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THE RING SPECTRUM $P(n)$ FOR THE PRIME 2

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Abstract

This paper is a companion to the Boardman–Wilson paper on the ring spectrum $P(n)$. When the prime is 2, this spectrum is not commutative, which introduces several complications. Here, we supply the necessary details of the relevant Hopf algebroids and Hopf ring for this case.

1. Introduction

It is well known that the ring spectrum $P(n)$ for the prime 2 is not commutative. The purpose of this paper is to deal with the resulting complications by supplying the details that were deferred from [**BW07**]. While some results carry over from the odd prime case (though sometimes non-obviously), other results require extensive modification. We adopt much notation from [ibid.].

We recall that the Brown–Peterson spectrum BP for the prime 2 has the coefficient ring $BP_* = \mathbb{Z}_{(2)}[v_1, v_2, v_3, \dots]$, a polynomial ring over $\mathbb{Z}_{(2)}$ (the integers localized at 2) on generators v_i in degree $2(2^i - 1)$. The ring spectrum $P(n)$ is constructed [**BW07**, §2] from BP to have the coefficient ring

$$P(n)_* = BP_*/I_n = \mathbb{F}_2[v_n, v_{n+1}, v_{n+2}, \dots],$$

where I_n denotes the ideal $(2, v_1, v_2, \dots, v_{n-1})$ and $0 < n < \infty$. (We are not concerned with the commutative ring spectra $P(0) = BP$ and $P(\infty) = H(\mathbb{F}_2)$.) We work in characteristic 2. Tensor products are taken over $P(n)_*$. Because it occurs frequently, we find it useful to write $N = 2^n - 1$.

As most of the work is in homology, we use *homology* degrees throughout (unlike [**BJW95**]): elements of the coefficient group $P(n)_i$ and homology classes in $P(n)_i(X)$ have degree i, which forces cohomology classes in $P(n)^i(X)$ to have degree $-i$.

In §2, we verify, following Nassau [**Na02**], that there are exactly two good multiplications on $P(n)$ and discuss their properties. In §3, we examine what happens to the Künneth formula and universal coefficient theorem in some generality. In §4, we specialize to $P(n)$; most standard results survive unchanged, *but not all*.

In §5, we recall from [**BW07**] the Hopf algebroid Γ that encodes the stable operations in $P(n)$-cohomology, the bigraded Hopf algebroid A^*_* that encodes the additive operations, and the Hopf ring that encodes the unstable operations. This Hopf ring

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is not commutative, which makes it necessary in §6 to be more precise about its structure. In §7, we prove the theorems stated in §6.

In §8, we develop a nonadditive idempotent cohomology operation to complete a proof in [**BW07**]. To establish its existence, in §9 we develop a concrete description of the Hopf ring.

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2. The spectrum $P(n)$

Here, we work in the graded stable homotopy category Stab_*. The construction [**BW07**, §2] of $P(n)$ makes it canonically a BP-module, equipped with an action of the exterior algebra $E(Q_0, Q_1, \ldots, Q_{n-1})$ over $P(n)_*$, where $Q_i\colon P(n) \to P(n)$ is a map of BP-modules of degree $-(2^{i+1}-1)$. The most important of these, Q_{n-1}, has degree $-N = -(2^n-1)$ and is easily described [**Na02**] in terms of the exact triangle

$$P(n-1) \xrightarrow{v_{n-1}} P(n-1) \xrightarrow{\rho} P(n) \xrightarrow{\delta} P(n-1)$$

as the composite $Q_{n-1} = \rho \circ \delta$. Given a multi-index $I = (i_0, i_1, \ldots i_{n-1})$, where each i_r is 0 or 1, we write Q^I for the composite $Q_0^{i_0} \circ Q_1^{i_1} \circ \cdots \circ Q_{n-1}^{i_{n-1}}$. If we put $|I| = -\deg Q^I$, we see that

$$|I| = \sum_{r=0}^{n-1} i_r(2^{r+1}-1) \leqslant 2^{n+1} - 2 - n = 2N - n < 2N = \deg v_n. \tag{1}$$

We have the canonical unit $\eta\colon S \to BP \to P(n)$, where S denotes the sphere spectrum. As in [**BW07**], we choose a fixed multiplication $\phi\colon P(n) \wedge P(n) \to P(n)$ that satisfies all the axioms 2.1 of [ibid.] except commutativity; this includes the derivation formula

$$Q_i \circ \phi = \phi \circ (Q_i \wedge \mathrm{id} + \mathrm{id} \wedge Q_i)\colon P(n) \wedge P(n) \longrightarrow P(n). \tag{2}$$

Since ϕ is associative, the iterated multiplication $\phi_k\colon P(n) \wedge P(n) \wedge \cdots \wedge P(n) \to P(n)$ is well defined for $k \geqslant 3$.

