|; \mathcal{H}^1) = 0$.*

Proposition 4.3. *$H^0_G(|\mathcal{D}_2|;\mathcal{H}^1)\cong L$ and $H^{*\geqslant 1}_G(|\mathcal{D}_2|;\mathcal{H}^1)=0$.*

Proposition 4.4. *$H^0_G(|\mathcal{D}_1|;\mathcal{H}^1)\cong C_2$ and $H^{*\geqslant 1}_G(|\mathcal{D}_1|;\mathcal{H}^1)=0$.*

Propositions 4.2–4.4 together with the exact sequence (3) immediately imply (1) because by hypothesis $p\geqslant 5$, whence $|L|\geqslant 4$.

Recall that k has characteristic p. Therefore the kernel of any group homomorphism $H\to k^\times$ contains the commutator subgroup of H and any p-subgroup of H. We will use this fact repeatedly.

Proof of Proposition 4.2. Since $\mathcal{D}_3$ is a single orbit of G with isotropy group $N_G(V_{1,1})$ it follows from (**R1**) that $H^*_G(|\mathcal{D}_3|;\mathcal{H}^1)=\operatorname{Hom}(N_G(V_{1,1}),k^\times)=L\times L$. □

Proof of Proposition 4.3. Since $|\mathcal{B}_p(G)|$ is G-equivalent to $|\Delta_p(G)|$, Symond's resolution of Webb's conjecture in [**14**] shows that the orbit space $|\mathcal{B}_p(G)|/G$ is contractible. But (2) shows that $|\mathcal{B}_p(G)|/G=(|\mathcal{D}_1|/G)\vee(|\mathcal{D}_2|/G)$. It follows that the CW-complex $|D_2|/G$, namely $[S(\mathcal{D}_2)]$, is contractible and since it is 1-dimensional with two 0-simplices $[V_2]$ and $[V_{1,1}]$, the poset $[S(\mathcal{D}_2)]$ must have the form

$$[V_2]\ \to\ [V_2<V_{1,1}]\ \leftarrow\ [V_{1,1}].$$

Now, $V_2\leqslant V_{1,1}=C_p\wr C_p$ is generated by the copy of C_p at the top and the diagonal copy of C_p in the base group $C_p\times\cdots\times C_p$ which is the centre of $V_{1,1}$. One easily deduces from (**R1**) and (**R2**) that $N_G(V_2<V_{1,1})/N_{V_{1,1}}(V_2)\cong\mathrm{GL}_1(p)^2$ as a diagonal subgroup of $\mathrm{GL}_2(p)$. With the notation of Definition 3.2 we have

$$\mathcal{A}_{\mathcal{H}^1}([V_2<V_{1,1}])\cong\operatorname{Hom}(\mathrm{GL}_1(p)^2,k^\times)\cong\mathcal{A}_{\mathcal{H}^1}([V_{1,1}]),$$

and $\mathcal{A}_{\mathcal{H}^1}([V_2])=\operatorname{Hom}(\mathrm{GL}_2(p),k^\times)\cong L$ because $\mathrm{GL}_2(p)_{\mathrm{ab}}=\mathbb{F}_p^\times$. By Lemma 3.3, the groups $H^*_G(|\mathcal{D}_2|;\mathcal{H}^1)$ are isomorphic to $H^*([S(\mathcal{D}_2)];\mathcal{A}_{\mathcal{H}^1})$, namely to the derived functors of the diagram $L\xrightarrow{\Delta}L\times L\xleftarrow{\ \mathrm{id}\ }L\times L$. This completes the proof. □

Lemma 4.5. *The inclusion $\mathcal{V}_1\subseteq\mathcal{D}_1$, see Definition 4.1, induces a G-equivariant homotopy equivalence $|\mathcal{V}_1|\to|\mathcal{D}_1|$.*

Proof. Given a subgroup P of G let $\delta_1(P)$ denote the subgroup of P generated by all the permutations $g\in P$ whose support contains at most p elements. Observe that δ_1 is invariant under conjugation, namely $\delta_1(gPg^{-1})=g\delta_1(P)g^{-1}$. By inspection $\delta_1({V_1}^{\times k})={V_1}^{\times k}$ and $\delta_1(V_{1,1})={V_1}^{\times p}$. We obtain a G-equivariant morphism of posets $\delta_1\colon\mathcal{D}_1\to\mathcal{V}_1$. Clearly, $|\delta_1|\circ i^{|\mathcal{D}_1|}_{|\mathcal{V}_1|}=\mathrm{Id}_{|\mathcal{V}_1|}$. The inclusions $\delta_1(P)\leqslant P$ give a G-equivariant homotopy $i^{|\mathcal{D}_1|}_{|\mathcal{V}_1|}\circ|\delta_1|\simeq\mathrm{Id}_{|\mathcal{D}_1|}$, cf. [**11**, 1.3]. The result follows. □

We leave the following result as an easy exercise for the reader.

