Homology, Homotopy and Applications, vol. 11(2), 2009, pp.229–250

REMARKS ON FINITE SUBSET SPACES

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(*communicated by J. Daniel Christensen*)

Abstract

This paper expands on and refines some known and less well-known results about the finite subset spaces of a simplicial complex X including their connectivity and manifold structure. It also discusses the inclusion of the singletons into the three-fold subset space and shows that this subspace is weakly contractible but generally non-contractible unless X is a cogroup. Some homological calculations are provided.

1. Statement of results

Let X be a topological space (always assumed to be path-connected), and k a positive integer. It has become increasingly useful in recent years to study the space

$$\mathrm{Sub}_n X := \{\{x_1, \ldots, x_\ell\} \subset X \mid \ell \leqslant n\}$$

of all finite subsets of X of cardinality at most n [**1, 3, 9, 15, 19, 23**]. This space is topologized as the identification space obtained from X^n by identifying two n-tuples if and only if the sets of their coordinates coincide [**4**]. The functors $\mathrm{Sub}_n(-)$ are homotopy functors in the sense that if $X \simeq Y$, then $\mathrm{Sub}_n(X) \simeq \mathrm{Sub}_n(Y)$. If $k \leqslant n$, then $\mathrm{Sub}_k X$ naturally embeds in $\mathrm{Sub}_n X$. We write $j_n \colon X \hookrightarrow \mathrm{Sub}_n X$ for the inclusion given by $j_n(x) = \{x\}$.

This paper takes advantage of the close relationship between finite subset spaces and symmetric products to deduce a number of useful results about them.

As a starting point, we discuss cell structures on finite subset spaces. We observe in Section 3 that if X is a finite d-dimensional simplicial complex, then $\mathrm{Sub}_n X$ is an nd-dimensional CW-complex and of which $\mathrm{Sub}_k X$ for $k \leqslant n$ is a subcomplex (Proposition 3.1). Furthermore, $\mathrm{Sub}\, X := \coprod_{n \geqslant 1} \mathrm{Sub}_n X$ has the structure of an abelian CW-monoid (without unit) whenever X is a simplicial complex.

In Section 4 we address a connectivity conjecture stated in [**25**]. We recall that a space X is r-connected if $\pi_i(X) = 0$ for $i \leqslant r$. A contractible space is r-connected for all positive r. In [**25**] Tuffley proves that $\mathrm{Sub}_n X$ is $n-2$-connected and conjectures that it is $n + r - 2$-connected if X is r-connected. We are able to confirm his conjecture for the three-fold subset spaces. In fact we show

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Research supported by NSERC grant A7218.
Received March 2, 2009, revised July 26, 2009; published on November 29, 2009.
2000 Mathematics Subject Classification: 55U10, 55S15.
Key words and phrases: finite subsets, connectivity, homology, manifold structure.
This article is available at `http://intlpress.com/HHA/v11/n2/a12`

Theorem 1.1. *If X is r-connected, $r \geqslant 1$ and $n \geqslant 3$, then $\mathrm{Sub}_n X$ is $r+1$-connected.*

In Section 5 we address a somewhat surprising fact about the embeddings

$$\mathrm{Sub}_k X \hookrightarrow \mathrm{Sub}_n X, \quad k \leqslant n.$$

A theorem of Handel [**9**] asserts that the inclusion $j\colon \mathrm{Sub}_k(X) \hookrightarrow \mathrm{Sub}_{2k+1}(X)$ for any $k \geqslant 1$ is trivial on homotopy groups (i.e. "weakly trivial"). This is, of course, not enough to conclude that j is the trivial map, and in fact it need not be. Let $\mathrm{Sub}_k(X, x_0)$ be the subspace of $\mathrm{Sub}_k X$ of all finite subsets containing the basepoint $x_0 \in X$. Handel's result is deduced from the more basic fact that the inclusion $j_{x_0}\colon \mathrm{Sub}_k(X, x_0) \hookrightarrow \mathrm{Sub}_{2k-1}(X, x_0)$ is weakly trivial. The following theorem implies that these maps are often not null-homotopic.

Theorem 1.2. *The embeddings*

$$j_{x_0}\colon X \hookrightarrow \mathrm{Sub}_3(X, x_0), \quad x \mapsto \{x, x_0\}$$

and

$$j\colon X \hookrightarrow \mathrm{Sub}_3(X), \qquad x \mapsto \{x\},$$

are both null-homotopic if X is a cogroup. If $X = S^1 \times S^1$ is the torus, then both j and j_{x_0} are non-trivial in homology and are hence essential.

For a definition of a cogroup, see Section 5. In particular, suspensions are cogroups. The second half of Theorem 1.2 follows from a general calculation given in Section 5 which exhibits a model for $\mathrm{Sub}_3(X, x_0)$ and uses it to show that its homology is an explicit quotient of the homology of the symmetric square $\mathrm{SP}^2 X$ by a submodule determined by the coproduct on $H_*(X)$. One deduces, in particular, a homotopy equivalence between $\mathrm{Sub}_3(\Sigma X, x_0)$ and the *reduced* symmetric square $\overline{\mathrm{SP}}^2(\Sigma X)$ (cf. Section 2.1 and Proposition 5.6). The methods in Section 5 are taken up again in [**12**] where an explicit spectral sequence is devised to compute $H_*(\mathrm{Sub}_n X)$ for any finite simplicial complex X and any $n \geqslant 1$.

The final two sections of this paper deal with manifold structures on $\mathrm{Sub}_n X$ and top homology groups. It is known that $\mathrm{Sub}_2 X = \mathrm{SP}^2 X$ is a closed manifold if and only if X is closed of dimension 2. This is a consequence of the fact that $\mathrm{SP}^2(\mathbb{R}^d)$ is not a manifold if $d > 2$, while $\mathrm{SP}^2(\mathbb{R}^2) \cong \mathbb{R}^4$ [**20**]. The following complete description is due to Wagner [**26**]:

Theorem 1.3. *Let X be a closed manifold of dimension $d \geqslant 1$. Then $\mathrm{Sub}_n X$ is a closed manifold if and only if either*

(i) $d = 1$ and $n = 3$, or

(ii) $d = 2$ and $n = 2$.

This result is established in Section 7 where we use, in the case $d \geqslant 2$, the connectivity result of Theorem 1.1, one observation from [**17**] and some homological calculations from [**13**]. In the case $d = 1$, we reproduce Wagner's cute argument. Furthermore in that section, we refine a result of Handel's [**9**] on the top homology groups of $\mathrm{Sub}_n X$ when X is a manifold. We point out that if X is a closed orientable

manifold of dimension $d \geqslant 2$, then the top homology group $H_{nd}(\mathrm{Sub}_n X)$ is trivial if d is odd and is $\mathbb{Z}$ if d is even. This group is always trivial if X is not orientable (see Section 6).

Acknowledgements

This work was initiated at PIMS in Vancouver and the first author would like to thank the institute for its hospitality.

2. Basic constructions

All spaces X in this paper are path-connected, paracompact, and have a chosen basepoint x_0.

The way we will think of $\mathrm{Sub}_n X$ is as a quotient of the n-th symmetric product $\mathrm{SP}^n X$. This symmetric product is the quotient of X^n by the permutation action of the symmetric group $\mathfrak{S}_n$. The quotient map $\pi\colon X^n \longrightarrow \mathrm{SP}^n X$ sends $(x_1, \ldots, x_n)$ to the equivalence class $[x_1, \ldots, x_n]$. It will be useful sometimes to write such an equivalence class as an an abelian product $x_1 \cdots x_n$, $x_i \in X$. There are topological embeddings

$$j_n\colon X \hookrightarrow \mathrm{SP}^n X, \qquad x \mapsto x x_0^{n-1}. \tag{1}$$

The finite subset space $\mathrm{Sub}_n X$ is obtained from $\mathrm{SP}^n X$ through the identifications

$$[x_1, \ldots, x_n] \sim [y_1, \ldots, y_n] \Longleftrightarrow \{x_1, \ldots, x_n\} = \{y_1, \ldots, y_n\}.$$

In multiplicative notation, elements of $\mathrm{Sub}_n X$ are products $x_1 x_2 \cdots x_k$ with $k \leqslant n$, and subject to the identifications $x_1^2 x_2 \cdots x_k \sim x_1 x_2 \cdots x_k$.

The topology of $\mathrm{Sub}_n X$ is the quotient topology inherited from $\mathrm{SP}^n X$ or X^n [**9**]. When X is Hausdorff, this topology is equivalent to the so-called *Vietoris finite* topology whose basis of open sets are sets of the form

$$[U_1, \ldots, U_k] := \{S \in \mathrm{Sub}_n X \mid S \subset \bigcup_{i=1}^{k} U_i \text{ and } S \cap U_i \neq \emptyset \text{ for each } i\},$$

where U_i is open in X [**26**]. When X is a metric space, $\mathrm{Sub}_k X$ is again a metric space under the Hausdorff metric, and hence it inherits a third and equivalent topology [**26**]. In all cases, for any topology we use, continuous maps between spaces induce continuous maps between their finite subset spaces.

