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HOCHSCHILD HOMOLOGY WITH COEFFICIENTS IN AN H-UNITAL IDEAL

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Abstract

Let k be a field and I an H-unital two-sided ideal of an associative unital k-algebra A. Then, based on Wodzicki's work, we show that the Hochschild homology groups of A with coefficients in I are isomorphic to the Hochschild homology groups of I.

Definitions and conventions

Let k be a commutative unital ring, A an associative unital algebra over k and M a bimodule over A. Consider the following complex, called the Hochschild complex,

$$C(A,M) := \cdots \to M \otimes A^{\otimes n} \xrightarrow{b} M \otimes A^{\otimes n-1} \xrightarrow{b} \cdots \xrightarrow{b} M \otimes A \xrightarrow{b} M,$$

where $\otimes = \otimes_k$ and the Hochschild boundary $b\colon M \otimes A^{\otimes n} \to M \otimes A^{\otimes n-1}$ is the k-linear map given by the formula

$$b(m, a_1, \ldots, a_n) := (ma_1, a_2, \ldots, a_n) + \sum_{i=1}^{n} (-1)^i (m, a_1, \ldots, a_i a_{i+1}, \ldots, a_n) + (-1)^n (a_n m, a_1, \ldots, a_{n-1}).$$

By definition, the nth Hochschild homology group of A with coefficients in M is the nth homology group of $C(A,M)$ denoted by $H_n(A,M)$. In the case where $M = A$, the Hochschild complex is denoted by $C(A)$ and its nth homology is called the nth Hochschild homology group of A denoted by $HH_n(A)$.

Let I be a k-algebra (not necessarily unital). Denote by I_+ the unital k-algebra $k \oplus I$, where the multiplication structure is given by

$$(k_1, x)(k_2, y) = (k_1 k_2, k_1 y + x k_2 + xy).$$

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Then the nth Hochschild homology group of I is defined as follows:

$$HH_n(I) := H_n(\mathrm{Coker}\{C(k) \to C(I_+)\}).$$

Consider now the following complex, called the bar complex,

$$C^{\mathrm{bar}}(I)\colon \ \cdots \to I^{\otimes n+1} \xrightarrow{b'} I^{\otimes n} \xrightarrow{b'} \cdots \xrightarrow{b'} I^{\otimes 2} \xrightarrow{b'} I,$$

where the differential $b'\colon I^{\otimes n+1} \to I^{\otimes n}$ is given by the formula

$$b'(x_0, x_1, \dots, x_n) := \sum_{i=0}^{n-1} (-1)^i (x_0, x_1, \dots, x_i x_{i+1}, \dots, x_n).$$

Definition (M. Wodzicki)**.** The k-algebra I is said to be H-unital, if

$$H_*(V \otimes C^{\mathrm{bar}}(I)) = 0$$

for any k-module V.

Main result

We prove the following:

Proposition 1. *Let k be a field, A a unital k-algebra and I an H-unital two-sided ideal of A. Then, for each $n \geqslant 0$, there are isomorphisms*

$$H_n(A, I) = HH_n(I) \qquad \textit{and} \qquad H_n(A, A/I) = HH_n(A/I).$$

Proof. Consider the following commutative diagram of complexes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C(A,I) & \longrightarrow & C(A) & \longrightarrow & C(A,A/I) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle 1_{C(A)}} & & \Big\downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & \mathrm{Ker}\{C(A) \to C(A/I)\} & \longrightarrow & C(A) & \longrightarrow & C(A/I) & \longrightarrow & 0,
\end{array} \tag{1}$$

where f and g are naturally induced homomorphisms of complexes. We will show that f and g induce the isomorphisms:

$$f_*\colon H_n(A, I) \xrightarrow{\cong} H_n(\mathrm{Ker}\{C(A) \to C(A/I)\}),$$
$$g_*\colon H_n(A, A/I) \xrightarrow{\cong} HH_n(A/I),$$

which will complete the proof, since, according to [**2**], there is an isomorphism

$$H_n(\mathrm{Ker}\{C(A) \to C(A/I)\}) = HH_n(I).$$

The upper row of (1) is exact, since k is a field. By definition, the lower row of (1) is also exact. Thus, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H_{n+1}(A,A/I) & \longrightarrow & H_n(A,I) & \longrightarrow & HH_n(A) & \longrightarrow & H_n(A,A/I) & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle g_*} & & \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle 1_{HH_n(A)}} & & \Big\downarrow{\scriptstyle g_*} & & \\
\cdots & \longrightarrow & HH_{n+1}(A/I) & \longrightarrow & HH_n(I) & \longrightarrow & HH_n(A) & \longrightarrow & HH_n(A/I) & \longrightarrow & \cdots .
\end{array} \tag{2}$$

From (2) and by the Five Lemma it suffices to show that g is a quasi-isomorphism. Since $g_n\colon (A/I) \otimes A^{\otimes n} \to (A/I) \otimes (A/I)^{\otimes n}$ is an epimorphism for each n, the key is to prove acyclicity for the complex $\mathrm{Ker}\, g$.

We need the following fact which is proved in [**2**] (see also [**1**]):

Lemma 1. *Let* $L = \operatorname{Ker}\{C^{\text{bar}}(A) \to C^{\text{bar}}(A/I)\}$. *Define the following filtration on* L*:*

$$F_pL_n = \textit{ linear span of } \{(a_0, a_1, \dots, a_n) \in L_n \mid \textit{ at least } n-p+1 \ a_j\textit{'s belong to } I\}.$$

Then the associated spectral sequence $E^r_{pq} (r \geqslant 0)$ *is in the first quadrant and* $E^1_{pq} = 0$.