Lemma 4.6. *Let K be a finite group, fix an integer $n\geqslant 1$ and set $G_n=K\wr\Sigma_n$. Then $(G_n)_{\mathrm{ab}}\cong K_{\mathrm{ab}}\times(\Sigma_n)_{\mathrm{ab}}$. The restriction of $G_n\to(G_n)_{\mathrm{ab}}$ to any one of the factors K of $K^n\leqslant G_n$ is the canonical projection $K\to K_{\mathrm{ab}}$ and the restriction of $G_n\to(G_n)_{\mathrm{ab}}$ to Σ_n is the projection onto $(\Sigma_n)_{\mathrm{ab}}$.*

If $n,m\geqslant 1$ then $G_n\times G_m\leqslant G_{n+m}$. The resulting $(G_n)_{\mathrm{ab}}\times(G_m)_{\mathrm{ab}}\to(G_{n+m})_{\mathrm{ab}}$ is induced by the fold map $K_{\mathrm{ab}}\times K_{\mathrm{ab}}\to K_{\mathrm{ab}}$ and by $(\Sigma_n)_{\mathrm{ab}}\times(\Sigma_m)_{\mathrm{ab}}\to(\Sigma_{n+m})_{\mathrm{ab}}$.

Notation 4.7. The following non-standard description of the $(n-1)$-simplex Δ^{n-1} will be used throughout. The r-simplices of Δ^{n-1} are sequences $i_0 < \cdots < i_r$ where $1 \leqslant i_0, \dots, i_r \leqslant n$. Face maps are obtained by inclusion of sequences. (The usual convention is $0 \leqslant i_0, \dots, i_r \leqslant n-1$.)

Proof of Proposition 4.4. In light of Lemma 4.5 and Lemma 3.3, we must prove that $H^*_G([S(\mathcal{V}_1)]; \mathcal{A}_{\mathcal{H}^1}) \cong C_2$.

The high transitivity of the symmetric groups and the description of $N_G(V_1^{\times k})$ in (**R3**) imply that every r-simplex of $S(\mathcal{V}_1)$ is conjugate in G to a simplex of the form $V_1^{\times i_0} < \cdots < V_1^{\times i_r}$ where $1 \leqslant i_0 < \cdots < i_r \leqslant p$. With the notation of 4.7 we see that $[S(\mathcal{V}_1)] = \Delta^{p-1}$.

For any group K let $\widehat{K}$ denote the abelian group $\mathrm{Hom}(K, k^\times)$. Let N denote the normalizer of C_p in Σ_p. Thus, $N = C_p \rtimes \mathrm{GL}_1(p)$ and observe that $\mathrm{GL}_1(p) \leqslant \Sigma_p$ is generated by an odd permutation, in fact a cycle of even length (p is odd). Set

$$L = \widehat{N} = \mathrm{Hom}(N, k^\times) = \mathrm{Hom}(\mathrm{GL}_1(p), k^\times) \cong C_{p-1}.$$

Consider the following functor $\Phi\colon (\Delta^{p-1})^{\mathrm{op}} \to \{\text{Groups}\}$. On objects

$$\Phi(i_0 < \cdots < i_r) = \big(\prod_{t=0}^{r} N \wr \Sigma_{i_t - i_{t-1}}\big) \times \Sigma_{p^2 - i_r p}, \qquad \text{(by convention } i_{-1} = 0).$$

For an r-simplex $\mathbf{i}$ and for $0 \leqslant j \leqslant r$, the effect of $\Phi(\mathbf{i}) \to \Psi(\partial_j \mathbf{i})$ is induced by the inclusions

$$\begin{array}{ll} (N \wr \Sigma_{i_j - i_{j-1}}) \times (N \wr \Sigma_{i_{j+1} - i_j}) \leqslant (N \wr \Sigma_{i_{j+1} - i_{j-1}}) & \text{if } 0 \leqslant j < r \\ (N \wr \Sigma_{i_r - i_{r-1}}) \times \Sigma_{p(p-i_r)} \leqslant \Sigma_{p(p-i_{r-1})} & \text{if } j = r. \end{array}$$

