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HOMOTOPY THEORY OF PRESHEAVES OF Γ-SPACES

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(*communicated by J. F. Jardine*)

Abstract

We consider the category of presheaves of Γ-spaces, or equivalently, of Γ-objects in simplicial presheaves. Our main result is the construction of stable model structures on this category parametrised by local model structures on simplicial presheaves. If a local model structure on simplicial presheaves is monoidal, then the corresponding stable model structure on presheaves of Γ-spaces is monoidal and satisfies the monoid axiom. This allows us to lift the stable model structures to categories of algebras and modules over a monoid.

Introduction

In his paper [21], Segal introduced Γ-spaces as a way to describe commutative monoids up to homotopy, and showed that they give rise to infinite loop spaces. Segal's original definition of a Γ-space, as a functor from the category of finite sets to spaces satisfying certain conditions, is what is now called a special Γ-space. In [4], Bousfield and Friedlander considered the category of all based functors from finite sets to simplicial sets; in particular they constructed a stable model structure on it, in which the fibrant objects are given by the very special Γ-spaces, and the weak equivalences are the stable equivalences of the associated spectra. As a consequence they show that the homotopy category of this model category is equivalent to the homotopy category of connective spectra.

Lydakis introduced a smash product for Γ-spaces in [17], making the category of Γ-spaces into a symmetric monoidal category. This smash product is compatible with the smash product of spectra after passage to the respective homotopy categories, thus making the category of Γ-spaces a convenient category for modeling multiplicative structures on connective spectra on a point set level. In [20], Schwede introduced a different model structure for Γ-spaces, Quillen equivalent to the one considered by Bousfield and Friedlander. This model structure satisfies the monoid axiom, an axiom first formulated by Schwede and Shipley in [22], which implies the existence of model structures on the categories of monoids and modules of Γ-spaces.

The main result of this paper is the construction of stable model structures on the category of presheaves of Γ-spaces, or equivalently, of Γ-objects in simplicial

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presheaves over an arbitrary small Grothendieck site. There are several model structures on simplicial presheaves, and we are focusing on the ones with local weak equivalences (Definition 1.1) as weak equivalences. We carry out the arguments without assuming any particular choice of model structure on simplicial presheaves, but we have to impose a cofibrancy condition on the domains of the generating sets (Hypothesis 3.1). When the site consists of one morphism only, our model structure will specialize to the one in [20].

The following theorem states the main results appearing as Theorem 4.12, Proposition 4.17 and Theorem 5.1 in the main body of the paper.

Theorem 0.1. *Let $\mathcal{C}$ be a small Grothendieck site and let* Spc *be the category of simplicial presheaves on $\mathcal{C}$ given a model structure according to Hypothesis 3.1. Let* $\Gamma\,\mathrm{Spc}$ *denote the category of based functors* $\Gamma \to \mathrm{Spc}$*, where* Γ *is the category of based finite ordinals.*

1. *There is a cofibrantly generated left proper model structure on the category* $\Gamma\,\mathrm{Spc}$ *with stable equivalences (Definition 4.10) as weak equivalences. The fibrant objects in this model structure coincides with the very special (Definition 4.7)* Γ*-spaces.*
2. *If the category* Spc *is a monoidal model category, then the stable model structure on* $\Gamma\,\mathrm{Spc}$ *is monoidal and satisfies the monoid axiom. Consequently, the category of module objects over a monoid in* $\Gamma\,\mathrm{Spc}$*, and the category of algebra objects over a commutative monoid in* $\Gamma\,\mathrm{Spc}$*, inherits model structures from* $\Gamma\,\mathrm{Spc}$ *by the results of* [22].

As a part of the construction, we compare our Γ-spaces to presheaves of spectra, and also show that the homotopy category of (presheaves of) Γ-spaces is equivalent to the homotopy category of connective (presheaves of) spectra. This equivalence is induced by a left Quillen functor from Γ-spaces to spectra, which maps very special Γ-spaces to Ω-spectra, thereby producing infinite loop objects in the category of simplicial presheaves.

As an application of the last part of Theorem 0.1 we construct an Eilenberg-Mac Lane functor H from presheaves of simplicial abelian groups to Γ-spaces and show that it is a Quillen equivalence between the categories of presheaves of simplicial abelian groups and the category of $H\mathbb{Z}$-modules. Corresponding results for presheaves of simplicial rings, and presheaves of simplicial modules over presheaves of commutative simplicial rings are also included.

Here is a quick outline of the paper. In Section 1 we recall some basic theory of simplicial presheaves, in particular the relevant model structures. Section 2 introduces the category of Γ-spaces, and in Section 3 we establish the strict model structure on this category. We apply Bousfield localization to this model structure in Section 4 to obtain the stable model structure on Γ-spaces, and compare its homotopy category to the homotopy category of connective presheaves of spectra. In Section 5 the stable model structure is lifted to the categories of modules and algebras over a (commutative) Γ-ring. Here we also obtain a Quillen equivalence between presheaves of simplicial modules over a presheaf of simplicial rings and modules over a Γ-ring. To this end, we first construct a model structure on presheaves of simplicial modules, and similarly for algebras.

We assume familiarity with the theory of model categories, as described in, e.g., Goerss and Jardine [7], Hirschhorn [8] or Hovey [9]. Some knowledge of classical Γ-spaces and simplicial presheaves is also assumed, but we recall what we need about simplicial presheaves in the first section. To prove the main theorem we make use of enriched left Bousfield localization as described in Barwick [2]. A quick review of this theory, together with some notes on bisimplicial presheaves, is located in an appendix.

In this paper, we use $\mathcal{M}(X,Y)$ to denote the set of morphisms between X and Y in the category $\mathcal{M}$, while $\mathrm{Map}(X,Y)$ and $\underline{\mathrm{Hom}}(X,Y)$ will denote respectively simplicial function complex and internal hom. More generally, when $\mathcal{M}$ is enriched in a category $\mathcal{V}$, the enriched hom objects will be denoted $\mathcal{V}\mathrm{Hom}(X,Y)$. When more than one category is under consideration, these objects will often be subscripted by the categories.

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1. Preliminaries on spaces

In this section we recall some facts about simplicial presheaves. Let $\mathcal{S}_*$ be the category of pointed simplicial sets. Fix a small site $\mathcal{C}$, i.e., a small category $\mathcal{C}$ with a Grothendieck topology. The functor category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S}_*)$, which we denote Spc, is the category of pointed simplicial presheaves on $\mathcal{C}$. As the notation suggests, we will call the objects in this category "spaces".

Each $U \in \mathcal{C}$ represents a discrete simplicial presheaf $\mathcal{C}(-,U)_+$, and we will write U for this space. Also, a simplicial set K defines a constant simplicial presheaf and we will use K to denote this space.

The category of spaces is closed symmetric monoidal, with monoidal product $\wedge$ defined sectionwise by

$$(X \wedge Y)(U) = X(U) \wedge Y(U)$$

for all $U \in \mathcal{C}$. Here we are using $\wedge$ to denote both the monoidal product of spaces X and Y and the smash product of based simplicial sets. Let K be a based simplicial set. Simplicial tensor $K \wedge -$ and cotensor $(-)^K$ are defined as

$$(K \wedge X)(U) = K \wedge X(U)$$
$$X^K(U) = X(U)^K$$

for each $U \in \mathcal{C}$.

The simplicial function complex $\mathrm{Map}(X,Y)$ of two spaces X and Y is defined in simplicial degree n to be

$$\mathrm{Map}(X,Y)_n = \mathrm{Spc}(X \wedge \Delta^n_+, Y)\,,$$

with face and degeneracy maps induced from Δ^n_+. There is also an internal hom-object

$\underline{\mathrm{Hom}}(X,Y)$ of spaces defined sectionwise by

$$\underline{\mathrm{Hom}}(X,Y)(U) = \mathrm{Map}(X|U, Y|U)\,,$$

where $X|U$ means X restricted to the local site $\mathcal{C} \downarrow U$.

We define homotopy groups of a space X as follows. First, let

$$L^2\colon \mathrm{Pre}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{C})$$

be the associated sheaf functor from the category of presheaves to the category of sheaves, which is left adjoint to the inclusion functor. Let $\pi_0^p(X)$ be the presheaf $U \mapsto \pi_0(X(U))$; the sheaf of path components is the associated sheaf $\pi_0(X) = L^2\pi_0^p(X)$. For $n \geqslant 1$, each $U \in \mathcal{C}$ and 0-simplex $x \in X(U)$, define the presheaf $\pi_n^p(X,x)$ on $\mathcal{C} \downarrow U$ as

$$\pi_n^p(X,x)(V) = \pi_n(|X(V)|, x|V)\,,$$

where $|-|$ denotes geometric realization of simplicial sets and $x|V$ denotes the restriction of x along $X(U) \to X(V)$. The sheaf $\pi_n(X,x) = L^2\pi_n^p(X,x)$ is the sheaf of homotopy groups of X over U with basepoint x.

Definition 1.1. A morphism $f\colon X \to Y$ of spaces is a *local weak equivalence* if the induced map of sheaves $\pi_0(X) \to \pi_0(Y)$ is a bijection, and the induced maps

$$\pi_n(X,x) \to \pi_n(Y, f(x))$$

are isomorphisms for all $n \geqslant 1, U \in \mathcal{C},\ x \in X(U)_0$. It is a *sectionwise weak equivalence* if $f(U)\colon X(U) \to Y(U)$ is a weak equivalence of simplicial sets for each $U \in \mathcal{C}$, and a sectionwise equivalence is in particular a local weak equivalence. Sectionwise cofibrations and fibrations are defined similarly.

There are several known model structures on Spc. We will only consider model structures on Spc in which the weak equivalences are given by the local weak equivalences of spaces.

Theorem 1.2 (Jardine [12]). *There is a cofibrantly generated proper simplicial model structure on* Spc *with sectionwise cofibrations (i.e., monomorphisms) as cofibrations and local weak equivalences as weak equivalences. This is the local injective model structure on* Spc.

To formulate the next theorem, let us define a *projective cofibration* of spaces to be a map that has the left lifting property with respect to maps that are both sectionwise fibrations and sectionwise weak equivalences.

Theorem 1.3 (Blander [3]). *There is a cofibrantly generated proper simplicial model structure on* Spc *with cofibrations as the projective cofibrations of spaces, and local weak equivalences as weak equivalences. This is the local projective model structure on* Spc.