We need the *opposite* ring spectrum $\overline{P(n)}$, which is the same spectrum $P(n)$ but equipped with the opposite multiplication

$$\bar{\phi}\colon P(n) \wedge P(n) \xrightarrow{T} P(n) \wedge P(n) \xrightarrow{\phi} P(n),$$

where T denotes the switch map. Mironov [**Mi78**, Thm. 4.7] computed it explicitly as

$$\bar{\phi} = \phi \circ T = \phi + v_n\, \phi \circ (Q_{n-1} \wedge Q_{n-1})\colon P(n) \wedge P(n) \longrightarrow P(n). \tag{3}$$

Consequently, whenever two factors $P(n)$ are interchanged, this equation introduces an extra term. From now on, we write $Q = Q_{n-1}$, as the other Q_i's are of lesser interest. It follows immediately from equation (2) that Q_i is also a derivation for $\bar{\phi}$,

$$Q_i \circ \bar{\phi} = \bar{\phi} \circ (Q_i \wedge \mathrm{id} + \mathrm{id} \wedge Q_i)\colon P(n) \wedge P(n) \longrightarrow P(n). \tag{4}$$

We next verify that there are exactly two good multiplications on $P(n)$, of which neither appears to be preferred. For $n \geqslant 3$, there are many uninteresting nonassociative multiplications on $P(n)$, such as

$$\phi + v_3\, \phi \circ (Q^{1,1,0} \wedge Q^{0,1,1})\colon P(3) \wedge P(3) \longrightarrow P(3).$$

Proposition 2.1. *There are exactly two associative BP-bilinear multiplications on $P(n)$ that have η as unit, namely ϕ and $\bar{\phi}$.*

Proof. By [**BW07**, Lemma 2.2(ii)], any BP-bilinear multiplication on $P(n)$ that has η as unit has the form

$$\phi' = \phi + \sum_{I,J} c_{I,J}\, \phi \circ (Q^I \wedge Q^J)\colon P(n) \wedge P(n) \longrightarrow P(n)$$

with $c_{I,J} \in P(n)_*$, where all multi-indices I etc. are understood to be nonzero. (The term ϕ must be present to make η a unit, as $Q_i \circ \eta = 0$ trivially. By (1), there are no terms with $c_{I,0}$ or $c_{0,J}$; further, each $c_{I,J}$ must be v_n or zero.)

With the help of (2), we expand the iterated product of ϕ' in the standard form [**BW07**, (2.4)] as

$$\begin{aligned}\phi' \circ (\phi' \wedge \mathrm{id}) = \phi_3 &+ \sum_{I,J} c_{I,J}\, \phi_3 \circ (Q^I \wedge Q^J \wedge \mathrm{id} + Q^I \wedge \mathrm{id} \wedge Q^J \\ &\qquad\qquad + \mathrm{id} \wedge Q^I \wedge Q^J) \\ &+ \sum_{I,J,K} c_{I+J,K}\, \phi_3 \circ (Q^I \wedge Q^J \wedge Q^K) \quad (I \text{ and } J \text{ disjoint}) \\ &+ \text{terms with } v_n^2,\end{aligned} \tag{5}$$

where we call I and J *disjoint* if Q^I and Q^J have no common factor. Similarly, $\phi' \circ (\mathrm{id} \wedge \phi')$ has the same form, except that the second sum is replaced by

$$\sum_{I,J,K} c_{I,J+K}\, \phi_3 \circ (Q^I \wedge Q^J \wedge Q^K) \quad (J \text{ and } K \text{ disjoint}). \tag{6}$$

For ϕ' to be associative, we must have $c_{I+J,K} = c_{I,J+K}$ for all I, J, K.

