A RELATIVE VERSION OF THE FINITENESS OBSTRUCTION THEORY OF C. T. C. WALL

ANNA DAVIS

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Abstract

In his 1965 paper C. T. C. Wall demonstrated that if a CW complex Y is finitely dominated, then the reduced projective class group of Y contains an obstruction which vanishes if and only if Y is homotopy equivalent to a finite CW complex. Wall also demonstrated that such an obstruction is invariant under homotopy equivalences. Subsequently Sum and Product Theorems for this obstruction were proved by L. C. Siebenmann.

In his second paper on the subject Wall gives an algebraic definition of the relative finiteness obstruction. If a CW complex Y is finitely dominated rel. a subcomplex X, then the reduced projective class group of Y contains an obstruction which vanishes if and only if Y is homotopy equivalent to a finite complex rel. X.

In this paper we will use a geometric construction to reduce the relative finiteness obstruction to the non-relative version. We will demonstrate that the relative finiteness obstruction is invariant under certain types of homotopy equivalences. We will also prove the relative versions of the Sum and the Product Theorems.

1. Introduction

The purpose of this paper is to provide a geometric reduction of Wall's relative finiteness obstruction [14] to his non-relative version [13]. The idea that such a reduction is possible as well as the choice of construction techniques used in this paper are due to Chapman. We will employ a modification of the infinite mapping cylinder construction used by Ferry in [7] to obtain a geometric description of the non-relative obstruction. Our modification is similar to the technique used by Chapman in [5] and involves a truncated version of the infinite mapping cylinder construction. As a result we obtain relative versions of the sum and product theorems of [10].

We will start by reviewing the non-relative theory. In what follows all spaces are assumed to be locally compact, separable and metric and all *maps* are continuous functions.

A map $d\colon X \longrightarrow Y$ is a *homotopy domination* if there exists a map $u\colon Y \longrightarrow X$ such that $du \simeq id_Y$. We say that u is an inverse of d and that Y is *dominated* by X. A CW complex is said to be *finitely dominated* if it is dominated by a finite complex.

Wall's finiteness obstruction theory arises from the question: *If Y is finitely dominated, when is Y homotopy equivalent to some finite CW complex?* Wall defines the finiteness obstruction of a finitely dominated CW complex Y to be an element $\sigma(Y)$ of the reduced projective class group $\tilde{K}_0(Z\pi_1(Y))$ and shows that $\sigma(Y) = 0$ if and only if Y is homotopy equivalent to a finite complex [13]. Wall also shows that $\sigma(Y)$ is an invariant of homotopy type. Siebenmann used finiteness obstruction to solve the problem of putting a boundary on an open manifold [10]. He also obtained product and sum theorems for the obstruction [10].

By a *CW pair* (Y, X) we mean a CW complex Y together with a subcomplex X closed in Y. Let (Y, X) be a CW pair. Y is said to be *finitely dominated rel. X* if there exists a compact CW complex K and a map $d\colon X \cup K \to Y$ such that d is a homotopy domination rel. X (i.e., there exists a homotopy inverse u such that $du \overset{h_t}{\simeq} id$ with $h_t(x) = x$ for all $x \in X$). Equivalently, Y is finitely dominated rel. X if the inclusion $i\colon X \cup C \hookrightarrow Y$ is a homotopy domination rel. X for some compact subcomplex $C \subset Y$. If Y is finitely dominated rel. X we say that (Y, X) is finitely dominated.

Let (Y_1, X) and (Y_2, X) be CW pairs. Suppose there is a homotopy equivalence $f\colon Y_1 \to Y_2$ such that $f|_X\colon X \to X$ is the identity. If there exists an inverse g such that $g|_X\colon X \to X$ is the identity, and homotopies F_t and G_t such that $fg \overset{F_t}{\simeq} id$, $gf \overset{G_t}{\simeq} id$ and $F_t|_X = id$, $G_t|_X = id$ for all t, then we say that f is a *homotopy equivalence rel. X*. If $Y_2 = X \cup K$, where K is a finite complex, we say that Y_1 is homotopy equivalent rel. X to a finite complex.

Relative finiteness obstruction theory arises from the question: *If Y is finitely dominated rel. X, when is Y homotopy equivalent to a finite complex rel. X?* Wall uses relative chain complexes to define the relative finiteness obstruction of a finitely dominated pair (Y, X) to be an element $\sigma(Y, X)$ of $\tilde{K}_0(Z\pi_1(Y))$ [14]. The obstruction vanishes if and only if Y is homotopy equivalent rel. X to some $X \cup K$, where K is finite. The relative finiteness obstruction plays a key role in infinite simple homotopy theory [11].

We will reduce the relative finiteness obstruction to the ordinary finiteness obstruction and derive the relative versions of the sum and product formulas. The ideas for the sum and product formulas came from Cohen who proves similar results for Whitehead torsion in [6]. Statements of Theorem 4.15 and Lemma 4.17 are analogous to Cohen's 20.2 and 20.3. The striking similarity between formulas and results for the relative finiteness obstruction and Whitehead torsion can be partially explained by the Bass-Heller-Swan result which states that the reduced projective class group of Y injects into the Whitehead group of $Y \times S^1$ [1].

Acknowledgements

2.   Preliminaries

2.1.   Finiteness Obstruction

Let X be a connected complex. Let $Z\pi_1(X) = Z\pi$ denote the integral group ring. The *reduced projective class group of X*, $\tilde{K}_0(Z\pi_1(X))$, can be thought of as the Grothendieck group of finitely generated projective $Z\pi$-modules, modulo free modules [**13**]. Henceforth we will refer to $\tilde{K}_0(Z\pi_1(X))$ as $\tilde{K}_0(X)$. The reduced projective class group of a non-connected complex X is the direct sum of the reduced projective class groups of the components of X [**10**].

$\tilde{K}_0$ is a functor from the category of topological spaces and continuous functions to the category of abelian groups and homomorphisms [**10**]. $\tilde{K}_0$ satisfies the following properties:

1. For any space X, $\tilde{K}_0(X)$ is an abelian group.

2. If $f\colon X \to Y$ is a continuous function then there is an induced homomorphism $f_*\colon \tilde{K}_0(X) \to \tilde{K}_0(Y)$.

3. $id_* = id$ and $(gf)_* = g_* f_*$.

4. $\tilde{K}_0$ is a homotopy functor i.e., $f \simeq g$ then $f_* = g_*$.

5. $\tilde{K}_0$ has compact support i.e.,

    (a) If $\sigma \in \tilde{K}_0(X)$ then there exists a compact $C \subset X$ and $\tau \in \tilde{K}_0(C)$ such that $i_*(\tau) = \sigma$, where $i\colon C \hookrightarrow X$ is an inclusion.

    (b) If $C \subset X$ is compact, $\tau \in \tilde{K}_0(C)$, and $i_*(\tau) = 0 \in \tilde{K}_0(X)$ then there exists a compact $D \subset X$ such that $C \subset D$ and $j_*(\tau) = 0 \in \tilde{K}_0(D)$, where $j\colon C \hookrightarrow D$ is an inclusion.

    If X is a CW complex we can ensure that C and D are finite subcomplexes.

6. Let X, X', Y and Y' be finitely dominated spaces. Let $f\colon X \to X'$ and $g\colon Y \to Y'$ be maps. There exists a pairing $\cdot\colon \tilde{K}_0(X) \times \tilde{K}_0(Y) \longrightarrow \tilde{K}_0(X \times Y)$. This pairing is natural in the sense that the following diagram commutes.

$$
\begin{array}{ccccc}
\tilde{K}_0(X) & \times & \tilde{K}_0(Y) & \overset{\cdot}{\longrightarrow} & \tilde{K}_0(X \times Y) \\
f_* \downarrow & & \downarrow g_* & & \downarrow (f \times g)_* \\
\tilde{K}_0(X') & \times & \tilde{K}_0(Y') & \overset{\cdot}{\longrightarrow} & \tilde{K}_0(X' \times Y')
\end{array}
$$

Let $d\colon K \to X$ be a map. We will use $\pi_n(d)$ to denote $\pi_n(M(d), K \times 1)$, where $M(d)$ is the mapping cylinder of d. We say that d is *n-connected* if K and X are connected and $\pi_i(d) = 0$ for $1 \leqslant i \leqslant n$.

Definition 2.1 ([**13**]).   1. We say that X satisfies F_1 if π is finitely generated.

2. We say that X satisfies F_2 if π is finitely presented, and for any finite complex K^2 and map $\phi\colon K^2 \to X$ inducing an isomorphism of fundamental groups, $\pi_2(\phi)$ is a finitely generated module over $Z\pi$.

3. We say that X satisfies F_n ($n \geqslant 3$) if it satisfies F_{n-1} and for any finite complex K^{n-1} and $(n-1)$-connected map $\phi\colon K^{n-1} \to X$, $\pi_n(\phi)$ is a finitely generated $Z\pi$-module.

Definition 2.2 ([**13**]). We say that X satisfies D_n if $H_i(\tilde{X}) = 0$ for $i > n$, and $H^{n+1}(X; \mathbf{B}) = 0$ for all coefficient bundles $\mathbf{B}$.

The following is a statement of Wall's theorem of [**13**] as it appears in [**12**].

Theorem 2.3. *X is dominated by a finite complex if and only if X satisfies F_N and D_N for some $N \geqslant 3$. In this case X satisfies F_n and D_n for all $n \geqslant N$. Moreover, given any $n \geqslant N$ we can find an n-connected map $D \colon L = L^n \to X$ with L finite. In this case $\pi_{n+1}(D)$ is finitely generated, projective over $Z\pi$ and the element $\sigma(X) = (-1)^{n+1}[\pi_{n+1}(D)] \in \tilde{K}_0(X)$ depends only on the homotopy type of X. The vanishing of $\sigma(X)$ is necessary and sufficient for X to be homotopy equivalent to a finite complex.*

In the above theorem, D is an extension of a finite domination $d \colon K \to X$, and L is obtained by attaching a finite number of cells to K [**13**].

The *finiteness obstruction* is defined to be $\sigma(X) = (-1)^{n+1}[\pi_{n+1}(D)]$. Obstruction may also be defined geometrically as in [**7**].

The finiteness obstruction satisfies the following properties:

1. Invariance (Wall [**13**])
   If $f \colon X \to Y$ is a homotopy equivalence and both spaces are finitely dominated, then $f_*(\sigma(X)) = \sigma(Y)$.