Inspection of (**R3**) shows that $\mathcal{A}_{\mathcal{H}^1} = \widehat{\Phi}$, namely $\mathcal{A}_{\mathcal{H}^1} = \mathrm{Hom}(\Phi, k^\times)$. Having identified $[S(\mathcal{V}_1)]$ with Δ^{p-1}, it remains to prove that

$$H^*(\Delta^{p-1}; \widehat{\Phi}) \cong C_2. \tag{4}$$

Consider the following functor $\Psi\colon \Delta^{p-1} \to \mathbf{Ab}$ defined by

$$\Psi(i_0 < \cdots < i_r) = \big(\prod_{t=0}^{r} N \wr \Sigma_{i_t - i_{t-1}}\big) \times (N \wr \Sigma_{p - i_r}), \qquad \text{(by convention } i_{-1} = 0).$$

It is a subfunctor of Φ via the inclusions $N \wr \Sigma_{p-i_r} \leqslant \Sigma_{p(p-i_r)}$. We obtain a morphism of functors $\widehat{\Phi} \to \widehat{\Psi}$ of abelian groups. Our goal now is to prove that it is a monomorphism and to calculate its cokernel. Fix an r-simplex $\mathbf{i} = (i_0 < \cdots < i_r)$ in Δ^{p-1} and consider $\widehat{\Phi}(\mathbf{i}) \to \widehat{\Psi}(\mathbf{i})$. Note that $(\Sigma_n)_{\mathrm{ab}} = C_2$ if $n \geqslant 2$ and that if $H \leqslant \Sigma_n$ contains an odd permutation then $H_{\mathrm{ab}} \to (\Sigma_n)_{\mathrm{ab}}$ is surjective.

Case (a). If $i_r = p$ then $\Sigma_{p^2 - i_r p}$ and $N \wr \Sigma_{p-i_r}$ are the trivial group and therefore $\widehat{\Phi}(\mathbf{i}) \to \widehat{\Psi}(\mathbf{i})$ is an isomorphism.

Case (b). If $i_r = p-1$ then $N \wr \Sigma_{p-i_r} = N$ and $\Sigma_{p(p-i_r)} = \Sigma_p$. Since $N = C_p \rtimes C_{p-1}$ contains an odd permutation, by applying $\mathrm{Hom}(-, k^\times)$ to the inclusion $N \leqslant \Sigma_p$ we obtain the monomorphism $C_2 \to L$ and therefore $\widehat{\Phi}(\mathbf{i}) \to \widehat{\Psi}(\mathbf{i})$ is injective with cokernel L/C_2.

Case (c). Assume that $i_r \leqslant p-2$. The inclusion of $N^{p-i_r} \leqslant \Sigma_{p(p-i_r)}$ contains odd permutations. Since p is odd, also the diagonal inclusion $\Sigma_{p-i_r} \leqslant \Sigma_{p(p-i_r)}$ contains odd permutations. By Lemma 4.6 the induced map $\widehat{\Sigma_{p(p-i_r)}} \to N \wr \widehat{\Sigma_{p-i_r}}$ is the diagonal inclusion $C_2 \to L \oplus C_2$ into $C_2 \oplus C_2$. It follows that $\widehat{\Phi}(\mathbf{i}) \to \widehat{\Psi}(\mathbf{i})$ is injective with cokernel L.

We obtain a short exact sequence of functors $\Delta^{p-1} \to \mathbf{Ab}$

$$0 \to \widehat{\Phi} \to \widehat{\Psi} \to \Gamma \to 0, \tag{5}$$

where the functor Γ has the form

$$\Gamma(\mathbf{i}) = \begin{cases} 0 & \text{if } i_r = p \\ L/C_2 & \text{if } i_r = p-1 \\ L & \text{if } i_r \leqslant p-2. \end{cases}$$

By Lemma 4.6, $\Gamma(\mathbf{j}) \to \Gamma(\mathbf{i})$ are induced by the quotient maps $L \to L/C_2 \to 0$.