Example 2.1. Of course $\mathrm{Sub}_1 X = X$ and $\mathrm{Sub}_2 X = \mathrm{SP}^2 X$. Generally, if $\mathbf{\Delta}^{n+1} X \subset \mathrm{SP}^{n+1} X$ denotes the image of the fat diagonal in X^{n+1}, that is

$$\mathbf{\Delta}^{n+1} X := \{x_1^{i_1} \ldots x_r^{i_r} \in \mathrm{SP}^{n+1} X \mid r \leqslant n, \sum i_j = n+1 \text{ and } \ i_j > 0\},$$

then there is a map

$$q\colon \mathbf{\Delta}^{n+1} X \longrightarrow \mathrm{Sub}_n X, \quad x_1^{i_1} \ldots x_r^{i_r} \longrightarrow \{x_1, \ldots, x_r\},$$

and a pushout diagram

$$\begin{array}{ccc} \mathbf{\Delta}^{n+1}X & \xrightarrow{\ i\ } & \mathrm{SP}^{n+1}X \\ \big\downarrow q & & \big\downarrow \\ \mathrm{Sub}_n X & \longrightarrow & \mathrm{Sub}_{n+1} X. \end{array} \tag{2}$$

This is quite clear since we obtain $\mathrm{Sub}_{n+1} X$ by identifying points in the fat diagonal to points in $\mathrm{Sub}_n X$. In particular, when $n = 2$, we have the pushout

$$\begin{array}{ccc} X \times X & \xrightarrow{\ i\ } & \mathrm{SP}^3 X \\ \big\downarrow q & & \big\downarrow \\ \mathrm{SP}^2 X & \longrightarrow & \mathrm{Sub}_3 X, \end{array} \tag{3}$$

where $q(x, y) = xy$ and $i(x, y) = x^2y$. The homology of $\mathrm{Sub}_3(X)$ can then be obtained from a Mayer-Vietoris sequence. Some calculations for the three-fold subset spaces are in Section 5.

There are two immediate and non-trivial consequences of the above pushouts. Albrecht Dold shows in [**7**] that the homology of the symmetric products of a CW-complex X only depends on the homology of X. The pushout diagram in (2) shows that, in the case of the finite subset spaces, this homology also depends on the *cohomology structure of* X. This general fact for the three- and four-fold subset spaces is further discussed in [**22**].

The second consequence of (2) is that it yields an important corollary.

Corollary 2.2. $\mathrm{Sub}_n X$ *is simply connected for* $n \geqslant 3$.

Proof. We use the following known facts about symmetric products: $\pi_1(\mathrm{SP}^n X) \cong H_1(X; \mathbb{Z})$ whenever $n \geqslant 2$, and the inclusion $j_n \colon X \hookrightarrow \mathrm{SP}^n X$ induces the abelianization map at the level of fundamental groups. (P.A. Smith [**21**] proves this for $n = 2$, but his argument applies for $n > 2$ [**22**].) For $n \geqslant 3$, consider the composite

$$X \xrightarrow{\alpha} \mathbf{\Delta}^n X \xrightarrow{i} \mathrm{SP}^n X$$

with $\alpha(x) = [x, x_0, \ldots, x_0]$. The induced map $j_{n*} = i_* \circ \alpha_*$ on π_1 is surjective, as we pointed out, and hence so is i_*. Assume we know that $\pi_1(\mathrm{Sub}_3(X)) = 0$. Then the fact that i_* is surjective implies immediately, by the Van-Kampen theorem and the pushout diagram in (2), that $\pi_1(\mathrm{Sub}_4 X) = 0$. By induction, we see that $\pi_1(\mathrm{Sub}_n X) = 0$ for larger n. Therefore, we need only establish the claim for $n = 3$. For that we apply Van Kampen to diagram (3). Consider the maps

$$\tau \colon x_0 \times X \hookrightarrow X \times X \xrightarrow{i} \mathrm{SP}^3 X$$

and

$$\beta \colon X \times x_0 \to X \times X \xrightarrow{q} \mathrm{SP}^2 X.$$

Now $i(x, y) = x^2y$ so that $\tau(x_0, x) = x_0^2 x = j_3(x)$ and $\beta(x, x_0) = x x_0 = j_2(x)$. Since the j_k's are surjective on π_1 it follows that τ and β are surjective on π_1. Therefore,

for any classes $u \in \pi_1(\mathrm{SP}^3 X)$ and $v \in \pi_1(\mathrm{SP}^2 X)$, $\exists$ a class $w \in \pi_1(X \times X)$ such that $i_*(w) = u$ and $q_*(w) = v$. This shows that $\pi_1(\mathrm{Sub}_3 X) = 0$. □

This corollary also follows from [**5, 25**], where it is shown that $\mathrm{Sub}_n X$ is $(n-2)$-connected for $n \geqslant 3$. However, the proof above is completely elementary.

2.1. Reduced constructions

For the spaces under consideration, the natural inclusion $\mathrm{Sub}_{n-1} X \subset \mathrm{Sub}_n X$ is a cofibration [**9**]. We write $\overline{\mathrm{Sub}}_n X := \mathrm{Sub}_n X / \mathrm{Sub}_{n-1} X$ for the cofiber. Similarly, $\mathrm{SP}^{n-1} X$ embeds in $\mathrm{SP}^n X$ as the closed subset of all configurations $[x_1, \ldots, x_n]$ with x_i at the basepoint for some i. We set $\overline{\mathrm{SP}}^n X := \mathrm{SP}^n X / \mathrm{SP}^{n-1} X$, the symmetric smash product.

Note that even though $\mathrm{SP}^2 X$ and $\mathrm{Sub}_2 X$ are the same, there is an essential difference between their reduced analogs. The difference here comes from the fact that the inclusion $X \hookrightarrow \mathrm{Sub}_2 X$ is the composite $X \xrightarrow{\Delta} X \times X \longrightarrow \mathrm{SP}^2 X \cong \mathrm{Sub}_2 X$, where Δ is the diagonal, while $j_2 \colon : X \hookrightarrow \mathrm{SP}^2 X$ is the basepoint inclusion.

Example 2.3. When $X = S^1$, $\mathrm{SP}^2(S^1)$ is the closed Möbius band. If we view this band as a square with two sides identified along opposite orientations, then $S^1 = \mathrm{SP}^1(S^1) \hookrightarrow \mathrm{SP}^2(S^1)$ embeds into this band as an edge (see figures on p. 1124 of [**23**]). Hence this embedding is homotopic to the embedding of an equator, and so $\overline{\mathrm{SP}}^2(S^1)$ is contractible. On the other hand, $S^1 = \mathrm{Sub}_1(S^1)$ embeds into $\mathrm{Sub}_2(S^1) = \mathrm{SP}^2(S^1)$ as the diagonal $x \mapsto \{x, x\} = [x, x]$, which is the boundary of the Möbius band, and so $\overline{\mathrm{Sub}}_2(S^1) = \mathbb{R}P^2$.

Example 2.4. When $X = S^2$, $\mathrm{SP}^2(S^2)$ is the complex projective plane $\mathbb{P}^2$, $\mathrm{SP}^1(S^2) = \mathbb{P}^1$ is a hyperplane, and $\overline{\mathrm{SP}}^2(S^2) = S^4$. On the other hand, $\overline{\mathrm{Sub}}_2(S^2)$ has the following description: Write $\mathbb{P}^1$ for $\mathbb{C} \cup \{\infty\}$. Then $\overline{\mathrm{Sub}}_2(S^2)$ is the quotient of $\mathbb{P}^2$ by the image of the Veronese embedding $\mathbb{P}^1 \longrightarrow \mathbb{P}^2$, $z \mapsto [z^2 : -2z : 1]$, $\infty \mapsto [1 : 0 : 0]$. To see this, identify $\mathrm{SP}^n(\mathbb{C})$ with $\mathbb{C}^n$ by sending $(z_1, \ldots, z_n)$ to the coefficients of the polynomial $(x - z_1) \cdots (x - z_n)$. This extends to the compactifications to give an identification of $\mathrm{SP}^n(S^2)$ with $\mathbb{P}^n$ ([**10**, Chapter 4]). When $n = 1$, (z, z) is mapped to the coefficients of $(x - z)(x - z)$, that is to $(z^2, -2z)$. Note that the diagonal $S^2 \longrightarrow \mathrm{SP}^2(S^2) = \mathbb{P}^2$ is multiplication by 2 on the level of H_2 so that, in particular, $H_4(\overline{\mathrm{Sub}}_2(S^2)) = \mathbb{Z}$, $H_2(\overline{\mathrm{Sub}}_2(S^2)) = \mathbb{Z}_2$, and all other reduced homology groups are zero.

3. Cell decomposition

If X is a simplicial complex, then there is a standard way to pick a $\mathfrak{S}_n$-equivariant simplicial decomposition for the product X^n so that the quotient map $X^n \longrightarrow \mathrm{SP}^n X$ induces a cellular structure on $\mathrm{SP}^n X$. We argue that this same cellular structure descends to a cell structure on $\mathrm{Sub}_n X$. The construction of this cell structure for the symmetric products is fairly classical [**14, 18**]. The following is a review and slight expansion:

Proposition 3.1. *Let X be a simplicial complex. For $n \geqslant 1$, there exist cellular decompositions for X^n, $\mathrm{SP}^n X$ and $\mathrm{Sub}_n X$ so that all of the quotient maps*

$$X^n \to \mathrm{SP}^n X \to \mathrm{Sub}_n X$$

and the concatenation pairings $+$ are cellular

$$\begin{array}{ccc} \mathrm{SP}^r X \times \mathrm{SP}^s X & \xrightarrow{+} & \mathrm{SP}^{r+s} X \\ \downarrow & & \downarrow \\ \mathrm{Sub}_r X \times \mathrm{Sub}_s X & \xrightarrow{+} & \mathrm{Sub}_{r+s} X. \end{array} \tag{4}$$

Furthermore, the subspaces $\mathbf{\Delta}^n, \mathrm{SP}^{n-1} X \subset \mathrm{SP}^n X$ and $\mathrm{Sub}_{n-1} X \subset \mathrm{Sub}_n X$ are subcomplexes.

Proof. Both $\mathrm{SP}^n X$ and $\mathrm{Sub}_n X$ are obtained from X^n via identifications. If for some simplicial (hence cellular) structure on X^n, derived from that on X, these identifications become simplicial (i.e. they identify simplices to simplices), then the quotients will have a cellular structure and the corresponding quotient maps will be cellular with respect to these structures.