2. Sum Theorem (Siebenmann [**10**])
   If $X = X_1 \cup X_2$, where X_1, X_2, $X_0 = X_1 \cap X_2$ are finitely dominated, then X is finitely dominated and

   $$\sigma(X) = j_{1*}\sigma(X_1) + j_{2*}\sigma(X_2) - j_{0*}\sigma(X_0),$$

   where $j_i \colon X_j \hookrightarrow X$ are inclusions.

3. Product Theorem (Siebenmann [**10**])
   If X_1 and X_2 are finitely dominated, connected CW complexes then $X_1 \times X_2$ is finitely dominated and

   $$\sigma(X_1 \times X_2) = \sigma(X_1) \cdot \sigma(X_2) + \chi(X_2)j_{1*}\sigma(X_1) + \chi(X_1)j_{2*}\sigma(X_2),$$

   where $j_i \colon X_i \hookrightarrow X_1 \times X_2$ are inclusions, and χ is the Euler Characteristic function.

2.2. Relative Finiteness Obstruction

We will reduce the relative finiteness obstruction to the ordinary obstruction as follows. Given a finitely dominated pair (Y, X) we construct a CW pair $(X \cup D', X)$ and a homotopy equivalence $u \colon X \cup D' \to Y$ rel. X, where D' is finitely dominated and $u(D')$ is contained in a compact subset of Y. Then $\sigma(Y, X)$ is defined to be the image of $\sigma(D')$ in $\tilde{K}_0(Z\pi_1(Y))$. This definition allows us to naturally deduce the relative versions of the sum and product theorems from the non-relative versions of [**10**].

1. Fundamental Property of Relative Finiteness Obstruction
   $\sigma(Y, X) = 0$ if and only if Y is homotopy equivalent rel. X to a finite complex.

2. Relative Invariance
   If (Y, X) is finitely dominated and $f\colon (Y, X) \to (Y_1, X_1)$ is a homotopy equivalence such that $f|_X\colon X \to X_1$ is a homotopy equivalence, then (Y_1, X_1) is finitely dominated and $f_*(\sigma(Y, X)) = \sigma(Y_1, X_1)$.

3. Relative Sum Theorem
   Suppose $Y = Y_1 \cup Y_2$, $Y_0 = Y_1 \cap Y_2$ and $X = X_1 \cup X_2$, $X_0 = X_1 \cap X_2$, where $X_1 \subset Y_1$, $X_2 \subset Y_2$, and $X_0 \subset Y_0$, and each (Y_i, X_i) is finitely dominated ($i = 0, 1, 2$). Then (Y, X) is finitely dominated and

   $$\sigma(Y, X) = j_{1*}(\sigma(Y_1, X_1)) + j_{2*}(\sigma(Y_2, X_2)) - j_{0*}(\sigma(Y_0, X_0)),$$

   where $j_i\colon Y_i \hookrightarrow Y$ are inclusions.

4. Relative Product Theorem
   If (Y_1, X_1) and (Y_2, X_2) are finitely dominated, path connected CW pairs, then $(Y_1 \times Y_2, (Y_1 \times X_2) \cup (X_1 \times Y_2))$ is finitely dominated and

   $$\sigma(Y_1 \times Y_2, (Y_1 \times X_2) \cup (X_1 \times Y_2)) = \sigma(Y_1, X_1) \cdot \sigma(Y_2, X_2) +$$
   $$\chi(Y_1, X_1) j_{2*} \sigma(Y_2, X_2) + \chi(Y_2, X_2) j_{1*} \sigma(Y_1, X_1),$$

   where $j_i\colon Y_i \hookrightarrow Y_1 \times Y_2$ are inclusions.

2.3. Miscellaneous

A *CW isomorphism* between two CW complexes Y_1 and Y_2 is a homeomorphism of Y_1 onto Y_2 such that the image of every cell of Y_1 is a cell of Y_2. A map is *cellular* provided that it sends every cell into cell(s) of equal or lower dimension.

A *direct mapping cylinder* $D(f)$ of a map $f\colon X \to X$ is the space formed from the disjoint union of $X \times [n, n+1]$, where n is an integer, by identifying $(x, n) \in X \times [n-1, n]$ with $(f(x), n) \in X \times [n, n+1]$ for each $x \in X$.
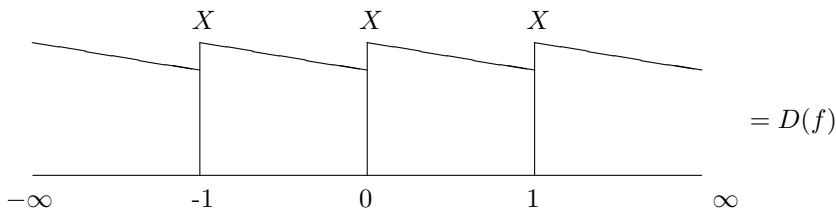
The *Hilbert Cube* is the countable infinite product

$$Q = \Pi_{i=1}^{\infty} I_i,$$

where each $I_i = [-1, 1]$. Every separable metric space can be embedded into Q. Let Q_0 denote the Hilbert Cube with one point removed. It is easy to show that if a space Z is an ANR, then there exists a proper embedding $k\colon Z \to Q_0$. (Let $\tilde{Z}$ be a one-point compactification of Z. Embed $\tilde{Z}$ into Q. Remove the image of ∞ from Q.) For more information on the Hilbert Cube, see [4].

3.   Preliminary Constructions

Definition 3.1. Let (Y, X) be a finitely dominated CW pair. Let $D = D' \cup X$, where D' is a finitely dominated subcomplex of D. If there exists a homotopy equivalence $\hat{u} \colon D \to Y$ rel. X with homotopy inverse $\hat{v}$ such that $\hat{u}(D')$ is contained in a compact set, then we say that (Y, X) is stable with respect to $(D', \hat{u}, \hat{v})$. When inverse $\hat{v}$ of $\hat{u}$ is irrelevant, we will say that (Y, X) is stable with respect to $(D', \hat{u})$ or simply, stable.

The construction that we carry out will result in $\hat{u}(D')$ being compact.

This section is devoted to the proof of the following theorem.

Theorem 3.2. *Every finitely dominated CW pair is stable.*

Let $C \subset Y$ be a finite subcomplex such that $X \cup C \hookrightarrow Y$ is a homotopy domination rel. X (i.e., $\exists$ a homotopy $h_t \colon Y \to Y$ rel. X such that $h_0 = id$ and $h_1(Y) \subset X \cup C$). Let $e = h_1|_{X \cup C} \colon X \cup C \to X \cup C$. Observe that e is the identity on X, so e is cellular on X. By [**9**] e is homotopic rel. X to a cellular map. Thus we can assume that e is cellular. Form the direct mapping cylinder $D(e)$. By 3.5 of [**6**] $D(e)$ is a CW complex.

For the purpose of Lemma 3.3 and definitions of u and v we will refer to a point of the direct mapping cylinder $D(e)$ as $(x, n + t)$, where $x \in X \cup C$, $0 \leqslant t < 1$ and n is an integer. Subsequently we will refer to points of $D(e)$ as (x, t), where $t \in R$.

Define maps

$$u \colon D(e) \longrightarrow Y, \quad u(x, n + t) = h_t(x)$$
$$v \colon Y \longrightarrow D(e), \quad v(y) = (h_1(y), 0).$$

Lemma 3.3. $uv \simeq id_Y$ *and* $vu \simeq id_{D(e)}$.

Proof. Observe that

$$uv = h_1 \simeq id.$$

Define $T \colon D(e) \longrightarrow D(e)$ by

$$T(x, n + t) = (x, n + t - 1).$$

We will first show that $vu \simeq T$.

Define $\tilde{u} \colon D(e) \longrightarrow Y \times R$ and $\tilde{v} \colon Y \times R \longrightarrow D(e)$ by

$$\tilde{u}(x, n + t) = \begin{cases} (x, n + 2t) & 0 \leqslant t \leqslant \frac{1}{2} \\ (h_{2t-1}(x), n + 1) & \frac{1}{2} \leqslant t < 1 \end{cases}$$

$$\tilde{v}(y, n + t) = \begin{cases} (h_1(y), n + 2t) & 0 \leqslant t < \frac{1}{2} \\ (h_1 h_{2-2t}(y), n + 1) & \frac{1}{2} \leqslant t < 1. \end{cases}$$

We want to show that $\tilde{v}\tilde{u} \simeq T$.

Define a homotopy $\psi_s \colon D(e) \longrightarrow D(e)$ by

$$\psi_s(x, n+t) = \begin{cases} (h_1(x), n+4t) & 0 \leqslant t < \frac{1}{4} \\ (h_1 h_{2-4t+s(4t-1)}(x), n+1) & \frac{1}{4} \leqslant t \leqslant \frac{1}{2} \\ (h_1 h_{2t-1+s(2-2t)}(x), n+1) & \frac{1}{2} \leqslant t < 1. \end{cases}$$

Then

$$\psi_0 = \tilde{v}\tilde{u}$$

$$\psi_1(x, n+t) = \begin{cases} (h_1(x), n+4t) & 0 \leqslant t < \frac{1}{4} \\ (h_1 h_1(x), n+1) & \frac{1}{4} \leqslant t < 1. \end{cases}$$

Let

$$\theta(x, n+t) = (h_1(x), n+t).$$

Then $\psi_1 \simeq \theta$ via the homotopy

$$H_s^3(x, n+t) = \begin{cases} (h_1(x), n+t+3ts) & 0 \leqslant t \leqslant \frac{1}{4} \\ r_s(h_1(x), n+t) & \frac{1}{4} \leqslant t < 1, \end{cases}$$

where

$$r_s(x, n+t) = \begin{cases} (x, n+t+s(1-t)) & 0 \leqslant s < 1 \\ (h_1(x), n+1) & s = 1. \end{cases}$$

Now we need to show that $T \simeq \theta$. Define $G_s \colon D(e) \to D(e)$ by

$$G_s(x, n+t) = \begin{cases} (x, n+t+s) & 0 \leqslant s < 1-t \\ (h_1(x), n+t+s) & 1-t \leqslant s \leqslant 1. \end{cases}$$

Observe that

$$G_0(x, n+t) = (x, n+t) = id(x, n+t)$$
$$G_1(x, n+t) = (h_1(x), n+t+1) = \theta T^{-1}(x, n+t).$$

Assuming that G is continuous we have $id \simeq \theta T^{-1}$. Therefore $T \simeq \theta$.