Return to the proof of Proposition 1. Define the following filtration on $\operatorname{Ker} g$:

$$\begin{aligned}\bar{F}_p(\operatorname{Ker} g_n) = \ & \text{linear span of } \{(a_0, a_1, \dots, a_n) \\ & \qquad \in \operatorname{Ker} g_n \mid \text{ at least } \ n-p \ a_j\text{'s } (j \geqslant 1) \ \text{ belong to } I\}.\end{aligned}$$

Denote by $\bar{E}^r_{pq}$ $(r \geqslant 0)$ the associated spectral sequence. We will show that $\bar{E}^1_{p*} = 0$. By definition,

$$\bar{F}_p(\operatorname{Ker} g_n) = (A/I) \otimes F_p(L_{n-1}) \Rightarrow \bar{E}^0_{pq} = (A/I) \otimes E^0_{p\ q-1},$$

where L, F and E are defined as in Lemma 1. Moreover, if $(a_0, \dots, a_n) \in \bar{F}_p(\operatorname{Ker} g_n)$, then it is easy to check that

$$(a_0a_1, a_2, \dots, a_n) \qquad \text{and} \qquad (a_na_0, a_1, \dots, a_{n-1}) \in \bar{F}_{p-1}(\operatorname{Ker} g_{n-1}).$$

Thus,

$$b(a_0, \dots, a_n) = -a_0 \otimes b'(a_1, \dots, a_n) \bmod \bar{F}_{p-1}(\operatorname{Ker} g_{n-1}).$$

Therefore, the following square

$$\begin{array}{ccc} \bar{E}^0_{pq} & \xrightarrow{\simeq} & (A/I) \otimes E^0_{p\ q-1} \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle 1\otimes(-b')} \\ \bar{E}^0_{p\ q-1} & \xrightarrow{\simeq} & (A/I) \otimes E^0_{p\ q-2} \end{array} \tag{3}$$

is commutative. Hence $\bar{E}^1_{pq} = (A/I) \otimes E^1_{p\ q-1} = 0$. $\square$

Given a k-algebra A (not necessarily unital) and an A-bimodule M, the complex $C(A, M)$ can be defined in the same way.

Let I be a two-sided ideal of a k-algebra A. The inclusion $I \subset A$ induces the map of complexes $C(I) \to C(A, I)$ denoted by $\alpha_{(A,I)}$.

Proposition 2. *Let k be a field. Then the following are equivalent:*

(i) I is H-unital;

(ii) $\alpha_{(A,I)}$ is a quasi-isomorphism for any k-algebra A which contains I as a two-sided ideal.

Proof of (i) $\Rightarrow$ (ii). Consider the following sequence:

$$C(I) \xrightarrow{\alpha_{(A,I)}} C(A, I) \xrightarrow{f} \operatorname{Ker}\{C(A) \to C(A/I)\},$$

where f is naturally defined as in diagram (1). As in the previous proposition we can show that f is a quasi-isomorphism, since in Lemma 1 it is not necessary to demand that A is unital (see [**1**] or [**2**]). Moreover, by [**2**], $f \circ \alpha_{(A,I)}$ is a quasi-isomorphism.

Proof of (ii) ⇒ (i). We have to show that, if $\alpha_{(A,I)}$ is a quasi-isomorphism for any A, then I is H-unital. We use the well-known trick. Namely, for any vector space V consider the k-algebra $I \oplus V$ with multiplication given by $(x, v)(y, u) = (xy, 0)$. The quasi-isomorphism $C(I) \to C(I \oplus V, I)$ implies that $C(I \oplus V, I)/C(I)$ is acyclic. But $C(I \oplus k, I)/C(I)$ contains $V \otimes C^{\mathrm{bar}}(I)[1]$ as a direct summand, where $C^{\mathrm{bar}}(I)[1]$ means that the complex $C^{\mathrm{bar}}(I)$ is shifted by 1. Thus, $V \otimes C^{\mathrm{bar}}(I)$ is acyclic. □

Remark. In the case when k is not a field, using the same arguments we can prove that a k-algebra I is H-unital if and only if $\alpha_{(A,I)}$ is a quasi-isomorphism for any k-algebra A which contains I as a two-sided ideal and $A \to A/I$ is k-split.

Corollary 1. *Let k be a field and A a unital algebra over k. If I is an H-unital ideal of A, then for each $m \geqslant 1$ and $n \geqslant 0$, we have $HH_n(\mathcal{M}_m(A), \mathcal{M}_m(I)) = HH_n(I)$, where $\mathcal{M}_m(A)$ and $\mathcal{M}_m(I)$ denote the algebras of all $m \times m$ matrices with entries in A and I, respectively.*

Proof. Straightforward from Proposition 2 and [**1**, 1.4.14 Theorem]. □

Let A be a unital algebra over k. Denote by $\bar{\mathcal{M}}(A)$ the algebra of all infinite matrices whose columns and rows contain a finite number of non-trivial elements. Put $\mathcal{M}(A) = \varinjlim_m \mathcal{M}_m(A)$, $m \geqslant 1$. Then $\mathcal{M}(A)$ is a two-sided ideal of $\bar{\mathcal{M}}(A)$ and we have the following:

Corollary 2. *If k is a field, then $HH_n(\bar{\mathcal{M}}(A), \mathcal{M}(A)) = HH_n(A)$, for each $n \geqslant 0$.*

Proof. It is known that $\mathcal{M}(A)$ is H-unital, since it has local units (see [**1**]). Hence $HH_n(\bar{\mathcal{M}}(A), \mathcal{M}(A)) = HH_n(\mathcal{M}(A))$, but by Morita invariance we have

$$HH_n(\mathcal{M}(A)) = \varinjlim_m HH_n(\mathcal{M}_m(A)) = \varinjlim_m HH_n(A) = HH_n(A). \qquad \square$$

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