As we know, one obtains a nice and natural $\mathfrak{S}_n$-equivariant simplicial structure on the product if one works with *ordered* simplicial complexes [**8, 14, 18**]. We write $X_\bullet$ for the abstract simplicial (i.e. triangulated) complex of which X is the realization. So we assume $X_\bullet$ to be endowed with a partial ordering on its vertices which restricts to a total ordering on each simplex. Let $\prec$ be that ordering. A point $w = (v_1, \ldots, v_n)$ is a vertex in $X^n_\bullet$ if and only if v_i is a vertex of $X_\bullet$. Different vertices

$$w_0 = (v_{01}, v_{02}, \ldots, v_{0n}), \ldots, w_k = (v_{k1}, v_{k2}, \ldots, v_{kn}) \tag{5}$$

span a k-simplex in $X^n_\bullet$ if, and only if, for each i, the $k+1$ vertices $v_{0i}, v_{1i}, \ldots, v_{ki}$ are contained in a simplex of X and $v_{0i} \prec v_{1i} \prec \cdots \prec v_{ki}$. We write $\varpi := [w_0, \ldots, w_k]$ for such a simplex.

The permutation action of $\tau \in \mathfrak{S}_n$ on $\varpi = [w_0, \ldots, w_k]$ is given by

$$\tau\varpi = [\tau w_0, \ldots, \tau w_n].$$

This is a well-defined simplex since the factors of each vertex

$$w_j = (v_{j_1 1}, v_{j_2 2}, \ldots, v_{j_n n})$$

are permuted simultaneously according to τ, and hence the order $\prec$ is preserved. The permutation action is then simplicial and $\mathrm{SP}^n X$ inherits a CW-structure by passing to the quotient.

Fact 1. If a point $p := (x_1, x_2, \ldots, x_n) \in X^n$ is such that $x_{i_1} = x_{i_2} = \cdots = x_{i_r}$, then p lies in some k-simplex ϖ whose vertices $[w_0, \ldots, w_k]$ are such that $v_{ji_1} = v_{ji_2} = \cdots = v_{ji_r}$ for $j = 0, \ldots, k$. This implies that the fat diagonal is a simplicial subcomplex. It also implies that any permutation that fixes such a point p must fix the vertices of the simplex it lies in and hences fixes it pointwise. In other words, if a permutation leaves a simplex invariant then it must fix it pointwise.

Fact 2. If $p = (x_1, x_2, \ldots, x_n) \in \varpi$ is a simplex with vertices $w_0, \ldots, w_k$ as in (5), and if $\pi \colon X^n \longrightarrow X^i$ is any projection, then $\pi(p)$ lies in the simplex with vertices

$\pi(w_0), \ldots, \pi(w_k)$ (which may or may not be equal). For instance, $\pi(p) := (x_1, \ldots, x_i)$ lies in the simplex with vertices $(v_{01}, v_{02}, \ldots, v_{0i}), \ldots, (v_{k1}, v_{k2}, \ldots, v_{ki})$.

We are now in a position to see that $\operatorname{Sub}_n X$ is a CW-complex. Recall that $\operatorname{Sub}_n X = X^n/_\sim$, where

$$(x_1, \ldots, x_n) \sim (y_1, \ldots, y_n) \Longleftrightarrow \{x_1, \ldots, x_n\} = \{y_1, \ldots, y_n\}.$$

Clearly, if $(x_1, \ldots, x_n) \sim (y_1, \ldots, y_n)$, then $\tau(x_1, \ldots, x_n) \sim \tau(y_1, \ldots, y_n)$ for $\tau \in \mathfrak{S}_n$. We wish to show that these identifications are simplicial. Let us argue through an example (the general case being identical). We have the identifications in $\operatorname{Sub}_6 X$:

$$p := (x, x, x, y, y, z) \sim (x, x, y, y, y, z) =: q. \tag{6}$$

By using Fact 2 applied to the projection, skipping the third coordinate and then Fact 1, we can see that p and q lie in simplices with vertices of the form

$$(v_1, v_1, ?, v_2, v_2, v_3).$$

By using Fact 1 again, p lies in a simplex σ_p with vertices of the form

$$(v_1, v_1, v_1, v_2, v_2, v_3),$$

while q lies in a simplex σ_q with vertices of the form $(v_1, v_1, v_2, v_2, v_2, v_3)$. It follows that the identification (6) identifies vertices of σ_p with vertices of σ_q, and hence identifies σ_p with σ_q as desired.

In conclusion, the quotient $\operatorname{Sub}_n X$ inherits a cellular structure and the composite

$$X^n \xrightarrow{\pi} \mathrm{SP}^n X \xrightarrow{q} \operatorname{Sub}_n X$$

is cellular. Since the pairing (4) is covered by $X^r \times X^s \longrightarrow X^{r+s}$, which is simplicial (by construction), and since the projections are cellular, the pairing (4) must be cellular. □

Remark 3.2. We could have worked with simplicial sets instead [**5**]. Similarly, Mostovoy (private communication) indicates how to construct a simplicial set $\operatorname{Sub}_n X$ out of a simplicial set X such that $|\operatorname{Sub}_n X| = \operatorname{Sub}_n |X|$. This approach will be further discussed in [**12**].

The following corollary is also obtained in [**5**].

Corollary 3.3. *For X a simplicial complex, $\operatorname{Sub}_k X$ has a* CW*-decomposition with top cells in $k \dim X$, so that $H_*(\operatorname{Sub}_k X) = 0$ for $* > k \dim X$.*

We collect a couple more corollaries

Corollary 3.4. *If X is a d-dimensional complex with $d \geqslant 2$, then the quotient map $\mathrm{SP}^n X \to \operatorname{Sub}_n X$ induces a homology isomorphism in top dimension nd.*

Proof. When X is as in the hypothesis, $\operatorname{Sub}_{n-1} X$ is a codimension d subcomplex of $\operatorname{Sub}_n X$ and since $d \geqslant 2$, $H_{nd}(\operatorname{Sub}_n X) = H_{nd}(\operatorname{Sub}_n X, \operatorname{Sub}_{n-1} X)$. On the other hand, Proposition 3.1 implies that $\mathbf{\Delta}^n X$ is a codimension d subcomplex of $\mathrm{SP}^n X$ so that $H_{nd}(\mathrm{SP}^n X) \cong H_{nd}(\mathrm{SP}^n X, \mathbf{\Delta}^n X)$ as well. But according to diagram (2), we have the homeomorphism

$$\mathrm{SP}^n X/\mathbf{\Delta}^n X \cong \operatorname{Sub}_n X/\operatorname{Sub}_{n-1} X.$$

Combining these facts yields the claim. □

Corollary 3.5. *Both* $\mathrm{SP}^k X$ *and the fat diagonal* $\mathbf{\Delta}^k \subset \mathrm{SP}^k X$ *have the same connectivity as* X*, and this is sharp.*

Proof. If X is an r-connected ordered simplicial complex, then X admits a simplicial structure so that the r-skeleton X_r is contractible in X to some point $x_0 \in X$. With such a simplicial decomposition we can consider Liao's induced decomposition $X^k_\bullet$ on X^k and its r-skeleton X^k_r. Note that

$$X^k_r \subset \bigcup_{i_1+\cdots+i_k \leqslant r} X_{i_1} \times X_{i_2} \times \cdots \times X_{i_k} \subset (X_r)^k.$$

If $F\colon X_r \times I \longrightarrow X$ is a deformation of X_r to x_0, then F^k is a deformation of $(X_r)^k$; hence X^k_r, to $(x_0, \ldots, x_0)$ in X^k, and this deformation is $\mathfrak{S}_k$ equivariant. Since the r-skeleton of $\mathrm{SP}^k X$ is the $\mathfrak{S}_k$-quotient of X^k_r, it is then itself contractible in $\mathrm{SP}^k X$, and this proves the first claim. Similarly, the simplicial decomposition we have introduced on X^k includes the fat diagonal Λ^k as a subcomplex with r-skeleton $\Lambda^k_r := \Lambda^k \cap X^k_r$. The deformation F^k preserves the fat diagonal and so it restricts to Λ^k and to an equivariant deformation $F^k\colon \Lambda^k_r \times I \longrightarrow \Lambda^k$. This means that the r-skeleton of $q(\Lambda^k) =: \mathbf{\Delta}^k \subset \mathrm{SP}^k X$ is itself contractible in $\mathbf{\Delta}^k$, and the second claim follows. This bound is sharp for symmetric products since when $X = S^2$, $\mathrm{SP}^2(S^2) = \mathbb{P}^2$. It is sharp for the fat diagonal as well since $\mathbf{\Delta}^3 X \cong X \times X$ has exactly the same connectivity of X. □

4. Connectivity

As we have established in Corollary 2.2, finite subset spaces $\mathrm{Sub}_n X$, $n \geqslant 3$, are always simply connected. In this section we further relate the connectivity of $\mathrm{Sub}_k X$ to that of X. We first need the following useful result proved in [**11**]:

Theorem 4.1. *If* X *is* r*-connected with* $r \geqslant 1$*, then* $\overline{\mathrm{SP}}^n X$ *is* $2n + r - 2$*-connected.*

Example 5.7 shows that $\overline{\mathrm{SP}}^2(S^k)$ is $k+1$-connected as asserted. Note that

$$\overline{\mathrm{SP}}^2(S^2) = S^4$$

is 3-connected, so Theorem 4.1 is sharp.

