Homology, Homotopy and Applications, vol. 11(1), 2009, pp.203–218

ON $H^*(\mathcal{C}; k^\times)$ FOR FUSION SYSTEMS

MARKUS LINCKELMANN

(*communicated by J.P.C. Greenlees*)

Abstract

We give a cohomological criterion for the existence and uniqueness of solutions of the 2-cocycle gluing problem in block theory. The existence of a solution for the 2-cocycle gluing problem is further reduced to a property of fusion systems of certain finite groups associated with the fusion system of a block.

1. Introduction

Given a block b of a finite group G over an algebraically closed field k of characteristic p with defect group P and associated fusion system $\mathcal{F}$, there is, for any $\mathcal{F}$-centric subgroup Q of P, a canonically determined element $\alpha_Q \in H^2(\mathrm{Aut}_{\mathcal{F}}(Q); k^\times)$, by [**13**, 1.12]. It is conjectured in [**14**, 4.2] that this family can be glued together to a class $\alpha \in H^2(\mathcal{F}^c; k^\times)$ satisfying $\alpha|_{\mathrm{Aut}_{\mathcal{F}}(Q)} = \alpha_Q$ for any Q belonging to the full subcategory $\mathcal{F}^c$ of $\mathcal{F}$-centric subgroups of P. We describe a cohomological criterion for the existence and uniqueness of α. This can be formulated more generally for EI-categories. Following [**18**, 9.2], an EI-category is a small category $\mathcal{C}$ with the property $\mathrm{End}_{\mathcal{C}}(X) = \mathrm{Aut}_{\mathcal{C}}(X)$ for any object X in $\mathcal{C}$. The set $[\mathcal{C}]$ of isomorphism classes of objects in $\mathcal{C}$ then becomes a partially ordered set via $[X] \leqslant [Y]$ if $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is non-empty, for X, Y objects in $\mathcal{C}$ and $[X]$, $[Y]$ their respective isomorphism classes. A morphism $\varphi\colon X \to Y$ in $\mathcal{C}$ need not induce a map between the automorphism groups of X, Y; the *subdivision* of $\mathcal{C}$ is a tool to address this issue. This category is defined as follows: The objects of $S(\mathcal{C})$ are faithful functors $\sigma\colon [m] \to \mathcal{C}$, where m is a non-negative integer and the totally ordered set $[m] = \{0 < 1 < \cdots < m\}$ is viewed as a category in the obvious way; a morphism in $S(\mathcal{C})$ from σ to another object $\tau\colon [n] \to \mathcal{C}$ is a pair (α, μ) consisting of an injective order-preserving map $\alpha\colon [m] \to [n]$ and an isomorphism of functors $\mu\colon \sigma \cong \tau \circ \alpha$. The composition of (α, μ) with another morphism (β, ν) from τ to $\rho\colon [r] \to \mathcal{C}$ is defined by $(\beta, \nu) \circ (\alpha, \mu) = (\beta \circ \alpha, (\nu\alpha) \circ \mu)$, where $\nu\alpha\colon \tau \circ \alpha \cong \rho \circ \beta \circ \alpha$ is induced by precomposing ν with α. Loosely speaking, $S(\mathcal{C})$ consists of chains of non-isomorphisms in $\mathcal{C}$. It is easy to see that (α, μ) induces a group homomorphism $\mathrm{Aut}_{S(\mathcal{C})}(\tau) \to \mathrm{Aut}_{S(\mathcal{C})}(\sigma)$ mapping $(\mathrm{Id}_{[n]}, \gamma)$ to $(\mathrm{Id}_{[m]}, \mu^{-1} \circ (\gamma\alpha) \circ \mu)$, for any automorphism γ of the functor τ, where $\gamma\alpha$ is the induced automorphism of $\tau \circ \alpha$. Clearly, $S(\mathcal{C})$ is again an EI-category.

Received September 26, 2007, revised October 14, 2008, November 25, 2008; published on May 25, 2009.
2000 Mathematics Subject Classification: 16E40, 18G40, 20C20.
Key words and phrases: fusion system, block, 2-cocycle.
This article is available at `http://intlpress.com/HHA/v11/n1/a10`

The partially ordered set $[S(\mathcal{C})]$, viewed as topological space, is called the *orbit space* of $\mathcal{C}$. We denote by **Ab** the category of abelian groups.

Theorem 1.1. *Let $\mathcal{C}$ be a finite* EI*-category and k an algebraically closed field. For any positive integer i, there is a canonical functor $\mathcal{A}^i\colon [S(\mathcal{C})] \to \mathbf{Ab}$ sending $[\sigma] \in [S(\mathcal{C})]$ to $H^i(\mathrm{Aut}_{S(\mathcal{C})}(\sigma); k^\times)$. If $H^1([S(\mathcal{C})]; k^\times) = H^2([S(\mathcal{C})]; k^\times) = 0$, then $H^1(\mathcal{C}; k^\times) \cong \lim_{[S(\mathcal{C})]}(\mathcal{A}^1)$, and if also $H^3([S(\mathcal{C})]; k^\times) = H^4([S(\mathcal{C})]; k^\times) = 0$, then there is an exact sequence of abelian groups*

$$0 \to H^1([S(\mathcal{C})]; \mathcal{A}^1) \to H^2(\mathcal{C}; k^\times) \to \lim_{[S(\mathcal{C})]}(\mathcal{A}^2) \to H^2([S(\mathcal{C})]; \mathcal{A}^1) \to H^3(\mathcal{C}; k^\times).$$

In particular, the group $H^2(\mathcal{C}; k^\times)$ is finite, of order coprime to $\mathrm{char}(k)$ *if* $\mathrm{char}(k)$ *is positive.*

As mentioned above, the motivation for considering the map

$$H^2(\mathcal{C}; k^\times) \to \lim_{[S(\mathcal{C})]}(\mathcal{A}^2)$$

comes from block theory, which is the reason for stating the above theorem for $k^\times$, but it is worth noting that the exact sequence in the theorem holds for any abelian group instead of $k^\times$. In order to be more precise, let b be a block of a finite group G over an algebraically closed field k of positive characteristic p, let P be a defect group of b and $\mathcal{F}$ an associated fusion system on P. Set $\mathcal{C} = \mathcal{F}^c$, the full subcategory $\mathcal{F}$ consisting of all $\mathcal{F}$-centric subgroups of P. Then $\mathcal{C}$ is a right ideal in $\mathcal{F}$, and hence $[S(\mathcal{C})]$ is contractible by [**16**, 1.1]. In particular, $H^i([S(\mathcal{C})]; k^\times) = 0$ for $i > 0$. Thus $\mathcal{C}$ satisfies the hypotheses of Theorem 1.1. As a consequence of work of Külshammer and Puig [**13**, 1.8, 1.12] in conjunction with Dade's splitting theorem on fusion, the block b determines for every $\sigma \in S(\mathcal{C})$ a class $\alpha_\sigma \in H^2(\mathrm{Aut}_{S(\mathcal{C})}; k^\times)$, and the family (α_σ) of these classes determines an element β in $\lim_{[S(\mathcal{C})]}(\mathcal{A}^2)$. Denote by γ the image of β in $H^2([S(\mathcal{C})]; \mathcal{A}^1)$. Then, by Theorem 1.1 above, the gluing problem [**14**, 4.2] has a solution if $\gamma = 0$, and the solution is unique if $H^1([S(\mathcal{C})]; \mathcal{A}^1) = 0$. In particular, if $H^1([S(\mathcal{C})]; \mathcal{A}^1) = H^2([S(\mathcal{C})]; \mathcal{A}^1) = 0$ then we have an isomorphism $H^2(\mathcal{C}; k^\times) \cong \lim_{[S(\mathcal{C})]}(\mathcal{A}^2)$, and so the gluing problem [**14**, 4.2] would have a solution for any block b with fusion system $\mathcal{F}$. This isomorphism holds trivially if $\mathcal{F} = N_\mathcal{F}(P)$ (which includes the case where P is abelian) and if P is a tame 2-group, in which case the right side is well known to be zero, and the left side is zero by a result of S. Park [**21**]. In general, this isomorphism is relevant for the block theoretic reformulation of Alperin's weight conjecture in terms of Bredon cohomology in [**15**, 4.3]. The purpose of the next result is to reduce the vanishing of $H^2([S(\mathcal{F}^c)]; \mathcal{A}^1)$ further to a statement on finite groups. For any finite group G, denote by $\Delta_p(G)$ the partially ordered G-set of chains $\sigma = Q_0 < Q_1 < \cdots < Q_m$ of non-trivial p-subgroups Q_i of G and denote by $[\Delta_p(G)]$ the set of G-conjugacy classes of chains in $\Delta_p(G)$, viewed as a partially ordered set via taking subchains. Denote by $\mathcal{N}_G\colon [\Delta_p(G)] \to \mathbf{Ab}$ the covariant functor sending the G-conjugacy class $[\sigma]$ of the chain $\sigma \in \Delta_p(G)$ as above to the abelian group $\mathcal{N}_G([\sigma]) = \mathrm{Hom}(N_G(\sigma); k^\times)$, where $\mathcal{N}_G(\sigma)$ is the intersection of the normalisers $N_G(Q_i)$, $0 \leqslant i \leqslant m$. As before, one checks that this is a well-defined functor which does not depend, up to unique isomorphism of functors, on the choice of a representative σ of $[\sigma]$.