Let $\Gamma', \Gamma'' \colon \Delta^{p-1} \to \mathbf{Ab}$ be the functors defined for objects $\mathbf{i} = (i_0 < \cdots < i_r)$ by

$$\Gamma'(\mathbf{i}) = \begin{cases} L & \text{if } 1 \leqslant i_r \leqslant p-1 \\ 0 & \text{if } i_r = p \end{cases} \qquad \Gamma''(\mathbf{i}) = \begin{cases} C_2 & \text{if } i_r = p-1 \\ 0 & \text{if } i_r \neq p-1. \end{cases}$$

Face maps $\mathbf{i} \subseteq \mathbf{j}$ induce either the identity or the trivial homomorphisms $\Gamma'(\mathbf{i}) \to \Gamma'(\mathbf{j})$ and $\Gamma''(\mathbf{i}) \to \Gamma''(\mathbf{j})$. We get a short exact sequence of functors

$$0 \to \Gamma'' \to \Gamma' \to \Gamma \to 0.$$

We view Δ^{p-2} as the $(p-1)$st face of Δ^{p-1}, that is, Δ^{p-2} consist of the simplices $\mathbf{i} = (i_0 < \cdots < i_r)$ of Δ^{p-1} with $i_r \leqslant p-1$. Similarly Δ^{p-3} is the $(p-2)$nd face of Δ^{p-2}. Thus, Δ^{p-3} is the subcomplex of Δ^{p-1} of the simplices $\mathbf{i}$ with $i_r \leqslant p-2$. At this point we should recall that $p \geqslant 5$.

By inspection of Definition 2.1 we see that $C^*(\Gamma'')$ is isomorphic to the cochain complex $C^*(\Delta^{p-2}, \Delta^{p-3}; C_2)$ of the relative simplicial complex $(\Delta^{p-2}, \Delta^{p-3})$. Since $p \geqslant 5$, the contractibility of the standard simplices and Lemma 2.2 imply that

$$H^*(\Delta^{p-1}; \Gamma'') \cong H^*(\Delta^{p-2}, \Delta^{p-3}; C_2) = 0.$$

The acyclicity of Γ'' now shows that $\Gamma' \to \Gamma$ induces an isomorphism

$$H^*(\Delta^{p-1}; \Gamma') \xrightarrow{\cong} H^*(\Delta^{p-1}; \Gamma). \tag{6}$$

By Lemma 4.6 we see that $\widehat{\Psi} \colon \Delta^{p-1} \to \mathbf{Ab}$ has the following form

$$\widehat{\Psi}(i_0 < \cdots < i_r) = \Big(\prod_{t=0}^{r} L \times \widehat{\Sigma_{i_t - i_{t-1}}} \Big) \times \begin{cases} 0 & \text{if } i_r = p \\ L \times \widehat{\Sigma_{p-i_r}} & \text{if } i_r < p. \end{cases}$$

We obtain a constant subfunctor $\Psi'(\mathbf{i}) = L$ of $\widehat{\Psi}$ via the diagonal inclusion and it is easy to check that the following square commutes

$$\begin{array}{ccc} \Psi' & \longrightarrow & \widehat{\Psi} \\ \downarrow & & \downarrow \\ \Gamma' & \longrightarrow & \Gamma. \end{array}$$

By inspection of Definition 2.1, there are isomorphisms $C^*(\Psi') \cong C^*(\Delta^{p-1}; L)$ and $C^*(\Gamma') \cong C^*(\Delta^{p-2}; L)$. The map $\Psi' \to \Gamma'$ gives rise to the map of cochain complexes induced by $\Delta^{p-2} \subseteq \Delta^{p-1}$. We deduce from Lemma 2.2 and the contractibility of the standard simplices that $\Psi' \to \Gamma'$ induces an isomorphism

$$H^*(\Delta^{p-1};\Psi') \xrightarrow{\cong} H^*(\Delta^{p-1};\Gamma') \cong \begin{cases} L & \text{if } * = 0 \\ 0 & \text{if } * = 0. \end{cases} \tag{7}$$

The commutative square above, together with (6) and (7) imply that $\widehat{\Psi} \to \Gamma$ induces an epimorphism $H^*(\Delta^{p-1};\widehat{\Psi}) \to H^*(\Delta^{p-1};\Gamma)$. By (6) and (7) and the long exact sequence associated to (5), the proof of (4), whence the proof of this proposition, will be complete if we prove that $H^*(\Delta^{p-1};\widehat{\Psi}) \cong L \oplus C_2$ (cohomology concentrated in degree 0).