Corollary 4.2 ([**18**, Corollary 4.7])**.** *If* X *is* r*-connected,* $r \geqslant 1$*, then*

$$H_*(X) \cong H_*(\mathrm{SP}^n X)$$

for $* \leqslant r + 2$*. This isomorphism is induced by the map* j_n *adjoining the basepoint.*

Proof. We give a short proof based on Theorem 4.1. By Steenrod's homological splitting [**18**]

$$H_*(\mathrm{SP}^n X) \cong \bigoplus_{k=1}^{n} H_*(\mathrm{SP}^k X, \mathrm{SP}^{k-1} X) = \bigoplus_{k=2}^{n} \tilde{H}_*(\overline{\mathrm{SP}}^k X) \oplus H_*(X) \tag{7}$$

with $\mathrm{SP}^0 X = \emptyset$, but $\tilde{H}_*(\overline{\mathrm{SP}}^k X) = 0$ for $* \leqslant 2k + r - 2$. The result follows. □

Remark 4.3. Note that Corollary 4.2 cannot be improved to $r = 0$ (i.e. X-connected). It fails already for the wedge $X = S^1 \vee S^1$ and $n = 2$ since $\mathrm{SP}^2(S^1 \vee S^1) \simeq S^1 \times S^1$ (see [**13**]) and hence $H_2(\mathrm{SP}^2(S^1 \vee S^1)) \ncong H_2(S^1 \vee S^1)$. Note also that (7) implies that $H_*(X)$ embeds into $H_*(\mathrm{SP}^n X)$ for all $n \geqslant 1$, a fact we will find useful below.

Proposition 4.4. *Suppose X is r-connected, $r \geqslant 1$. Then $\mathrm{Sub}_k X$ is $r+1$-connected whenever $k \geqslant 3$.*

Proof. Write $x_0 \in X$ for the basepoint and assume $k \geqslant 3$. Remember that the $\mathrm{Sub}_k X$ are simply connected for $k \geqslant 3$ (Corollary 2.2) so by the Hurewicz theorem if they have trivial homology up to degree $r+1$, then they are connected up to that level. We will now show by induction that $H_*(\mathrm{Sub}_k X) = 0$ for $* \leqslant r+1$. The first step is to show that $H_*(\mathrm{SP}^k X, \mathbf{\Delta}^k) = H_*(\mathrm{Sub}_k X, \mathrm{Sub}_{k-1} X) = 0$ for $* \leqslant r+1$. We write $i\colon \mathbf{\Delta}^k \hookrightarrow \mathrm{SP}^k X$ for the inclusion.

From the fact that $\mathbf{\Delta}^k$ and $\mathrm{SP}^k X$ have the same connectivity as X (Corollary 3.5), their homology vanishes up to degree r which implies similarly that the relative groups are trivial up to that degree. On the other hand, X embeds in $\mathbf{\Delta}^k$ via $x \mapsto [x, x_0, \ldots, x_0]$ (this is a well-defined map since $k \geqslant 3$) and, since the composite $j_k\colon X \longrightarrow \mathbf{\Delta^k} \xrightarrow{i} \mathrm{SP}^k X$ is an isomorphism on H_{r+1} (Corollary 4.2), we see that the map $i_*\colon H_{r+1}(\mathbf{\Delta}^k) \longrightarrow H_{r+1}(\mathrm{SP}^k X)$ is surjective. Hence, $H_{r+1}(\mathrm{SP}^k X, \mathbf{\Delta}^k) = 0$.

Now since $0 = H_*(\mathrm{SP}^k X, \mathbf{\Delta}^k) = H_*(\mathrm{Sub}_k X, \mathrm{Sub}_{k-1} X)$ for $* \leqslant r+1$, it follows that

$$H_*(\mathrm{Sub}_{k-1} X) \cong H_*(\mathrm{Sub}_k X) \qquad \text{for } * \leqslant r$$

and that

$$H_{r+1}(\mathrm{Sub}_{k-1} X) \to H_{r+1}(\mathrm{Sub}_k X) \quad \text{is surjective.}$$

So if we prove that $H_*(\mathrm{Sub}_3 X) = 0$ for $* \leqslant r+1$, then by induction we will have proved our claim.

Consider the homology long exact sequences for

$$(\mathrm{Sub}_3 X, \mathrm{Sub}_2 X) \qquad \text{and} \qquad (\mathrm{SP}^3 X, \mathbf{\Delta}^3 X),$$

where again we identify $\mathbf{\Delta}^3 X$ with $X \times X$. We obtain commutative diagrams

$$\begin{array}{ccccccccc}
\longrightarrow & H_{r+2}(\mathrm{Sub}_3 X, \mathrm{Sub}_2 X) & \longrightarrow & H_{r+1}(\mathrm{Sub}_2 X) & \xrightarrow{i_*} & H_{r+1}(\mathrm{Sub}_3 X) & \longrightarrow & 0 \\
 & \cong\uparrow & & q_*\uparrow & & \pi_*\uparrow & & \\
\longrightarrow & H_{r+2}(\mathrm{SP}^3 X, X^2) & \longrightarrow & H_{r+1}(X^2) & \xrightarrow{\alpha_*} & H_{r+1}(\mathrm{SP}^3 X) & \longrightarrow & 0,
\end{array}$$

where $\alpha(x, y) = x^2 y$ and $\pi\colon \mathrm{SP}^3 X \longrightarrow \mathrm{Sub}_3 X$ is the quotient map. We want to show that $i_* = 0$ so that by exactness $H_{r+1}(\mathrm{Sub}_3 X) = 0$. Now q_* is surjective since the composite

$$X \longrightarrow X \times \{x_0\} \hookrightarrow X \times X \longrightarrow \mathrm{SP}^2 X = \mathrm{Sub}_2 X$$

induces an isomorphism on H_{r+1} by Corollary 4.2. Showing that $i_* = 0$ comes down, therefore, to showing that $\pi_* \circ \alpha_* = 0$. But note that for $r \geqslant 1$, which is the connectivity of X, classes in $H_{r+1}(X \times X)$ are necessarily spherical and we have the

following commutative diagram:

$$\begin{array}{ccccc} \pi_{r+1}X \times \pi_{r+1}(X) & \xrightarrow{\cong} & \pi_{r+1}(X \times X) & \longrightarrow & \pi_{r+1}(\mathrm{Sub}_3(X)) \\ & & \downarrow h & & \downarrow h \\ & & H_{r+1}(X \times X) & \xrightarrow{\pi_* \circ \alpha_*} & H_{r+1}(\mathrm{Sub}_3(X)), \end{array}$$

where h is the Hurewicz homomorphism. The top map is trivial since when restricted to each factor $\pi_{r+1}(X)$ it is trivial according to the useful Theorem 5.1 below (or to Corollary 5.2). Since h is surjective, $\pi_* \circ \alpha_* = 0$ and $H_{r+1}(\mathrm{Sub}_3 X) = 0$ as desired. □

5. The three-fold finite subset space

There are many subtle points that come up in the study of finite subset spaces. We illustrate several of them through the study of the pair $(\mathrm{Sub}_3 X, X)$. The three-fold subset space has been studied in [**17, 19, 23**] for the case of the circle and in [**24**] for topological surfaces.

Again all spaces below are assumed to be connected. We say a map is weakly contractible (or weakly trivial) if it induces the trivial map on all homotopy groups. The following is based on a cute argument well explained in [**9**] or ([**3**, §3.4]).

Theorem 5.1 ([**9**])**.** $\mathrm{Sub}_k(X)$ *is weakly contractible in* $\mathrm{Sub}_{2k+1}(X)$.

Caveat 1. A map $f\colon A \longrightarrow Y$ being weakly contractible does not generally imply that f is null homotopic. Indeed let T be the torus and consider the projection $T \longrightarrow S^2$ which collapses the one-skeleton. Then this map induces an isomorphism on H_2 but is trivial on homotopy groups since $T = K(\mathbb{Z}^2, 1)$. Of course, if $A = S^k$ is a sphere, then "weakly trivial" and "null-homotopic" are the same since the map $A \longrightarrow Y$ represents the zero element in $\pi_k Y$. For example, in ([**6**, Lemma 3]), the authors explicitly construct an extension of the inclusion $S^n \hookrightarrow \mathrm{Sub}_3(S^n)$ to the disk $B^{n+1} \longrightarrow \mathrm{Sub}_3(S^n)$, $\partial B^{n+1} = S^n$. This section argues that this implication does not generally hold for non-suspensions.

Caveat 2. When comparing symmetric products to finite subset spaces, one has to watch out for the fact that the basepoint inclusion $\mathrm{SP}^k(X) \longrightarrow \mathrm{SP}^{k+1}(X)$ *does not commute* via the projection maps with the inclusion $\mathrm{Sub}_k(X) \longrightarrow \mathrm{Sub}_{k+1}(X)$. This has already been pointed out in Example 2.3 and is further illustrated in the corollary below.

Corollary 5.2. *The composite*

$$SP^k(X) \longrightarrow SP^{2k+1}(X) \longrightarrow \mathrm{Sub}_{2k+1}(X)$$

is weakly trivial.

Proof. This map is equivalent to the composite

$$\mathrm{SP}^k(X) \longrightarrow \mathrm{Sub}_k(X) \xrightarrow{\mu} \mathrm{Sub}_{k+1}(X, x_0) \hookrightarrow \mathrm{Sub}_{2k+1}(X), \tag{8}$$

where $\mu(\{x_1, \ldots, x_k\}) = \{x_0, x_1, \ldots, x_k\}$, x_0 is the basepoint of X and $\mathrm{Sub}_{k+1}(X, x_0)$ is the subspace of $\mathrm{Sub}_{k+1}(X)$ of all subsets containing this basepoint. Note that μ is

not an embedding as pointed out in [**24**] but is one-to-one away from the fat diagonal. The key point here is again ([**9**, Theorem 4.1]) which asserts that the inclusion

$$\mathrm{Sub}_{k+1}(X, x_0) \hookrightarrow \mathrm{Sub}_{2k+1}(X, x_0)$$

is weakly contractible. This in turn implies that the last map in (8) is weakly trivial as well and the claim follows. □

Caveat 3. For $n \geqslant 2$, one can embed $X \hookrightarrow \mathrm{Sub}_n(X)$ in several ways. There is of course the natural inclusion j giving X as the subspace of singletons. There is also, for any choice of $x_0 \in X$, the embedding $j_{x_0} \colon x \mapsto \{x, x_0\}$. Any two such embeddings for different choices of x_0 are equivalent when X is path-connected (any choice of a path between x_0 and x_0' gives a homotopy between j_{x_0} and $j_{x_0'}$). It turns out, however, that j and j_{x_0} are fundamentally different. The simplest example was already pointed out for S^1, where $\mathrm{Sub}_2(S^1)$ was the Möbius band with j being the embedding of the boundary circle while j_{x_0} is the embedding of an equator.