Theorem 1.2. *Let $\mathcal{F}$ be a fusion system on a finite p-group P. Suppose that for any finite group G isomorphic to $\mathrm{Aut}_{\mathcal{F}}(Q)/\mathrm{Inn}(Q)$ for some $\mathcal{F}$-centric subgroup Q of P, we have $H^1([\Delta_p(G)];\mathcal{N}_G)=0$. Then $H^2([S(\mathcal{F}^c)];\mathcal{A}^1)=0$; in particular, the canonical map $H^2(\mathcal{F}^c;k^\times)\to\lim_{[S(\mathcal{F}^c)]}(\mathcal{A}^2)$ is surjective.*

Theorem 1.2 will follow from a spectral sequence using a filtration indexed by isomorphism classes of $\mathcal{F}$-centric subgroups, by making use of the fact that the cohomology of $\mathcal{A}^1$ may be calculated using *normal* chains of subgroups of P. After briefly reviewing some basic facts on functor cohomology in Section 2, we consider regular EI-categories and prove Theorem 1.1 in Section 3. This is followed by a section proving Theorem 1.2 and Section 5 on regular functors between EI-categories, which in turn is used in the last section to show that in order to calculate $H^*(\mathcal{C};k^\times)$ for a right ideal $\mathcal{C}$ of a fusion system, we may replace $\mathcal{C}$ by its image in the orbit category or its inverse image in a centric linking system.

Acknowledgements

The author would like to thank Jesper Grodal for many valuable comments.

2. Background

We collect, mostly from [**5, 6, 9, 10, 11, 23, 24**], background material on functor cohomology, which we will use without further comment. See also [**25**] for a broader exposition and further references. A *right ideal* in a category $\mathcal{C}$ is a full subcategory $\mathcal{D}$ of $\mathcal{C}$ with the property that if $\varphi\colon X\to Y$ is a morphism in $\mathcal{C}$ and if X belongs to $\mathcal{D}$, then Y also belongs to $\mathcal{D}$. For $\mathcal{C}$ a small category, denote by $\hat{\mathcal{C}}$ and $\check{\mathcal{C}}$ the categories of covariant and contravariant functors from $\mathcal{C}$ to **Ab**, respectively. The cohomology $H^*(\mathcal{C};\mathcal{A})$, or $H^*(\mathcal{C}^{op};\mathcal{A})$ of a functor $\mathcal{A}$ in $\hat{\mathcal{C}}$ or $\check{\mathcal{C}}$, respectively, is the graded abelian group defined as follows: for any non-negative integer n denote by $H^n(\mathcal{C};\mathcal{A})$ or $H^n(\mathcal{C}^{op};\mathcal{A})$ the n-th right derived functor of the limit functor $\lim_{\mathcal{C}}$ over $\mathcal{C}$ from $\hat{\mathcal{C}}$ or $\check{\mathcal{C}}$ to **Ab**, respectively. If A is an abelian group and $\mathcal{A}$ the constant covariant functor on $\mathcal{C}$ taking the value A, then we write $H^n(\mathcal{C};A)$ instead of $H^n(\mathcal{C};\mathcal{A})$; similarly for the contravariant constant functor. It is well known that $H^*(\mathcal{C};A)\cong H^*(\mathcal{C}^{op};A)$. If $\Phi\colon\mathcal{C}\to\mathcal{D}$ is a covariant functor, then we denote by $\Phi^*\colon\hat{\mathcal{D}}\to\hat{\mathcal{C}}$ the induced restriction functor sending $\mathcal{A}$ in $\hat{\mathcal{C}}$ to $\mathcal{A}\circ\Phi$; we use (abusively) the same notation for the restriction functor $\check{\mathcal{D}}\to\check{\mathcal{C}}$. We denote by $\Phi_*,\Phi_!\colon\hat{\mathcal{C}}\to\hat{\mathcal{D}}$ the left and right adjoint (Kan extension functors) of Φ^*, and as before, use the same notation for the categories $\check{\mathcal{C}}$, $\check{\mathcal{D}}$. By [**24**, 1.4.(ii)], [**11**, 5.3] or [**5**, Appendix II, Thm. 3.6], there is a cohomology spectral sequence, called the *base change spectral sequence*

$$E_2^{p,q}=H^p(\mathcal{D}^{op};R^q\Phi_!(\mathcal{A}))\Rightarrow H^{p+q}(\mathcal{C}^{op};\mathcal{A})$$

for any $\mathcal{A}$ in $\check{\mathcal{C}}$, where $R^q\Phi_!$ is the q-th right derived functor of $\Phi_!$. It is well known (see e.g. [**5**, Appendix II, §3] for the homology version) that $R^q\Phi_!$ can be computed explicitly by

$$R^q\Phi_!(\mathcal{A})(Y)=H^q((\Phi^Y)^{op};\mathcal{A}^Y),$$

where $\mathcal{A}$ is in $\check{\mathcal{C}}$,Y is an object in $\mathcal{D}$, Φ^Y is the category with objects pairs (X,φ)

with X an object in $\mathcal{C}$ and $\varphi\colon \Phi(X) \to Y$ a morphism in $\mathcal{D}$. A morphism in Φ^Y from (X,φ) to (X',φ') is a morphism $\alpha\colon X \to X'$ in $\mathcal{C}$ satisfying $\varphi' \circ \Phi(\alpha) = \varphi$, and $\mathcal{A}^Y$ is the functor obtained by restricting $\mathcal{A}$ to Φ^Y via the forgetful functor $\Phi^Y \to \mathcal{C}$ sending (X,φ) to X. We will need the base change spectral sequence only for the case where $\mathcal{C}$ is an EI-category and $\mathcal{D}$ is a partially ordered set. In that case, for any Y in $\mathcal{D}$, the category Φ^Y can be identified with the category $\mathcal{C}_{\leqslant Y}$ consisting of all objects X in $\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{D}}(\Phi(X), Y)$ is non-empty. Hence the base change spectral sequence takes the well-known form

$$E_2^{p,q} = H^p(\mathcal{D}^{op}; Y \mapsto H^q(\mathcal{C}_{\leqslant Y}; \mathcal{A})) \Rightarrow H^{p+q}(\mathcal{C}^{op}; \mathcal{A})$$

for any $\mathcal{A}$ in $\check{\mathcal{C}}$, where in the expression $H^q(\mathcal{C}_{\leqslant Y}; \mathcal{A})$ the restriction of $\mathcal{A}$ to $\mathcal{C}_{\leqslant Y}$ is again denoted by $\mathcal{A}$. If $\Phi_!$ is exact, then the base change spectral sequence collapses. This happens, in particular, if Φ has a right adjoint $\Psi\colon \mathcal{D} \to \mathcal{C}$, for in that case the right adjoint $\Phi_!$ of Φ^* is isomorphic to the exact restriction functor Ψ^*. Thus, if Φ has a right adjoint, we have $H^*(\mathcal{D}^{op}; \Phi_!(\mathcal{A})) \cong H^*(\mathcal{C}^{op}; \mathcal{A})$ for any $\mathcal{A}$ in $\check{\mathcal{C}}$ (cf. [**9**, 3.1]).

3. Regular EI-categories and proof of Theorem 1.1

Following [**15**, 2.1], an EI-category $\mathcal{C}$ is called *regular* if for any two objects X, Y in $\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(X, Y) \neq \emptyset$, the group $\mathrm{Aut}_{\mathcal{C}}(X)$ acts regularly (i.e. transitively and freely) on $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. Any morphism in a regular EI-category is a monomorphism. For any EI-category $\mathcal{C}$ the subdivision $S(\mathcal{C})$ is regular, and if $\mathcal{C}$ is regular, there is a contravariant functor from $\mathcal{C}$ to the category of groups sending an object X to its automorphism group $\mathrm{Aut}_{\mathcal{C}}(X)$ and a morphism $\varphi\colon X \to Y$ in $\mathcal{C}$ to the unique map $\mathrm{Aut}_{\mathcal{C}}(Y) \to \mathrm{Aut}_{\mathcal{C}}(X)$, which sends $\sigma \in \mathrm{Aut}_{\mathcal{C}}(Y)$ to the unique $\rho \in \mathrm{Aut}_{\mathcal{C}}(X)$ satisfying $\varphi \circ \rho = \sigma \circ \varphi$. We use the regularity of $\mathcal{C}$ for the existence and uniqueness of ρ (cf. [**15**, 2.2]). For any EI-category $\mathcal{C}$, the subdivision $S(\mathcal{C})$ comes with canonical functors from $S(\mathcal{C})$ to $\mathcal{C}$ and $\mathcal{C}^{op}$ sending an object $\sigma\colon [m] \to \mathcal{C}$ in $S(\mathcal{C})$ to $\sigma(m)$ and $\sigma(0)$, respectively. If every isomorphism class of $\mathcal{C}$ has a unique element, then $S(\mathcal{C})$ is equivalent to the opposite of the category $s(\mathcal{C})$ defined in [**23**, §1], and hence [**23**, 1.5] implies that the canonical functor $S(\mathcal{C}) \to \mathcal{C}$ induces, for any abelian group A, an isomorphism $H^*(S(\mathcal{C}); A) \cong H^*(\mathcal{C}; A)$. For regular EI-categories the appropriate base change spectral sequence specialises to the following spectral sequence:

Theorem 3.1. *Let $\mathcal{C}$ be a regular* EI-*category and A an abelian group. There is a cohomology spectral sequence*

$$E_2^{p,q} = H^p([\mathcal{C}]^{op}; [X] \mapsto H^q(\mathrm{Aut}_{\mathcal{C}}(X); A)) \Rightarrow H^{p+q}(\mathcal{C}; A).$$

Proof. Denote by $\Phi\colon S(\mathcal{C}) \to [\mathcal{C}]^{op}$ the canonical functor sending an object $\sigma\colon [m] \to \mathcal{C}$ in $S(\mathcal{C})$ to the isomorphism class $[\sigma(0)]$ in $[\mathcal{C}]$. One checks that for any X in $\mathcal{C}^{op}$ we have $S(\mathcal{C})_{\leqslant [X]} = S(\mathcal{C}_{\geqslant X})$. Thus the base change spectral sequence associated with Φ takes the form

$$E_2^{p,q} \;=\; H^p([\mathcal{C}]^{op}; [X] \mapsto H^q(S(\mathcal{C}_{\geqslant X}); \mathcal{A})) \Rightarrow H^{p+q}(S(\mathcal{C}); \mathcal{A}).$$