Each projective cofibration $i\colon A \to B$ can be factored as a monomorphism $j\colon A \to C$ followed by a local injective trivial fibration $p\colon C \to B$. Since p is also a sectionwise trivial fibration, there is a lift in the diagram

$$\begin{array}{ccc} A & \xrightarrow{j} & C \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B & \longrightarrow & B\,, \end{array}$$

from which we see that i is a retraction of j; hence i is a monomorphism. This shows that the class of projective cofibrations is contained in the class of local injective cofibrations. In fact any set I of monomorphisms containing the set of generating projective cofibrations determines a local model structure on Spc.

Theorem 1.4 (Jardine [16])**.** *Let I be a set of monomorphisms containing the set of generating projective cofibrations. There is a cofibrantly generated proper simplicial model structure on* Spc *with I as the set of generating cofibrations and local weak equivalences as weak equivalences.*

An example of an intermediate model structure which differs from the local injective and local projective ones is the flasque model structure constructed by Isaksen [11].

Remark 1.5. To each model structure on the category of spaces, there is a corresponding model structure on the category of simplicial sheaves, such that the inclusion functor from simplicial sheaves to spaces becomes a Quillen equivalence. By application of this Quillen equivalence, all the homotopy theoretic results in this paper have corresponding results for simplicial sheaves.

Proposition 1.6. *If $f\colon X \to Y$ is a local weak equivalence of spaces and Z is a space, then the induced map $f \wedge 1\colon X \wedge Z \to Y \wedge Z$ is a local weak equivalence.*

Proof. This is stated in [14, 2.46]. □

Proposition 1.7. *If $f\colon X \to Y$ is a local weak equivalence of spaces, where X and Y are fibrant in any of the model structures constructed in Theorem 1.4, then f is a sectionwise weak equivalence.*

Proof. By [3, 1.3], the local projective model structure on spaces is a Bousfield localization of the projective model structure consisting of the projective cofibrations and sectionwise fibrations and weak equivalences, so in this case the result follows from general properties of Bousfield localizations. But a space X which is fibrant in any intermediate model structure is in particular fibrant in the local projective structure and we are done. □

Let $\mathcal{M}$ be a monoidal model category with monoidal product $\wedge$ and let TC be the class of trivial cofibrations in $\mathcal{M}$. Recall that the monoid axiom is the statement that all maps in $(TC \wedge \mathcal{M})$-cell are weak equivalences, where X-cell denotes the closure under transfinite compositions of pushouts of maps in X. This axiom ensures that the categories of modules and algebras over a monoid in $\mathcal{M}$ inherit model structures from $\mathcal{M}$; we will elaborate somewhat on this in Section 5. If the category $\mathcal{M}$ is cofibrantly generated with generating trivial cofibrations J, then to show that the monoid axiom holds it suffices to check that every map in $(J \wedge \mathcal{M})$-cell is a weak equivalence. See [22] for further details.

Proposition 1.8. *The local injective model structure on* Spc *is monoidal.*

Proof. Note that when $X \to Y$ is a monomorphism, so is $X \wedge Z \to Y \wedge Z$. Given two monomorphisms $X_i \to Y_i$, consider the pushout diagram

$$\begin{array}{ccc} X_1 \wedge X_2 & \longrightarrow & Y_1 \wedge X_2 \\ \downarrow & & \downarrow \\ X_1 \wedge Y_2 & \longrightarrow & P\,. \end{array} \tag{1.1}$$

By evaluating in sections and quoting the corresponding result about simplicial sets, we get that the induced pushout product map $P \to Y_1 \wedge Y_2$ is a monomorphism.

If in addition $X_1 \to Y_1$ is a local weak equivalence, then so is the top horizontal map in the pushout diagram, by Proposition 1.6. Left properness of the local injective model structure implies that the bottom map in (1.1) is a local weak equivalence, and using the 2-out-of-3 axiom we conclude that $P \to Y_1 \wedge Y_2$ is a local weak equivalence, so Spc is monoidal. □

Proposition 1.9. *Let $\mathcal{C}$ be a site which has finite products, and consider the category of spaces with the local projective model structure. In this case,* Spc *is a monoidal model category.*

Proof. The pushout product of a monomorphism and a monomorphic local weak equivalence is a local weak equivalence by Proposition 1.8, so it suffices to check that the pushout product of two generating projective cofibrations is a projective cofibration.

The generating cofibrations in the local projective model structure can be chosen as the set of maps

$$\{K \wedge U \to L \wedge U\}_{K \to L,\, U \in \mathcal{C}}\,,$$

where $K \to L$ ranges over generating cofibrations of simplicial sets. Now recall the isomorphism

$$(K \wedge U) \wedge (L \wedge V) \cong (K \wedge L) \wedge (U \times V)\,,$$

where K and L are simplicial sets, and $U, V \in \mathcal{C}$. Let $f_i \colon K_i \wedge U_i \to L_i \wedge U_i$, $i = 1, 2$, be two generating projective cofibrations. We can identify the pushout product of f_1 and f_2 with the corresponding pushout product of $K_1 \to L_1$ and $K_2 \to L_2$ in simplicial sets, smashed with the product $U_1 \times U_2$. Since the functor $(U_1 \times U_2) \wedge -$ preserves cofibrations, the result follows. □

Proposition 1.10. *Assume that* Spc *is given any of the model structures constructed in Theorem 1.4, and assume in addition that the model structure is monoidal. Then it also satisfies the monoid axiom.*

Proof. Consider the class C of morphisms consisting of $X \wedge Z \to Y \wedge Z$, where Z is a space and $X \to Y$ is a trivial cofibration of spaces. By Proposition 1.6, this class is contained in the class of local injective trivial cofibrations, i.e., monomorphisms which are also local weak equivalences. Now C-cell is also contained in the class of local injective cofibrations, since trivial cofibrations are closed under the formation of cell objects, and in particular every morphism in C-cell is a local weak equivalence. □

In Section 4 we will apply Bousfield localization to the category of Γ-spaces. For this we need to know that our categories are combinatorial, in the sense of Jeff Smith. An account of this notion is given in Dugger [6]; we recall the relevant definitions below.

Definition 1.11. Let λ be a regular cardinal and $\mathcal{M}$ a category. An object $X \in \mathcal{M}$ is *λ-presentable* if the represented functor $\mathcal{M}(X, -)$ preserves λ-filtered colimits. The category $\mathcal{M}$ is *locally λ-presentable* if it is cocomplete, and there exists a set $\{G_i\}$ of λ-presentable objects in $\mathcal{M}$ such that every object in $\mathcal{M}$ can be written as a λ-filtered colimit of the G_i's. $\mathcal{M}$ is *locally presentable* if it is locally λ-presentable for some λ.

Definition 1.12. A model category is *combinatorial* if it is locally presentable and cofibrantly generated.

Remark 1.13. There is another notion which assures the applicability of Bousfield localization developed in Hirschhorn's book [8], called cellularity, which is more suitable for categories built from topological spaces.

The following basic result is found in e.g. [1, 1.12].

Proposition 1.14. *Let $\mathcal{I}$ be a small category. Then the functor category* $\mathrm{Fun}(\mathcal{I}, \mathrm{Set}_*)$ *is locally presentable.*

Since Spc is isomorphic to $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \Delta^{\mathrm{op}}, \mathrm{Set}_*)$, we have the following result.

Corollary 1.15. *The category of spaces, given any of the intermediate model structures in Theorem 1.4, is combinatorial.*

2. The category of Γ-spaces

Let Γ be the full subcategory of the category of pointed sets with objects $n^+ = \{0, 1, \ldots, n\}$, for $n \geqslant 0$, where 0 is the basepoint in n^+. Let $\mathcal{M}$ be a pointed category. The full subcategory of $\mathrm{Fun}(\Gamma, \mathcal{M})$ consisting of functors that send 0^+ to the basepoint in $\mathcal{M}$ is the category of Γ-objects in $\mathcal{M}$, denoted $\Gamma\mathcal{M}$. The inclusion of Γ in the category of finite based sets, fSet_*, is an equivalence of categories, and $\Gamma\mathcal{M}$ is equivalent to the subcategory of $\mathrm{Fun}(\mathrm{fSet}_*, \mathcal{M})$ consisting of based functors.

When $\mathcal{M}$ is the category $\mathcal{S}_*$ of pointed simplicial sets, objects in $\Gamma\mathcal{S}_*$ are classically called Γ-spaces; model structures on this category are constructed in the articles Bousfield and Friedlander [4] and Schwede [20]. Our objects of study will be Γ-objects in Spc, which we also call Γ-spaces. Alternatively, our Γ-spaces can be thought of as presheaves of ordinary Γ-spaces, i.e., $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \Gamma\mathcal{S}_*)$. Note that when $\mathcal{C}$ consists of one morphism only, we recover the category $\Gamma\mathcal{S}_*$, and our stable model structure will be constructed so that we recover the stable model structure in [20].

To start with, we want to define a closed symmetric monoidal structure on $\Gamma\,\mathrm{Spc}$. Observe that Γ is symmetric monoidal under the operation $\wedge\colon \Gamma \times \Gamma \to \Gamma$ given by $(m^+, n^+) \mapsto mn^+$. Given two Γ-spaces F and G, the smash product $F \wedge G$ is defined as the left Kan extension filling out the following diagram:

$$\begin{array}{ccccc} \Gamma \times \Gamma & \xrightarrow{(F,G)} & \mathrm{Spc} \times \mathrm{Spc} & \xrightarrow{-\wedge-} & \mathrm{Spc}\,. \\ \wedge\big\downarrow & & & \nearrow & \\ \Gamma & - & - & - & \end{array}$$

More explicitly, the smash product is the pointwise colimit

$$(F \wedge G)(n^+) = \operatorname*{colim}_{i^+ \wedge j^+ \to n^+} F(i^+) \wedge G(j^+)\,.$$

It follows from the universal property of the colimit that maps of Γ-spaces $F \wedge G \to H$ are in one-to-one correspondence with maps $F(i^+) \wedge G(j^+) \to H(i^+ \wedge j^+)$ that are natural in i^+ and j^+, and that this property characterizes $F \wedge G$ up to isomorphism.