One might ask the question whether it is true that j is null-homotopic if and only if j_{x_0} is null-homotopic? This is at least true for suspensions as the next lemma illustrates.

Recall that a co-H space X is a space whose diagonal map factors up to homotopy through the wedge; that is there exists a δ such that the composite

$$X \xrightarrow{\delta} X \vee X \hookrightarrow X \times X$$

is homotopic to the diagonal $\Delta \colon X \longrightarrow X \times X, x \mapsto (x, x)$. A cogroup X is a co-H space that is co-associative with a homotopy inverse. This latter condition means there is a map $c \colon X \longrightarrow X$ such that $X \xrightarrow{\delta} X \vee X \xrightarrow{\nabla(c \vee 1)} X$ is null-homotopic. This is in fact the definition of a left inverse but it implies the existence of a right inverse as well [**2**]. If X is a cogroup, then for every based space Y, the set of based homotopy classes of based maps $[X, Y]$ is a group. The suspension of a space is a cogroup and there exist several interesting cogroups that are not suspensions ([**2**, §4]).

Write $j_{x_0} \colon X \hookrightarrow \mathrm{Sub}_3(X, x_0)$ for the map $x \mapsto \{x, x_0\}$. Its continuation to $\mathrm{Sub}_3(X)$ is also written j_{x_0}.

Lemma 5.3. *Suppose X is a cogroup. Then the embeddings $j_{x_0} \colon X \hookrightarrow \mathrm{Sub}_3(X, x_0)$ and $j \colon X \hookrightarrow \mathrm{Sub}_3(X)$ are null-homotopic.*

Proof. The argument in [**9**] extends to this situation. We deal with j_{x_0} first. This is a based map at x_0. Its homotopy class $[j_{x_0}]$ lives in the group $G = [X, \mathrm{Sub}_3(X, x_0)]$. The following composite is checked to be again j_{x_0}:

$$j_{x_0} \colon X \xrightarrow{\Delta} X \times X \xrightarrow{j_{x_0} + j_{x_0}} \mathrm{Sub}_3(X, x_0).$$

This factors up to homotopy through the wedge

$$\iota \colon X \xrightarrow{\delta} X \vee X \xrightarrow{j_{x_0} \vee j_{x_0}} \mathrm{Sub}_3(X, x_0).$$

Of course $[\iota] = [j_{x_0}]$, but observe that $[\iota] = 2[j_{x_0}]$ by definition of the additive structure of G. This means that $[j_{x_0}] = 2[j_{x_0}]$; thus $[j_{x_0}] = 0$ and j_{x_0} is trivial (through a homotopy fixing x_0)

Let us now apply this to the inclusion $j\colon X \hookrightarrow \mathrm{Sub}_3(X)$ which is assumed to be based at x_0. We also denote the composite $X \xrightarrow{j_{x_0}} \mathrm{Sub}_3(X, x_0) \longrightarrow \mathrm{Sub}_3 X$ by j_{x_0}. Using the co-H structure as before, we get the homotopy commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times X \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle j+j} \\ X \vee X & \xrightarrow{j_{x_0} \vee j_{x_0}} & \mathrm{Sub}_3(X). \end{array}$$

Since j_{x_0} was just shown to be null homotopic, then so is $j = (j + j) \circ \Delta$. □

Let us now turn to the second part of Theorem 1.2.

5.1. The space $\mathrm{Sub}_3(X, x_0)$

The preceding discussion shows the usefulness of looking at the based finite subset space $\mathrm{Sub}_n(X, x_0)$. We start with a key computation. Write Δ for the diagonal $X \longrightarrow \mathrm{SP}^2 X$, $x \mapsto [x, x]$, and identify the image of $j_*\colon H_*(X) \hookrightarrow H_*(\mathrm{SP}^2(X))$ with $H_*(X)$ by the Steenrod homological splitting (7).

Lemma 5.4. *Let X be a compact cell complex. Then*

$$H_*(\mathrm{Sub}_3(X, x_0)) = H_*(\mathrm{SP}^2 X)/I$$

where I is the submodule generated by $\Delta_* c - c, c \in H_*(X) \hookrightarrow H_*(\mathrm{SP}^2 X)$.

Proof. Start with the map $\alpha\colon \mathrm{SP}^2(X) \longrightarrow \mathrm{Sub}_3(X, x_0), [x, y] \mapsto \{x, y, x_0\}$, which is surjective and generically one-to-one (i.e. one-to-one on the subspace of points $[x, y]$ with $x \neq y$). Observe that $\alpha([x, x]) = \alpha([x, x_0])$. This implies that $\mathrm{Sub}_3(X, x_0)$ is homeomorphic to the identification space

$$\mathrm{SP}^2(X)/\sim, \quad [x, x] \sim [x, x_0], \quad \forall x \in X. \tag{9}$$

In order to compute the homology of this quotient we will replace it with the following space:

$$\begin{gathered} W_2(X) := \mathrm{SP}^2(X) \sqcup X \times I/\sim, \\ [x, x] \sim (x, 1), \quad [x, x_0] \sim (x, 0), \quad [x_0, x_0] \sim (x_0, t). \end{gathered} \tag{10}$$

It is not hard to see that (9) and (10) are homotopy equivalent. We can easily see that these spaces are homology equivalent as follows (this is enough for our purpose): There is a well-defined map

$$g\colon W_2(X) \longrightarrow \mathrm{SP}^2(X)/\sim$$

sending $[x, y] \mapsto [x, y], (x, t) \mapsto [x, x_0]$. The inverse image $g^{-1}([x, y]) = [x, y]$ if $x \neq y$ and both points are different from x_0. The inverse image of $[x, x]$ or $[x, x_0]$ is an interval when $x \neq x_0$, hence contractible, and it is a point when $x = x_0$. In all cases, preimages under g are acyclic and hence g is a homology equivalence by the Begle-Vietoris theorem. The homology structure of $\mathrm{Sub}_3(X, x_0)$ can be made much more apparent using the form (10) and this is why we have introduced it.

Let $(C_*(\mathrm{SP}^2(X)), \partial)$ be a chain complex for $\mathrm{SP}^2(X)$ containing $C_*(X)$ as a subcomplex and for which the diagonal map $X \longrightarrow \mathrm{SP}^2 X$ is cellular. Associate to

$c \in C_i(X)$ a chain $|c|$ in degree $i+1$ representing $I \times c \in C_{i+1}(I \times X)$ if $c \neq x_0$ (the 0-chain representing the basepoint). We write $|C_*(X)|$ for the set of all such chains. The geometry of our construction gives a chain complex for $W_2(X)$ as follows:

$$C_*(W_2(X)) = C_*(\mathrm{SP}^2(X)) \oplus |C_*(X)| \tag{11}$$

with boundary d such that $d(c) = \partial c$ and

$$d|c| = c - \Delta_*(c) - |\partial c|.$$

This comes from the formula for the boundary of the product of two cells which is in general given by $\partial(\sigma_1 \times \sigma_2) = \partial(\sigma_1) \times \sigma_2 + (-1)^{|\sigma_1|}\sigma_1 \times \partial(\sigma_2)$. We check indeed that $d \circ d = 0$. To compute the homology we need to understand cycles and boundaries in this chain complex. Write a general element of (11) as $\alpha + |c|$. The boundary of this element is $\partial\alpha + c - \Delta_*(c) - |\partial c|$ and it is zero, if and only if, $\partial\alpha = \Delta_*(c) - c$ and $|\partial c| = 0$. That is, if and only if, c is a cycle and $\Delta_*(c) - c$ is a boundary. This means that in $H_*(\mathrm{SP}^2(C))$, $\Delta_*(c) = c$. We claim this is not possible unless $c = 0$. Indeed, if c is a positive dimensional (homology) class, then $\Delta_*(c) = c \otimes 1 + \sum c' \otimes c'' + 1 \otimes c$ in $H_*(X \times X)$ and hence in $H_*(\mathrm{SP}^2(C))$, $\Delta_*(c) = 2c + \sum c' * c''$ where by definition $c' * c'' = q_*(c' \otimes c'')$ and $q\colon X \times X \longrightarrow \mathrm{SP}^2(X)$ is the projection. This can never be equal to c since $\sum c' * c'' \in H_*(\mathrm{SP}^2 X, X)$.

To recapitulate, $\alpha + |c|$ is a cycle if and only if α is a cycle and $c = 0$. The only cycles in $C_*(W_2(X))$ are those that are already cycles in the first summand $C_*(\mathrm{SP}^2(X))$. On the other hand, among these classes the only boundaries consist of boundaries in $C_*(\mathrm{SP}^2(X))$ and those of the form $\Delta_*(c) - c$ with c a cycle in $C_*(X)$ (in particular the only 0-cycle is represented by x_0). This proves our claim. □

Remark 5.5 (Added in revision). We could have noticed alternatively the existence of a pushout diagram

$$\begin{array}{ccc} X \vee X & \xrightarrow{f} & \mathrm{SP}^2 X \\ \big\downarrow{\scriptstyle \mathrm{fold}} & & \big\downarrow{\scriptstyle \alpha} \\ X & \xrightarrow{j_{x_0}} & \mathrm{Sub}_3(X, x_0), \end{array}$$

where $f(x, x_0) = [x, x]$ is the diagonal and $f(x_0, x) = [x, x_0]$. We can in fact deduce Lemma 5.4 from this pushout. We can also deduce that $\mathrm{Sub}_3(X, x_0)$ is simply connected if X is. This useful fact we use to establish Proposition 5.6 next.