As mentioned above, we have $H^{p+q}(S(\mathcal{C}); \mathcal{A}) \cong H^{p+q}(\mathcal{C}; \mathcal{A})$ and $H^q(S(\mathcal{C}_{\geqslant X}); \mathcal{A}) \cong H^q(\mathcal{C}_{\geqslant X}; \mathcal{A})$. By Lemma 3.3 below, this is isomorphic to $H^q(\mathrm{Aut}_{\mathcal{C}}(X); A)$, whence the result. ∎

Lemma 3.2. *Let $\mathcal{C}$ be a small category, X an object in $\mathcal{C}$ and denote by $\mathcal{D}$ the full subcategory of $\mathcal{C}$ having X as the unique object. Suppose that for any object Y in $\mathcal{C}$ there is a morphism $\iota_Y\colon X \to Y$ in $\mathcal{C}$ such that the map $\mathrm{Aut}_{\mathcal{C}}(X) \to \mathrm{Hom}_{\mathcal{C}}(X,Y)$ sending $\alpha \in \mathrm{Aut}_{\mathcal{C}}(X)$ to $\iota_Y \circ \alpha$ is a bijection. Then the inclusion functor $\Phi\colon \mathcal{D} \to \mathcal{C}$ has a right adjoint $\Psi\colon \mathcal{C} \to \mathcal{D}$.*

Proof. Define Ψ on objects by $\Psi(Y) = X$ for all Y in $\mathcal{C}$. For a morphism $\varphi\colon Y \to Z$ in $\mathcal{C}$, define the morphism $\Psi(\varphi) \in \mathrm{Aut}_{\mathcal{C}}(X) = \mathrm{Aut}_{\mathcal{D}}(X)$ as follows: By the assumptions there is a unique $\beta \in \mathrm{Aut}_{\mathcal{C}}(X)$ such that $\varphi \circ \iota_Y = \iota_Z \circ \beta$. We set $\Psi(\varphi) = \beta$. A trivial verification shows that this construction is functorial and yields a right adjoint for Φ. □

Lemma 3.3. *Let $\mathcal{C}$ be a regular* EI*-category, X an object in $\mathcal{C}$ and A an abelian group. Restriction induces an isomorphism $H^*((\mathcal{C}_{\geqslant X})^{op};A) \cong H^*(\mathrm{Aut}_{\mathcal{C}}(X);A)$.*

Proof. For any object Y in $\mathcal{C}_{\geqslant X}$ choose a morphism $\iota_Y : X \to Y$. Since $\mathcal{C}$ is regular, composition with ι_Y induces a bijection $\mathrm{Aut}_{\mathcal{C}}(X) \cong \mathrm{Hom}_{\mathcal{C}}(X,Y)$. Thus 3.2 applies, showing that the inclusion functor $\{X\} \to \mathcal{C}_{\geqslant X}$ has a right adjoint Ψ. But then the base change spectral sequence associated with this inclusion functor collapses and yields the isomorphism as stated. □

Proof of Theorem 1.1. We use the notation of 1.1. The spectral sequence 3.1 applied to the regular EI-category $S(\mathcal{C})$ and the abelian group $k^\times$ takes the form

$$E_2^{p,q} = H^p([S(\mathcal{C})];\mathcal{A}^q) \Rightarrow H^{p+q}(S(\mathcal{C});k^\times).$$

That is, the E_2-page has the form

$$\begin{array}{cccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
H^0([S(\mathcal{C})];\mathcal{A}^2) & H^1([S(\mathcal{C})];\mathcal{A}^2) & H^2([S(\mathcal{C})];\mathcal{A}^2) & H^3([S(\mathcal{C})];\mathcal{A}^2) & \cdots & \cdots \\
H^0([S(\mathcal{C})];\mathcal{A}^1) & H^1([S(\mathcal{C})];\mathcal{A}^1) & H^2([S(\mathcal{C})];\mathcal{A}^1) & H^3([S(\mathcal{C})];\mathcal{A}^1) & \cdots & \cdots \\
k^\times & H^1([S(\mathcal{C})];k^\times) & H^2([S(\mathcal{C})];k^\times) & H^3([S(\mathcal{C})];k^\times) & \cdots & \cdots .
\end{array}$$

This spectral sequence approximates $H^*(S(\mathcal{C});k^\times) \cong H^*(\mathcal{C};k^\times)$. Thus if the groups $H^1([S(\mathcal{C})];k^\times)$ and $H^2([S(\mathcal{C})];k^\times)$ are zero, then there is no differential starting or ending at the coordinates $(0,1)$, and hence we get the isomorphism

$$H^1(\mathcal{C};k^\times) \cong H^0([S(\mathcal{C})];\mathcal{A}^1) = \lim_{[S(\mathcal{C})]}(\mathcal{A}^1)$$

as stated in 1.1. In addition, suppose now that $H^3([S(\mathcal{C})];k^\times) = H^4([S(\mathcal{C})];k^\times) = 0$. There is no non-zero differential starting or ending at $E_2^{1,1} = H^1([S(\mathcal{C})];\mathcal{A}^1)$, and hence, again since this spectral sequence approximates $H^*(\mathcal{C},k^\times)$, we get an injective map $H^1([S(\mathcal{C})];\mathcal{A}^1) \to H^2(\mathcal{C};k^\times)$. The cokernel of this map is the kernel of the differential

$$E_2^{0,2} = \lim_{[S(\mathcal{C})]}(\mathcal{A}^2) \to E_2^{2,1} = H^2([S(\mathcal{C})];\mathcal{A}^1)$$

because all differentials starting at $(0,2)$ from page 3 onwards are zero by the assumptions. Since from page 3 onwards there is no non-zero differential ending or starting at $(2,1)$ it follows that the cokernel of the last map is a subgroup of $H^3(\mathcal{C};k^\times)$. This proves the exactness of the sequence stated in 1.1, and as pointed out earlier, this part of the argument is valid for any abelian group instead of $k^\times$. The groups $\mathcal{A}^1([\sigma]) \cong \mathrm{Hom}(\mathrm{Aut}_{S(\mathcal{C})}(\sigma), k^\times)$ and $\mathcal{A}^2([\sigma]) = H^2(\mathrm{Aut}_{S(\mathcal{C})}(\sigma); k^\times)$ are finite, of order coprime to $\mathrm{char}(k)$ if $\mathrm{char}(k)$ is positive; hence $H^1([S(\mathcal{C})]; \mathcal{A}^1)$ and $\lim_{[S(\mathcal{C})]}(\mathcal{A}^2)$ are finite, proving the finiteness of $H^2(\mathcal{C}; k^\times)$ with the properties as stated. □

4. Proof of Theorem 1.2

The terminology on fusion systems we use follows [**17**]; in particular, a fusion system means a saturated fusion system in the sense of [**3**]. See [**22**, §2], [**3**, Appendix], or [**17**, §3] for details regarding normalisers, centralisers and quotients in fusion systems. For $\mathcal{C}$ a right ideal in a fusion system $\mathcal{F}$ on a finite p-group P, denote by $S_<(\mathcal{C})$ the full subcategory of all $\sigma\colon [m] \to \mathcal{C}$ in $S(\mathcal{C})$ such that for $0 \leqslant i < j \leqslant m$ we have $\sigma(i) < \sigma(j)$, and the morphism $\sigma(i<j)$ from $\sigma(i)$ to $\sigma(j)$ is the inclusion map. By [**16**, 4.2], the subcategory $S_<(\mathcal{C})$ is equivalent to $S(\mathcal{C})$. We denote by $S_\lhd(\mathcal{C})$ the full subcategory of all $\sigma\colon [m] \to \mathcal{C}$ in $S_<(\mathcal{C})$ such that for $0 \leqslant i \leqslant j \leqslant m$ the subgroup $\sigma(j)$ is normal in $\sigma(i)$, or equivalently, such that $\sigma(i)$ is normal in the maximal subgroup $\sigma(m)$ of the chain of subgroups σ. The proof of 1.2 is based on the following spectral sequence.

Theorem 4.1. *Let $\mathcal{F}$ be a fusion system on a finite p-group P and $\mathcal{C}$ a right ideal in $\mathcal{F}$. Let $\{R_q \mid 0 \leqslant q \leqslant r\}$ be a set of representatives of the $\mathcal{F}$-isomorphism classes of subgroups of P belonging to $\mathcal{C}$ such that R_q is fully $\mathcal{F}$-normalised for $0 \leqslant q \leqslant r$ and such that $|R_q| \geqslant |R_{q+1}|$ for $1 \leqslant q \leqslant r$. Denote by $\mathcal{C}_q$ the right ideal in $N_\mathcal{F}(R_q)/R_q$ consisting of all non-trivial subgroups of $N_P(R_q)/R_q$, where $1 \leqslant q \leqslant r$. Let $\mathcal{A}\colon [S_\lhd(\mathcal{C})] \to \mathbf{Ab}$ be a covariant functor. Then, for $1 \leqslant q \leqslant r$, the functor $\mathcal{A}$ induces a covariant functor $\mathcal{N}^q\colon [S_\lhd(\mathcal{C}_q)] \to \mathbf{Ab}$, and there is a spectral sequence*

$$E_1^{p,q} \Rightarrow H^{p+q}([S_\lhd(\mathcal{C})]; \mathcal{A})$$

with the following properties:

- *(i) We have $E_1^{p,q} = H^{p+q-1}([S_\lhd(\mathcal{C}_q)]; \mathcal{N}^q)$ for $p+q \geqslant 2$ and $1 \leqslant q \leqslant r$.*
- *(ii) We have $E_1^{p,q} = \{0\}$ if $q < 0$, or $q > r$, or $p+q < 0$, we have $E_1^{p,0} = \{0\}$ for $p \neq 0$, and we have $E_1^{0,0} = \mathcal{A}([P])$.*
- *(ii) If there is an integer $m \geqslant 1$ such that $H^m([S_\lhd(\mathcal{C}_q)]; \mathcal{N}^q) = \{0\}$ for $1 \leqslant q \leqslant r$, then $H^{m+1}([S_\lhd(\mathcal{C})]; \mathcal{A}) = \{0\}$.*
- *(iv) If $\mathcal{A}$ vanishes on all chains of length zero in $[S_\lhd(\mathcal{C})]$, then also*

  $$E_1^{p,q} = H^0([S_\lhd(\mathcal{C}_q)]; \mathcal{N}^q)$$

  for $p+q = 1$ and $1 \leqslant q \leqslant r$, and $E_1^{p,q} = \{0\}$ if $p+q=0$ or $q=0$.