Simplicial function complexes of Γ-spaces are defined to be

$$\mathrm{Map}(F, G)_n = \Gamma\,\mathrm{Spc}(F \wedge \Delta^n_+, G)$$

in simplicial degree n; the face and degeneracy maps are the obvious ones. From this we define the simplicial presheaf-hom, or space-hom, in sections by

$$\mathrm{SpcHom}(F, G)(U) = \mathrm{Map}(F|U, G|U)\,,$$

where $|U$ denotes pointwise restriction to the local site $\mathcal{C} \downarrow U$. Finally, internal hom-Γ-spaces are defined by setting

$$\underline{\mathrm{Hom}}(F, G)(n^+) = \mathrm{SpcHom}(F, G(n^+ \wedge -))\,.$$

We have given the constructions of the objects involved in the following result, which is a special case of Day's work in [5].

Proposition 2.1. *The category* $\Gamma\,\mathrm{Spc}$ *is a simplicial closed symmetric monoidal category enriched over* Spc.

A set defines a discrete simplicial set, and therefore a constant simplicial presheaf. In particular, the sets $\Gamma(n^+, k^+)$ define the corepresented Γ-space Γ^n given pointwise by $\Gamma^n(k^+) = \Gamma(n^+, k^+)$. Let F be a Γ-space and let $F \circ \Gamma^n$ denote the Γ-space given pointwise by

$$(F \circ \Gamma^n)(k^+) = F(\Gamma(n^+, k^+))\,.$$

Note that the smash product of two Γ-spaces defined above coincides with the one given sectionwise by Lydakis' smash product of classical Γ-spaces [17]. Hence the following two lemmas follow immediately from the corresponding natural isomorphisms for classical Γ-spaces in [17].

Lemma 2.2. *There are natural isomorphisms:*

1. $\mathrm{SpcHom}(\Gamma^n, F) \cong F(n^+)$,
2. $\Gamma^m \wedge \Gamma^n \cong \Gamma^{mn}$,
3. $F \wedge \Gamma^n \cong F \circ \Gamma^n$.

Lemma 2.3. *Smashing with a* Γ*-space preserves monomorphisms of* Γ*-spaces.*

There are functors

$$\mathrm{L}_n : \mathrm{Spc} \rightleftarrows \Gamma\,\mathrm{Spc} : \mathrm{Ev}_n \tag{2.1}$$

for each $n \geqslant 0$, where Ev_n is evaluation at n^+ and $\mathrm{L}_n(X) = X \wedge \Gamma^n$. From Lemma 2.2 we have a natural isomorphism

$$\mathrm{L}_m(X) \wedge \mathrm{L}_n(Y) \cong \mathrm{L}_{mn}(X \wedge Y)\,. \tag{2.2}$$

Proposition 2.4. *The functors in* (2.1) *form an adjoint pair.*

Proof. We need to provide a natural isomorphism

$$\Gamma\,\mathrm{Spc}(X \wedge \Gamma^n, F) \cong \mathrm{Spc}(X, F(n^+))\,.$$

Since $\Gamma\,\mathrm{Spc}$ is enriched over Spc, there is a natural isomorphism

$$\Gamma\,\mathrm{Spc}(X \wedge \Gamma^n, F) \cong \mathrm{Spc}(X, \mathrm{SpcHom}(\Gamma^n, F))\,,$$

which combined with part (1) of Lemma 2.2 gives the result. □

3. Strict model structures

In this section we establish basic results about the strict projective model structures on $\Gamma\,\mathrm{Spc}$.

Hypothesis 3.1. *For the rest of this paper we will assume, unless otherwise noted, that* Spc *is given one of the intermediate model structures described in Theorem 1.4, including the local injective and local projective structures. Suppose further that the sets of generating (trivial) cofibrations can be chosen with cofibrant domains.*

Definition 3.2. A map $F \to G$ of Γ-spaces is a

- *strict weak equivalence* if $F(n^+) \to G(n^+)$ is a local weak equivalence in Spc for all $n \geqslant 0$.
- *strict fibration* if $F(n^+) \to G(n^+)$ is a fibration in Spc for all $n \geqslant 0$.
- *cofibration* if it has the left lifting property with respect to the maps that are both strict weak equivalences and projective fibrations.

Theorem 3.3. *Let I and J be the sets of generating cofibrations and generating trivial cofibrations in* Spc. *Then* $\Gamma\,\mathrm{Spc}$ *with the classes of strict weak equivalences, cofibrations and strict fibrations is a cofibrantly generated proper* Spc*-model category, with generating cofibrations*

$$I_\Gamma = \bigcup_{n \geqslant 0} \mathrm{L}_n(I)$$

and generating trivial cofibrations

$$J_\Gamma = \bigcup_{n \geqslant 0} \mathrm{L}_n(J)\,.$$

We will refer to this model structure as the strict model structure *on* $\Gamma\,\mathrm{Spc}$.

Proof. This result is an application of more general results concerning strict projective model structures on diagram categories, which can be found in Hirschhorn's book, [8, 11.6.1, 11.7.3, 13.1.14]. The model structure is enriched in Spc by [2, 3.30]. □

Corollary 3.4. *The adjoint functor pair* (2.1) *is a Quillen pair, and* Ev_n *preserves cofibrations. In particular, cofibrations are monomorphisms.*

Proof. The first statement follows immediately from Theorem 3.3, the second statement follows from [8, 11.6.3]. □

Corollary 3.5. *The* Γ*-space* $X \wedge \Gamma^n$ *is cofibrant when* X *is a cofibrant space. In particular,* Γ^n *is cofibrant.*

Proof. This follows by applying L_n to the map $* \to X$. □

Since $\Gamma\,\mathrm{Spc}$ as a category is isomorphic to $\mathrm{Fun}(\Gamma \times \mathcal{C}^{\mathrm{op}} \times \Delta^{\mathrm{op}}, \mathrm{Set}_*)$, it is locally presentable by Proposition 1.14.

Corollary 3.6. *The category of* Γ*-spaces with the strict model structure is combinatorial.*

Proposition 3.7. *The category of* Γ*-spaces equipped with the strict model structure is a monoidal model category provided* Spc *is monoidal.*

Proof. Since the monoidal unit Γ^1 is cofibrant, it suffices to check the pushout product axiom. Let $F_i \to G_i$, where $i = 1, 2$, be two cofibrations, and construct the pushout diagram

$$\begin{array}{ccc} F_1 \wedge F_2 & \longrightarrow & G_1 \wedge F_2 \\ \downarrow & & \downarrow \\ F_1 \wedge G_2 & \longrightarrow & P\,. \end{array} \tag{3.1}$$

We may assume that the $F_i \to G_i$ are of the form

$$X_i \wedge \Gamma^{n_i} \to Y_i \wedge \Gamma^{n_i}\,,$$

where $X_i \to Y_i$ are cofibrations in Spc. Using the isomorphism (2.2), and the fact that $\mathrm{L}_{n_1 n_2}$ preserves colimits, we can apply $\mathrm{L}_{n_1 n_2}$ to the pushout constructed from the maps $X_i \to Y_i$ to obtain

$$\begin{array}{ccc} \mathrm{L}_{n_1 n_2}(X_1 \wedge X_2) & \longrightarrow & \mathrm{L}_{n_1 n_2}(Y_1 \wedge X_2) \\ \downarrow & & \downarrow \\ \mathrm{L}_{n_1 n_2}(X_1 \wedge Y_2) & \longrightarrow & \mathrm{L}_{n_1 n_2}(X_1 \wedge Y_2 \coprod_{X_1 \wedge X_2} Y_1 \wedge X_2)\,, \end{array}$$

which is isomorphic to (3.1). We know that

$$X_1 \wedge Y_2 \coprod_{X_1 \wedge X_2} Y_1 \wedge X_2 \to X_2 \wedge Y_2$$

is a cofibration of spaces, by the assumption that Spc is monoidal, so we see that the map $P \to \mathrm{L}_{n_1 n_2}(Y_1 \wedge Y_2)$ is a cofibration. The same argument gives the corresponding result about trivial cofibrations. □

Proposition 3.8. *The strict model structure on* $\Gamma\,\mathrm{Spc}$ *satisfies the monoid axiom when* Spc *does.*

Proof. We need to show that the maps in $(J_\Gamma \wedge \Gamma\,\mathrm{Spc})$-cell are weak equivalences. Consider first a map f of the form

$$\mathrm{L}_n(X) \wedge F \to \mathrm{L}_n(Y) \wedge F$$

where $X \to Y$ is a generating trivial cofibration in Spc. Evaluating at k^+, we get

$$X \wedge (\Gamma^n \wedge F)(k^+) \xrightarrow{f(k^+)} Y \wedge (\Gamma^n \wedge F)(k^+)\,,$$

so $f(k^+)$ is in $J \wedge \mathrm{Spc}$ for all k^+. Now, if g is in $(J_\Gamma \wedge \Gamma\,\mathrm{Spc})$-cell, then it is a transfinite composition of pushouts of maps f_i in $J_\Gamma \wedge \Gamma\,\mathrm{Spc}$. Since each $f_i(k^+)$ is in $J \wedge \mathrm{Spc}$, and colimits in $\Gamma\,\mathrm{Spc}$ are computed pointwise, $g(k^+)$ is in $(J \wedge \mathrm{Spc})$-cell. Using the assumption that the monoid axiom holds in Spc, we see that $g(k^+)$ is a weak equivalence for all k^+. □

Lemma 3.9. *A filtered colimit of strict equivalences is a strict equivalence.*

Proof. Local weak equivalences of spaces are preserved under filtered colimits, since sheaves of homotopy groups commute with filtered colimits, and the lemma follows since colimits of Γ-spaces are defined pointwise. □

Proposition 3.10. *Strict equivalences of* Γ*-spaces are preserved when smashed with a cofibrant* Γ*-space.*

Proof. Let $f\colon F \to G$ be a strict equivalence. The induced map $F \circ \Gamma^n \to G \circ \Gamma^n$ is clearly a strict equivalence, so by Lemma 2.2, the map $f \wedge 1\colon F \wedge \Gamma^n \to G \wedge \Gamma^n$ is a strict equivalence.