Note that Lemma 5.4 above says that $H_*(\mathrm{Sub}_3(X, x_0))$ only depends on $H_*(X)$ and on its coproduct (i.e. on the cohomology of X). When X is a suspension the situation becomes simpler. The following result is a nice combination of Lemmas 5.3 and 5.4.

Proposition 5.6. *There is a homotopy equivalence* $\mathrm{Sub}_3(\Sigma X, x_0) \simeq \overline{\mathrm{SP}}^2(\Sigma X)$.

Proof. When X is a suspension, all classes are primitive so that $\Delta_*(c) = 2c$ for all $c \in H_*(X)$. Combining Steenrod's splitting (7),

$$H_*(\mathrm{SP}^2 X) \cong H_*(X) \oplus H_*(\mathrm{SP}^2 X, X),$$

with Lemma 5.4, we deduce immediately that $H_*(\mathrm{Sub}_3(\Sigma X, x_0)) \cong H_*(\overline{\mathrm{SP}}^2(\Sigma X))$. Both spaces are simply connected (by Remark 5.5 and Theorem 4.1) and so it is

enough to exhibit a map between them that induces this homology isomorphism. Consider the map $\alpha\colon \mathrm{SP}^2(\Sigma X) \longrightarrow \mathrm{Sub}_3(\Sigma X, x_0)$, $[x,y] \mapsto \{x,y,x_0\}$ as in the proof of Lemma 5.4. Its restriction to ΣX is null-homotopic according to Lemma 5.3 and hence it factors through the quotient $\overline{\mathrm{SP}}^2(\Sigma X) \longrightarrow \mathrm{Sub}_3(\Sigma X, x_0)$. By inspection of the proof of Lemma 5.4 we see that this map induces an isomorphism on homology. □

Example 5.7. A description of $\overline{\mathrm{SP}}^2(S^k)$ is given in ([**10**, Example 4K.5]) from which we infer that

$$\mathrm{Sub}_3(S^k, x_0) \simeq \Sigma^{k+1}\mathbb{R}P^{k-1}, \qquad k \geqslant 1.$$

This generalizes the calculation in [**24**] that $\mathrm{Sub}_3(S^2, x_0) \simeq S^4$.

5.2. Homology calculations

We determine the homology of $\mathrm{Sub}_3(T, x_0)$ and $\mathrm{Sub}_3(T)$ where T is the torus $S^1 \times S^1$. Symmetric products of surfaces are studied in various places (see [**13, 24**] and references therein). Their homology is torsion free and hence particularly simple to describe. We will write $q\colon X^n \longrightarrow \mathrm{SP}^n X$ throughout for the quotient map and

$$q_*(a_1 \otimes \ldots \otimes a_n) = a_1 * a_2 * \cdots * a_n$$

for its induced effect in homology. (Since our spaces are torsion free we identify $H_*(X \times Y)$ with $H_*(X) \otimes H_*(Y)$.)

Corollary 5.8. *The inclusion $j\colon \mathrm{Sub}_2(T, x_0) \hookrightarrow \mathrm{Sub}_3(T, x_0)$ is essential.*

Proof. We will show that j_* is non-trivial on $H_2(\mathrm{Sub}_2(T, x_0)) = H_2(T) = \mathbb{Z}$. Here $H_*(T)$ is generated by e_1, e_2 in dimension one, and by the orientation class $[T]$ in dimension two. The groups $H_*(\mathrm{SP}^2 T)$ are given as follows [**13**] (the generators are indicated between brackets):

$$\tilde{H}_*(\mathrm{SP}^2 T) = \begin{cases} \mathbb{Z}\{\gamma_2\}, & \dim 4 \\ \mathbb{Z}\{e_1 * [T], e_2 * [T]\}, & \dim 3 \\ \mathbb{Z}\{[T], e_1 * e_2\}, & \dim 2 \\ \mathbb{Z}\{e_1, e_2\}, & \dim 1, \end{cases} \tag{12}$$

where γ_2 is the orientation class $[\mathrm{SP}^2 T]$ ($\mathrm{SP}^2(T)$ is a compact complex surface). Then $[T] * [T] = 2\gamma_2$. Let Δ be the diagonal into the symmetric square

$$X \xrightarrow{\Delta} X \times X \xrightarrow{q} \mathrm{SP}^2(X).$$

Since

$$\Delta_*([T]) = [T] \otimes 1 + e_1 \otimes e_2 - e_2 \otimes e_1 + 1 \otimes [T],$$

$$q_*([T] \otimes 1) = q_*(1 \otimes [T]) = [T]$$

and

$$q_*(e_1 \otimes e_2) = -q_*(e_2 \otimes e_1) = e_1 * e_2,$$

we see that

$$\Delta_*([T]) = 2[T] + 2e_1 * e_2. \tag{13}$$

We can consider the composite

$$j_{x_0}\colon T \xrightarrow{\Delta} \mathrm{SP}^2T \xrightarrow{\alpha} \mathrm{Sub}_3(T,x_0) = \mathrm{SP}^2T/\sim,$$

where α is as in the proof of Lemma 5.4. According to Lemma 5.4, using the expression of the diagonal in (13), there are classes $a = \alpha_*[T], b = \alpha_*(e_1 * e_2)$ with $a = -2b \neq 0$. But $(j_{x_0})_*[T] = (\alpha\circ\Delta)_*[T] = \alpha_*([T]) = a$, and this is non-zero as desired. □

Remark 5.9. We can of course complete the calculation of $H_*(\mathrm{Sub}_3(T,x_0))$ from Lemma 5.4. Under α_*, $e_i \mapsto 0$ (primitive classes map to 0), $e_1 * e_2 \mapsto b$, $[T] \mapsto a = -2b$, $e_i * [T] \mapsto c_i$, and $\gamma_2 \mapsto d$, so that

$$H_1 = 0, \quad H_2 = \mathbb{Z}\{a\}, \quad H_3 = \mathbb{Z}\{c_1, c_2\}, \quad H_4 = \mathbb{Z}\{d\}.$$

It is equally easy to write down the homology groups for $\mathrm{Sub}_3(S,x_0)$ for any genus $g \geqslant 1$ surface, orientable or not.

Next we analyze the inclusion $T \hookrightarrow \mathrm{Sub}_3 T$ in the case of the torus (compare [**24**]). The starting point is the pushout (3) and the associated Mayer-Vietoris sequence

$$\cdots \longrightarrow H_*(T\times T) \xrightarrow{q_*\oplus i_*} H_*(\mathrm{SP}^2T)\oplus H_*(\mathrm{SP}^3T) \xrightarrow{g_*-\pi_*} H_*(\mathrm{Sub}_3 T) \longrightarrow H_{*-1}(T\times T) \longrightarrow \cdots,$$

where $q\colon T\times T \longrightarrow \mathrm{SP}^2T$ is the quotient map, $i(x,y) = x^2y$, $g\colon \mathrm{SP}^2T \hookrightarrow \mathrm{Sub}_3 T$ is the inclusion (here we have identified SP^2T with $\mathrm{Sub}_2 T$) and $\pi\colon \mathrm{SP}^3T \longrightarrow \mathrm{Sub}_3 T$ is the projection. We focus on degree 2 and follow [**13**] for the next computations.

We have $H_2(T\times T) = \mathbb{Z}^2$ generated by $[T]\otimes 1$ and $1\otimes[T]$, $H_2(\mathrm{SP}^2T) = \mathbb{Z}^2 = H_2(\mathrm{SP}^3T)$ generated by a class of the same name $[T] = q_*([T]\otimes 1) = q_*(1\otimes[T])$ and by $e_1 * e_2$; see (12). To describe the effect of i_* we write it as a composite

$$i\colon T\times T \xrightarrow{\Delta\times 1} T\times T\times T \xrightarrow{q} \mathrm{SP}^3T.$$

This gives $i_*([T]\otimes 1) = 2[T] + 2e_1 * e_2$ as in (13), while $i_*(1\otimes[T]) = [T]$. The Mayer-Vietoris then looks like

$$\begin{aligned}\cdots \longrightarrow \mathbb{Z}^2 &\xrightarrow{q_*\oplus i_*} \mathbb{Z}^2\oplus\mathbb{Z}^2 \xrightarrow{g_*-\pi_*} H_2(\mathrm{Sub}_3 T) \longrightarrow H_1(T\times T) \longrightarrow \cdots\\ (1,0) &\longmapsto ((1,0),(2,2))\\ (0,1) &\longmapsto ((1,0),(1,0)).\end{aligned}$$

This sequence is exact. Observe that the class $((2,2),(0,0))$ is not in the kernel of $g_* - \pi_*$ because it cannot be in the image of $q_*\oplus i_*$. This means that $g_*(2,2)\neq 0$. This is all we need to derive the non-nullity of the map $j\colon X \hookrightarrow \mathrm{Sub}_3 X$.

Corollary 5.10. $j_*([T]) \neq 0$.

Proof. The inclusion j is the composite

$$j\colon T \xrightarrow{\Delta} T\times T \xrightarrow{\pi} \mathrm{SP}^2T \xrightarrow{g} \mathrm{Sub}_3 T$$

so that $j_*([T]) = g_*(2,2)$, and this is non-trivial as asserted above. □

6. The top dimension

Using facts about orientability of configuration spaces of closed manifolds ([**11**] for example), we slightly elaborate on [**9**] and ([**24**, Theorem 3]).

Proposition 6.1. *Suppose M is a closed manifold of dimension $d \geqslant 2$. Then*

$$H_{nd}(SP^n M; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } d \text{ even and } M \text{ orientable} \\ 0, & \text{if } d \text{ odd or } M \text{ non-orientable.} \end{cases}$$

For mod-2 coefficients, $H_{nd}(SP^n M; \mathbb{F}_2) = \mathbb{F}_2$. In all cases, the map

$$H_{nd}(SP^n M) \longrightarrow H_{nd}(\mathrm{Sub}_n\, M)$$

is an isomorphism (Corollary 3.4).