We break up the proof in several steps. Let $\mathcal{F}$ be a fusion system on a finite p-group P, where p is a prime, and let $\mathcal{C}$ be a right ideal in $\mathcal{F}$. Following the notation in [**16**, 4.1], the category $S_\lhd(\mathcal{C})$ is the full subcategory of $S(\mathcal{C})$ whose elements can be denoted as chains

$$\sigma = Q_0 < Q_1 < \cdots < Q_m$$

of subgroups Q_i of P belonging to $\mathcal{C}$ such that Q_i is normal in Q_m for $0 \leqslant i \leqslant m$.

Chains of this type were introduced in [**12**]. Given such a chain

$$\sigma = Q_0 < Q_1 < \cdots < Q_m$$

we denote, for any integer i such that $0 \leqslant i \leqslant m$, by $\sigma\backslash i$, the chain

$$\sigma\backslash i = Q_0 < \cdots Q_{i-1} < Q_{i+1} < \cdots < Q_m$$

obtained from removing Q_i. Inclusion of chains yields a canonical morphism $\sigma\backslash i \to \sigma$ in $S_\lhd(\mathcal{C})$. For any fully $\mathcal{F}$-normalised subgroup R of P in $\mathcal{C}$, we denote by $S_{R\lhd}(\mathcal{C})$ the full subcategory of $S_\lhd(\mathcal{C})$ consisting of all chains $\sigma = Q_0 < Q_1 < \cdots < Q_m$ with $Q_0 \cong R$ in $\mathcal{F}$ and $m \geqslant 1$. We denote by $\mathcal{C}_R$ the right ideal in $N_\mathcal{F}(R)/R$ consisting of all non-trivial subgroups of $N_P(R)/R$. We keep this notation throughout this section.

Lemma 4.2. *Let $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ be a chain in $S_\lhd(\mathcal{F})$ and let R be a fully $\mathcal{F}$-normalised subgroup of P such that $R \cong Q_0$ in $\mathcal{F}$. Then there is a chain $\tau = (R_0 < R_1 < \cdots < R_m)$ in $S_\lhd(\mathcal{F})$ such that $\tau \cong \sigma$ and $R_0 = R$.*

Proof. By standard properties of fusion systems [**17**, 2.6], there is a morphism

$$\varphi\colon N_P(Q_0) \to P$$

such that $\varphi(Q_0) = R$, because R is fully $\mathcal{F}$-normalised. Taking

$$\tau = \varphi(\sigma) = (\varphi(Q_0) < \varphi(Q_1) < \cdots < \varphi(Q_m))$$

proves the lemma. □

Lemma 4.3. *Let $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ and $\tau = (R_0 < R_1 < \cdots < R_m)$ be chains in $S_\lhd(\mathcal{F})$ such that $Q = Q_0 = R_0$ is fully $\mathcal{F}$-normalised. Suppose that $m > 0$. Then $\sigma \cong \tau$ in $S_\lhd(\mathcal{F})$ if and only if $\sigma\backslash 0 \cong \tau\backslash 0$ in $S_\lhd(N_\mathcal{F}(Q))$.*

Proof. Any isomorphism $\sigma \cong \tau$ induces an automorphism on $Q = Q_0 = R_0$ and hence an isomorphism $\sigma\backslash 0 \cong \tau\backslash 0$ in $S_\lhd(N_\mathcal{F}(Q))$. Conversely, any isomorphism $\sigma\backslash 0 \cong \tau\backslash 0$ in $S_\lhd(N_\mathcal{F}(Q))$ induces an automorphism on Q, hence an isomorphism $\sigma \cong \tau$. □

Lemma 4.4. *Suppose that $\mathcal{F} = N_\mathcal{F}(Q)$ for some normal subgroup Q of P. Let $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ and $\tau = (R_0 < R_1 < \cdots < R_m)$ be two chains in $S_\lhd(\mathcal{F})$ such that $Q \subseteq Q_0$ and $Q \subseteq R_0$. Then the chains $\bar\sigma = (Q_0/Q < Q_1/Q < \cdots < Q_m/Q)$ and $\bar\tau = (R_0/Q < R_1/Q < \cdots < R_m/Q)$ belong to the category $S_\lhd(\mathcal{F}/Q)$, and we have $\sigma \cong \tau$ in $S_\lhd(\mathcal{F})$ if and only if $\bar\sigma \cong \bar\tau$ in $S_\lhd(\mathcal{F}/Q)$.*

Proof. Clearly, the chains $\bar\sigma$, $\bar\tau$ are in $S_\lhd(\mathcal{F}/Q)$, and if $\sigma \cong \tau$ then $\bar\sigma \cong \bar\tau$. Suppose conversely that we have an isomorphism $\bar\mu = (\bar\mu_i)_{0\leqslant i\leqslant m}\colon \bar\sigma \cong \bar\tau$ given by a family of isomorphisms $\bar\mu_i\colon Q_i/Q \cong R_i/Q$ in $\mathcal{F}/Q$. Then any representative μ_m of $\bar\mu_m$ is an isomorphism $Q_m \cong R_m$ which sends Q_i to R_i, for $0 \leqslant i \leqslant m$, and hence setting $\mu_i = \mu_m|_{Q_i}$ yields an isomorphism $\sigma \cong \tau$. □

Proposition 4.5. *Let R be a fully $\mathcal{F}$-normalised subgroup of P. The map sending a chain $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ of positive length m in $S_{R\lhd}(\mathcal{C})$ to the chain $\bar\sigma\backslash 0 = (Q_1/R < Q_2/R < \cdots < Q_m/R)$ in $S_\lhd(\mathcal{C}_R)$ induces an isomorphism of posets $[S_{R\lhd}(\mathcal{C})] \cong [S_\lhd(\mathcal{C}_R)]$.*

Proof. This follows from combining the three previous lemmas. □

Proof of Theorem 4.1. We use the notation from the statement of 4.1. In particular, $\{R_q \mid 0 \leqslant q \leqslant r\}$ is a system of representatives of the isomorphism classes in $\mathcal{C}$ with all R_q fully $\mathcal{F}$-normalised and ordered in such a way that $|R_q| \geqslant |R_{q+1}|$ for $0 \leqslant q < r$. Note that since $\mathcal{C}$ is a right ideal in $\mathcal{F}$ this implies, in particular, that $R_0 = P$. By [**16**, 3.7] applied to the left ideal $[S_\lhd(\mathcal{C})]$ in $[S(\mathcal{C})]$, the cohomology of $\mathcal{A}$ is that of a cochain complex of abelian groups $C_\lhd = C_\lhd(\mathcal{A})$ which is in degree $n \geqslant 0$ equal to

$$C_\lhd^n = \oplus_{[\sigma]\in[S_\lhd(\mathcal{C})],|\sigma|=n}\mathcal{A}([\sigma]).$$

The differential of this complex is an alternating sum of maps

$$\mathcal{A}([\sigma] < [\tau])\colon \mathcal{A}([\sigma]) \to \mathcal{A}([\tau]),$$

where $[\sigma], [\tau]$ are isomorphism classes of chains σ, τ in $S_\lhd(\mathcal{C})$ such that $[\sigma] < [\tau]$ and $|\sigma| + 1 = |\tau|$. For any integer q such that $0 \leqslant q \leqslant r$ we define a full subcategory $S_\lhd^{(q)}(\mathcal{C})$ of $S_\lhd(\mathcal{C})$ as follows: a chain $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ in $S_\lhd(\mathcal{C})$ belongs to $S_\lhd^{(q)}(\mathcal{C})$ if and only if $Q_0 \cong R_j$ for some $j \geqslant q$. The subcategory $S_\lhd^{(q)}(\mathcal{C})$ is in fact a right ideal in $S_\lhd(\mathcal{C})$: if there is a morphism from a chain $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ in $S_\lhd^{(q)}(\mathcal{C})$ to a chain $\sigma' = (Q'_0 < Q'_1 < \cdots < Q'_{m'})$ in $S_\lhd(\mathcal{C})$, then, in particular, either $Q_0 \cong Q'_0$ or $|Q'_0| < |Q_0|$. Since $Q_0 \cong R_j$ for some $j \geqslant q$ this implies $Q'_0 \cong R_{j'}$ for some $j' \geqslant j$. Thus σ' belongs to the category $S_\lhd^{(q)}(\mathcal{C})$ as well. It follows that for $0 \leqslant q \leqslant r$ there is a subcomplex $C_\lhd^{(q)}$ of $C_\lhd$ defined by

$$(C_\lhd^{(q)})^n = \oplus_{\sigma\in[S_\lhd^{(q)}(\mathcal{C})],|\sigma|=n}\mathcal{A}([\sigma])$$

for $n \geqslant 0$. Since the category $S_\lhd(\mathcal{C})$ is filtered by the subcategories

$$S_\lhd^{(r)}(\mathcal{C}) \subseteq S_\lhd^{(r-1)}(\mathcal{C}) \subseteq \cdots \subseteq S_\lhd^{(0)}(\mathcal{C}) = S_\lhd(\mathcal{C})$$

it follows that the cochain complex $C_\lhd$ has a filtration of the form

$$\{0\} \subseteq C_\lhd^{(r)} \subseteq C_\lhd^{(r-1)} \subseteq \cdots \subseteq C_\lhd^{(0)} = C_\lhd.$$

For notational convenience we set $C_\lhd^{(q)} = 0$ for $q > r$. By [**19**, 2.6], the spectral sequence associated by this filtration takes the form

$$E_1^{p,q} = H^{p+q}(C_\lhd^{(q)}/C_\lhd^{(q+1)}) \Rightarrow H^{p+q}(C_\lhd).$$