Now let C be a cofibrant Γ-space. Since $\Gamma\,\mathrm{Spc}$ is cofibrantly generated with generating cofibrations I_Γ, C is a retract of an I_Γ-cell complex, where by I_Γ-cell complex we mean that the unique map $* \to C$ is a transfinite composition of pushouts of maps in I_Γ. Weak equivalences are closed under retracts, so it suffices to consider $C = \mathrm{colim}_{\alpha<\gamma}\, C_\alpha$, γ an ordinal, where the maps $C_\alpha \to C_{\alpha+1}$ are given by pushout diagrams

$$\begin{array}{ccc} X \wedge \Gamma^n & \longrightarrow & C_\alpha \\ {\scriptstyle i\wedge 1}\downarrow & & \downarrow \\ Y \wedge \Gamma^n & \longrightarrow & C_{\alpha+1}\,. \end{array} \tag{3.2}$$

Here $i\colon X \to Y$ is a cofibration of spaces.

Smashing (3.2) with F and G gives us two pushout diagrams as the top and bottom faces of a cubical diagram. Assuming by induction that $F \wedge C_\alpha \to G \wedge C_\alpha$ is a strict equivalence, we see that the gluing lemma (see [7, II.8.12]) can be applied to conclude that $F \wedge C_{\alpha+1} \to G \wedge C_{\alpha+1}$ is a strict equivalence. Since $F \wedge C \to G \wedge C$ is the colimit of the maps $F \wedge C_\alpha \to G \wedge C_\alpha$ we can conclude by applying Lemma 3.9. □

4. Stable model structures

In this section we will construct the stable model structures for (presheaves of) Γ-spaces and compare it to the model category of (presheaves of) spectra. In fact, parts of our construction relies on this comparison; we will begin by recalling the theory of spectra on a site.

For us, a spectrum is a sequence of objects $E^k \in \mathrm{Spc}$ indexed by non-negative integers k together with structure maps

$$S^1 \wedge E^k \to E^{k+1}$$

for each k. Maps of spectra are sequences of maps $f^k\colon E^k \to F^k$ compatible with the structure maps in the sense that the diagram

$$\begin{array}{ccc} S^1 \wedge E^k & \longrightarrow & E^{k+1} \\ {\scriptstyle 1\wedge f^k}\big\downarrow & & \big\downarrow{\scriptstyle f^{k+1}} \\ S^1 \wedge F^k & \longrightarrow & F^{k+1} \end{array}$$

commutes for all k. Denote the category of spectra by Spt.

A spectrum E is levelwise fibrant if each E^k is fibrant, and is an Ω-spectrum if the adjoints $E^k \to \Omega_f E^{k+1}$ of the structure maps are weak equivalences. The (derived) loop functor $\Omega_f\colon \mathrm{Spc} \to \mathrm{Spc}$ is by definition a fibrant replacement $(-)_f$ followed by the simplicial cotensor $(-)^{S^1}$ on spaces. Note that we do not require our Ω-spectra to be levelwise fibrant. A map $f\colon E \to F$ of spectra is a cofibration if $f^0\colon E^0 \to F^0$ is a cofibration of spaces and the induced maps

$$(S^1 \wedge F^k) \bigcup_{S^1\wedge E^k} E^{k+1} \to F^{k+1}$$

are cofibrations of spaces for all $k \geqslant 0$. The map f is a stable equivalence of spectra if it induces isomorphisms $\pi_n(E) \to \pi_n(F)$ of stable homotopy sheaves for all integers n and $U \in \mathcal{C}$, where the stable homotopy sheaf $\pi_n(E)$ is by definition the colimit of the system

$$\cdots \to \pi_{n+k}(E^k) \to \pi_{n+k+1}(S^1 \wedge E^k) \to \pi_{n+k+1}(E^{k+1}) \to \dots .$$

The following result was first proved by Jardine in [13, 2.8] for the local injective model structure on Spc; Hovey has results for spectra in more general model categories in [10, 3.3].

Theorem 4.1. *Let* Spc *be given any intermediate model structure. With the above notions of stable cofibrations and stable equivalences, the category* Spt *of spectra is a cofibrantly generated proper* Spc*-model category. A spectrum is stably fibrant if and only if it is a levelwise fibrant Ω-spectrum.*

Let F be a Γ-space, which we now consider as a based functor from all finite based sets to Spc. The functor F induces a functor $\bar{F}\colon \mathcal{S}_* \to \mathrm{s\,Spc}$ from simplicial sets to simplicial spaces, by applying F in each simplicial degree. We can compose F with the diagonal functor $d\colon \mathrm{s\,Spc} \to \mathrm{Spc}$ to get a functor

$$d\bar{F}\colon \mathcal{S}_* \to \mathrm{Spc} .$$

Proposition 4.2. *Let $K \to L$ be a weak equivalence of simplicial sets. Then the induced map $d\bar{F}(K) \to d\bar{F}(L)$ is a sectionwise equivalence, and in particular a local weak equivalence.*

Proof. This follows from the corresponding result for classical Γ-spaces in [4, 4.9], since $d\bar{F}(K)(U)$ coincides with the corresponding construction for the classical Γ-space $F(U)$. □

Each pair of based sets A, B induces natural maps

$$A \wedge F(B) \to F(A \wedge B)$$

whose adjoints $A \to \mathrm{Spc}(F(B), F(A \wedge B))$ are described by sending an element a to the map $F(a \wedge -)$. These maps induce simplicial maps

$$X \wedge \bar{F}(Y) \to \bar{F}(X \wedge Y),$$

where X and Y are based simplicial sets. By applying the diagonal functor this results in maps

$$X \wedge d\bar{F}(Y) \to d\bar{F}(X \wedge Y)\,. \tag{4.1}$$

The spectrum associated to a Γ-space F, which we denote $\mathrm{Sp}(F)$, is defined on each level as $\mathrm{Sp}(F)^n = d\bar{F}(S^n)$. Here $S^n = S^1 \wedge \cdots \wedge S^1$ (n times.) As a special case of (4.1), we have

$$S^m \wedge d\bar{F}(S^n) \to d\bar{F}(S^{m+n})$$

which gives us the structure maps for $\mathrm{Sp}(F)$.

Lemma 4.3. *The functor* $\mathrm{Sp}(F)$ *has the following properties:*

1. $\mathrm{Sp}(F)^0 = F(1^+)$
2. $\mathrm{Sp}(\Gamma^n) = \mathbb{S}^{\times n}$
3. $\mathrm{Sp}(X \wedge F) = X \wedge \mathrm{Sp}(F)$*, for spaces* X*.*

Let E be a spectrum. We obtain a Γ-space $\Phi(E)$ by defining

$$\Phi(E)(n^+) = \mathrm{SpcHom}_{\mathrm{Spt}}(\mathbb{S}^{\times n}, E)\,,$$

where $\mathbb{S}$ denotes the sphere spectrum. Here $\mathrm{SpcHom}_{\mathrm{Spt}}(-,-)$ denotes the space of morphisms in the category of spectra, defined sectionwise in the same way as for Γ-spaces; i.e.,

$$\mathrm{SpcHom}_{\mathrm{Spt}}(E, F)(U) = \mathrm{Map}_{\mathrm{Spt}}(E|U, F|U)$$

for all $U \in \mathcal{C}$. A morphism $\theta\colon m^+ \to n^+$ induces a map $\theta^*\colon \mathbb{S}^{\times n} \to \mathbb{S}^{\times m}$ by copying the $\theta(i)$'th factor into the i'th factor. This map in turn induces $\Phi(E)(m^+) \to \Phi(E)(n^+)$.

Lemma 4.4. *The spectrum* $\mathrm{Sp}(F)$ *coincides with the coequalizer of the diagram*

$$\bigvee_{\theta\colon m^+ \to n^+} \mathbb{S}^{\times n} \wedge F(m^+) \underset{\theta^* \wedge 1}{\overset{1 \wedge F(\theta)}{\rightrightarrows}} \bigvee_{k^+} \mathbb{S}^{\times k} \wedge F(k^+)\,.$$

Proof. Since colimits in Spt, Spc and $\mathcal{S}_*$ are computed pointwise, it suffices to show that the following diagram

$$\bigvee_{\theta\colon m^+\to n^+} (S^i_q)^{\times n} \wedge F(m^+)(U)_q \underset{\theta^*\wedge 1}{\overset{1\wedge F(\theta)}{\rightrightarrows}} \bigvee_{k^+} (S^i_q)^{\times k} \wedge F(k^+)(U)_q \xrightarrow{f} F(S^i_q)(U)_q$$

is a coequalizer of sets, for all $i, q \geqslant 0$, where f is described as follows. A collection of k ordered elements x_j in S^i_q specifies a map $k^+ \to S^i_q$, and by applying F we get a map $(S^i_q)^{\times k} \to \mathrm{Set}_*(F(k^+)(U)_q, F(S^i_q)(U)_q)$. Take the adjoint of this and sum over k^+ to get f. We omit the straightforward element chase. □

Proposition 4.5. *The functors*

$$\mathrm{Sp} : \Gamma\,\mathrm{Spc} \rightleftarrows \mathrm{Spt} : \Phi$$

constitute an adjoint pair. Furthermore, this adjunction can be extended to a Spc-*adjunction*

$$\mathrm{SpcHom}_{\mathrm{Spt}}(\mathrm{Sp}(F), E) \cong \mathrm{SpcHom}_{\Gamma\,\mathrm{Spc}}(F, \Phi(E))\,.$$

Proof. First note that we have an adjunction

$$\mathrm{Spt}(X \wedge E, F) \cong \mathrm{Spc}(X, \mathrm{SpcHom}(E, F)), \tag{4.2}$$

where X is a space and E, F are spectra. Now take the coequalizer in Lemma 4.4 and apply the functor $\mathrm{Spt}(-, E)$ and the isomorphism (4.2). The result is that $\mathrm{Spt}(\mathrm{Sp}(F), E)$ is the equalizer of

$$\prod_{\theta\colon m^+\to n^+} \mathrm{Spc}(F(m^+), \mathrm{SpcHom}(\mathbb{S}^{\times n}, E)) \leftleftarrows \prod_{k^+} \mathrm{Spc}(F(k^+), \mathrm{SpcHom}(\mathbb{S}^{\times k}, E))\,.$$

Any map $\mathrm{Sp}(F) \to E$ thus corresponds to a collection of maps

$$F(k^+) \to \mathrm{SpcHom}(\mathbb{S}^{\times k}, E) = \Phi(E)(n^+)$$

natural in k^+, i.e., a map of Γ-spaces $F \to \Phi(E)$. □

Proposition 4.6. *The functor* Sp *preserves cofibrations.*

Proof. It suffices to consider generating cofibrations in $\Gamma\,\mathrm{Spc}$. Let $X \wedge \Gamma^n \to Y \wedge \Gamma^n$ be a generating cofibration, where $X \to Y$ is a cofibration of spaces. By Lemma 4.3 we need to show that

$$X \wedge \mathrm{Sp}(\Gamma^n) \to Y \wedge \mathrm{Sp}(\Gamma^n)$$

is a cofibration of spectra, but this is immediate since $\mathrm{Sp}(\Gamma^n) = \mathbb{S}^{\times n}$ is a cofibrant spectrum and Spt is a Spc-model category. □

Definition 4.7. A Γ-space F is *special* if the maps

$$F(n^+) \to F(1^+) \times \cdots \times F(1^+)$$

induced by the n usual projections from n^+ to 1^+ are weak equivalences for all $n \geqslant 1$. If, in addition, the map

$$F(2^+) \to F(1^+) \times F(1^+)$$

induced by a projection and the fold map is a weak equivalence, then F is *very special*.