Suppose $c_{L,M} = v_n$. Then L and M cannot be disjoint, because if they were, (1) would imply $|L| + |M| = |L + M| < \deg v_n$. Suppose Q^L and Q^M have the common factor Q_k. We claim that $Q^L = Q_k$; otherwise, $Q^L = Q_k \circ Q^P$ with $P \neq 0$, and the term $v_n\, \phi_3 \circ (Q^P \wedge Q_k \wedge Q^M)$ would appear in (5) but not in (6). Similarly, $Q^M = Q_k$. Finally, to make $|L| + |M| = \deg v_n$, we need $k = n - 1$, which forces $\phi' = \bar{\phi}$. □

Complex conjugation

There is an alternate explanation for why $P(n)$ should behave differently when $p = 2$.

Given any complex-oriented ring spectrum E with a natural first Chern class $x(\xi) \in E^2(-)$ for complex line bundles, one can define the *complex conjugate* Chern class $\bar{x}(\xi) = -x(\bar{\xi})$, where $\bar{\xi}$ denotes the complex conjugate line bundle (with a sign to make $\bar{x}$ a strict Chern class, if we are not in characteristic 2). Since $\xi \otimes \bar{\xi}$ is a trivial line bundle, we have the formula

$$\bar{x}(\xi) = -[-1](x(\xi)),$$

where the series $[-1](x)$ is defined in terms of the formal group law $F(x, y)$ of $x(-)$ by $F(x, [-1](x)) = 0$.

For BP, taking formal logarithms, we have $\log x + \log([-1](x)) = 0$. If p is odd, we have simply $[-1](x) = -x$, because all the exponents in the series $\log x$ are odd; then $\bar{x}(\xi) = x(\xi)$, leaving nothing to discuss. But for $p = 2$, the series $[-1](x)$ is highly nontrivial. Now $P(n)$ inherits its complex orientation and Chern classes from BP, and $[-1](x)$ remains nontrivial for $P(n)$.

Nassau proved [**Na02**, Thm. 3] that the two good multiplications on $P(n)$ are abstractly isomorphic.

Theorem 2.2 (Nassau)**.** *There is a canonical antiautomorphism Ξ of the ring spectrum $P(n)$, i.e. isomorphism $\Xi\colon P(n) \cong \overline{P(n)}$ of ring spectra, which is characterized by $\Xi\, x(\xi) = \bar{x}(\xi)$ in cohomology for any complex line bundle ξ.*

In words, as a complex-oriented ring spectrum, $\overline{P(n)}$ is isomorphic to $P(n)$ equipped with the multiplication ϕ and the conjugate complex orientation.

More generally, we determine all automorphisms and antiautomorphisms. We recall from [**BW07**, Thm. 6.4(ii)] that when $p = 2$, the Hopf algebroid for $P(n)$ is the polynomial ring

$$\Gamma = \overline{P(n)}_*(P(n)) = P(n)_*[a_{(0)}, a_{(1)}, \ldots, a_{(n-1)}, b_{(n+1)}, b_{(n+2)}, \ldots],$$

where the missing elements $b_{(j)}$ are given by

$$a_{(i)}^2 = b_{(i+1)} \qquad (\text{for } 0 \leqslant i \leqslant n-1). \tag{7}$$

As an abelian group, $\overline{P(n)}_*(P(n))$ is identical to $P(n)_*(P(n))$ (and even has the same $P(n)_*$-module structure, as we see in §4). The novelty here is that the multiplication is given by $cd = \phi_*(c \mathbin{\overline{\times}} d)$, where $c \mathbin{\overline{\times}} d$ denotes the cross product in $\overline{P(n)}$-homology.

As in [**Bo95**, (11.1)], we identify a stable operation $r\colon P(n) \to P(n)$ with its associated $P(n)_*$-linear functional $\langle r, -\rangle\colon \Gamma \to P(n)_*$. (We check in Proposition 4.5 that it still *is* linear.) The homomorphism $r_*\colon \Gamma \to \Gamma$ induced by r on $\overline{P(n)}$-homology is also of interest. equation (11.33) of [ibid.] uses the coalgebra structure (ψ_S, ϵ_S) on Γ to express r_* in terms of $\langle r, -\rangle$ as

$$r_*\colon \Gamma \xrightarrow{\psi_S} \Gamma \otimes \Gamma \xrightarrow{\mathrm{id} \otimes \langle r, -\rangle} \Gamma \otimes P(n)_* \cong \Gamma, \tag{8}$$

where we use the *right* action of $P(n)_*$ on Γ. Conversely, we recover $\langle r, -\rangle$ from r_* by equation (11.30) of [ibid.] as

$$\langle r, -\rangle\colon \Gamma \xrightarrow{r_*} \Gamma \xrightarrow{\epsilon_S} P(n)_*. \tag{9}$$