The quotient complexes of this filtration are zero unless $0 \leqslant q \leqslant r$, and they vanish in negative degrees; thus we get $E_1^{p,q} = \{0\}$ for $q < 0$ or $q > r$ or $p + q < 0$ as claimed in *(ii)*. The right side in this spectral sequence is $H^{p+q}(C_\lhd) = H^{p+q}([S_\lhd(\mathcal{C})]; \mathcal{A})$ by [**16**, 3.7] applied to the left ideal $[S_\lhd(\mathcal{C})]$ in $[S(\mathcal{C})]$. We need to identify $E_1^{p,q}$ in the remaining cases. For $1 \leqslant q \leqslant r$ we set $\mathcal{D}_q = S_{R_q\lhd}(\mathcal{C})$, the full subcategory of $S_\lhd(\mathcal{C})$ consisting of all chains $\sigma = (Q_0 < Q_1 < \cdots < Q_m)$ of positive length m such that $Q_0 \cong R_q$. By convention, $\mathcal{D}_0$ is the empty category. That is, on object sets, we have a disjoint union of full subcategories

$$S_\lhd^{(q)}(\mathcal{C}) = S_\lhd^{(q+1)}(\mathcal{C}) \cup \mathcal{D}_q \cup [R_q].$$

Thus, for $0 \leqslant q \leqslant r$, the quotient complex

$$C_\lhd^{(q)}/C_\lhd^{(q+1)}$$

is equal to

$$(C_\triangleleft^{(q)}/C_\triangleleft^{(q+1)})^n \cong \oplus_{[\sigma]\in[\mathcal{D}_q],|\sigma|=n}\mathcal{A}([\sigma])$$

for $n>0$, and

$$(C_\triangleleft^{(q)}/C_\triangleleft^{(q+1)})^0 \cong \mathcal{A}([R_q])$$

with $[R_q]$ viewed as an isomorphism class of chains of length zero. Note that by our conventions, for $q=0$ this quotient complex is concentrated in degree zero, where it is equal to $\mathcal{A}([P])$. Thus, in particular, $E_1^{p,0}=\{0\}$ for $p\neq 0$ and $E_1^{0,0}=\mathcal{A}([P])$, which completes the proof of *(ii)*. Set $\mathcal{C}_q=\mathcal{C}_{R_q}$ for $1\leqslant q\leqslant r$; that is, $\mathcal{C}_q$ is the full subcategory of $N_\mathcal{F}(R_q)/R_q$ consisting of all non-trivial subgroups of $N_P(R_q)/R_q$; by convention, $\mathcal{C}_0$ is empty. By Proposition 4.5 we have an isomorphism of posets $[\mathcal{D}_q]\cong[S_\triangleleft(\mathcal{C}_{R_q})]$, where $1\leqslant q\leqslant r$. This isomorphism induces a functor

$$\mathcal{N}^q\colon [S_\triangleleft(\mathcal{C}_q)]\to \mathbf{Ab}$$

such that

$$\mathcal{N}^q([Q_1/R_q<Q_2/R_q<\cdots<Q_m/R_q])=\mathcal{A}([R_q<Q_1<Q_2<\cdots<Q_m]),$$

where $Q_1<Q_2<\cdots<Q_m$ is a chain in $S_\triangleleft(N_\mathcal{F}(R_q))$ with $R_q<Q_1$ and $m\geqslant 1$. Note that the isomorphism of posets $[\mathcal{D}_q]\cong[S_\triangleleft(\mathcal{C}_q)]$ sends a chain of positive length m to a chain of length $m-1$; this accounts for the degree shift in the statement of 4.1. Thus we can reformulate our description of the quotient complex $C_\triangleleft^{(q)}/C_\triangleleft^{(q+1)}$ as follows: in degree $n>0$ we have

$$(C_\triangleleft^{(q)}/C_\triangleleft^{(q+1)})^n \cong \oplus_{[\tau]\in[S_\triangleleft(\mathcal{C}_q)],|\tau|=n-1}\mathcal{N}^q([\tau]),$$

(which by our conventions is zero if $q=0$ and $n>0$) and the degree zero term is equal to $\mathcal{A}([R_q])$, as pointed out before. It follows again from the description of the cohomology of posets associated with subdivisions in [**16**, §3] that for $n>0$ we have

$$(C_\triangleleft^{(q)}/C_\triangleleft^{(q+1)})^n \cong C_\triangleleft(\mathcal{N}^q)^{n-1},$$

where $C_\triangleleft(\mathcal{N}^q)$ is the complex which computes the cohomology of the functor $\mathcal{N}^q$ as in [**16**, 3.1], with $\mathcal{C}_q$ instead of $\mathcal{C}$. In other words, $(C_\triangleleft^{(q)}/C_\triangleleft^{(q+1)})$ is the mapping cone of a chain map of the form $\mathcal{A}([R_q])\to C_\triangleleft(\mathcal{N}^q)$, with $\mathcal{A}([R_q])$ viewed as complex concentrated in degree zero. In particular, we have

$$H^n(C_\triangleleft^{(q)}/C_\triangleleft^{(q+1)})\cong H^{n-1}([S_\triangleleft(\mathcal{C}_q)];\mathcal{N}^q)$$

for $n\geqslant 2$ and $1\leqslant q\leqslant r$. This completes the proof of *(i)*. Furthermore, if all functors $\mathcal{N}^q$, for $1\leqslant q\leqslant r$, have vanishing cohomology in a fixed positive degree m, then the cohomology of $C_\triangleleft$ vanishes in degree $m+1$. Statement *(iii)* follows. Finally, if $\mathcal{A}$ vanishes on chains of length zero, then

$$C_\triangleleft^{(q)}/C_\triangleleft^{(q+1)}=C_\triangleleft(\mathcal{N}^q)[-1]$$

where $0\leqslant q\leqslant r$, and so in this case we see that

$$H^n(C_\triangleleft^{(q)}/C_\triangleleft^{(q+1)})\cong H^{n-1}([S_\triangleleft(\mathcal{C}_q)];\mathcal{N}^q)$$

for all integers n. This completes the proof of *(iv)*. □

Proof of Theorem 1.2. Let $\mathcal{F}$ be a fusion system on a finite p-group P, set $\mathcal{C} = \mathcal{F}^c$, and denote by $\{R_q \mid 0 \leqslant q \leqslant r\}$ a system of representatives of the isomorphism classes in $\mathcal{C}$ with all R_q fully $\mathcal{F}$-normalised and ordered in such a way that $|R_q| \geqslant |R_{q+1}|$ for $0 \leqslant q < r$. The plan is to apply 4.1 to the functor $\mathcal{A}^1$ restricted to $S_\triangleleft(\mathcal{C})$. Note that by [**16**, 4.7, 4.11] we have an isomorphism

$$H^*([S(\mathcal{C})]; \mathcal{A}^1) \cong H^*([S_\triangleleft(\mathcal{C})]; \mathcal{A}^1),$$

where we use the same notation $\mathcal{A}^1$ for the restriction of $\mathcal{A}^1$ to $S_\triangleleft(\mathcal{C})$. Note further that $\mathcal{A}^1([\sigma]) = H^1(\mathrm{Aut}_{S(\mathcal{C})}(\sigma); k^\times) = \mathrm{Hom}(\mathrm{Aut}_{S(\mathcal{C})}; k^\times)$ for any σ in $S(\mathcal{C})$. Similarly, for any finite group G the cohomology of the functor $\mathcal{N}_G$ on $[\Delta_p(G)]$ is invariant under restriction to the subset of normal chains in $\Delta_p(G)$. Moreover, $[\Delta_p(G)]$ is isomorphic, as partially ordered set, to $[S(\mathcal{F}_S(G))]$, where S is a Sylow-p-subgroup of G. By [**1**, Prop. C], for $0 \leqslant q \leqslant r$, there is a finite group L_q such that L_q has $N_P(R_q)$ as Sylow-p-subgroup, $R_q = O_p(L_q)$, $C_{L_q}(R_q) = Z(R_q)$ and $N_\mathcal{F}(R_q) = \mathcal{F}_{R_q}(L_q)$. Thus $N_\mathcal{F}(R_q)/R_q$ is the fusion system of the finite group $L_q/R_q \cong \mathrm{Aut}_\mathcal{F}(R_q)/\mathrm{Inn}(R_q)$ on $N_P(R_q)/R_q$. It follows that if $\mathcal{C}_q$ is the right ideal in $N_\mathcal{F}(R_q)/R_q$, as defined in the statement of 4.1, then $S_\triangleleft(\mathcal{C}_q)$ can be identified with the partially ordered subset of normal chains in $\Delta_p(L_q/R_q)$, and the cohomology of the functor $\mathcal{N}_{L_q/R_q}$ remains invariant under restriction to $[S_\triangleleft(\mathcal{C}_q)]$, and this restriction coincides with the functor $\mathcal{N}^q$ in the statement of 4.1. Thus statement *(iii)* in 4.1 implies Theorem 1.2. □

5. Regular functors

Definition 5.1. Let $\mathcal{C}$, $\mathcal{D}$ be EI-categories. A covariant functor $\Phi\colon \mathcal{C} \to \mathcal{D}$ is called *regular* if Φ induces an isomorphism $[\mathcal{C}] \cong [\mathcal{D}]$, for any two objects X, Y in $\mathcal{C}$ the map from $\mathrm{Hom}_\mathcal{C}(X, Y)$ to $\mathrm{Hom}_\mathcal{D}(\Phi(X), \Phi(Y))$ induced by Φ is surjective, and for any two objects X, Y in $\mathcal{C}$ such that $\mathrm{Hom}_\mathcal{C}(X, Y)$ is non-empty, the group $K(X) = \ker(\mathrm{Aut}_\mathcal{C}(X) \to \mathrm{Aut}_\mathcal{D}(\Phi(X))$ acts freely on $\mathrm{Hom}_\mathcal{C}(X, Y)$ through composition of morphisms and induces a bijection

$$\mathrm{Hom}_\mathcal{C}(X, Y)/K(X) \cong \mathrm{Hom}_\mathcal{D}(\Phi(X), \Phi(Y)).$$

If $\mathcal{C} \to \mathcal{D}$ is an extension of $\mathcal{D}$ by a functor $\mathcal{Z}\colon \mathcal{D}^{op} \to \mathbf{Ab}$ then the structural functor $\mathcal{C} \to \mathcal{D}$ is regular. Clearly, an EI-category is regular if and only if the canonical functor $\mathcal{C} \to [\mathcal{C}]$ is regular. The following lemma on lifting commutative diagrams through regular functors is used below to show that regular functors induce regular functors on subdivisions.