Note that when F is special the maps

$$F(1^+) \times F(1^+) \xleftarrow{\sim} F(2^+) \xrightarrow{\nabla} F(1^+)$$

induce a commutative monoid structure on the sheaf $\pi_0(F(1^+))$. If F is very special, then $\pi_0(F(1^+))$ is in fact a sheaf of abelian groups.

Proposition 4.8. *The functor* Sp *sends very special* Γ*-spaces to* Ω*-spectra. The functor* Φ *sends fibrant spectra to strictly fibrant very special* Γ*-spaces.*

Proof. Let F be a very special Γ-space, and let $F \to F_f$ be a strictly fibrant replacement. Since $\bar{F}(S^n) \to \bar{F}_f(S^n)$ is a strict equivalence of simplicial spaces, the induced map $d\bar{F}(S^n) \to d\bar{F}_f(S^n)$ of spaces is a local equivalence by Proposition 6.1. We need to show that the map $d\bar{F}_f(S^n) \to \Omega_f d\bar{F}_f(S^{n+1})$ is a local weak equivalence. Since F is very special, so is F_f, and in fact the maps

$$F_f(n^+) \to F_f(1^+) \times \cdots \times F_f(1^+)$$

and

$$F_f(2^+) \to F_f(1^+) \times F_f(1^+)$$

are sectionwise equivalences by Proposition 1.7 since each $F_f(n^+)$ is fibrant. Thus $F_f(U)$ is a very special Γ-space in the classical sense, for each $U \in \mathcal{C}$, and by [4, 4.2] each map $d\bar{F}_f(U)(S^n) \to \Omega_f d\bar{F}_f(U)(S^{n+1})$ is a weak equivalence of simplicial sets. This implies in particular that $d\bar{F}_f(S^n) \to \Omega_f d\bar{F}_f(S^{n+1})$ is a local weak equivalence.

For the second statement, let E be a fibrant spectrum. The canonical stable equivalence $\mathbb{S} \vee \cdots \vee \mathbb{S} \to \mathbb{S} \times \cdots \times \mathbb{S}$ has cofibrant domain and codomain; hence the induced map

$$\mathrm{SpcHom}(\mathbb{S}^{\times n}, E) \to \mathrm{SpcHom}(\mathbb{S}^{\vee n}, E) \cong \mathrm{SpcHom}(\mathbb{S}, E)^{\times n}$$

is a local weak equivalence, i.e., $\Phi(E)$ is special. Similarly, the map $\mathbb{S} \vee \mathbb{S} \to \mathbb{S} \times \mathbb{S}$ induced by an inclusion and the diagonal map is a stable equivalence, so $\Phi(E)$ is very special. □

Definition 4.9. The *n-th homotopy sheaf* $\pi_n(F)$ of a Γ-space F is the n-th homotopy sheaf of the associated spectrum $\mathrm{Sp}(F)$. We write $\pi_*(F)$ for the $\mathbb{Z}$-graded abelian sheaf $\oplus_n \pi_n(F)$.

Note that the sheaf $\pi_n(F)$ is isomorphic to the sheaf associated to the presheaf given sectionwise as $\pi_n(F(U))$, for $U \in \mathcal{C}$, where $\pi_n(F(U))$ is the homotopy group of the classical Γ-space $F(U)$ as defined in [20, §1].

Definition 4.10. A map $F \to G$ in $\Gamma\,\mathrm{Spc}$ is a

- *stable equivalence* if the induced map $\pi_*(F) \to \pi_*(G)$ is an isomorphism.
- *stable fibration* if it has the right lifting property with respect to the maps that are both cofibrations and stable equivalences.

Recall that a spectrum E is called connective if $\pi_n(E) = 0$ for $n < 0$. Since the k-simplices of $\Delta^n/\partial\Delta^n$ for $k < n$ consist of the basepoint only, and since $\Delta^n/\partial\Delta^n$ is weakly equivalent to S^n, it follows from Proposition 4.2 that $d\bar{F}(S^n)$ is $(n-1)$-connected, and that $\mathrm{Sp}(F)$ is a connective spectrum.

Lemma 4.11. *The following holds for the adjunction in Proposition 4.5:*

1. *The composition* $F \to \Phi(\mathrm{Sp}(F)) \to \Phi(\mathrm{Sp}(F)_f)$ *of the unit map and* Φ *applied to a fibrant replacement of* $\mathrm{Sp}(F)$*, is a strict weak equivalence for special* Γ*-spaces* F*.*
2. *When* E *is a fibrant spectrum, the counit map* $\mathrm{Sp}(\Phi(E)) \to E$ *induces isomorphisms* $\pi_n(\mathrm{Sp}(\Phi(E))) \to \pi_n(E)$ *for all* $n \geqslant 0$*. In particular,* $\mathrm{Sp}(\Phi(E)) \to E$ *is a stable equivalence when* E *is a fibrant connective spectrum.*

Proof. Let F be special. The commutative diagram

$$\begin{array}{ccc} F(n^+) & \longrightarrow & \mathrm{SpcHom}(\mathbb{S}^{\times n}, \mathrm{Sp}(F)_f) \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\ F(1^+)^{\times n} & & \mathrm{SpcHom}(\mathbb{S}^{\vee n}, \mathrm{Sp}(F)_f) \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \cong} \\ (\mathrm{Sp}(F)^0_f)^{\times n} & \xrightarrow{\cong} & \mathrm{SpcHom}(\mathbb{S}, \mathrm{Sp}(F)_f)^{\times n} \end{array}$$

shows that the top map is a local weak equivalence for each $n \geqslant 0$.

When E is a fibrant spectrum, $\pi_n(E) \cong \pi_n(E^0)$ for all $n \geqslant 0$, so the second statement of the lemma is reduced to the statement that $\mathrm{Sp}(\Phi(E))^0 \to E^0$ is a local weak equivalence of spaces. But this map coincides with the canonical weak equivalence

$$\mathrm{Sp}(\Phi(E))^0 = (\Phi E)(1^+) = \mathrm{SpcHom}_{\mathrm{Spt}}(\mathbb{S}, E) \to E^0 . \qquad \square$$

We let $\mathrm{Ho}(\mathrm{Spt})_{\geqslant 0}$ denote the full subcategory of $\mathrm{Ho}(\mathrm{Spt})$ consisting of the connective spectra.

Theorem 4.12. *The category* $\Gamma\,\mathrm{Spc}$ *with the classes of stable equivalences, cofibrations and stable fibrations is a cofibrantly generated left proper* Spc*-model category, such that the functor pair in Proposition 4.5 induces an equivalence of categories*

$$L\,\mathrm{Sp} : \mathrm{Ho}(\Gamma\,\mathrm{Spc}) \simeq \mathrm{Ho}(\mathrm{Spt})_{\geqslant 0} : R\Phi .$$

The stably fibrant objects in $\Gamma\,\mathrm{Spc}$ *are the very special* Γ*-spaces that are also strictly fibrant. A strict fibration of stably fibrant* Γ*-spaces is necessarily a stable fibration. A stable equivalence between stably fibrant* Γ*-spaces is a strict equivalence.*

Proof. Let Σ be the set of maps consisting of

$$\Gamma^1 \vee \cdots \vee \Gamma^1 \to \Gamma^n$$

for all $n \geqslant 1$, and the shear map

$$\Gamma^1 \vee \Gamma^1 \to \Gamma^2 .$$

These morphisms are induced by the same morphisms in Γ as in Definition 4.7, and corepresent the morphisms displayed there. Since the strict model structure on $\Gamma\,\mathrm{Spc}$ is combinatorial, left proper and enriched over Spc, we can apply enriched left Bousfield localization (see Theorem 6.4) with respect to Σ to obtain a new combinatorial and left proper model structure on $\Gamma\,\mathrm{Spc}$. For the remainder of this proof we will refer to this model structure as the "localized model structure".

The localized fibrant objects are given by the Σ-local objects. A Γ-space H is Σ-local if and only if it is strictly fibrant and the maps

$$\mathrm{SpcHom}(\Gamma^n, H) \to \mathrm{SpcHom}(\Gamma^1 \vee \cdots \vee \Gamma^1, H)$$

and

$$\mathrm{SpcHom}(\Gamma^2, H) \to \mathrm{SpcHom}(\Gamma^1 \vee \Gamma^1, H)$$

are weak equivalences of spaces, for $n \geqslant 1$. Composing with the isomorphism

$$\mathrm{SpcHom}(\Gamma^1 \vee \cdots \vee \Gamma^1, H) \to \mathrm{SpcHom}(\Gamma^1, H) \times \cdots \times \mathrm{SpcHom}(\Gamma^1, H)$$

and using the isomorphism (1) in Lemma 2.2, it is clear that the Σ-local objects coincide with the strictly fibrant very special Γ-spaces.

The localized weak equivalences are defined to be those maps $f\colon F \to G$ that have a cofibrant replacement $f_c\colon F_c \to G_c$ (in the strict model structure) that induces local weak equivalences

$$f_c^*\colon\ \mathrm{SpcHom}(G_c, H) \to \mathrm{SpcHom}(F_c, H)$$

of spaces for all Σ-local H. We have to identify the localized weak equivalences with the stable equivalences.