Theorem 2.3. *An automorphism or antiautomorphism r of the ring spectrum $P(n)$ is uniquely determined by the values $\langle r, b_{(j)}\rangle \in P(n)_{2(2^j-1)}$ for $j > n$ and the value $\langle r, b_{(n)}\rangle = 0$ (for an automorphism) or $\langle r, b_{(n)}\rangle = v_n$ (for an antiautomorphism). The values may be chosen arbitrarily.*

Proof of Theorem 2.2. Since $\bar{x}$ is a 2-typical Chern class on BP, there is a unique map of ring spectra $\Xi\colon BP \to BP$ that induces $\Xi\, x = \bar{x}$. For BP, we have $[-1](x) \equiv x + v_n x^{2^n} + \dots \bmod I_n$; this follows from the standard fact [**RW77**, Thm. 3.11(b)] that $[2](x) \equiv v_n x^{2^n} + \dots \bmod I_n$ and the formal group law identity $F([2](x), [-1](x)) = [1](x) = x$. From [**Bo95**, (13.3)] we have $\Xi\, x = \sum_j \langle \Xi, b_j \rangle x^j$. We compare this with $\bar{x} = [-1](x)$ to deduce $\langle \Xi, b_{(n)} \rangle = v_n$; then Theorem 2.3 shows that $\Xi\colon BP \to BP$ descends to an antiautomorphism of $P(n)$. □

Theorem 2.3 will follow immediately from Lemma 2.4, below. For this, we need information on the induced homomorphism $r_*\colon P(n)_* \to P(n)_*$ of homotopy groups. The discussion in [**BW07**, §6.6] shows that for each $k \geqslant n$, the stable main relation $(\mathcal{R}_k)$ reduces to $w_k \equiv v_k 1 \bmod I_k\Gamma$, where $w_k = \eta_R v_k$ and I_k denotes the ideal $(v_n, v_{n+1}, \dots, v_{k-1})$ in $P(n)_*$. Then by [**Bo95**, Prop. 11.22(c)], on homotopy groups

$$r_* v_k = \langle r, w_k \rangle \equiv \langle r, v_k 1 \rangle = v_k \langle r, 1 \rangle \qquad \bmod I_k. \tag{10}$$

Lemma 2.4. *Suppose that $r\colon P(n) \to P(n)$ is a map of spectra (or operation) of degree zero whose linear functional $\langle r, - \rangle\colon \Gamma \to P(n)_*$ satisfies:*

(i) $\langle r, 1 \rangle = 1$*;*

(ii) $\langle r, b^{J+K} \rangle = \langle r, b^J \rangle \langle r, b^K \rangle$ *for all multi-indices J and K;*

(iii) $\langle r, a^I b^J \rangle = 0$ *whenever I is nonzero.*

(a) *If $\langle r, b_{(n)} \rangle = 0$, r is an automorphism of the ring spectrum $P(n)$, and every automorphism has this form;*

(b) *If $\langle r, b_{(n)} \rangle = v_n$, r is an antiautomorphism of the ring spectrum $P(n)$, and every antiautomorphism has this form.*

Proof. It is clear from diagrams (8) and (9) that r is multiplicative if and only if $\langle r, - \rangle$ is. Because $a_{(i)}$ has odd degree, $\langle r, a_{(i)} \rangle = 0$ for all i. For dimensional reasons, $\langle r, - \rangle$ automatically preserves the relation (7) for $i \leqslant n-2$, leaving only the case $i = n-1$ to check. If $\langle r, b_{(n)} \rangle = 0$, $\langle r, - \rangle$ is multiplicative. Then (10) is enough to guarantee that r is invertible, and we have (a).