Proof. When $d = 2$ the claim is immediate since, as is well known, $\mathrm{SP}^n M$ is a closed manifold (orientable if and only if M is; see [**26**]). Generally our statement follows from the fact that $\mathrm{SP}^n(X)$ is an orbifold with codimension > 1 singularities, and hence its top homology group is that of a manifold. More explicitly, in our case, let us denote by $B(M, n)$ the configuration space of finite sets of cardinality n in M; that is

$$B(M, n) = \mathrm{SP}^n M - \mathbf{\Delta}^n = \mathrm{Sub}_n\, M - \mathrm{Sub}_{n-1}\, M,$$

where $\mathbf{\Delta}^n$ is the singular set consisting of tuples with at least one repeated entry (the image of the fat diagonal as defined in Section 2). By Poincaré duality suitably applied ([**11**, Lemma 3.5])

$$H^i(B(M, n); \pm\mathbb{Z}) \cong H_{nd-i}(\mathrm{SP}^n M, \mathbf{\Delta}^n; \mathbb{Z}), \tag{14}$$

where $\pm\mathbb{Z}$ is the orientation sheaf. By definition,

$$H^i(B(M, n), \pm\mathbb{Z}) = H^i(\mathrm{Hom}_{Br_n(M)}(C_*(\tilde{B}(M, n)), \mathbb{Z}))$$

where $Br_n(M) = \pi_1(B(M, n))$ is the braid group of M, $\tilde{B}(M, n)$ is the universal cover of $B(M, n)$ and the action of the class of a loop on $\mathbb{Z}$ is multiplication by ± 1 according to whether the loop preserves or reverses orientation. It is known that $B(M, n)$ is orientable if and only if M is orientable and even-dimensional ([**11**, Lemma 2.6]). That is, we can replace $\pm\mathbb{Z}$ by $\mathbb{Z}$ if M is orientable and d is even.

Since $\mathbf{\Delta}^n$ is a subcomplex of codimension d in $\mathrm{SP}^n M$, we have

$$H_{nd-i}(\mathrm{SP}^n M, \mathbf{\Delta}^n) \cong H_{nd-i}(\mathrm{SP}^n M) \quad \text{for} \quad i < d - 1.$$

In particular, for $i = 0$ we obtain

$$H^0(B(M, n); \pm\mathbb{Z}) \cong H_{nd}(\mathrm{SP}^n M; \mathbb{Z}). \tag{15}$$

If M is even-dimensional and orientable, then

$$H^0(B(M, n); \pm\mathbb{Z}) \cong H^0(B(M, n); \mathbb{Z}) = \mathbb{Z},$$

since $B(M, n)$ is connected if $\dim M \geqslant 2$. If $\dim M$ is odd or M is non-orientable, then $B(M, n)$ is not orientable and $H^0(B(M, n); \pm\mathbb{Z}) = 0$, because $H^0(B(M, n); \pm\mathbb{Z})$ is the subgroup $\{m \in \mathbb{Z} \mid gm = m, \forall g \in \mathbb{Z}[\pi_1(B(M, n))]\}$. This establishes the claim for the

symmetric products and hence for the finite subset spaces according to Corollary 3.4. □

Example 6.2. For $k \geqslant 2$ we have $H_{2k}(\mathrm{SP}^2 S^k) = H_{2k}(\overline{\mathrm{SP}}^2 S^k) = H_{k-1}(\mathbb{R}P^{k-1})$ (see Example 5.7) and this is $\mathbb{Z}$ or 0 depending on whether k is even or odd as predicted by Proposition 6.1.

6.1. The case of the circle

When $M = S^1$, Proposition 6.1 is not true anymore since $\mathrm{SP}^n S^1 \simeq S^1$ for all $n \geqslant 1$, while $\mathrm{Sub}_n(S^1)$ is either S^n or S^{n-1} depending on whether n is odd or even [**15, 23**]. In this case, it is still possible to explicitly describe the quotient map $\mathrm{SP}^n(S^1) \longrightarrow \mathrm{Sub}_n(S^1)$.

A beautiful theorem of Morton asserts that the multiplication map

$$\mathrm{SP}^{n+1}(S^1) \longrightarrow S^1$$

is an n-disc bundle η_n over S^1 which is orientable if and only if n is even [**16**]. A close scrutiny of Morton's proof shows that the sphere bundle associated to η_n consists of the image of the fat diagonal $\mathbf{\Delta}^{n+1}$, i.e. the singular set. If $\mathrm{Th}(\eta_n)$ is the Thom space of η_n, then

$$\mathrm{Th}(\eta_n) = \mathrm{SP}^{n+1}(S^1)/\mathbf{\Delta}^{n+1} = \mathrm{Sub}_{n+1} S^1 / \mathrm{Sub}_n S^1. \tag{16}$$

Since η_n is trivial when $n = 2k$ is even, it follows that

$$\mathrm{Th}(\eta_{2k}) = S^{2k} \wedge S^1_+ = S^{2k+1} \vee S^{2k}. \tag{17}$$

However, as pointed out above, $\mathrm{Sub}_{2k+1}(S^1) \simeq S^{2k+1}$. The map

$$\mathrm{SP}^{2k+1}(S^1) \longrightarrow \mathrm{Sub}_{2k+1}(S^1)$$

factors through the Thom space (17) and the top cell maps to the top cell. Combining (16) and (17), it is immediate to see that

Lemma 6.3. *The map* $\mathrm{Th}(\eta_{2k}) \longrightarrow \mathrm{Sub}_{2k+1}(S^1)$*, restricted to the first wedge summand in (17), induces a map* $S^{2k+1} \longrightarrow \mathrm{Sub}_{2k+1}(S^1)$ *which is a homotopy equivalence.*

7. Manifold structure

In this last section we prove Theorem 1.3. We distinguish three cases: when the dimension of the manifold is $d > 2$, $d = 2$ or $d = 1$.

Lemma 7.1. *Suppose X is a manifold of dimension $d > 2$. Then* $\mathrm{Sub}_n X$ *is never a manifold if $n \geqslant 2$.*

Proof. Consider the projection $X^n \longrightarrow \mathrm{Sub}_n X$ given by identifying tuples whose sets of coordinates are the same. This projection restricts to an $n!$ regular covering between the complements $\pi_n : X^n - \Lambda^n \longrightarrow \mathrm{Sub}_n X - \mathrm{Sub}_{n-1} X$, where Λ^n as before is the fat diagonal in X^n. Suppose $\mathrm{Sub}_n X$ is a manifold of dimension nd (necessarily). Pick a point in $\mathrm{Sub}_{n-1} X$ and an open chart U around it. Now $U \cong \mathbb{R}^{nd}$ and

$Y = U \cap \mathrm{Sub}_{n-1} X$ is a closed subset in U. We can apply Alexander duality to the pair (Y, U) and obtain

$$H_{nd-i-1}(U - Y) \cong H^i(Y).$$

But $Y \subset \mathrm{Sub}_{n-1}(X)$ is an open subspace in a simplicial complex of dimension $(n-1)d$; therefore $H^{nd-2}(Y) = 0$ (since $d > 2$) and so $H_1(U - Y) = 0$. We can now use an elementary observation of Mostovoy [**17**] to the effect that since $U - Y$ is covered by $\pi_n^{-1}(U - Y)$, a connected étale cover of degree $n!$, then it is impossible for $H_1(U - Y)$ to be trivial since the monodromy gives a surjection $\pi_1(U - Y) \longrightarrow \mathfrak{S}_n$, and hence a non-trivial map $H_1(U - Y) \longrightarrow \mathbb{Z}_2$. □

Theorem 2.4 of [**26**] shows that our Lemma 7.1 is valid if $d = 2$ and $n > 2$ as well. As opposed to the geometric approach of Wagner, we provide below a short homological proof of this result.

Lemma 7.2. *Suppose X is a closed topological surface. Then $\mathrm{Sub}_n X$ is a manifold if and only if $n = 2$.*

Proof. We will show that if $n \geqslant 3$, then $\mathrm{Sub}_n(X)$ cannot even have the homotopy type of a closed manifold by showing that it does not satisfy Poincaré duality. We rely on results of [**13**] that give a simple description of a CW-decomposition of a space $\widehat{\mathrm{SP}}^n X$ homotopy equivalent to $\mathrm{SP}^n X$ when X is a two-dimensional complex. Since X is a closed two-dimensional manifold, it has a cell structure of the form $X = \bigvee^r S^1 \cup D^2$ where D^2 is a two-dimensional cell attached to a bouquet of circles. Each circle corresponds in the cellular chain complex for $\widehat{\mathrm{SP}}^n X$ to a one-dimensional cell generator e_i, $1 \leqslant i \leqslant r$, while the two-dimensional cell is represented by D. This chain complex has a concatenation product $*\colon C_*(\widehat{\mathrm{SP}}^r X) \otimes C_*(\widehat{\mathrm{SP}}^s X) \longrightarrow C_*(\widehat{\mathrm{SP}}^{r+s} X)$ under which these cells map to product cells. The full cell complex for $\widehat{\mathrm{SP}}^n X$ is made up of all products of the form

$$e_{i_1} * \cdots * e_{i_\ell} * \mathrm{SP}^k D, \qquad i_1 + \cdots + i_\ell + k \leqslant n,$$

where $i_r \neq i_s$ if $r \neq s$, and where $\mathrm{SP}^k D$ is a $2k$-dimensional cell represented geometrically by the k-th symmetric product of D^2. The boundary ∂ is a derivation and is completely determined on generators by $\partial e_i = 0$ and $\partial \mathrm{SP}^n D = \partial D * \mathrm{SP}^{n-1} D$.