We now need to show that G is continuous. We will demonstrate that G is continuous on a closed segment of the direct mapping cylinder. Continuity of G will follow from the Pasting Lemma.

Let M_1 be the mapping cylinder of $e \colon X \cup C \to e(X \cup C)$. Let M_2 be the mapping cylinder of $e|_{e(X \cup C)} \colon e(X \cup C) \to e(e(X \cup C))$. Let $M = M_1 \cup M_2$ (1-level of M_1 is identified with the 0-level of M_2).

Consider the following diagram:

$$\begin{array}{ccc} ((X \cup C) \times [0,1]) \times I & \xrightarrow{F} & (X \cup C) \times [0,2] \\ \Big\downarrow{q = q_1 \times id} & & \Big\downarrow{q_2} \\ M_1 \times I & \xrightarrow{G|_{M_1 \times I}} & M \end{array}$$

where the maps are defined as follows

$$F(x, n + t, s) = (x, n + t + s)$$

$$q_1(x, n + t) = \begin{cases} (x, n + t) & n = 0 \\ (h_1(x), 1) & n = 1 \end{cases}$$

$$q_2(x, n + t) = \begin{cases} (x, n + t) & n = 0 \\ (h_1(x), n + t) & n = 1 \\ (h_1 h_1(x), 2) & n = 2. \end{cases}$$

Observe that the diagram commutes. Since q is a proper continuous surjection it follows that q is a quotient map.

To show continuity of G pick an open set $U \subset M$, then

$$q^{-1}(G^{-1}(U)) = F^{-1}(q_2^{-1}(U)) = (q_2 F)^{-1}(U)$$

but $(q_2 F)^{-1}(U)$ is open since $q_2 F$ is continuous. Therefore $q^{-1}(G^{-1}(U))$ is open. But then $G^{-1}(U)$ is open because q is a quotient map. So G is continuous.

Let $p = proj\colon Y \times R \longrightarrow Y$. Define $i\colon Y \hookrightarrow Y \times R$ by $i(y) = (y, 0)$. Let $u'(x, n + t) = (h_t(x), n + t)$.

We have the following sequence of homotopies:

$$vu = \tilde{v} i u = \tilde{v} i p u' \overset{H_s^1}{\simeq} \tilde{v} i p \tilde{u} \overset{H_s^2}{\simeq} \tilde{v} \tilde{u} = \psi_0 \overset{\psi_s}{\simeq} \psi_1 \overset{H_s^3}{\simeq} \theta \overset{G_s}{\simeq} T,$$

where

$$H_s^1(x, n + t) = \begin{cases} \tilde{v}(h_{ts}(x), 0) & 0 \leqslant t \leqslant \frac{1}{2} \\ \tilde{v}(h_{(2t-1)(1-s)+st}(x), 0) & \frac{1}{2} \leqslant t < 1 \end{cases}$$

$$H_s^2(x, n + t) = \begin{cases} \tilde{v}(x, n + 2t(1 - s)) & 0 \leqslant t \leqslant \frac{1}{2} \\ \tilde{v}(h_{(2t-1)}(x), n + (1 - s)) & \frac{1}{2} \leqslant t < 1. \end{cases}$$

Finally we need to show that $T \simeq id_{D(e)}$. Observe that T is a homeomorphism. Thus $T = T^{-1} T T \simeq T^{-1} v(uv) u \simeq T^{-1} vu \simeq T^{-1} T = id.$ $\qquad\square$

Lemma 3.4. *There exists a compact subcomplex V of $X \cup C$ such that $C \subset Int(V)$ and $h_1 h_{t_1} h_{t_2}(C) \subset Int(V)$ for all $t_1, t_2 \in I$.*

Proof. Recall that $h\colon Y \times I \to Y$ is a homotopy. Then $h(C \times I) \subset A$ for some compact $A \subset Y$. Thus $h_{t_2}(C) \subset A$ for all $t_2 \in I$.

Also, $h(A \times I) \subset B$ for some compact $B \subset Y$. Thus $h_{t_1}(A) \subset B$ for all $t_1 \in I$. Therefore there exists a compact subcomplex V such that $C \subset Int(V)$ and

$$h_1 h_{t_1} h_{t_2}(C) \subset h_1 h_{t_1}(A) \subset h_1(B) \subset Int(V) \subset X \cup C. \qquad\square$$

Constructing D'

Choose a subcomplex V of $X \cup C$ as in Lemma 3.4. Let D'' be the direct mapping cylinder of $e|_V$. Then $D'' \subset D(e)$.

Let X' be a finite subcomplex of $X \cup C$ containing V in its interior. Let D' be the direct mapping cylinder of $e' = e|_{X'}$. Then $D' \subset D(e)$. Since e' is cellular, D' is

a subcomplex of $D = X \cup D'$, where the union is taken by identifying $X' \subset X$ with $X' \times \{0\} \subset D'$. Note that since $vu(x,t) = v(h_t(x)) = (h_1 h_t(x), 0)$, we have $vu(D') \subset X' \times \{0\}$. Thus, D' is finitely dominated.

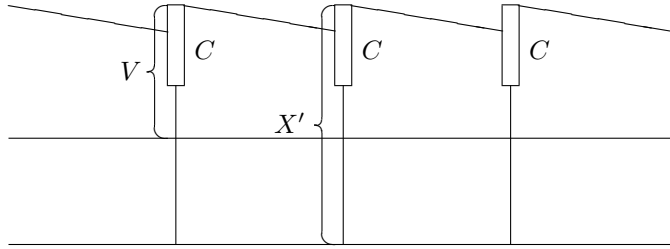Figure 1: D'

Lemma 3.5. *The homotopy* $vu \simeq id_{D(e)}$ *restricts to* $vu|_{D'} \simeq id_{D'}$.

Proof. If we use homotopies of Lemma 3.3 together with Lemma 3.4 it is easy to check that the path of each element of D' lies in D'. $\qquad\square$

Theorem 3.6. *There exists a homotopy equivalence* $\hat{u}\colon D' \cup X \to Y$ *such that* $\hat{u}(D')$ *is compact.*

Proof. We will treat $D(e)$ as $D'' \cup ((X \setminus V) \times R)$.

Choose a map $\phi\colon X' \setminus Int(V) \to [0,1]$ such that $\phi(Bd(X')) = 1$ and $\phi(Bd(V)) = 0$. Define $r\colon D(e) \to D(e)$ by

$$r(x,t) = \begin{cases} (x,t) & x \in V \\ (x,0) & x \in X \setminus X' \\ (x, \frac{1}{\phi(x)} - 1) & x \in X' \setminus V,\, t > \frac{1}{\phi(x)} - 1 \\ (x, -\frac{1}{\phi(x)} + 1) & x \in X' \setminus V,\, t < -\frac{1}{\phi(x)} + 1 \\ (x,t) & x \in X' \setminus V,\, |t| \leqslant \frac{1}{\phi(x)} - 1. \end{cases}$$

Clearly r is a strong deformation retraction, i.e., there exists a map (a horizontal push)

$$g_s\colon D(e) \longrightarrow D(e)$$

such that $g_0 = id$, $g_1(D(e)) = r(D(e))$ and $g_t|_{r(D(e))} = id$. (See Figure 2) Thus,

$$X \cup D' \overset{k}{\hookrightarrow} D(e)$$

is a homotopy equivalence with inverse r. ($kr \overset{g_t}{\simeq} id$, $rk \overset{g_t|_{X \cup D'}}{\simeq} id$.)

Let

$$\hat{u} = uk = u|_{X \cup D'}$$

then

$$\hat{u}\colon X \cup D' \longrightarrow Y$$

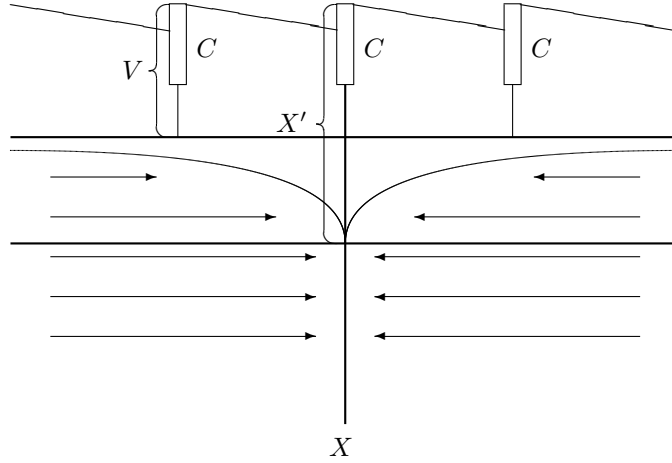is a homotopy equivalence with a homotopy inverse $v' = rv$.

Figure 2: Action of g_s on $D(e)$.

Each "cell" $X' \times I$ of the direct mapping cylinder D' is compact. Therefore $\hat{u}(X' \times I)$ is compact. Because of the way u is defined, the images of all these "cells" coincide in Y. Thus, $\hat{u}(D')$ is compact in Y. $\qquad\square$

To complete the proof of Theorem 3.2 we need to show that $\hat{u}$ is a bounded homotopy equivalence rel. X.

Theorem 3.7. *Let (Y, X) be a CW pair. Let Z be a locally compact ANR containing X' as a closed subset. Suppose $f\colon Z \to Y$ is a homotopy equivalence such that $f|_{X'}\colon X' \to X$ is a homeomorphism. Then there exists a homotopy inverse g such that $g|_X\colon X \to X'$ is a homeomorphism and $gf \simeq id_Z$ rel. X' and $fg \simeq id_Y$ rel. X.*

Proof. Let $k\colon Z \to Q_0$ be a proper embedding. Define $f'\colon Z \to Y \times Q_0$ by

$$f'(z) = (f(z), k(z)).$$

It can be shown that f' is a proper embedding. Thus $f'(Z)$ is closed in $Y \times Q_0$.

There exists some homotopy inverse g' of f. Define $r'\colon Y \times Q_0 \to f'(Z) \subset Y \times Q_0$ by

$$r' = f'g'\pi_Y,$$

where π_Y is a projection map in the Y coordinate. Let $i\colon f'(Z) \hookrightarrow Y \times Q_0$ be an inclusion. Since $ir'(y, t) = (fg'(y), kg'(y))$ and $fg' \simeq id$ we have $ir' \simeq id$.