Consider the following diagram

$$\begin{array}{ccc} \mathrm{SpcHom}_{\Gamma\,\mathrm{Spc}}(G_c, \Phi(E)) & \xrightarrow{\cong} & \mathrm{SpcHom}_{\mathrm{Spt}}(\mathrm{Sp}(G_c), E) \\ \downarrow{\scriptstyle f_c^*} & & \downarrow{\scriptstyle \mathrm{Sp}(f_c)^*} \\ \mathrm{SpcHom}_{\Gamma\,\mathrm{Spc}}(F_c, \Phi(E)) & \xrightarrow{\cong} & \mathrm{SpcHom}_{\mathrm{Spt}}(\mathrm{Sp}(F_c), E), \end{array} \tag{4.3}$$

where the horizontal maps come from the adjunction in Proposition 4.5. Note that $\mathrm{Sp}(f_c)$ is a map between cofibrant objects by Proposition 4.6. Since Spt is a Spc-model category, $\mathrm{Sp}(f_c)\colon\ \mathrm{Sp}(F_c) \to \mathrm{Sp}(G_c)$ is a stable equivalence of spectra if and only if $\mathrm{Sp}(f_c)^*$ is a local weak equivalence of spaces for all fibrant spectra E. It follows that f_c is a stable equivalence of Γ-spaces if and only if f_c^* is a local weak equivalence for all fibrant E. In particular, a localized weak equivalence is a stable equivalence since by Proposition 4.8 we know that $\Phi(E)$ is a Σ-local Γ-space.

When H is a very special Γ-space the map $H \to \Phi(\mathrm{Sp}(H)_f)$ is a strict weak equivalence by Lemma 4.11, and hence induces local weak equivalences of spaces in the diagram

$$\begin{array}{ccc} \mathrm{SpcHom}(G_c, H) & \xrightarrow{\sim} & \mathrm{SpcHom}(G_c, \Phi(\mathrm{Sp}(H)_f)) \\ \downarrow{\scriptstyle f_c^*} & & \downarrow{\scriptstyle f_c^*} \\ \mathrm{SpcHom}(F_c, H) & \xrightarrow{\sim} & \mathrm{SpcHom}(F_c, \Phi(\mathrm{Sp}(H)_f))\,. \end{array} \tag{4.4}$$

It follows from (4.3) and (4.4) that a stable equivalence is a localized weak equivalence.

Now that we have identified the localized weak equivalences with the stable equivalences, Sp is a left Quillen functor by Proposition 4.6 since the localization process does not change the class of cofibrations. The Quillen pair Sp and Φ induces derived adjoint functors $L\,\mathrm{Sp}$ and $R\Phi$ on the homotopy categories of $\Gamma\,\mathrm{Spc}$ and Spt, which by Proposition 4.8 restrict to functors

$$L\,\mathrm{Sp} : \mathrm{Ho}(\Gamma\,\mathrm{Spc}) \rightleftarrows \mathrm{Ho}(\mathrm{Spt})_{\geqslant 0} : R\Phi\,.$$

To show that $L\,\mathrm{Sp}$ is an equivalence, it is enough to note that Sp detects weak equivalences, and that the counit map $\mathrm{Sp}(\Phi(E)) \to E$ is a stable equivalence for connective fibrant spectra E by Lemma 4.11. □

Proposition 4.13. *Smashing with a cofibrant* Γ*-space preserves stable equivalences.*

Proof. First note that $\underline{\mathrm{Hom}}(C, H)$ is very special when C is cofibrant and H is fibrant, since $\Gamma\,\mathrm{Spc}$ is a Spc-model category. Let $f\colon F \to G$ be stable equivalence with cofibrant replacement $f_c\colon F_c \to G_c$, and C a cofibrant Γ-space. We have that $\mathrm{Map}(G_c, H) \to \mathrm{Map}(F_c, H)$ is a weak equivalence for all fibrant H, so in particular

$$\mathrm{Map}(G_c, \underline{\mathrm{Hom}}(C, H)) \to \mathrm{Map}(F_c, \underline{\mathrm{Hom}}(C, H))$$

is a weak equivalence for all cofibrant C and fibrant H. Together with the isomorphism $\mathrm{Map}(F_c, \underline{\mathrm{Hom}}(C, H)) \cong \mathrm{Map}(F_c \wedge C, H)$ this implies that $f_c \wedge 1$ is a stable equivalence. The commutative diagram

$$\begin{array}{ccc} F_c \wedge C & \longrightarrow & F \wedge C \\ {\scriptstyle f_c \wedge 1}\downarrow & & \downarrow{\scriptstyle f \wedge 1} \\ G_c \wedge C & \longrightarrow & G \wedge C\,, \end{array}$$

where the horizontal maps are strict weak equivalences by Proposition 3.10, implies that $f \wedge 1$ is a stable equivalence. □

Lemma 4.14. *Let $F \to G$ be a monomorphism of* Γ*-spaces. Then there is an exact sequence of abelian sheaves*

$$\cdots \to \pi_{n+1}(G/F) \to \pi_n(F) \to \pi_n(G) \to \pi_n(G/F) \to \pi_{n-1}(F) \to \ldots\,.$$

Proof. This follows from [20, 1.3] by evaluating in sections and applying the exact sheafification functor. □

Proposition 4.15. *Pushouts of* Γ*-spaces preserve monomorphic stable equivalences.*

Proof. Consider the pushout diagram

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ F' & \longrightarrow & G', \end{array}$$

where $F \to G$ is a monomorphic stable equivalence. It follows that the map $F' \to G'$ is a monomorphism, and that $G'/F' \cong G/F$, so by Lemma 4.14 the map $F' \to G'$ is also a stable equivalence. □

Proposition 4.16. *The stable model structure on* $\Gamma\,\mathrm{Spc}$ *is monoidal when* Spc *is monoidal.*

Proof. The first part of the pushout product axiom is immediate from Proposition 3.7. Given a pushout diagram

$$\begin{array}{ccc} \mathrm{L}_n(X) \wedge F & \longrightarrow & \mathrm{L}_n(Y) \wedge F \\ \downarrow & & \downarrow \\ \mathrm{L}_n(X) \wedge G & \longrightarrow & P\,, \end{array}$$

it suffices to check that the induced map $P \to \mathrm{L}_n(Y) \wedge G$ is a trivial cofibration when $X \to Y$ is a generating cofibration of spaces and $F \to G$ is a generating trivial cofibration of Γ-spaces.

First note that $\mathrm{L}_n(X)$ and $\mathrm{L}_n(Y)$ are cofibrant. The left vertical map in the pushout diagram is a monomorphism by Lemma 2.3, and a stable equivalence by Proposition 4.13. By Proposition 4.15 the right vertical map is a stable equivalence; the pushout product map is now seen to be a stable equivalence by the 2-out-of-3 property of stable equivalences. □

Proposition 4.17. *The stable model structure on* $\Gamma\,\mathrm{Spc}$ *satisfies the monoid axiom when* Spc *is monoidal.*

Proof. Let $F \to G$ be a trivial cofibration and let H be a Γ-space. The induced map $F \wedge H \to G \wedge H$ is a monomorphism by Lemma 2.3, and we claim that the cofibre $(G/F) \wedge H$ is stably contractible, which by 4.14 implies that $F \wedge H \to G \wedge H$ is a stable equivalence. First take a cofibrant replacement $H_c \to H$. Since $* \to G/F$ is a stable equivalence, $(G/F) \wedge H_c$ is stably contractible by Proposition 4.13, and also stably equivalent to $(G/F) \wedge H$, which proves the claim.

By Proposition 4.15, it remains to show that a transfinite composition of stable equivalences is a stable equivalence. Note first that homotopy groups of Γ-spaces commute with filtered colimits, since this is true for spectra of simplicial sets and sheafification is exact. A transfinite composition $F_0 \to \mathrm{colim}_\alpha F_\alpha$, where each $F_\alpha \to F_{\alpha+1}$ is a stable equivalence, induces an isomorphism

$$\pi_* F_0 \to \operatorname*{colim}_\alpha \pi_*(F_\alpha) \cong \pi_*(\operatorname*{colim}_\alpha F_\alpha).$$ □

A symmetric spectrum is a spectrum E with a Σ_n-action on each E^n such that the iterated structure maps

$$S^m \wedge E^n \to S^{m-1} \wedge E^{1+n} \to \cdots \to E^{m+n}$$

are $\Sigma_m \times \Sigma_n$-equivariant, where $\Sigma_m \times \Sigma_n$ is identified with a subgroup of Σ_{m+n} in the usual way. Morphisms of symmetric spectra are morphisms of spectra that are equivariant at each level. We denote the category of symmetric spectra by Spt^Σ.

Let $\mathcal{U}\colon \mathrm{Spt}^\Sigma \to \mathrm{Spt}$ denote the forgetful functor, which is right adjoint to a "free symmetric spectrum" functor $\mathcal{F}\colon \mathrm{Spt} \to \mathrm{Spt}^\Sigma$. A map $f\colon E \to F$ of symmetric spectra is a fibration if $\mathcal{U}(f)\colon \mathcal{U}(E) \to \mathcal{U}(F)$ is a fibration of spectra. There are simplicial mapping spaces of symmetric spectra, and weak equivalences of symmetric spectra are those maps f which induce weak equivalences of simplicial sets $\mathrm{Map}(F,H) \to \mathrm{Map}(E,H)$ for all fibrant symmetric spectra H. If $\mathcal{U}(f)$ is a stable equivalence of spectra, then f is a weak equivalence of symmetric spectra, but the converse is not true.

The following theorem is a special case of a result by Hovey [10, 8.7].

Theorem 4.18. *With the above definitions of fibrations and stable equivalences* Spt^Σ *is a cofibrantly generated proper* Spc*-model category, such that*

$$\mathcal{F} : \mathrm{Spt} \rightleftarrows \mathrm{Spt}^\Sigma : \mathcal{U}$$

defines a Quillen equivalence.