Set $K = N \wr \Sigma_p$ and let it act on the poset Ω of the non-empty subsets of $\{1,\dots,p\}$ via the projection onto Σ_p. One easily checks that $[S(\Omega)] = \Delta^{p-1}$ and that, by choosing appropriate representatives, the isotropy groups of the r-simplices of $S(\Omega)$ are

$$\mathrm{Iso}_K(i_0 < \cdots < i_r) = \Psi(i_0 < \cdots < i_r).$$

Thus, if $\mathcal{H}^1_K$ is the coefficient functor for K defined in 3.1 with $A = k^\times$, we see that $C^*(\widehat{\Psi}) \cong C^*(\mathcal{A}_{\mathcal{H}^1_K})$, whence by Lemma 3.3,

$$H^*(\Delta^{p-1};\widehat{\Psi}) \cong H^*([S(\Omega)];\mathcal{A}_{\mathcal{H}^1_K}) \cong H^*_K(|\Omega|;\mathcal{H}^1_K).$$

Now, $|\Omega|$ is K-equivalent to a point because $\{1,\dots,p\}$ is a maximal element of Ω fixed by K. Therefore $H^*_K(|\Omega|;\mathcal{H}^1_K) \cong \mathcal{H}^1_K(\mathrm{pt}) = \widehat{N \wr \Sigma_p} = L \oplus C_2$ by Lemma 4.6. This completes the proof. □

References

[1] J. L. Alperin, P. Fong, Weights for symmetric and general linear groups. *J. Algebra* **131** (1990), no. 1, 2–22.

[2] D. J. Benson, *Representations and cohomology. II. Cohomology of groups and modules.* Cambridge Studies in Advanced Mathematics, 31. Cambridge University Press, Cambridge, 1991.

[3] A. K. Bousfield, D. M. Kan, *Homotopy limits, completions and localizations.* Lecture Notes in Mathematics, Vol. 304. Springer-Verlag, Berlin-New York, 1972.

[4] Glen E. Bredon, *Equivariant cohomology theories.* Lecture Notes in Mathematics, No. 34. Springer-Verlag, Berlin-New York 1967

[5] W. G. Dwyer, Sharp homology decompositions for classifying spaces of finite groups. Group representations: cohomology, group actions and topology (Seattle, WA, 1996), 197–220, *Proc. Sympos. Pure Math.*, **63**, Amer. Math. Soc., Providence, RI, 1998.

[6] Jesper Grodal, Higher limits via subgroup complexes. *Ann. of Math.* (2) **155** (2002), no. 2, 405–457.

[7] Reinhard Knörr, Geoffrey R. Robinson, Some remarks on a conjecture of Alperin. *J. London Math. Soc.* (2) **39** (1989), no. 1, 48–60.

[8] Burkhard Külshammer, Lluís Puig, Extensions of nilpotent blocks. *Invent. Math.* **102** (1990), no. 1, 17–71.

[9] Markus Linckelmann, On $H^*(\mathcal{C}; k^\times)$ for fusion systems. *Homology, Homotopy and Applications* **11**(1), 203–218.

[10] Markus Linckelmann, Alperin's weight conjecture in terms of equivariant Bredon cohomology. *Math. Z.* **250** (2005), no. 3, 495–513.

[11] Daniel Quillen, Homotopy properties of the poset of nontrivial p-subgroups of a group. *Adv. in Math.* **28** (1978), no. 2, 101–128.

[12] Jolanta Słomińska, Some spectral sequences in Bredon cohomology. *Cahiers Topologie Géom. Différentielle Catég.* **33** (1992), no. 2, 99–133.

[13] Edwin H. Spanier, *Algebraic topology.* McGraw-Hill Book Co., New York-Toronto, Ont.-London 1966

[14] Peter Symonds, The orbit space of the p-subgroup complex is contractible. *Comment. Math. Helv.* **73** (1998), no. 3, 400–405.

[15] Weibel, Charles A. *An introduction to homological algebra.* Cambridge Studies in Advanced Mathematics, 38. Cambridge University Press, Cambridge, 1994.

Assaf Libman `a.libman@abdn.ac.uk`

Mathematics Department, University of Aberdeen, Fraser Noble building, King's College, Aberdeen, AB24 3UE, U.K.