Let G be a homotopy such that $G_0 = id$ and $G_1 = g'f$. Let $F_t = f'G_t(f')^{-1}|_{f'(Z)}$ then for any $(f(z), k(z)) \in f'(Z)$ we have:

$$F_0(f(z), k(z)) = f'G_0(f')^{-1}|_{f'(Z)}(f(z), k(z)) = f'G_0(z) = f'(z) = (f(z), k(z))$$

$$\begin{aligned}
F_1(f(z), k(z)) &= f'G_1(z) = f'g'f(z) = f'g'\pi_Y(f(z), k(z)) \\
&= r'|_{f'(Z)}(f(z), k(z)) = r'i(f(z), k(z)).
\end{aligned}$$

Therefore $r'i \simeq id$.

$f'(Z)$ is closed in $Y \times Q_0$ therefore, by the Homotopy Extension Theorem, there exists a homotopy $H \colon (Y \times Q_0) \times I \to Y \times Q_0$ such that $H_1 = r'$ and $H_t|_{f'(Z)} = F_t$.

Let

$$r = H_0.$$

Then $r' \simeq r$ and $ri = id_{f'(Z)}$. Thus, r is a retraction homotopic to the identity. Using the proof of Theorem 3.1 of [**3**] we can ensure that $r \simeq id$ rel. $f'(Z)$.

Let $w \colon Y \to Q_0$ be a map which extends $kf|_X^{-1}$. Define $j \colon Y \to Y \times Q_0$ by

$$j(y) = (y, w(y)).$$

Since $Q_0 \cong Q \times [0,1)$ is convex and $wf|_{X'} = k|_{X'}$, there is a straight line homotopy $wf \simeq k$ rel. X'. $f'(z) = (f(z), k(z))$ and $jf(z) = (f(z), wf(z))$. Since the first coordinates agree and other coordinates are joined by a homotopy rel. X' we conclude that $jf \simeq f'$ rel. X'.

Define $g \colon Y \to Z$ by

$$g = (f')^{-1}rj.$$

Then for any $x \in X \subset Y$ we have

$$
\begin{aligned}
g(x) &= (f')^{-1}rj(x) \\
&= (f')^{-1}r(x, w(x)) \\
&= (f')^{-1}r(f(f|_X^{-1}(x)), k(f|_X^{-1}(x))) \\
&= (f')^{-1}rf'(f|_X^{-1}(x)) \\
&= (f')^{-1}f'(f|_X^{-1}(x)) \\
&= f|_X^{-1}(x).
\end{aligned}
$$

Thus, $g|_X \colon X \to X'$ is a homeomorphism.

We will now show that $fg \simeq id_Y$ rel. X and $gf \simeq id_Z$ rel. X'. rj takes Y to $f'(Z)$. Since f' is an embedding, every point of $f'(Z)$ can be expressed as $(f(z), k(z))$ in terms of some $z \in Z$. But then we have

$$f(f')^{-1}(f(z), k(z)) = f(z) = \pi_Y(f(z), k(z));$$

therefore,

$$fg = f(f')^{-1}rj = \pi_Y rj.$$

$r \simeq id$ therefore

$$\pi_Y rj \simeq \pi_Y j.$$

This homotopy can be chosen to be rel. X because for all $x \in X$ we have

$$j(x) = (x, w(x)) = (f(f|_X^{-1}(x)), k(f|_X^{-1}(x))) = f'(f|_X^{-1}(x)) \in f'(Z)$$

and the homotopy $r \simeq id$ is rel. $f'(Z)$. But $\pi_Y j = id$; therefore, $fg \simeq id$ rel. X.

Recall that $jf \simeq f'$ rel. X' and that $r|_{f'(Z)} = id$. We have

$$gf = (f')^{-1}rjf \simeq (f')^{-1}rf' = (f')^{-1}f' = id$$

with the homotopy taking place rel. X'. □

Let $\hat{v}$ be a homotopy inverse of $\hat{u}$ as in Theorem 3.7. Then (Y, X) is stable with respect to $(D', \hat{u}, \hat{v})$. This completes the proof of Theorem 3.2.

4. Relative Finiteness Obstruction: Definition and Properties

Definition 4.1. Let (Y, X) be a finitely dominated CW pair stable with respect to (D', u, v). Then we have

$$D' \xrightarrow{\ i\ } D \xrightarrow{\ u\ } Y,$$

where i is an inclusion map. This induces

$$\tilde{K}_0(D') \xrightarrow{\ i_*\ } \tilde{K}_0(D) \xrightarrow{\ u_*\ } \tilde{K}_0(Y).$$

D' is finitely dominated, so $\sigma(D') \in \tilde{K}_0(D')$. Define the relative finiteness obstruction $\sigma(Y, X)$ by

$$\sigma(Y, X) = u_* i_* \sigma(D').$$

Lemma 4.2. *Let* $D = D' \cup X$ *be as in Definition 4.1. Then there exists a map* $\alpha \colon D \to D$ *such that* $\alpha \overset{F_t}{\simeq} id$ *rel.* X *and* $F_t(D') \subset D'$, $\alpha(D) \subset X \cup C$, *where* C *is compact in* D'.

Proof. D' is finitely dominated, therefore there exists a finite subcomplex P of D' and a map $\beta \colon D' \to P$ such that $i\beta \overset{f_t}{\simeq} id$ ($f_0 = id$, $f_1 = i\beta$).

Let $X' = D' \cap X$. We can modify $\beta = i\beta \colon D' \to D'$ so that $\beta \simeq id_{D'}$ rel. X'. Choose finite subcomplexes V and W of D' so that $X' \subset Int(V) \subset V \subset Int(W)$. Choose a map $\phi \colon W \setminus Int(V) \to [0, 1]$ such that $\phi(Bd(W)) = 1$ and $\phi(Bd(V)) = 0$. Define $\tilde{f}_t \colon D' \to D'$ by

$$\tilde{f}_t(x) = \begin{cases} f_t(x) & x \in D' \setminus Int(W) \\ x & x \in V \\ f_{\phi(x)t}(x) & x \in W \setminus Int(V). \end{cases}$$

Observe that $\tilde{f}_1 \simeq id$ rel. X'. Extend $\tilde{f}_1$ to $\alpha \colon D \to D$ by the identity. Then $\alpha \simeq id$ rel. X. Let $C = P \cup W$, then $\alpha(D) \subset X \cup C$. $\square$

Lemma 4.3. *Let* $D_1 = D'_1 \cup X_1$, $D_2 = D'_2 \cup X_2$ *be as in Definition 4.1. Suppose* $\gamma \colon D_1 \to D_2$ *is a homotopy equivalence such that* $\gamma|_{X_1} \colon X_1 \to X_2$ *is a homeomorphism. Then there exists a homotopy equivalence* $f' \colon D_1 \to D_2$ *with inverse* g *such that*

(1) $gf' \overset{F_t}{\simeq} id_{D_1}$ *rel.* X_1, *and* $F_t(D'_1) \subset D'_1 \cup K_1$, *where* K_1 *is compact in* X_1.

(2) $f'g \overset{G_t}{\simeq} id_{D_2}$ *rel.* X_2, *and* $G_t(D'_2) \subset D'_2 \cup K_2$, *where* K_2 *is compact in* X_2.

Proof. We will prove (1), the proof of (2) is similar. For $i = 1, 2$ let $\alpha_i \colon D_i \to D_i$ be a homotopy equivalence rel. X_i such that $\alpha_i(D'_i) \subset C_i$, where C_i is compact in D'_i as in Lemma 4.2. Let λ^i_t be a homotopy joining each α_i with id_{D_i} ($\lambda^i_0 = id$, $\lambda^i_1 = \alpha_i$).

Let τ be a homotopy inverse of γ as in Theorem 3.7 with $\tau\gamma \overset{T_t}{\simeq} id_{D_1}$ rel. X_1 ($T_0 = id$, $T_1 = \tau\gamma$) and $\gamma\tau \simeq id_{D_2}$ rel. X_2. Let

$$f' = \gamma\alpha_1$$
$$g = \tau\alpha_2.$$

Clearly $gf' = \tau\alpha_2\gamma\alpha_1 \simeq id$ rel. X_1 and $f'g = \gamma\alpha_1\tau\alpha_2 \simeq id$ rel. X_2.

Define $F_t \colon D_1 \to D_1$ by

$$F_t(x) = \begin{cases} \lambda^1_{3t}(x) & 0 \leqslant t \leqslant 1/3 \\ T_{3t-1}\alpha_1(x) & 1/3 \leqslant t \leqslant 2/3 \\ \tau\lambda^2_{3t-2}\gamma\alpha_1(x) & 2/3 \leqslant t \leqslant 1. \end{cases}$$

It is easy to check that F_t is continuous and that $F_0 = id$ and $F_1 = \tau\alpha_2\gamma\alpha_1 = gf'$.

Now we show that $F_t(D_1') \subset D_1' \cup K_1$ for some compact $K_1 \subset X_1$. For $t \in [0, \frac{1}{3}]$ the homotopy stays in D_1', as in Lemma 4.2. For $t \in [\frac{1}{3}, \frac{2}{3}]$ observe that $\alpha_1(D_1') \subset C_1'$. Therefore $T_t\alpha_1(D_1') \subset T(C_1 \times I)$ which is compact. Finally, $\gamma\alpha_1(D_1')$ is contained in some compact set $C \subset D_2$. Therefore $\lambda^2_t\gamma\alpha_1(D_1') \subset \lambda^2(C \times I)$ which is compact in D_2. Thus, $\tau\lambda^2_t\gamma\alpha_1(D_1') \subset \tau\lambda^2(C \times I)$ which is compact in D_1. $\qquad\square$

Corollary 4.4. *Let (Y_1, X_1) and (Y_2, X_2) be finitely dominated CW pairs. Suppose (Y_1, X_1) is stable with respect to (D_1', u_1, v_1), and (Y_2, X_2) is stable with respect to (D_2', u_2, v_2), and suppose that there exists a homotopy equivalence $f \colon Y_1 \to Y_2$ such that $f|_{X_1} \colon X_1 \to X_2$ is a homeomorphism. Then there exists a homotopy equivalence $f' \colon D_1 \to D_2$ with homotopy inverse g such that conditions (1) and (2) of Lemma 4.3 are satisfied.*

Proof. Let $\gamma = v_2fu_1$ and apply Lemma 4.3. $\qquad\square$

Corollary 4.5. *Let all spaces and maps be as in Lemma 4.3 and Corollary 4.4. In addition assume that $f|_{X_1} \colon X_1 \to X_2$ is a CW isomorphism. Then there exist a finite subcomplex $P_1 \subset X_1$ and a finite subcomplex $P_2 \subset X_2$ such that $f'|_{D_1' \cup P_1} \colon D_1' \cup P_1 \to D_2' \cup P_2$ is a homotopy equivalence with homotopy inverse*

$$g|_{D_2' \cup P_2} \colon D_2' \cup P_2 \to D_1' \cup P_1.$$

Proof. By Lemma 4.3 and Corollary 4.4 $f' = v_2fu_1\alpha_1$. Find a compact subcomplex P_1 of X_1 such that $K_1 \subset P_1$ and $K_2 \subset f'(P_1)$. Observe that $\alpha_1|_{X_1}$, $u_1|_{X_1}$ and $v_2|_{X_2}$ are identity maps. Thus $f'(P_1)$ is a subcomplex. Let $P_2 = f'(P_1)$. $\qquad\square$

Theorem 4.6. *Let all spaces and maps be as in Corollary 4.5. Then*

$$f_*u_{1*}i_{1*}\sigma(D') = u_{2*}i_{2*}\sigma(D_1'),$$

where i_1, i_2 are inclusions.