If $X = \bigvee^r S^1 \cup D$ is a closed manifold, then in mod-2 homology, $\partial D = 0$ (the top cell). This implies of course that $\partial \mathrm{SP}^n D = 0$ (the top cell of $\mathrm{SP}^n X$), while $H_{2n-1}(\mathrm{SP}^n X, \mathbb{Z}_2) \cong \mathbb{Z}_2^r$ with generators $e_i * \mathrm{SP}^{n-1} D$. This shows, in particular, that $H_{2n-1}(\mathrm{SP}^n X; \mathbb{Z}_2) \neq 0$ if $r \geqslant 1$, that is if X is not the two sphere. Observe that this calculation is compatible with Theorem 2 of [**24**].

Now we know that $\mathrm{Sub}_n X$ is simply connected if $n \geqslant 3$. Suppose $\mathrm{Sub}_n X$ is a closed manifold, then by Poincaré duality, $H_{2n-1}(\mathrm{Sub}_n X; \mathbb{Z}_2) = H_1(\mathrm{Sub}_n X; \mathbb{Z}_2) = 0$. But recall the pushout diagram (2) and its associated Mayer-Vietoris exact sequence

$$\begin{aligned} H_{2n-1}(\Delta_n) \longrightarrow H_{2n-1}(\mathrm{Sub}_{n-1} X) \oplus H_{2n-1}(\mathrm{SP}^n X) \\ \longrightarrow H_{2n-1}(\mathrm{Sub}_n X) \longrightarrow H_{2n-2}(\Delta_n) \longrightarrow \cdots . \end{aligned}$$

Since Δ_n and $\mathrm{Sub}_{n-1} X$ are $(2n-2)$-dimensional subcomplexes of $\mathrm{Sub}_n X$, their

homology in degree $2n-1$ vanishes. The sequence above becomes

$$0 \longrightarrow H_{2n-1}(\mathrm{SP}^n X) \longrightarrow H_{2n-1}(\mathrm{Sub}_n X) \longrightarrow H_{2n-2}(\Delta_n) \longrightarrow \cdots$$

and $H_{2n-1}(\mathrm{SP}^n X)$ injects into $H_{2n-1}(\mathrm{Sub}_n X)$. When $H_1(X) \neq 0$, that is when X is not the sphere, $H_{2n-1}(\mathrm{Sub}_n X)$ is non-trivial contradicting Poincaré duality.

We are left with the case $\mathrm{Sub}_n(S^2)$ and $n \geqslant 3$. Here we have to rely on a calculation of Tuffley [**24**] who shows that

$$H_{2n-2}(\mathrm{Sub}_n(S^2)) = \mathbb{Z} \oplus \mathbb{Z}_{n-1}. \tag{18}$$

But $\mathrm{Sub}_n(S^2)$ is 2-connected according to Theorem 1.1 and Poincaré duality is violated in this case as well. □

Remark 7.3. A computation of the homology of $\mathrm{Sub}_n(S^2)$ for all n and various field coefficients will appear in [**12**]. It is however straightforward using the Mayer-Vietoris sequence for the pushout (3) to show that

$$\tilde{H}_*(\mathrm{Sub}\, 3S^2) \cong \begin{cases} \mathbb{Z}, & *=6 \\ \mathbb{Z} \oplus \mathbb{Z}_2, & *=4. \end{cases} \tag{19}$$

Similar computations appear in [**5, 22, 24**].

Finally we address the case $d = 1$. Write $I = [0,1], \dot{I} = (0,1)$. First of all $\mathrm{SP}^n(I) \cong I^n$. In fact, this is precisely the n-simplex since any point of $\mathrm{SP}^n(I)$ can be written uniquely as an n-tuple $(x_1, \ldots, x_n)$ with $0 \leqslant x_1 \leqslant \cdots \leqslant x_n \leqslant 1$. The quotient map $q_2 \colon \mathrm{SP}^2(I) \longrightarrow \mathrm{Sub}_2(I)$ is a homeomorphism and hence every interior point of $\mathrm{Sub}_2(I)$ has a manifold neighborhood. The same for $n = 3$ since $\mathrm{SP}^3(I)$ is the three simplex

$$\{(x_1, x_2, x_3) \mid 0 \leqslant x_1 \leqslant x_2 \leqslant x_3 \leqslant 1\}$$

with four faces: $F_1 : \{x_1 = 0\}$, $F_2 : \{x_1 = x_2\}$, $F_3 : \{x_2 = x_3\}$ and $F_4 : \{x_3 = 1\}$, and the quotient map $q_3 \colon \mathrm{SP}^3(I) \to \mathrm{Sub}_3(I)$ identifies the faces F_2 and F_3. Such an identification gives again I^3 and $\mathrm{Sub}_3(\dot{I})$ is this simplex with two faces removed [**19**]. For $n > 3$, the corresponding map q_n identifies various faces of the simplex $\mathrm{SP}^n(I)$ to obtain $\mathrm{Sub}_n(I)$, but this fails to give a manifold structure on the quotient for there are just too many "branches" that come together at a single point in the image of the boundary of this simplex. This is made precise below.

Lemma 7.4. $\mathrm{Sub}_n(S^1)$ *is a closed manifold if and only if* $n = 1, 3$.

Observe that if n is even, then $\mathrm{Sub}_n S^1$ cannot be a closed manifold for a simple reason: no closed manifold of dimension n can be homotopic to a sphere of dimension $n-1$.

*Proof of Lemma 7.4 following [**26**, Theorem 2.3].* Let M be a manifold and D a disc neighborhood of a point $x \in M$. Then an open neighborhood of $x \in \mathrm{Sub}_n(M)$ is $\mathrm{Sub}_n(D)$. So if $\mathrm{Sub}_n(D)$ is not a manifold, then neither is $\mathrm{Sub}_n(M)$. To prove Lemma 7.4 we will argue as in [**26**] that $\mathrm{Sub}_n(\mathbb{R})$ is not a manifold for $n \geqslant 4$.

For a metric space X (with metric d), non-empty subsets $S, T \subset X$, and fixed elements $s \in S, t \in T$, we define

$$d(s,T) = \inf\{d(s,t) \mid t \in T\},$$
$$d(S,t) = \inf\{d(s,t) \mid s \in S\}.$$

Then the Hausdorff metric D on $\mathrm{Sub}_n(X)$ is defined to be

$$D(S,T) := \sup\{d(s,T), d(t,S) \mid s \in S, t \in T\}.$$

Thus $D(S,T) < \epsilon$ means that each $s \in S$ is within an ϵ-neighborhood of some point in T and each $t \in T$ is within an ϵ-neighborhood of some point in S.

We wish to show that $\mathrm{Sub}_n(\mathbb{R})$ for $n \geqslant 4$ is not homemorphic to $\mathbb{R}^n$. Pick $S = \{1, 2, \ldots, n-1\}$ in $\mathrm{Sub}_{n-1}(\mathbb{R})$ and for each i consider the open set C_i (in the Hausdorff metric) of all subsets $\{p_1, \ldots, p_{n-1}, q_i\} \in \mathrm{Sub}_n(\mathbb{R})$ such that $p_j \in (j - \frac{1}{2}, j + \frac{1}{2})$ and $q_i \in (i - \frac{1}{2}, i + \frac{1}{2})$. We then see that C_i is the subset with one or two points in the $\frac{1}{2}$-neighborhood of i and a single point in the $\frac{1}{2}$-neighborhood of j for $i \neq j$. Note that $C_i \subset U$ where $U = \{T \in \mathrm{Sub}_n(\mathbb{R}) \mid D(S,T) < 1/2\}$. Observe that

$$C_1 = \mathrm{Sub}_2\left(\frac{1}{2}, \frac{3}{2}\right) \times \left(\frac{3}{2}, \frac{5}{2}\right) \times \cdots \times \left(n - 1 - \frac{1}{2}, n - 1 + \frac{1}{2}\right).$$

This is an n-dimensional manifold with boundary $V = U \cap \mathrm{Sub}_{n-1}(\mathbb{R})$, and in fact one has

$$C_i = \left\{T \in U : T \cap \left(i - \frac{1}{2}, i + \frac{1}{2}\right) \text{ has 1 or 2 points}\right\} \cup V.$$

Clearly $C_1 \cup C_2 \cup \cdots \cup C_{n-1} = U$ and, more importantly, all these open sets have a common boundary at V; i.e. $C_i \cap C_j = V$. If $n \geqslant 4$, we can choose at least three such C_i, say C_1, C_2, C_3. Then $C_1 \cup C_2$ is an open n-dimensional manifold (union over the common boundary V). It must be contained in the interior of $\mathrm{Sub}_n(\mathbb{R})$ and hence must be open there if $\mathrm{Sub}_n(\mathbb{R})$ were to be an n-dimensional manifold. But $C_1 \cup C_2$ is not open in $\mathrm{Sub}_n(\mathbb{R})$ since every neighborhood of $\{1, 2, \ldots, n-1\}$ must meet $C_3 - V$ which is disjoint from $C_1 \cup C_2$ (i.e. "too many" branches come together at that point). □

We conclude this paper with the following cute theorem of Bott, which is the most significant early result on the subject:

Corollary 7.5 (Bott)**.** *There is a homeomorphism* $\mathrm{Sub}_3(S^1) \cong S^3$.

Proof. It has been known since Seifert that the Poincaré conjecture holds for Seifert manifolds; that is, if a Seifert 3-manifold is simply connected then it is homeomorphic to S^3.[1] Clearly $\mathrm{Sub}_3(S^1)$ is a Seifert manifold where the action of S^1 on a subset is by multiplication on elements of that subset. Since it is simply connected (Corollary 2.2), the claim follows. Note that the S^1-action has two exceptional fibers consisting of the orbits of $\{1, -1\}$ and $\{1, j, j^2\}$ where $j = e^{2\pi i/3}$ (compare [**23**]). □

[1]We thank Peter Zvengrowski for reminding us of this fact.

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