Lemma 5.2. *Let $\mathcal{C}$, $\mathcal{D}$ be* EI*-categories and $\Phi\colon \mathcal{C} \to \mathcal{D}$ a regular functor. If*

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\nu} \\ V & \xrightarrow[\psi]{} & W \end{array}$$

is a diagram in $\mathcal{C}$ *such that the diagram in* $\mathcal{D}$

$$\begin{array}{ccc} \Phi(X) & \xrightarrow{\Phi(\varphi)} & \Phi(Y) \\ {\scriptstyle \Phi(\mu)}\downarrow & & \downarrow{\scriptstyle \Phi(\nu)} \\ \Phi(V) & \xrightarrow[\Phi(\psi)]{} & \Phi(W) \end{array}$$

is commutative, then there is a unique automorphism $\rho \in \mathrm{Aut}_{\mathcal{C}}(X)$ *such that* $\Phi(\rho) = \mathrm{Id}_{\Phi(X)}$ *and such that the diagram in* $\mathcal{C}$

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ {\scriptstyle \mu\circ\rho}\downarrow & & \downarrow{\scriptstyle \nu} \\ V & \xrightarrow[\psi]{} & W \end{array}$$

is commutative.

Proof. The morphisms $\nu \circ \varphi$ and $\psi \circ \mu$ are two morphisms from X to W in $\mathcal{C}$ whose images in the morphism set $\mathrm{Hom}_{\mathcal{D}}(\Phi(X), \Phi(W))$ are equal. Since the kernel $K(X)$ of the map $\mathrm{Aut}_{\mathcal{C}}(X) \to \mathrm{Aut}_{\mathcal{D}}(\Phi(X))$ acts freely on $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, inducing a bijection $\mathrm{Hom}_{\mathcal{C}}(X, W)/K(X) \cong \mathrm{Hom}_{\mathcal{D}}(\Phi(X), \Phi(W))$, there is a unique $\rho \in K(X)$ with the required property. □

Proposition 5.3. *Let* $\mathcal{C}$, $\mathcal{D}$ *be* EI*-categories and* $\Phi\colon \mathcal{C} \to \mathcal{D}$ *a regular functor. Then* Φ *induces a regular functor* $S(\mathcal{C}) \to S(\mathcal{D})$.

Proof. There is an obvious functor $S(\mathcal{C}) \to S(\mathcal{D})$ sending an object $\sigma\colon [m] \to \mathcal{C}$ in $S(\mathcal{C})$ to the object $\Phi \circ \sigma\colon [m] \to \mathcal{D}$ in $S(\mathcal{D})$. Set $\bar{\sigma} = \Phi \circ \sigma$ and $|\sigma| = m$. Let $\tau\colon n \to \mathcal{C}$ be another object in $S(\mathcal{C})$. We show that the map $\mathrm{Hom}_{S(\mathcal{C})}(\sigma, \tau) \to \mathrm{Hom}_{S(\mathcal{D})}(\bar{\sigma}, \bar{\tau})$ induced by Φ is surjective; we proceed by induction over $|\sigma|$. For $|\sigma| = 0$ this follows from the regularity of the functor $\Phi\colon \mathcal{C} \to \mathcal{D}$. Suppose that $|\sigma| = m$ is positive; denote by σ' the object in $S(\mathcal{C})$ obtained by deleting $\sigma(0)$. Then $|\sigma'| = m - 1$; hence by induction, the map from $\mathrm{Hom}_{S(\mathcal{C})}(\sigma', \tau)$ to $\mathrm{Hom}_{S(\mathcal{D})}(\bar{\sigma}', \bar{\tau})$ is surjective. Using the previous lemma one sees that the map from $\mathrm{Hom}_{S(\mathcal{C})}(\sigma, \tau)$ to $\mathrm{Hom}_{S(\mathcal{D})}(\bar{\sigma}, \bar{\tau})$ is surjective as well. The rest is an easy verification. □

Lemma 5.4. *Let* $\mathcal{C}$, $\mathcal{D}$ *be* EI*-categories and* $\Phi\colon \mathcal{C} \to \mathcal{D}$ *a regular functor. Let* X *be an object in* $\mathcal{C}$ *and set* $Y = \Phi(X)$.

(i) The partially ordered set $[\Phi^Y]$ *has* $[(X, \mathrm{Id}_Y)]$ *as the unique maximal element.*

(ii) We have $\mathrm{Aut}_{\Phi^Y}(X, \mathrm{Id}_Y) = \ker(\mathrm{Aut}_{\mathcal{C}}(X) \to \mathrm{Aut}_{\mathcal{D}}(Y))$.

Proof. For *(i)*, let (Z, ψ) be an object in Φ^Y; that is, Z is an object in $\mathcal{C}$ and $\psi\colon \Phi(Z) \to Y$ is a morphism in $\mathcal{D}$. Since Φ is regular, there is a morphism $\beta\colon Z \to X$ such that $\Phi(\beta) = \psi = \mathrm{Id}_Y \circ \psi$. Thus β is a morphism in Φ^Y from (Z, ψ) to (X, Id_Y). In other words, $[(Z, \psi)] \leqslant [(X, \mathrm{Id}_Y)]$ for any object (Z, ψ) in Φ^Y. This proves *(i)*, and *(ii)* is trivial. □

Lemma 5.5. *Let $\mathcal{C}$, $\mathcal{D}$ be* EI-*categories and $\Phi\colon \mathcal{C} \to \mathcal{D}$ a regular functor. Let X be an object in $\mathcal{C}$ and set $Y = \Phi(X)$. Suppose that every morphism in $\mathcal{C}$ is a monomorphism.*

(i) The category Φ^Y is regular.

(ii) For any object (Z, ψ) in Φ^Y, we have

$$\mathrm{Aut}_{\Phi^Y}(Z, \psi) = \ker(\mathrm{Aut}_{\mathcal{C}}(Z) \to \mathrm{Aut}_{\mathcal{D}}(\Phi(Z)).$$

Proof. Let $\alpha, \beta\colon (Z, \psi) \to (Z', \psi')$ be two morphisms in Φ^Y. That is, Z, Z' are objects in $\mathcal{C}$ and $\psi \in \mathrm{Hom}_{\mathcal{D}}(\Phi(Z), Y)$, $\psi' \in \mathrm{Hom}_{\mathcal{D}}(\Phi(Z'), Y)$ such that

$$\psi' \circ \Phi(\alpha) = \psi = \psi' \circ \Phi(\beta).$$

Since Φ is regular there are morphisms $\varphi \in \mathrm{Hom}_{\mathcal{C}}(Z, X)$, $\varphi' \in \mathrm{Hom}_{\mathcal{C}}(Z', X)$ such that $\Phi(\varphi) = \psi$ and $\Phi(\varphi') = \psi'$. Then $\Phi(\varphi' \circ \alpha) = \Phi(\varphi) = \Phi(\varphi' \circ \beta)$. The regularity of Φ implies that there is a unique automorphism $\gamma \in \mathrm{Aut}_{\mathcal{C}}(X)$ such that $\varphi' \circ \alpha \circ \gamma = \varphi' \circ \beta$ and such that $\Phi(\gamma) = \mathrm{Id}_Y$. Since φ' is a monomorphism we get that $\alpha \circ \gamma = \beta$. It follows that γ is the unique automorphism of (Z, ψ) in Φ^Y satisfying $\alpha \circ \gamma = \beta$. This proves *(i)*. When applied to the case $(Z, \psi) = (Z', \psi')$ and $\beta = \mathrm{Id}_Z$, this argument proves *(ii)*. □

Theorem 5.6. *Let $\mathcal{C}$, $\mathcal{D}$ be* EI-*categories and $\Phi\colon \mathcal{C} \to \mathcal{D}$ a regular functor. Suppose that every morphism in $\mathcal{C}$ is a monomorphism. Let A be an abelian group and for any object X in $\mathcal{C}$ denote by $K(X)$ the kernel of the group homomorphism $\mathrm{Aut}_{\mathcal{C}}(X) \to \mathrm{Aut}_{\mathcal{D}}(\Phi(X))$ induced by Φ. Suppose that $H^q(K(X); A) = \{0\}$ for $q \geqslant 0$ and any object X in $\mathcal{C}$, with respect to the trivial action of $K(X)$ on A. Then Φ induces an isomorphism $H^*(\mathcal{C}; A) \cong H^*(\mathcal{D}; A)$.*

Proof. Let Y be an object in $\mathcal{D}$. By 5.5, the category Φ^Y is regular, and by 5.4, the partially ordered set $[\Phi^Y]$ has a unique maximal element. By 3.1 applied to Φ^Y there is a spectral sequence of the form

$$H^p([\Phi^Y]; [(Z, \psi)] \mapsto H^q(\mathrm{Aut}_{\Phi^Y}(Z, \psi); A)) \Rightarrow H^{p+q}(\Phi^Y; A).$$

Now $\mathrm{Aut}_{\Phi^Y}(Z, \psi) = K(Z)$ by 5.5, and hence the assumptions imply that this spectral sequence collapses to an isomorphism $H^p([\Phi^Y]; A) \cong H^p(\Phi^Y; A)$. However, this group is zero for q positive as $[\Phi^Y]$ has a unique maximal element. This shows that $R^q\Phi_!(A) = 0$ for q positive, and hence the base change spectral sequence of Φ collapses to an isomorphism $H^*(\mathcal{C}; A) \cong H^*(\mathcal{D}; A)$ as stated. □