Proof. By Corollary 4.5 there exist compact $P_i \subset X_i$, $i = 1, 2$, such that

$$v_2fu_1\alpha_1|_{D_1' \cup P_1} \colon D_1' \cup P_1 \to D_2' \cup P_2$$

is a homotopy equivalence rel. X_1. Consider the following diagram where j_1, j_2, k_1,

k_2 are inclusions. (Note that $i_1 = k_1 j_1$ and $i_2 = k_2 j_2$.)

$$
\begin{array}{ccc}
D_1' & & D_2' \\
j_1 \downarrow & & \downarrow j_2 \\
D_1' \cup P_1 & \xrightarrow{v_2 f u_1 \alpha_1|_{D_1' \cup P_1}} & D_2' \cup P_2 \\
k_1 \downarrow & & \downarrow k_2 \\
D_1 & \xrightarrow{v_2 f u_1 \alpha_1} & D_2 \\
u_1 \alpha_1 \downarrow \uparrow v_1 & & u_2 \downarrow \uparrow v_2 \\
Y_1 & \xrightarrow{f} & Y_2
\end{array}
$$

This gives rise to

$$
\begin{array}{ccc}
\tilde{K}_0(D_1') & & \tilde{K}_0(D_2') \\
j_{1*} \downarrow & & \downarrow j_{2*} \\
\tilde{K}_0(D_1' \cup P_1) & \xrightarrow{(v_2 f u_1 \alpha_1|_{D_1' \cup P_1})_*} & \tilde{K}_0(D_2' \cup P_2) \\
k_{1*} \downarrow & & \downarrow k_{2*} \\
\tilde{K}_0(D_1) & \xrightarrow{(v_2 f u_1 \alpha_1)_*} & \tilde{K}_0(D_2) \\
(u_1 \alpha_1)_* \downarrow \uparrow v_{1*} & & u_{2*} \downarrow \uparrow v_{2*} \\
\tilde{K}_0(Y_1) & \xrightarrow{f_*} & \tilde{K}_0(Y_2)
\end{array}
$$

Each P_i is compact, therefore $\sigma(P_i) = \sigma(D_i' \cap P_i) = 0$. Thus by the Sum Theorem [10] $\sigma(D_i' \cup P_i) = j_{i*}\sigma(D_i')$. Since $v_2 f u_1 \alpha_1|_{D_1' \cup P_1}$ is a homotopy equivalence $(v_2 f u_1 \alpha_1|_{D_1' \cup P_1})_* \sigma(D_1' \cup P_1) = \sigma(D_2' \cup P_2)$ which gives us:

$$(v_2 f u_1 \alpha_1|_{D_1' \cup P_1})_* j_{1*} \sigma(D_1') = j_{2*} \sigma(D_2')$$
$$k_{2*}(v_2 f u_1 \alpha_1|_{D_1' \cup P_1})_* j_{1*} \sigma(D_1') = k_{2*} j_{2*} \sigma(D_2') = i_{2*} \sigma(D_2')$$
$$(v_2 f u_1 \alpha_1)_* k_{1*} j_{1*} \sigma(D_1') = i_{2*} \sigma(D_2')$$
$$v_{2*} f_* u_{1*} i_{1*} \sigma(D_1') = i_{2*} \sigma(D_2')$$
$$u_{2*} v_{2*} f_* u_{1*} i_{1*} \sigma(D_1') = u_{2*} i_{2*} \sigma(D_2')$$
$$f_* u_{1*} i_{1*} \sigma(D_1') = u_{2*} i_{2*} \sigma(D_2'). \qquad \square$$

The following corollary shows that $\sigma(Y, X)$ is well defined by demonstrating that $\sigma(Y, X)$ is independent of the choice of D'.

Corollary 4.7. *Suppose (Y, X) is a finitely dominated CW pair stable with respect to (D', u, v) and (D_1', u_1, v_1). Then $u_* i_* \sigma(D') = u_{1*} i_{1*} \sigma(D_1')$.*

Proof. In the above proof let $Y_1 = Y_2 = Y$ and let $f = id$. □

Corollary 4.8 (Weak Invariance). *Let (Y_1, X_1) and (Y_2, X_2) be finitely dominated CW pairs. Suppose there exists a homotopy equivalence $f \colon Y_1 \to Y_2$ such that*

$$f|_{X_1} \colon X_1 \to X_2$$

is a CW isomorphism. Then $f_* \sigma(Y_1, X_1) = \sigma(Y_2, X_2)$.

Lemma 4.9. *Let (Z, Z_0) be a CW pair such that Z_0 is finite. Suppose Z is homotopy equivalent to some finite CW complex K. Then there exists a finite CW complex P such that $Z_0 \subset P$ and there is a homotopy equivalence $P \to Z$ rel. Z_0.*

Proof. There exist maps $f \colon Z \to K$, $g \colon K \to Z$ such that $gf \simeq id$ and $fg \simeq id$. Let $P = M(f|_{Z_0})$. Consider maps

$$P \xrightarrow{\ r\ } K \xrightarrow{\ g\ } Z,$$

where r is the collapse to the base along the rays of the mapping cylinder. Since g and r are both homotopy equivalences, gr is also a homotopy equivalence.

Observe that $gr|_{Z_0} = gf|_{Z_0} \simeq id_{Z_0}$. By the Homotopy Extension Theorem gr is homotopic to some g' such that $g'|_{Z_0} = id_{Z_0}$. $g'|_{Z_0} \colon Z_0 \to Z_0$ is a homeomorphism. Thus, by Proposition 3.2 of [**3**], there exists f' such that $f'g' \simeq id$ rel. Z_0 and $g'f' \simeq id$ rel. Z_0. □

Theorem 4.10 (Fundamental Property of the Relative Finiteness Obstruction). *Let (Y, X) be a finitely dominated CW pair. Then $\sigma(Y, X) = 0$ if and only if $X \hookrightarrow Y$ can be extended to a homotopy equivalence $X \cup P \to Y$ rel. X where P is a finite CW complex such that $P \cap X$ is a subcomplex of both X and P.*

Proof. Suppose $X \hookrightarrow Y$ can be extended to a homotopy equivalence $u \colon X \cup P \to Y$. Since P is a finite CW complex, (Y, X) is stable with respect to (P, u). Thus $\sigma(Y, X) = u_* i_* \sigma(P)$. But P is finite, therefore $\sigma(P) = 0$ and $\sigma(Y, X) = 0$.

Conversely, assume $\sigma(Y, X) = 0$. Suppose that (Y, X) is stable with respect to (D', u, v). Then we have:

$$u_* i_* \sigma(D') = 0$$
$$v_* u_* i_* \sigma(D') = 0$$
$$(vu)_* i_* \sigma(D') = 0$$
$$i_* \sigma(D') = 0 \in \tilde{K}_0(D).$$

Since D' is finitely dominated, $\sigma(D') \in \tilde{K}_0(D')$. There exists a compact subset A of D', an inclusion map $j \colon A \hookrightarrow D'$ and $\tau \in \tilde{K}_0(A)$ such that $\sigma(D') = j_*(\tau)$. Thus $i_* \sigma(D') = i_* j_*(\tau) = 0 \in \tilde{K}_0(D)$.

There exists a compact set B such that $A \subset B \subset D$ and an inclusion map $\alpha \colon A \hookrightarrow B$ such that $\alpha_*(\tau) = 0$. Without loss of generality we may assume that $X \cap D' \subset B$.

By the Sum Theorem

$$\sigma(D' \cup B) = \omega_{1*}\sigma(D') + \omega_{2*}\sigma(B) - \omega_{3*}\sigma(D' \cap B),$$

where ω_1, ω_2 and ω_3 are appropriate inclusions. But B and $D' \cap B$ are compact therefore $\omega_{2*}\sigma(B) = \omega_{3*}\sigma(D' \cap B) = 0$.

To show that $\omega_{1*}\sigma(D') = 0$ recall that $\sigma(D') = j_*(\tau)$. Observe that $\omega_1 j = \beta\alpha$, where $\beta\colon B \hookrightarrow D' \cup B$ is an inclusion. Thus,

$$\beta_*\alpha_*(\tau) = \omega_{1*}j_*(\tau)$$

but $\alpha_*(\tau) = 0$; therefore,

$$\omega_{1*}\sigma(D') = \omega_{1*}j_*(\tau) = \beta_*\alpha_*(\tau) = 0.$$

Thus, $\sigma(D' \cup B) = 0$ and $D' \cup B$ is homotopy equivalent to some finite CW complex K.

Let $Z = D' \cup B$ and $Z_0 = (D' \cup B) \cap X$. By Lemma 4.9 there exists a finite CW complex P such that $Z_0 \subset P$, and there is a homotopy equivalence $P \to Z$ rel. Z_0. We can extend this homotopy equivalence to the rest of X by the identity to obtain a homotopy equivalence $P \cup X \to Z \cup X = D$ rel. X. But $D \simeq Y$ rel. X. Therefore $P \cup X \simeq Y$ rel. X. $\square$

Theorem 4.11. *Let (Y, X) be a CW pair. Suppose Y is finitely dominated rel. X, and X is finitely dominated, then Y is finitely dominated and*

$$\sigma(Y) = \sigma(Y, X) + i_*\sigma(X),$$

where $i\colon X \hookrightarrow Y$ is an inclusion map.