As the Σ_n-action on S^n induces an action on $d\bar{F}(S^n)$, the functor Sp factors through the category of symmetric spectra in the sense that we have a commutative diagram

$$\begin{array}{ccc} \Gamma\,\mathrm{Spc} & \xrightarrow{\mathrm{Sp}} & \mathrm{Spt} \\ & {\scriptstyle \mathrm{Sp}^\Sigma}\searrow \quad \nearrow{\scriptstyle \mathcal{U}} & \\ & \mathrm{Spt}^\Sigma\ . & \end{array}$$

Proposition 4.19. *The functor* Sp^Σ *is lax monoidal.*

Proof. The corresponding functor for classical Γ-spaces (which we also denote Sp) is lax monoidal by [19, 3.3]. We can apply this functor sectionwise and conclude, using the fact that $\mathrm{Sp}(F)(U) = \mathrm{Sp}(F(U))$, for a Γ-space F and $U \in \mathcal{C}$. □

Note that Sp^Σ is not strong monoidal since $\mathrm{Sp}^\Sigma(\Gamma^m \wedge \Gamma^n) = \mathrm{Sp}^\Sigma(\Gamma^{mn}) = \mathbb{S}^{\times mn}$, while $\mathrm{Sp}^\Sigma(\Gamma^m) \wedge \mathrm{Sp}^\Sigma(\Gamma^n) = \mathbb{S}^{\times m} \wedge \mathbb{S}^{\times n}$. Nor is Sp^Σ a left Quillen functor, since the symmetric spectrum $\mathrm{Sp}^\Sigma(\Gamma^n) = \mathbb{S}^{\times n}$ is not cofibrant when $n \geqslant 2$.

5. Algebras and modules

A Γ-ring is a monoid in the category of Γ-spaces, i.e., a Γ-space R equipped with a unit map $\mathbb{S} \to R$ and a multiplication map $R \wedge R \to R$ making the usual diagrams commute (see e.g. Mac Lane [18, VII.3].) Given a Γ-ring R, we can consider the category of modules over R. A left R-module is a Γ-space M with an action $R \wedge M \to M$, again making certain obvious diagrams commute, and maps of R-modules are maps of Γ-spaces that respect the action. We let $\Gamma\,\mathrm{Mod}_R$ denote the category of left R-modules. Given a commutative Γ-ring R, we have the category of algebras over R. An R-algebra is a monoid in the category of R-modules, and maps of R-algebras are maps of R-modules respecting the monoid structure. Let $\Gamma\,\mathrm{Alg}_R$ denote the category of R-algebras.

Since Γ Spc satisfies the monoid axiom, we can apply [22, 4.1] and immediately get model structures on the categories of modules and algebras over a monoid. Here we are assuming the stable model structure on Γ Spc. Of course, the result is also true for the strict model structure.

Theorem 5.1. *Suppose the model structure on* Spc *is monoidal, and let R be a* Γ*-ring.*

1. *The category* $\Gamma\,\mathrm{Mod}_R$ *inherits a cofibrantly generated model structure from* Γ Spc*, which is monoidal and satisfies the monoid axiom if R is commutative.*

2. If R is commutative, then the category $\Gamma\,\mathrm{Alg}_R$ inherits a cofibrantly generated model structure from $\Gamma\,\mathrm{Spc}$, and every cofibrant R-algebra is also cofibrant as an R-module.

The model structures in Theorem 5.1 are created by forgetful functors: a map f of R-modules is a weak equivalence (fibration) if and only if its image $\mathcal{U}(f)$ under the forgetful functor $\mathcal{U}\colon \Gamma\,\mathrm{Mod}_R \to \Gamma\,\mathrm{Spc}$ is a weak equivalence (fibration), and similarly for R-algebras.

As an application we now establish some results about the Eilenberg-Mac Lane Γ-spaces, and the correspondence with presheaves of simplicial abelian groups and rings. The following are the presheaf versions of results in Schwede [20]. Let sAbPre be the category of presheaves of simplicial abelian groups, which is symmetric monoidal under a sectionwise tensor product. For a monoid A in sAbPre, let $\mathrm{sModPre}_\mathrm{A}$ be the category of A-modules, and for a commutative monoid B, let $\mathrm{sAlgPre}_\mathrm{B}$ be the category of B-algebras. A map in sAbPre is a weak equivalence (fibration) if the underlying map of spaces is a local weak equivalence (fibration.) In the same way, weak equivalences and fibrations in $\mathrm{sModPre}_\mathrm{A}$ and $\mathrm{sAlgPre}_\mathrm{B}$ are defined on the underlying spaces.

Let

$$\mathbb{Z}\colon\ \mathrm{Spc} \rightleftarrows \mathrm{sAbPre}\colon \mathcal{U} \tag{5.1}$$

be the adjoint pair consisting of the free simplicial abelian presheaf functor $\mathbb{Z}$ and the forgetful functor $\mathcal{U}$.

Theorem 5.2. *With the above definitions of weak equivalences and fibrations the category* sAbPre *is a cofibrantly generated model category, with generating cofibrations $\mathbb{Z}(I)$ and generating trivial cofibrations $\mathbb{Z}(J)$, such that the adjoint pair* (5.1) *is a Quillen pair. If* Spc *is monoidal, then the categories* $\mathrm{sModPre}_\mathrm{A}$ *and* $\mathrm{sAlgPre}_\mathrm{B}$ *are cofibrantly generated model categories as well.*

Proof. To prove the statement for sAbPre, we apply a general result found in [8, 11.3.2] concerning lifts of model structures via adjoint functors. To apply this result, we need to check that the maps in $\mathbb{Z}(J)$-cell are weak equivalences. Since local weak equivalences commute with filtered colimits, it suffices to check that a pushout in sAbPre of a map in $\mathbb{Z}(J)$ is a weak equivalence. The functor $\mathbb{Z}$ preserves monomorphisms and weak equivalences by [15, 2.1]; we proceed by showing that a pushout of a monomorphic weak equivalence is a weak equivalence.

We need the fact that a map $f\colon A \to B$ of simplicial abelian presheaves is a weak equivalence if and only if the induced map $(Mf)_*\colon H_*(MA) \to H_*(MB)$ of homology sheaves is an isomorphism, where M denotes the Moore complex functor. Let

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ B & \longrightarrow & D \end{array}$$

be a pushout diagram in sAbPre, where f is trivial cofibration. Applying M gives a pushout diagram of presheaves of chain complexes, and the homomorphism Mf

induces the following long exact sequence of homology sheaves.

$$\cdots \to H_n(MA) \to H_n(MB) \to H_n(\operatorname{coker} Mf) \to H_{n-1}(MA) \to \dots .$$

Since f is a weak equivalence, $H_*(\operatorname{coker} Mf) = 0$, which implies $H_*(\operatorname{coker} Mg) = 0$, and the corresponding long exact sequence for Mg implies that g is a weak equivalence.

The model structures for $\mathrm{sModPre_A}$ and $\mathrm{sAlgPre_B}$ follow from [22, 4.1]. □

Let A be a presheaf of simplicial abelian groups. The Eilenberg-Mac Lane Γ-space HA associated to A is defined as follows. For each n^+ in Γ let $HA(n^+) = A^{\times n}$, and for each map $f\colon n^+ \to m^+$ let the induced map $HA(n^+) \to HA(m^+)$ be defined by

$$(a_1, \dots, a_n) \mapsto (\sum_{f(i)=1} a_i, \dots, \sum_{f(i)=m} a_i)$$

in each section. A map of simplicial abelian presheaves $A \to B$ induces a map of Γ-spaces $HA \to HB$. Note that HA is very special, and its associated spectrum is a generalized Eilenberg-Mac Lane spectrum for A since $\pi_n(HA) = \pi_n(HA(1^+)) = \pi_n(A)$.

A functor L in the opposite direction is described as follows. Let F be a Γ-space, and consider the map

$$p_{1*} + p_{2*} - \nabla_* \colon \widetilde{\mathbb{Z}}F(2^+) \to \widetilde{\mathbb{Z}}F(1^+)\,, \tag{5.2}$$

where p_1 and p_2 are the two projections $2^+ \to 1^+$ in Γ, ∇ is the fold map, and $\widetilde{\mathbb{Z}}$ denotes the reduced free simplicial abelian presheaf associated to a space. The value of L on F is now defined to be the cokernel of (5.2).

The following result is just a sectionwise application of [20, 1.2].

Lemma 5.3. *The functor L is strong symmetric monoidal, while H is lax symmetric monoidal. There is an adjunction*

$$L : \Gamma\,\mathrm{Spc} \rightleftarrows \mathrm{sAbPre} : H\,.$$

Both L and H preserve modules, rings, and commutative rings. Let A be a presheaf of simplicial rings and B be a presheaf of commutative simplicial rings. The functors L and H induce adjunctions

$$L : \Gamma\,\mathrm{Mod}_{HA} \rightleftarrows \mathrm{sModPre_A} : H$$
$$L : \Gamma\,\mathrm{Alg}_{HB} \rightleftarrows \mathrm{sAlgPre_B} : H\,.$$

Lemma 5.4. *All three adjunctions in Lemma 5.3 are Quillen adjunctions.*

Proof. Let us consider the first adjunction; the results for the other two follow by the same argument. Since trivial fibrations of spaces are closed under finite products, H takes trivial fibrations of simplicial abelian presheaves to strictly trivial fibrations of Γ-spaces, which coincide with the stably trivial fibrations of Γ-spaces.

The functor H also takes fibrations of simplicial abelian presheaves to strict fibrations of Γ-spaces between stably fibrant Γ-spaces, which coincide with stable fibrations between stably fibrant Γ-spaces. □

Theorem 5.5. *Let A be a presheaf of simplical rings. Then the adjoint functors H and L constitute a Quillen equivalence between the categories of presheaves of simplicial A-modules and HA-modules.*

Proof. The following proof is an adaption of Schwede's argument given in [20, 4.2]. The functor H preserves weak equivalences, and detects weak equivalences since a stable equivalence $HM \to HN$ is a strict equivalence, and, in particular, $M = HM(1^+) \to HN(1^+) = N$ is a local weak equivalence. It remains to show that for every cofibrant HA-module M the unit map $M \to HL(M)$ is a stable equivalence.

We first consider Γ-spaces of the form $HA \wedge X$, where X is a space, and we claim that the presheaf map $\pi_*^p(HA \wedge X) \to \pi_*^p(HL(HA \wedge X))$ is a sectionwise isomorphism. After evaluating in sections, we are led to consider the map

$$\pi_*(HA(U) \wedge K) \to \pi_*(HL(HA \wedge K)(U))$$

as a natural transformation of functors of the simplicial set K. But this is easily seen to be an isomorphism for the case $K = S^0$, and both functors are homology theories with coefficients in A, since $L(HA \wedge K)(U)$ is just the free $A(U)$-module generated by K. Thus the map is an isomorphism for all K and in particular for $X(U)$.