6. Further invariance properties of $H^*(\mathcal{C}; k^\times)$

Let p be a prime. A fusion systems $\mathcal{F}$ of a block with defect group P of a finite p-solvable group G is, by a result of Puig, always the fusion system of a finite group L having P as Sylow-p-subgroup such that $Q = O_p(L)$ is an $\mathcal{F}$-centric subgroup of P. The results of this section, besides providing some reduction techniques for calculating $H^*(\mathcal{F}^c; k^\times)$, imply that $H^2(\mathcal{F}^c; k^\times)$ is the p'-part of the Schur multiplier of L in that case (see 6.6 below). It is well known that any element α in the p'-part of the Schur multiplier of L arises as a Külshammer-Puig 2-cocycle of the fusion system $\mathcal{F}$ at Q in a suitable block of a finite central p'-extension $\hat{L}$ of L determined by α. We first show the invariance of $H^*(\mathcal{F}^c;k^\times)$ with respect to taking quotients by central p-subgroups. If $\mathcal{F}$ is a fusion system on a finite p-group P and Z is a subgroup of $Z(P)$, then the equality $\mathcal{F} = C_\mathcal{F}(Z)$ means that every morphism $\varphi\colon Q \to R$ in $\mathcal{F}$ can be extended to a morphism $\psi\colon QZ \to RZ$ in $\mathcal{F}$ such that $\psi|_Z = \mathrm{Id}_Z$. If $\mathcal{F} = C_\mathcal{F}(Z)$, then $\mathcal{F}$ induces a fusion system on P/Z, denoted by $\mathcal{F}/Z$. Given a fusion system $\mathcal{F}$ on a finite p-group P, we denote as before by $\mathcal{F}^c$ the full subcategory in $\mathcal{F}$ of all $\mathcal{F}$-centric subgroups of P; this is a right ideal in $\mathcal{F}$. We denote by $\bar{\mathcal{F}}$ the quotient category of $\mathcal{F}$ having as objects the subgroups of P and as morphism sets the orbits $\mathrm{Aut}_R(R)\backslash\mathrm{Hom}_\mathcal{F}(Q,R)$ of the morphism set $\mathrm{Hom}_\mathcal{F}(Q,R)$ with respect to the action of the group of inner automorphisms $\mathrm{Aut}_R(R)$ of R via the composition of group homomorphisms, called the *orbit category of* $\mathcal{F}$. Denote by $\bar{\mathcal{F}}^c$ the image of $\mathcal{F}^c$ in $\bar{\mathcal{F}}$. While all morphisms in $\mathcal{F}$ are monomorphisms, this is no longer true in $\bar{\mathcal{F}}$; however, all morphisms in $\bar{\mathcal{F}}^c$ are epimorphisms. For the centric linking system $\mathcal{L}$ of $\mathcal{F}$ we refer to [**3**, 1.7]. The following invariance properties are essentially consequences of 5.6.

Proposition 6.1. *Let $\mathcal{F}$ be a fusion system on a finite p-group P such that $\mathcal{F} = C_\mathcal{F}(Z)$ for some subgroup Z of $Z(P)$ and let $\mathcal{C}$ be a right ideal in $\mathcal{F}$ consisting of subgroups of P containing Z such that the image $\mathcal{C}/Z$ of $\mathcal{C}$ in $\mathcal{F}/Z$ is contained in $(\mathcal{F}/Z)^c$. Let k be an algebraically closed field of characteristic p. The canonical functor $\mathcal{C} \to w\mathcal{C}/Z$ is regular and induces an isomorphism $H^*(\mathcal{C};k^\times) \cong H^*(\mathcal{C}/Z;k^\times)$.*

Proof. Let Q, R be subgroups of P belonging to $\mathcal{C}$. Then Z is contained in Q, R by the assumptions. The morphism set $\mathrm{Hom}_{\mathcal{F}/Z}(Q/Z, R/Z)$ is the canonical image of the morphism set $\mathrm{Hom}_\mathcal{F}(Q,R)$; in particular, the canonical functor $\mathcal{C} \to \mathcal{C}/Z$ is surjective on morphisms. This also implies that if $Q/Z \cong R/Z$ in $\mathcal{C}/Z$ then $Q \cong R$ in $\mathcal{C}$, and hence we have the isomorphism of posets $[\mathcal{C}] \cong [\mathcal{C}/Z]$. If $\varphi, \psi\colon Q \to R$ are two morphisms in $\mathcal{C}$ whose images in $\mathcal{C}/Z$ are equal, then, in particular, $\varphi(Q) = \psi(Q) \subseteq R$, and with the obvious abuse of notation we get an automorphism $\kappa = \psi^{-1} \circ \varphi$ of Q whose image in $\mathcal{C}/Z$ is the identity on Q/Z. Thus κ is the unique element of the group $K(Q) = \ker(\mathrm{Aut}_\mathcal{F}(Q) \to \mathrm{Aut}_{\mathcal{F}/Z}(Q/Z))$ satisfying $\varphi = \psi \circ \kappa$. This shows that the canonical functor $\mathcal{C} \to \mathcal{C}/Z$ is regular. Moreover, the group $K(Q)$ is an abelian p-group and since k is algebraically closed of characteristic p this implies that $H^q(K(Q);k^\times) = \{0\}$ for all $q > 0$. Thus 5.6 applies, proving the theorem. □

Proposition 6.2. *Let $\mathcal{F}$ be a fusion system on a finite group P, let $\mathcal{C}$ be a right ideal in $\mathcal{F}^c$ and let $\bar{\mathcal{C}}$ be the canonical image of $\mathcal{C}$ in the orbit category $\bar{\mathcal{F}}$. Let k be an algebraically closed field of characteristic p. The canonical functor $\mathcal{C} \to \bar{\mathcal{C}}$ induces a regular functor $S(\mathcal{C}) \to S(\bar{\mathcal{C}})$ and an isomorphism $H^*(\mathcal{C};k^\times) \cong H^*(\bar{\mathcal{C}};k^\times)$.*

Proof. Two subgroups Q, P of P are isomorphic in $\mathcal{F}$ if and only if they are isomorphic in $\bar{\mathcal{F}}$. Thus the canonical functor $\mathcal{C} \to \bar{\mathcal{C}}$ sends non-isomorphisms to non-isomorphisms; hence it induces a functor $S(\mathcal{C}) \to S(\bar{\mathcal{C}})$, which in turn induces an isomorphism of partially ordered sets $[S(\mathcal{C})] \cong [S(\bar{\mathcal{C}})]$. Let $\sigma\colon [m] \to \mathcal{C}$ and $\tau\colon [n] \to \mathcal{C}$ be objects in $S(\mathcal{C})$ and denote by $\bar\sigma$, $\bar\tau$ their images in $S(\bar{\mathcal{C}})$. Let $(\alpha, \bar\mu)\colon \bar\sigma \to \bar\tau$ be a morphism in $S(\bar{\mathcal{C}})$; that is, $\alpha\colon [m] \to [n]$ is an order-preserving map and $\bar\mu\colon \bar\sigma \cong \bar\tau \circ \alpha$ is a natural isomorphism. Explicitly, $\bar\mu$ consists of a compatible family of group isomorphisms $\bar\mu_i\colon \bar\sigma(i) \cong \bar\tau(\alpha(i))$; that is, for $0 \leqslant i < m$ we have $\bar\mu_{i+1} \circ \bar\sigma(i < i+1) = \bar\tau(\alpha(i) < \alpha(i+1)) \circ \bar\mu_i$. Since the functor $\mathcal{C} \to \bar{\mathcal{C}}$ is surjective on morphisms, there are group isomorphisms $\mu_i\colon \sigma(i) \to \tau(\alpha(i))$ in $\mathcal{C}$, but this family need not be a natural transformation. More precisely, for $0 \leqslant i < m$ there is a group element $v_i \in \tau(\alpha(i))$ such that if we denote by c_i the inner automorphism of $\tau(\alpha(i))$ given by conjugation with v_i, we have $c_{i+1} \circ \mu_{i+1} \circ \sigma(i < i+1) = \tau(\alpha(i) < \alpha(i+1)) \circ \mu_i$. An easy inductive argument shows that after replacing μ_{i+1} by $c_{i+1} \circ \mu_{i+1}$ we get a natural isomorphism $\mu\colon \sigma \cong \tau \circ \alpha$ lifting $\bar{\mu}$, which shows that the functor $S(\mathcal{C}) \to S(\bar{\mathcal{C}})$ is surjective on morphisms. Finally, if μ, μ' are two morphisms in $S(\mathcal{C})$ from σ to τ, then by the regularity of $S(\mathcal{C})$ there is a unique automorphism β of σ such that $\mu' = \mu \circ \beta$. Thus, if, in addition, the images $\bar{\mu}$, $\bar{\mu}'$ of μ, μ' in $S(\bar{\mathcal{C}})$ are equal, then the image $\bar{\beta}$ in $S(\bar{\mathcal{C}})$ satisfies $\mu = \mu \circ \beta$. Since every morphism in $S(\bar{\mathcal{C}})$ is a monomorphism this implies that $\bar{\beta} = \mathrm{Id}_{\bar{\sigma}}$, and hence β belongs to the kernel $K(\sigma)$ of the canonical map from $\mathrm{Aut}_{S(\mathcal{C})}(\sigma)$ to $\mathrm{Aut}_{S(\bar{\mathcal{C}})}(\bar{\sigma})$, which shows that indeed the functor $S(\mathcal{C}) \to S(\bar{\mathcal{C}})$ is regular. Moreover, by standard properties of central group extensions, the group $K(\sigma)$ is a p-group. It follows that $H^q(K(\sigma); k^\times) = 0$ for positive q. Thus 5.6 implies $H^*(S(\mathcal{C}); k^\times) \cong H^*(S(\bar{\mathcal{C}}); k^\times)$, whence the stated isomorphism $H^*(\mathcal{C}; k^\times) \cong H^*(\bar{\mathcal{C}}; k^\times)$. □