Proof. There exists a finite subcomplex C of Y such that the inclusion $i\colon X \cup C \hookrightarrow Y$ is a homotopy domination. C and $X \cap C$ are finite therefore they are finitely dominated. By the Sum Theorem $X \cup C$ must be finitely dominated. Thus there exists a finite CW complex P such that $P \xrightarrow{f} X \cup C$ is a homotopy domination. But then $if\colon P \to Y$ is also a homotopy domination.

$\sigma(Y, X) = u_*j_*\sigma(D')$ where D' and u are such that (Y, X) is stable with respect to (D', u), and $j\colon D' \hookrightarrow D' \cup X$ is an inclusion. It follows from the Sum Theorem that $D' \cup X$ is finitely dominated and

$$\sigma(D' \cup X) = j_*\sigma(D') + k_*\sigma(X) - 0,$$

where $k\colon X \hookrightarrow D' \cup X$ is an inclusion.

Applying u_* to both sides we get

$$u_*\sigma(D' \cup X) = u_*j_*\sigma(D') + u_*k_*\sigma(X).$$

But u is a homotopy equivalence therefore $u_*\sigma(D' \cup X) = \sigma(Y)$. Since u is the identity on X we have $u_*k_*\sigma(X) = i_*\sigma(X)$ which yields the desired result. $\square$

Corollary 4.12. *If Y is finitely dominated with respect to $\emptyset$, then Y is finitely dominated and $\sigma(Y, \emptyset) = \sigma(Y)$.*

Lemma 4.13. *Let (Y, X) and (Y', X') be CW pairs. Suppose Y' is finitely dominated rel. X' and there exists a homotopy equivalence $f\colon Y' \to Y$ such that $f|_{X'}\colon X' \to X$ is a homeomorphism. Then Y is finitely dominated rel. X.*

Proof. By Theorem 3.7 there exists a homotopy inverse $g\colon Y \to Y'$ such that $gf \simeq id_{Y'}$ rel. X', $fg \simeq id_Y$ rel. X, and $g|_X = f^{-1}|_X$.

Y' is finitely dominated rel. X', therefore there exists a compact subcomplex C' of Y' such that the inclusion $i\colon X' \cup C' \hookrightarrow Y'$ is a homotopy domination rel. X'. Let h be homotopy inverse of i. Then there exists a homotopy H_t such that $H_0 = id_{Y'}$, $H_1 = ih$ and $H_t|_{X'} = id$.

Define G_t by

$$G_t = fH_t g\colon Y \longrightarrow Y.$$

Then

$$G_0 = fg \simeq id_Y \quad rel.\, X$$

and

$$G_1(Y) = fhg(Y) \subset fh(Y') \subset f(X' \cup C') = X \cup f(C'). \qquad \square$$

Theorem 4.14 (Relative Sum Theorem)**.** *Let (Y_1, X_1), (Y_2, X_2) be finitely dominated CW pairs. Let $Y = Y_1 \cup Y_2$, $X = X_1 \cup X_2$, $Y_0 = Y_1 \cap Y_2$, $X_0 = X_1 \cap X_2$. Suppose that (Y_0, X_0) is finitely dominated. Then (Y, X) is finitely dominated and*

$$\sigma(Y, X) = j_{1*}\sigma(Y_1, X_1) + j_{2*}\sigma(Y_2, X_2) - j_{0*}\sigma(Y_0, X_0),$$

where $j_i\colon Y_i \hookrightarrow Y$ ($i = 0,\, 1,\, 2$) are inclusions.

Proof. Case 1. $X_0 = X \cap Y_0$.

For $i = 0,\, 1,\, 2$ we have $h_t^i\colon Y_i \to Y_i$ such that $h_0^i = id_{Y_i}$, $h_1^i(Y_i) \subset C_i \cup X_i$ where each C_i is compact in Y_i, and $h_t^i|_{X_i} = id_{X_i}$.

Using the Homotopy Extension Theorem we can extend h_t^0 to $k_t^i\colon Y_i \to Y_i$ ($i = 1, 2$) in such a way that $k_t^i|_{X_i} = id$.

Next we want to modify h_t^i ($i = 1,\, 2$) so that $h_t^i|_{X_0 \cup C_0} = id$. Observe that we already have $h_t^i|_{X_0} = id$ ($i = 1,\, 2$). We will perform the modification of h_t^1. Modification of h_t^2 is similar.

There exist finite subcomplexes V and K of Y_1 such that $C_0 \subset Int(V)$ and $V \subset Int(K)$. Choose a map $\phi\colon K \setminus Int(V) \to [0,1]$ such that $\phi(Bd(K)) = 1$ and $\phi(Bd(V)) = 0$.

Define

$$\tilde{h}_t^1(y) = \begin{cases} h_t^1(y) & y \in Y_1 \setminus Int(K) \\ y & y \in V \\ h_{\phi(y)t}^1(y) & y \in K \setminus Int(V). \end{cases}$$

Observe that $\tilde{h}_0^1 = id$, $\tilde{h}_t^1|_{X_1 \cup C_0} = id$ and $\tilde{h}_1^1(Y_1) \subset P_1 \cup X_1$ for some compact $P_1 \subset Y_1$. Similarly, we can modify h_t^2 so that $\tilde{h}_0^2 = id$, $\tilde{h}_t^2|_{X_2 \cup C_0} = id$ and $\tilde{h}_1^2(Y_2) \subset P_2 \cup X_2$ for some compact $P_2 \subset Y_2$.

Define

$$f_t^i(y) = \begin{cases} k_{2t}^i(y) & 0 \leqslant t \leqslant 1/2 \\ \tilde{h}_{2t-1}^i k_1^i(y) & 1/2 \leqslant t \leqslant 1. \end{cases}$$

Then f^i is continuous and $f_t^1 = f_t^2$ for all $y \in Y_0$.

Let

$$f_t(y) = \begin{cases} f_t^1(y) & y \in Y_1 \\ f_t^2(y) & y \in Y_2. \end{cases}$$

Observe that $f_1(Y) \subset (P_1 \cup P_2) \cup X$ and the inclusion $(P_1 \cup P_2) \cup X \hookrightarrow Y$ is a homotopy domination rel. X.

There exists a finite CW complex M such that $P_1 \cup P_2 \subset M$. Then $M \cup X \hookrightarrow Y$ is a homotopy domination rel. X. Let $M_i = M \cap Y_i$, then $M_i \cup X_i \hookrightarrow Y_i$ is a homotopy domination rel. X_i because $P_i \subset M_i$.

Using the mapping cylinder constructions we can define D_i and D_i' $(i = 0, 1, 2)$ such that (Y_i, X_i) is stable with respect to (D_i', u_i) and $D_1 \cap D_2 = D_0$.

Recall that $u_i(y, t) = f_t^i(y)$ $(i = 0, 1, 2)$. Since $f_t^1 = f_t^2$ for all $y \in Y_0$, u_1 agrees with u_2 on Y_0. Thus we can define u by

$$u(y) = \begin{cases} u_1(y) & y \in D_1 \\ u_2(y) & y \in D_2. \end{cases}$$

Then $u(y, t) = f_t(y)$. Let $D = D_1 \cup D_2$ and $D' = D_1' \cup D_2'$. Then (Y, X) is stable with respect to (D', u).

Let the following be inclusions:

$$D_i' \xrightarrow{\delta_i} D' \xrightarrow{\theta} D$$

$$D_i' \xrightarrow{\alpha_i} D_i \xrightarrow{\beta_i} D.$$

Then

$$\begin{aligned}
\sigma(Y_1 \cup Y_2, X_1 \cup X_2) &= u_* \theta_* \sigma(D') \\
&= u_* \theta_* \delta_{1*} \sigma(D_1') + u_* \theta_* \delta_{2*} \sigma(D_2') - u_* \theta_* \delta_{0*} \sigma(D_0') \\
&= u_* \beta_{1*} \alpha_{1*} \sigma(D_1') + u_* \beta_{2*} \alpha_{2*} \sigma(D_2') - u_* \beta_{0*} \alpha_{0*} \sigma(D_0') \\
&= j_{1*}(u|_{D_1})_* \alpha_{1*} \sigma(D_1') + j_{2*}(u|_{D_2})_* \alpha_{2*} \sigma(D_2') \\
&\quad - j_{0*}(u|_{D_0})_* \alpha_{0*} \sigma(D_0') \\
&= j_{1*} u_{1*} \alpha_{1*} \sigma(D_1') + j_{2*} u_{2*} \alpha_{2*} \sigma(D_2') - j_{0*} u_{0*} \alpha_{0*} \sigma(D_0') \\
&= j_{1*} \sigma(Y_1, X_1) + j_{2*} \sigma(Y_2, X_2) - j_{0*} \sigma(Y_0, X_0).
\end{aligned}$$

Case 2. $X_0 \neq X \cap Y_0$.

We will separate those parts of X_1 and X_2 which do not intersect in Y_0 and apply Case 1.

Consider $Y_1 \times [0, 1]$. We will identify Y_1 with $Y_1 \times \{0\}$. Construct a reduced product $Y_1 \times_{X_0} [0, 1]$ by identifying (x, t) with $(x, 0)$ for all $x \in X_0$ and $t \in [0, 1]$. Observe that $Y_1 \times_{X_0} [0, 1]$ is a CW complex and $X_1 \times \{0\}$, $X_1 \times \{1\}$ are subcomplexes. Clearly $Y_1 \times_{X_0} [0, 1]$ is finitely dominated rel. $X_1 \times \{0\}$.

There exists a homeomorphism $f \colon Y_1 \times_{X_0} [0, 1] \to Y_1 \times_{X_0} [0, 1]$ such that $f(y, 0) = (y, 1)$ for all $y \in Y_1$. $f|_{X_1 \times \{0\}} \colon X_1 \times \{0\} \to X_1 \times \{1\}$ is a homeomorphism. Thus, by Lemma 4.13, $Y_1 \times_{X_0} [0, 1]$ is finitely dominated rel. $X_1 \times \{1\}$.

We can repeat this process for $Y_2 \times_{X_0} [-1, 0]$. Then $Y_2 \times_{X_0} [-1, 0]$ is finitely dominated rel. $X_2 \times \{-1\}$.