The map $\Gamma^1 \wedge n^+ \to \Gamma^n$ induced by the n projections $n^+ \to 1^+$ is a stable equivalence, since the induced map of spectra is just the canonical inclusion $\mathbb{S}^{\vee n} \to \mathbb{S}^{\times n}$. This implies that $F \wedge n^+ \cong F \wedge \Gamma^1 \wedge n^+$ is stably equivalent to $F \wedge \Gamma^n$ for all Γ-spaces F. The composite functor HL preserves weak equivalences between cofibrant objects, so the unit map of $HA \wedge X \wedge \Gamma^n$ is a stable equivalence by the case already proved.

Let M be a cofibrant HA-module, i.e., a retract of a colimit $\operatorname{colim}_{\alpha<\gamma} M_\alpha$, where γ is an ordinal and the maps $M_\alpha \to M_{\alpha+1}$ are pushouts of generating cofibrations in $\Gamma\operatorname{Mod}_{HA}$. The generating cofibrations in $\Gamma\operatorname{Mod}_{HA}$ are of the form

$$HA \wedge X \wedge \Gamma^n \to HA \wedge Y \wedge \Gamma^n\,,$$

where $X \to Y$ is a (generating) cofibration of spaces. If we have a pushout diagram of the form

$$\begin{array}{ccc} HA \wedge X \wedge \Gamma^n & \longrightarrow & M_\alpha \\ \downarrow & & \downarrow \\ HA \wedge Y \wedge \Gamma^n & \longrightarrow & M_{\alpha+1} \end{array}$$

and assume that the map $M_\alpha \to HL(M_\alpha)$ is a stable equivalence, we can use the first part and the gluing lemma (see e.g. [7, II.8.12]) to show that the map $M_{\alpha+1} \to HL(M_{\alpha+1})$ is a stable equivalence. Now the induced map

$$\operatorname*{colim}_{\alpha<\gamma} M_\alpha \to \operatorname*{colim}_{\alpha<\gamma} HL(M_\alpha)$$

is a stable equivalence, and $\operatorname{colim} HL(M_\alpha)$ is stably equivalent to $HL(\operatorname{colim} M_\alpha)$ since L preserves colimits and

$$\pi_*(\operatorname{colim} HA_\alpha) \cong \operatorname{colim} \pi_*(HA_\alpha) \cong \operatorname{colim} \pi_*(A_\alpha) \cong \pi_*(\operatorname{colim} A_\alpha) \cong \pi_* H(\operatorname{colim} A_\alpha)\,.$$

Finally, since M is a retract of $\operatorname{colim} M_\alpha$, the unit map $M \to HL(M)$ is also a stable equivalence. □

Theorem 5.6. *Let B be a presheaf of commutative simplicial rings. Then the adjoint functors H and L are a Quillen equivalence between the categories of presheaves of simplicial B-algebras and HB-algebras.*

Proof. Since every cofibrant HB-algebra is cofibrant as an HB-module, the proof of Theorem 5.5 applies. □

6. Appendix

6.1. Simplicial spaces

Given a simplicial space X, i.e., a bisimplicial presheaf, we obtain a space $X_{m,*}$ by fixing the first simplicial degree m. We say that a map $X \to Y$ is a strict equivalence if $X_{m,*} \to Y_{m,*}$ is a local weak equivalence for all m.

Proposition 6.1. *Let $X \to Y$ be a strict equivalence of simplicial spaces. Then the induced diagonal map $dX \to dY$ is a local weak equivalence of spaces.*

Proof. The result only depends on the weak equivalences on simplicial presheaves, so we are free to choose the local injective model structure where every object is cofibrant. Now the proof in [7, IV.1.7] for bisimplicial sets carries over, mutatis mutandis. □

6.2. Enriched left Bousfield localization

Here we summarize the theory of enriched left Bousfield localization as developed in Barwick [2]. We will ignore the set-theoretic details that appear in these statements; they are treated carefully in Barwick's paper.

Definition 6.2. Let $\mathcal{V}$ be a symmetric monoidal model category and $\mathcal{M}$ a $\mathcal{V}$-model category. Suppose Σ is a set of morphisms in $\mathcal{M}$. A left Bousfield localization of $\mathcal{M}$ with respect to Σ enriched over $\mathcal{V}$ is a $\mathcal{V}$-model category $L_{\Sigma/\mathcal{V}}\mathcal{M}$, equipped with a left Quillen $\mathcal{V}$-functor $\mathcal{M} \to L_{\Sigma/\mathcal{V}}\mathcal{M}$ that is initial among left Quillen $\mathcal{V}$-functors $L\colon \mathcal{M} \to \mathcal{N}$ to $\mathcal{V}$-model categories $\mathcal{N}$ such that Lf is a weak equivalence in $\mathcal{N}$ for all f in Σ.

Definition 6.3. Let $\mathcal{V}$, $\mathcal{M}$ and Σ be as in Definition 6.2.

- An object Z in $\mathcal{M}$ is *$\Sigma/\mathcal{V}$-local* if it is fibrant, and for any morphism $A \to B$ in Σ the morphism
$$\mathcal{V}\mathrm{Hom}(B_c, Z) \to \mathcal{V}\mathrm{Hom}(A_c, Z)$$
is a weak equivalence in $\mathcal{V}$.
- A morphism $A \to B$ in $\mathcal{M}$ is a *$\Sigma/\mathcal{V}$-local equivalence* if for any $\Sigma/\mathcal{V}$-local object Z in $\mathcal{M}$, the morphism
$$\mathcal{V}\mathrm{Hom}(B_c, Z) \to \mathcal{V}\mathrm{Hom}(A_c, Z)$$
is a weak equivalence in $\mathcal{V}$.

The following result is proved in [2, 3.18].

Theorem 6.4. *Suppose that $\mathcal{V}$ is a combinatorial monoidal model category and $\mathcal{M}$ is a left proper and combinatorial $\mathcal{V}$-model category. Suppose further that the generating cofibrations and generating trivial cofibrations in $\mathcal{V}$ and $\mathcal{M}$ all have cofibrant domains. Let Σ be a set of morphisms in $\mathcal{M}$. Then the left Bousfield localization of $\mathcal{M}$ with respect to Σ enriched over $\mathcal{V}$ exists, and it has the following properties:*

- *As a category, $L_{\Sigma/\mathcal{V}}\mathcal{M}$ is just $\mathcal{M}$.*
- *The model category $L_{\Sigma/\mathcal{V}}\mathcal{M}$ is combinatorial and left proper.*
- *The cofibrations in $L_{\Sigma/\mathcal{V}}\mathcal{M}$ are the same as those of $\mathcal{M}$.*
- *The fibrant objects in $L_{\Sigma/\mathcal{V}}\mathcal{M}$ are the $\Sigma/\mathcal{V}$-local objects in $\mathcal{M}$.*
- *The weak equivalences in $L_{\Sigma/\mathcal{V}}\mathcal{M}$ are the $\Sigma/\mathcal{V}$-local equivalences.*

References

[1] J. Adámek and J. Rosický, *Locally presentable and accessible categories*, London Mathematical Society Lecture Note Series, vol. **189**, Cambridge University Press, Cambridge, 1994.

[2] C. Barwick, *On (enriched) left Bousfield localizations of model categories*, preprint at arXiv:0708.2067.

[3] B.A. Blander, *Local projective model structures on simplicial presheaves*, K-Theory **24** (2001), no. 3, 283–301.

[4] A.K. Bousfield and E.M. Friedlander, *Homotopy theory of* Γ*-spaces, spectra, and bisimplicial sets*, Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II, 1978, pp. 80–130.

[5] B. Day, *On closed categories of functors*, Lecture Notes in Math., vol. **137**, Springer-Verlag, New York, 1970, Reports of the Midwest Category Seminar, IV.

[6] D. Dugger, *Combinatorial model categories have presentations*, Adv. Math. **164** (2001), no. 1, 177–201.

[7] P.G. Goerss and J.F. Jardine, *Simplicial homotopy theory*, Progress in Mathematics, vol. **174**, Birkhäuser Verlag, Basel, 1999.

[8] P.S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. **99**, American Mathematical Society, Providence, RI, 2003.

[9] M. Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. **63**, American Mathematical Society, Providence, RI, 1999.

[10] M. Hovey, *Spectra and symmetric spectra in general model categories*, J. Pure Appl. Algebra **165** (2001), no. 1, 63–127.

[11] D.C. Isaksen, *Flasque model structures for simplicial presheaves*, K-Theory **36** (2005), no. 3–4, 371–395 (2006).

[12] J.F. Jardine, *Simplicial presheaves*, J. Pure Appl. Algebra **47** (1987), no. 1, 35–87.

[13] J.F. Jardine, *Stable homotopy theory of simplicial presheaves*, Canad. J. Math. **39** (1987), no. 3, 733–747.

[14] J.F. Jardine, *Generalized étale cohomology theories*, Progress in Mathematics, vol. **146**, Birkhäuser Verlag, Basel, 1997.

[15] J.F. Jardine, *Presheaves of chain complexes*, K-Theory **30** (2003), no. 4, 365–420.

[16] J.F. Jardine, *Intermediate model structures for simplicial presheaves*, Canad. Math. Bull. **49** (2006), no. 3, 407–413.

[17] M. Lydakis, *Smash products and* Γ*-spaces*, Math. Proc. Cambridge Philos. Soc. **126** (1999), no. 2, 311–328.

[18] S. Mac Lane, *Categories for the working mathematician*, Second Edition, Graduate Texts in Mathematics, vol. **5**, Springer-Verlag, New York, 1998.

[19] M.A. Mandell, J.P. May, S. Schwede, and B. Shipley, *Model categories of diagram spectra*, Proc. London Math. Soc. (3) **82** (2001), no. 2, 441–512.

[20] S. Schwede, *Stable homotopical algebra and* Γ*-spaces*, Math. Proc. Cambridge Philos. Soc. **126** (1999), no. 2, 329–356.

[21] G. Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312.

[22] S. Schwede and B.E. Shipley, *Algebras and modules in monoidal model categories*, Proc. London Math. Soc. (3) **80** (2000), no. 2, 491–511.

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