Proposition 6.3. *Let $\mathcal{F}$ be a fusion system on a finite p-group P having a centric linking system $\mathcal{L}$ with structural functor $\pi\colon \mathcal{L} \to \mathcal{F}^c$. Let $\mathcal{C}$ be a right ideal in $\mathcal{L}$ and let $\mathcal{D}$ be its image in $\mathcal{F}$ under π. Let k be an algebraically closed field of characteristic p. The functor π induces an isomorphism $H^*(\mathcal{C}; k^\times) \cong H^*(\mathcal{D}; k^\times)$.*

Proof. The centric linking system $\mathcal{L}$ is an extension of $\mathcal{F}^c$ by the *centre functor* $\mathcal{Z}\colon \mathcal{F}^c \to \mathbb{Z}_{(p)}$ sending an $\mathcal{F}$-centric subgroup Q of P to its centre $Z(Q)$. Thus π induces a regular functor $\mathcal{C} \to \mathcal{D}$ and for any $\mathcal{F}$-centric subgroup of P we have $K(Q) = \ker(\mathrm{Aut}_{\mathcal{L}}(Q) \to \mathrm{Aut}_{\mathcal{F}}(Q)) \cong Z(Q)$; hence $H^q(K(Q); k^\times) = \{0\}$ for $q > 0$. Thus 5.6 applies and yields the isomorphism as stated. □

Proposition 6.4. *Let $\mathcal{F}$ be a fusion system on a finite p-group P, let Q be a normal subgroup of P such that $\mathcal{F} = N_{\mathcal{F}}(Q)$ and let $\mathcal{C} = \mathcal{F}_{\geqslant Q}$ be the right ideal in $\mathcal{F}$ consisting of all subgroups of P containing Q. Let A be an abelian group. Restriction induces an isomorphism $H^*(\mathcal{C}; A) \cong H^*(\mathrm{Aut}_{\mathcal{F}}(Q); A)$.*

Proof. Let $\mathcal{D}$ be the full subcategory of $\mathcal{C}$ having Q as unique object and let $\Phi\colon \mathcal{D} \to \mathcal{C}$ be the inclusion functor. For any subgroup R belonging to $\mathcal{C}$ denote by $\iota_R\colon Q \subseteq R$ the inclusion homomorphism. For any morphism $\varphi\colon Q \to R$ we have $\varphi(Q) = Q$ because $\mathcal{F} = N_{\mathcal{F}}(Q)$. Thus composition with the inclusion morphism ι_R induces a bijection $\mathrm{Aut}_{\mathcal{F}}(Q) \cong \mathrm{Hom}_{\mathcal{F}}(Q, R)$. It follows from 3.2 that Φ has a right adjoint $\Psi\colon \mathcal{C} \to \mathcal{D}$. Then Ψ^* is right adjoint to Φ^*; in other words, $\Phi_! = \Psi^*$ is exact, and hence we have $H^*(\mathcal{D}; A) = H^*(\mathcal{C}; \Psi^*(A)) = H^*(\mathcal{C}; A)$ by [**9**, 3.1]. □

Proposition 6.5. *Let $\mathcal{F}$ be a fusion system on a finite p-group P, let Q be an $\mathcal{F}$-centric normal subgroup of P such that $\mathcal{F} = N_{\mathcal{F}}(Q)$ and let $\mathcal{C} = \mathcal{F}_{\geqslant Q}$ be the right ideal in $\mathcal{F}$ consisting of all subgroups of P containing Q. Let $\mathcal{A}\colon \bar{\mathcal{F}}^c \to$ **Ab** be a contravariant functor. Restriction induces an isomorphism $H^*((\bar{\mathcal{F}}^c)^{op}; \mathcal{A}) \cong H^*((\bar{\mathcal{C}})^{op}; \mathcal{A})$.*

Proof. The inclusion functor $\Psi\colon \bar{\mathcal{C}} \to \bar{\mathcal{F}}^c$ has a left adjoint Φ sending R in $\bar{\mathcal{F}}^c$ to QR. Indeed, for any morphism $\varphi\colon R \to S$ in $\mathcal{F}^c$ there is a morphism $\psi\colon QR \to QS$ in $\mathcal{C}$ extending φ, and the image of ψ in $\bar{\mathcal{C}}$ is unique since every morphism in $\bar{\mathcal{F}}^c$ is an epimorphism, which shows that there is a canonical functor Φ sending R to QR. Moreover, for R in $\mathcal{F}^c$ and S in $\mathcal{C}$ we have $\mathrm{Hom}_{\bar{\mathcal{C}}}(QR, S) \cong \mathrm{Hom}_{\bar{\mathcal{F}}^c}(R, S)$, which shows that Φ is left adjoint to Ψ. Thus restriction along Ψ induces the stated isomorphism by [**9**, 3.1]. □

Corollary 6.6. *Let $\mathcal{F}$ be a fusion system on a finite p-group P such that $\mathcal{F} = N_{\mathcal{F}}(Q)$ for some $\mathcal{F}$-centric normal subgroup Q of P. Let L be a finite group with P as Sylow-p-subgroup such that $Q = O_p(L)$, $C_L(Q) = Z(Q)$ and such that $\mathcal{F}_P(L) = \mathcal{F}$. Then $H^*(\mathcal{F}^c; k^\times) \cong H^*(L; k^\times)$.*

Proof. We have $L/Z(Q) \cong \mathrm{Aut}_{\mathcal{F}}(Q)$, and hence combining the above results yields

$$H^*(\mathcal{F}^c; k^\times) \cong H^*(\mathcal{C}; k^\times) \cong H^*(\mathrm{Aut}_{\mathcal{F}}(Q); k^\times) \cong H^*(L; k^\times).$$ □

References

[**1**] C. Broto, N. Castellana, J. Grodal, R. Levi, and B. Oliver, Subgroup families controlling p-local finite groups, *Proc. London Math. Soc. (3)* **91** (2005), 325–354.

[**2**] C. Broto, R. Levi, and R. Oliver, Homotopy equivalences of p-completed classifying spaces of finite groups, *Invent. Math.* **151** (2003), no. 3, 611–664.

[**3**] C. Broto, R. Levi, and B. Oliver, The homotopy theory of fusion systems, *J. Amer. Math. Soc.* **16** (2003), no. 4, 779–856.

[**4**] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, NJ (1956).

[**5**] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Ergeb. der Math. **35**, Springer-Verlag, New York (1967).

[**6**] J. Grodal, Higher limits via subgroup complexes, *Annals of Math.* **155** (2002), no. 2, 405–457.

[**7**] H. Herrlich and G.E. Strecker, *Category theory, An introduction*, Second edition, Sigma Ser. Pure Math. **1**, Heldermann Verlag, Berlin (1979).

[**8**] P. Hilton and U. Stammbach, *A course in homological algebra*, Second edition, Graduate Texts in Math. **4**, Springer-Verlag, New York (1971).

[**9**] S. Jackowski and J. McClure, Homotopy decomposition of classifying spaces via elementary abelian subgroups, *Topology* **31** (1992), no. 1, 113–132.

[**10**] S. Jackowski, J. McClure, and R. Oliver, Homotopy classification of self-maps of BG via G-actions I, II, *Annals of Math.* **135** (1992), no. 2, 184–226, 227–270.

[**11**] S. Jackowski and J. Słomińska, G-functors, G-posets and homotopy decompositions of G-spaces, *Fund. Math.* **169** (2001), no. 3, 249–287.

[**12**] R. Knörr and G.R. Robinson, Some remarks on a conjecture of Alperin, *J. London Math. Soc. (2)* **39** (1989), no. 1, 48–60.

[**13**] B. Külshammer and L. Puig, Extensions of nilpotent blocks, *Invent. Math.* **102** (1990), no. 1, 17–71.

[**14**] M. Linckelmann, Fusion category algebras, *J. Algebra* **277** (2004), no. 1, 222–235.

[15] M. Linckelmann, Alperin's weight conjecture in terms of equivariant Bredon cohomology, *Math. Z.* **250** (2005), no. 3, 495–513.

[16] M. Linckelmann, The orbit space of a fusion system is contractible, *Proc. London Math. Soc. (3)* **98** (2009), 191–216.

[17] M. Linckelmann, Introduction to fusion systems, in *Group representation theory* (M. Geck, D. Testerman, and J. Thévenaz, eds.), EPFL Press, Lausanne (2007), 79–113.

[18] W. Lück, *Transformation groups and algebraic K-theory*, Lecture Notes in Math. **1408** (1989), Springer-Verlag, New York.

[19] J. McCleary, *A user's guide to spectral sequences*, Second edition, Cambridge Studies in Adv. Math., Cambridge University Press, Cambridge (2001).

[20] B. Oliver, Higher limits via Steinberg representations, *Comm. Alg.* **22** (1994), no. 4, 1381–1393.

[21] S. Park, The gluing problem for some fusion systems, preprint (2008).

[22] L. Puig, Frobenius categories, *J. Algebra* **303** (2006), no. 1, 309–357.

[23] J. Słomińska, Homotopy colimits on EI-categories, in *Algebraic topology* (Poznań 1989), Lecture Notes in Math. **1474** (1991), 273–294, Springer-Verlag, New York..

[24] J. Słomińska, Decompositions of categories over posets and cohomology of categories, *Manuscripta Math.* **104** (2001), no. 1, 21–38.

[25] P.J. Webb, An introduction to the representations and cohomology of categories, in *Group representation theory* (M. Geck, D. Testerman, and J. Thévenaz, eds.), EPFL Press, Lausanne (2007), 149–173.

Markus Linckelmann `m.linckelmann@abdn.ac.uk`

University of Aberdeen, Department of Mathematical Sciences, Aberdeen AB24 3UE, United Kingdom