Let

$$Y_1' = Y_1 \times_{X_0} [0,1]$$
$$Y_2' = Y_2 \times_{X_0} [-1,0]$$
$$Y' = Y_1' \cup Y_2'$$
$$X_1' = X_1 \times \{1\}$$
$$X_2' = X_2 \times \{-1\}$$
$$X' = X_1' \cup X_2'.$$

Observe that $Y_1' \cap Y_2' = Y_0$ and $X_1' \cap X_2' = X_0$. Since Y_i' is finitely dominated rel. X_i' $(i = 1, 2)$ and Y_0 is finitely dominated rel. X_0, by Case 1 we know that Y' is finitely dominated rel. X'.

Let $p\colon Y' \to Y$ be the projection to the zero level. Then p is a homotopy equivalence such that its restriction to X' is a homeomorphism. By Lemma 4.13, Y is finitely dominated rel. X.

By Case 1,

$$\sigma(Y', X') = k_{1*}\sigma(Y_1', X_1') + k_{2*}\sigma(Y_2', X_2') - k_{0*}\sigma(Y_0', X_0').$$

Finally, by Weak Invariance

$$\begin{aligned}
\sigma(Y, X) &= p_*\sigma(Y', X') \\
&= j_{1*}(p|_{Y_1'})_*\sigma(Y_1', X_1') + j_{2*}(p|_{Y_2'})_*\sigma(Y_2', X_2') - j_{0*}(p|_{Y_0'})_*\sigma(Y_0', X_0') \\
&= j_{1*}\sigma(Y_1, X_1) + j_{2*}\sigma(Y_2, X_2) - j_{0*}\sigma(Y_0, X_0),
\end{aligned}$$

where k_i, j_i $(i = 0, 1, 2)$ are appropriate inclusions. $\square$

Theorem 4.15. *Let (Z, Y, X) be a CW triple. Suppose Z is finitely dominated rel. Y and Y is finitely dominated rel. X. Then Z is finitely dominated rel. X and*

$$\sigma(Z, X) = \sigma(Z, Y) + i_*\sigma(Y, X),$$

where $i\colon Y \hookrightarrow Z$ is an inclusion map.

Proof. There exists a compact subcomplex C of Z such that the inclusion $i\colon Y \cup C \hookrightarrow Z$ is a homotopy domination rel. Y. We are given that Y is finitely dominated rel. X. C is finite, therefore C is finitely dominated. Thus, by the Relative Sum Theorem, $Y \cup C$ is finitely dominated rel. X. $X \subset Y$ therefore Z is finitely dominated rel. X.

$\sigma(Z, Y) = u_*j_*\sigma(D')$ where D' and u are such that (Z, Y) is stable with respect to (D', u) and $j\colon D' \hookrightarrow D' \cup Y$ is an inclusion.

D' is finitely dominated, therefore D' is finitely dominated rel. $\emptyset$. Y is finitely dominated rel. X. $D' \cap Y$ is compact. So it follows from the Relative Sum Theorem that $D' \cup Y$ is finitely dominated rel. $\emptyset \cup X = X$ and

$$\sigma(D' \cup Y, X) = j_*\sigma(D') + k_*\sigma(Y, X) - 0,$$

where $k\colon Y \hookrightarrow D' \cup Y$ is an inclusion.

Applying u_* to both sides we get

$$u_*\sigma(D' \cup Y, X) = u_* j_* \sigma(D') + u_* k_* \sigma(Y, X).$$

But u is a homotopy equivalence such that $u|_X = id$, therefore by Weak Invariance $u_*\sigma(D' \cup Y, X) = \sigma(Z, X)$. Since u is the identity on Y we have $u_* k_* \sigma(Y, X) = i_*\sigma(Y, X)$ which yields the desired result. $\qquad\square$

Corollary 4.16. *Let $(Y_1, Y_2, \ldots, Y_k)$ be a CW k-tuple. Suppose that for $n = 1, \ldots, k-1$ each Y_n is finitely dominated rel. Y_{n+1}. Then Y_1 is finitely dominated rel. Y_k and*

$$\sigma(Y_1, Y_k) = \sigma(Y_1, Y_2) + \sum_{n=2}^{k-1} i_{n*}\sigma(Y_n, Y_{n+1}),$$

where $i_n \colon Y_n \hookrightarrow Y_1$ are inclusions.

Lemma 4.17. *Suppose (Y, X) and (Z, X) are CW pairs such that $Y \cap Z = X$ and Y is finitely dominated rel. X, then $(Y \cup Z, Z)$ is finitely dominated and $\sigma(Y \cup Z, Z) = i_*\sigma(Y, X)$ where $i \colon Y \hookrightarrow Y \cup Z$ is an inclusion.*

Proof. $(Y \cup Z, Z)$ can be written as $(Y \cup Z, X \cup Z)$. Observe that (Y, X), (Z, Z) and $(Y \cap Z, X \cap Z)$ are finitely dominated. Thus, by the Relative Sum Theorem, $(Y \cup Z, X \cup Z)$ is finitely dominated and we have:

$$\begin{aligned}
\sigma(Y \cup Z, Z) &= \sigma(Y \cup Z, X \cup Z) \\
&= i_*\sigma(Y, X) + j_*\sigma(Z, Z) - k_*\sigma(Y \cap Z, X \cap Z) \\
&= i_*\sigma(Y, X) + 0 - 0 \\
&= i_*\sigma(Y, X).
\end{aligned}$$
$\qquad\square$

Lemma 4.18. *Let (Y, X) and (Y_1, X_1) be CW pairs such that (Y, X) is finitely dominated. Suppose that X is a subcomplex of X_1 and $i \colon X \hookrightarrow X_1$ is a homotopy equivalence rel. X. If $f \colon Y \to Y_1$ is a homotopy equivalence rel. X, then (Y_1, X_1) is finitely dominated and $f_*\sigma(Y, X) = \sigma(Y_1, X_1)$.*

Proof. Define $\tilde{f} \colon Y \cup X_1 \to Y_1$ by

$$\tilde{f}|_Y = f, \quad \tilde{f}|_{X_1} = id$$

then $\tilde{f}$ is a homotopy equivalence. By Theorem 3.7 $\tilde{f}$ is a homotopy equivalence rel. X_1. By Lemma 4.17 $(Y \cup X_1, X_1)$ is finitely dominated. But $\tilde{f}$ is a homotopy equivalence such that $\tilde{f}|_{X_1} \colon X_1 \to X_1$ is a homeomorphism. Therefore by Lemma 4.13 (Y_1, X_1) is finitely dominated.

By Lemma 4.17 $\sigma(Y \cup X_1, X_1) = j_*\sigma(Y, X)$, where $j \colon Y \hookrightarrow Y \cup X_1$ is an inclusion. Since $\tilde{f}$ is a homotopy equivalence rel. X_1 and $\tilde{f}|_{X_1}$ is a CW isomorphism, by Weak Invariance we have

$$\tilde{f}_*\sigma(Y \cup X_1, X_1) = \sigma(Y_1, X_1),$$

but

$$\tilde{f}_*\sigma(Y \cup X_1, X_1) = \tilde{f}_* j_* \sigma(Y, X) = f_*\sigma(Y, X).$$
$\qquad\square$

Theorem 4.19 (Invariance). *Let (Y, X) and (Y_1, X_1) be CW pairs such that (Y, X) is finitely dominated. Suppose $f \colon Y \to Y_1$ is a homotopy equivalence such that $f|_X \colon X \to X_1$ is a homotopy equivalence. Then (Y_1, X_1) is finitely dominated and $f_* \sigma(Y, X) = \sigma(Y_1, X_1)$.*

Proof. $f \colon X \to X_1$ is a homotopy equivalence; therefore, $f \colon X \to X_1 \times \{0\} \subset X \times [0, \infty)$ is homotopic to a proper homotopy equivalence $f' \colon X \to X_1 \times [0, \infty)$ [**4**, Lemma 21.1]. Let M be the mapping cylinder of $f'|_X$. Let $F_t \colon X \to M$ be a homotopy such that $F_0 = f|_X$ and $F_1 = id_X$. Form $Y_1 \cup M$ by attaching M to Y_1 along $X_1 \times \{0\}$. Use the Homotopy Extension Theorem to extend F_t to $\tilde{F}_t \colon Y \to Y_1 \cup M$ so that $\tilde{F}_t|_X = F_t$ and $\tilde{F}_0 = f$.

Let g be a homotopy inverse of f. Let $c \colon Y_1 \cup M \to Y_1$ be a map such that $c|_{Y_1} = id$ and $c|_M$ is a collapse of M to $X_1 \times [0, \infty)$ followed by a collapse of $X_1 \times [0, \infty)$ to $X_1 \times \{0\}$. Then $\tilde{F}_0$ is a homotopy equivalence with a homotopy inverse gc. Since $\tilde{F}_0 \simeq \tilde{F}_1$, $\tilde{F}_1$ is also a homotopy equivalence. Observe that $\tilde{F}_1|_X = id$, therefore by Theorem 3.7, $\tilde{F}_1$ is a homotopy equivalence rel. X. Also, $X \hookrightarrow M$ is a homotopy equivalence rel. X.

By Lemma 4.18 $(Y_1 \cup M, M)$ is finitely dominated. Therefore there exists $h_t \colon Y_1 \cup M \to Y_1 \cup M$ such that $h_0 = id$, $h_t|_M = id$ and $h_1(Y_1 \cup M) \subset M \cup C$ for some compact subcomplex C. Consider

$$H_t = ch_t|_{Y_1} \colon Y_1 \to Y_1,$$

$H_0 = id_{Y_1}$, $H_t|_{X_1} = id$ and $H_1(Y_1) \subset X_1 \cup c(C)$, where $c(C)$ is compact. It follows that (Y_1, X_1) is finitely dominated.

Also by Lemma 4.18 we have

$$\tilde{F}_{1*} \sigma(Y, X) = \sigma(Y_1 \cup M, M).$$

But by Lemma 4.17,

$$\sigma(Y_1 \cup M, M) = i_* \sigma(Y_1, X_1),$$

where $i \colon Y_1 \hookrightarrow Y_1 \cup M$ is an inclusion. Thus, we have

$$\tilde{F}_{1*} \sigma(Y, X) = i_* \sigma(Y_1, X_1)$$
$$c_* \tilde{F}_{1*} \sigma(Y, X) = c_* i_* \sigma(Y_1, X_1),$$

but $c\tilde{F}_1 \simeq f$ because $c\tilde{F}_1 \simeq c\tilde{F}_0 = cf = f$, so

$$f_* \sigma(Y, X) = \sigma(Y_1, X_1). \qquad \square$$

Theorem 4.20 (Relative Product Theorem). *Let (Y_1, X_1) and (Y_2, X_2) be finitely dominated, path connected CW pairs. Then*

(1) $$(Y_1 \times Y_2, (Y_1 \times X_2) \cup (X_1 \times Y_2))$$

is finitely dominated and

(2) $\sigma(Y_1 \times Y_2, (Y_1 \times X_2) \cup (X_1 \times Y_2)) = \sigma(Y_1, X_1) \cdot \sigma(Y_2, X_2) +$
$$\chi(Y_1, X_1) j_{2*} \sigma(Y_2, X_2) + \chi(Y_2, X_2) j_{1*} \sigma(Y_1, X_1),$$

where $j_i \colon Y_i \hookrightarrow Y_1 \times Y_2$ $(i = 1, 2)$ are inclusions.

Proof. For $i = 1, 2$ suppose that (Y_i, X_i) is stable with respect to (D'_i, u_i). Let $D_i = D'_i \cup X_i$. Let $X'_i = X_i \cap D'_i$. Note that by following the original constructions we can insure that each X'_i is compact.

(1) We will start by showing that $(D_1 \times D_2, (D_1 \times X_2) \cup (X_1 \times D_2))$ is finitely dominated. The following figure will be helpful.
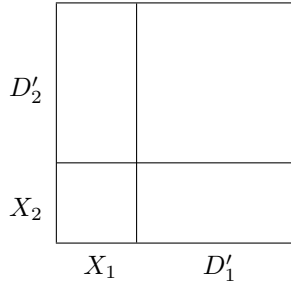

Figure 3: $D_1 \times D_2$

Observe that

$$D_1 \times D_2 = (D'_1 \times D'_2) \cup [(X_1 \times D_2) \cup (D_1 \times X_2)].$$

Each D'_i is finitely dominated by the definition; thus,

$$(D'_1 \times D'_2, \emptyset)$$

is finitely dominated.

Clearly

$$((X_1 \times D_2) \cup (D_1 \times X_2), (X_1 \times D_2) \cup (D_1 \times X_2))$$

is finitely dominated.

Finally we need to demonstrate that

$$((D'_1 \times D'_2) \cap [(X_1 \times D_2) \cup (D_1 \times X_2)], \emptyset)$$

is finitely dominated. But

$$(D'_1 \times D'_2) \cap [(X_1 \times D_2) \cup (D_1 \times X_2)] = (X'_1 \times D'_2) \cup (D'_1 \times X'_2).$$

This is finitely dominated by the Sum Theorem since each term of the union is finitely dominated by the Product Theorem [**10**] while the intersection is a compact set $X'_1 \times X'_2$.

It is clear that

$$u_1 \times u_2 \colon D_1 \times D_2 \to Y_1 \times Y_2$$

is a homotopy equivalence rel. $X_1 \times X_2$. Thus, it restricts to a homotopy equivalence

$$u_1 \times u_2|_{(D_1 \times X_2) \cup (X_1 \times D_2)} \colon (D_1 \times X_2) \cup (X_1 \times D_2) \to (Y_1 \times X_2) \cup (X_1 \times Y_2).$$

The desired result follows from Invariance.

(2) We will first derive the formula for $\sigma(D_1 \times D_2, (D_1 \times X_2) \cup (X_1 \times D_2))$. We will use inc to denote any appropriate inclusion.

$\sigma(D_1 \times D_2, (D_1 \times X_2) \cup (X_1 \times D_2))$
$\quad = inc_*\sigma(D_1' \times D_2', \emptyset) + inc_*\sigma((X_1 \times D_2) \cup (D_1 \times X_2), (X_1 \times D_2) \cup (D_1 \times X_2))$
$\quad\quad - inc_*\sigma((X_1' \times D_2') \cup (D_1' \times X_2'), \emptyset)$
$\quad = inc_*(\sigma(D_1') \cdot \sigma(D_2')) + \chi(D_1')inc_*\sigma(D_2') + \chi(D_2')inc_*\sigma(D_1')$
$\quad\quad - inc_*\sigma(X_1' \times D_2') - inc_*\sigma(D_1' \times X_2') + inc_*\sigma(X_1' \times X_2')$
$\quad = inc_*(\sigma(D_1') \cdot \sigma(D_2')) + \chi(D_1')inc_*\sigma(D_2') + \chi(D_2')inc_*\sigma(D_1')$
$\quad\quad - \chi(X_1')inc_*\sigma(D_2') - \chi(X_2')inc_*\sigma(D_1')$
$\quad = inc_*(\sigma(D_1') \cdot \sigma(D_2')) + (\chi(D_1') - \chi(X_1'))inc_*\sigma(D_2') + (\chi(D_2')$
$\quad\quad - \chi(X_2'))inc_*\sigma(D_1')$
$\quad = inc_*(\sigma(D_1') \cdot \sigma(D_2')) + \chi(D_1', X_1')inc_*\sigma(D_2') + \chi(D_2', X_2')inc_*\sigma(D_1')$
$\quad = inc_*(\sigma(D_1') \cdot \sigma(D_2')) + \chi(Y_1, X_1)inc_*\sigma(D_2') + \chi(Y_2, X_2)inc_*\sigma(D_1').$

Applying $(u_1 \times u_2)_*$ we get

$\sigma(Y_1 \times Y_2, (Y_1 \times X_2) \cup (X_1 \times Y_2))$
$\quad = (u_1 \times u_2)_*\sigma(D_1 \times D_2, (D_1 \times X_2) \cup (X_1 \times D_2))$
$\quad = (u_1 \times u_2)_*inc_*(\sigma(D_1') \cdot \sigma(D_2')) + \chi(Y_1, X_1)(u_1 \times u_2)_*inc_*\sigma(D_2')$
$\quad\quad + \chi(Y_2, X_2)(u_1 \times u_2)_*inc_*\sigma(D_1')$
$\quad = u_{1*}inc_*\sigma(D_1') \cdot u_{2*}inc_*\sigma(D_2') + \chi(Y_1, X_1)inc_*u_{2*}inc_*\sigma(D_2')$
$\quad\quad + \chi(Y_2, X_2)inc_*u_{1*}ic_*\sigma(D_1')$
$\quad = \sigma(Y_1, X_1) \cdot \sigma(Y_2, X_2) + \chi(Y_1, X_1)j_{2*}\sigma(Y_2, X_2) + \chi(Y_2, X_2)j_{1*}\sigma(Y_1, X_1). \quad \square$

If in the above theorem we let (Y_2, X_2) be $(K, \emptyset)$ where K is (1) finite or (2) finitely dominated, then the result reduces to

(1) $\qquad\qquad\qquad \sigma(Y_1 \times K, X_1 \times K) = \chi(K)j_{1*}\sigma(Y_1, X_1)$

(2) $\qquad\quad \sigma(Y_1 \times K, X_1 \times K) = \sigma(Y_1, X_1) \cdot \sigma(K) + \chi(Y_1, X_1)j_{2*}\sigma(K)$
$\qquad\qquad\qquad\qquad\qquad + \chi(K)j_{1*}\sigma(Y_1, X_1).$

It is interesting to note that if (Y_1, X_1) and (Y_2, X_2) are finitely dominated, $(Y_1 \times Y_2, X_1 \times X_2)$ may not be finitely dominated. For example, let

$$Z = \{(x, y) \,|\, (x - 2n)^2 + y^2 = 1, \, n = 0, 1, 2, \ldots\}.$$

Clearly, (Z, Z) is finitely dominated. Let (Y, X) be a finitely dominated pair satisfying the following condition: *For any homotopy domination $C \cup X \hookrightarrow Y$, X is not a strong deformation retract of $C \cup X$.* Then $(Z \times Y, Z \times X)$ is not finitely dominated.

References

[1] H. Bass, A. Heller, and R. Swan, The Whitehead group of a polynomial extension, *Inst. Hautes Études Sci. Publ. Math.* (1964), No. 22, 61–79.

[2]   C. Borsuk, *Theory of retracts*, Monografie Matematyczne, Tom 44 Państwowe Wydawnictwo Naukowe, Warsaw, 1967.

[3]   T. A. Chapman, Homotopy conditions which detect simple homotopy equivalences, *Pacific J. Math.*, **80** (1979), No. 1, 13–46.

[4]   T. A. Chapman, *Lectures on Hilbert cube manifolds*, Regional Conference Series in Mathematics, No. 28, American Mathematical Society, Providence, R. I., 1976.

[5]   T. A. Chapman, *Controlled simple homotopy theory and applications*, Lecture Notes in Mathematics, 1009. Springer-Verlag, Berlin, 1983.

[6]   M. M. Cohen, *A course in simple homotopy theory*, Springer-Verlag, New York-Berlin, 1973.

[7]   S. Ferry, *A simple-homotopy approach to finiteness obstruction*, Lecture Notes in Mathematics, 870, Springer-Verlag, Berlin-New York, 1981.

[8]   M. Greenberg, *Lectures on algebraic topology*, W. A. Benjamin, Inc., New York-Amsterdam, 1967.

[9]   H. Schubert, *Topology*, Allyn and Bacon, Boston, 1968.

[10] L. C. Siebenmann, *The obstruction to finding a boundary for an open manifold of dimension greater than five*, Princeton Thesis, 1965.

[11] L. C. Siebenmann, Infinite simple homotopy types, Nederl. *Akad. Wetensch. Proc. Ser. A* 73 No. 5 and Indag. Math. **32** (1970), 479–495.

[12] K. Varadarajan, *The finiteness obstruction of C. T. C. Wall*, J. Wiley and Sons, New York, 1989.

[13] C. T. C. Wall, Finiteness conditions for CW complexes, *Annals of Math. (2)*, **81** (1965), 56–69.

[14] C. T. C. Wall, Finiteness conditions for CW complexes II, *Proc. Roy. Soc. Ser. A*, **295** (1966), 129–139.

Anna Davis   davisa@ohiodominican.edu

Mathematics Department, Ohio Dominican University, 1216 Sunbury Rd., Columbus, OH, 43219, USA