For r to be an antiautomorphism, we require the diagram

$$\begin{array}{ccc} P(n) \wedge P(n) & \xrightarrow{r \wedge r} & P(n) \wedge P(n) \\ \downarrow{\scriptstyle \overline{\phi}} & & \downarrow{\scriptstyle \phi} \\ P(n) & \xrightarrow{\quad r \quad} & P(n) \end{array}$$

to commute. We apply $\overline{P(n)}$-homology to obtain the commutative diagram

$$\begin{array}{ccccc} \Gamma \otimes \Gamma & \xrightarrow{r_* \otimes r_*} & \Gamma \otimes \Gamma & \xrightarrow{\epsilon_S \otimes \epsilon_S} & P(n)_* \otimes P(n)_* \\ \downarrow{\scriptstyle \overline{\times}} & & \downarrow{\scriptstyle \overline{\times}} & & \\ \overline{P(n)}_*(P(n) \wedge P(n)) & \xrightarrow{(r \wedge r)_*} & \overline{P(n)}_*(P(n) \wedge P(n)) & & \downarrow{\scriptstyle \cong} \\ \downarrow{\scriptstyle \overline{\phi}_*} & & \downarrow{\scriptstyle \phi_*} & & \\ \Gamma & \xrightarrow{\quad r_* \quad} & \Gamma & \xrightarrow{\quad \epsilon_S \quad} & P(n)_* \end{array}$$

in which we recognize $\epsilon_S \circ r_*$ as $\langle r, -\rangle$ by (9). We evaluate on $c \otimes d \in \Gamma \otimes \Gamma$. The left side of the diagram is *not* the multiplication on Γ; instead, it sends $c \otimes d$ to $\bar{\phi}_*(c \mathbin{\overline{\times}} d) = cd + v_n(Q_*c)(Q_*d)$, by (3). We deduce that $\langle r, -\rangle$ must satisfy the identity

$$\langle r, cd\rangle + v_n\langle r, (Q_*c)(Q_*d)\rangle = \langle r, c\rangle\langle r, d\rangle$$

for all c and d. We take $c = d = a_{(n-1)}$; since $Q_*a_{(n-1)} = 1$ (see Proposition 5.1), we must have $\langle r, b_{(n)}\rangle = v_n$. Then (10) shows that r is invertible, to complete (b). □

3. Noncommutative ring spectra

In the following section, we consider what effects the noncommutativity of $P(n)$ for $p = 2$ has on standard results. As preparation, in this section we find it useful for clarity to be vastly more general. *We assume that E is a ring spectrum with characteristic 2, associative multiplication $\phi\colon E \wedge E \to E$, and two-sided unit $\eta\colon S \to E$*, without any commutation rules. (The restriction to characteristic 2 is merely for convenience, as it covers our current examples. It can be removed by inserting appropriate signs, of which a few are less than obvious.) The biggest change is that E-(co)homology must now be treated as an *E_*-bimodule*.

We continue to work in the graded stable homotopy category Stab_*, and understand all (co)homology in the *reduced* sense. Tensor products are now taken over E_*. We suppress the associativity equivalences for the smash product $\wedge$, but not those for commutativity $X \wedge Y \simeq Y \wedge X$ or unit $S \wedge X \simeq X \simeq X \wedge S$ equivalences. However, we do identify

$$E_i = \pi_i^S(E) = E^{-i}(S) = \mathit{Stab}_i(S, E) \cong \mathit{Stab}_i(S, E \wedge S) = E_i(S).$$

We economize by recycling the proofs in [**Bo95**], as most of the necessary commutative diagrams are unchanged. (Many of them can also be found in Adams [**Ad69**].) All we have to do is determine which actions are actually used in the old proofs, in the absence of commutativity.

Labeled (co)homology

Because E is not commutative, we must keep track of how the various copies of E are shuffled. To accomplish this, we adopt the Mattaponi[1] naming convention, by decorating each copy of E with a character string to indicate its provenance.

In detail, a given cohomology class $x \in E^*(X)$ becomes a map $x\colon X \to E_x$, a homology class $a \in E_*(X)$ becomes a map $a\colon S \to E_a \wedge X$, and an element $v \in E_*$ becomes a map $v\colon S \to E_v$. Further copies of E are introduced (recursively) by multiplications $\phi\colon E_A \wedge E_B \to E_{AB}$ or the opposite multiplications $\bar{\phi}\colon E_A \wedge E_B \to E_{BA}$, where A and B may be any character strings. For example, $\phi_3\colon E_a \wedge E_b \wedge E_c \to E_{abc}$ is well defined, and generally ϕ_k. (One could be even more general and replace selected copies of E by a left or right E-module, with no extra work.)

[1] In Virginia [**US69**], the Po and Ni Rivers join to form the Poni River, which in turn merges with the Matta River to form the Mattaponi River (pronounced Mattapon-EYE); the south-to-north order is respected throughout.

Cross products

Given cohomology classes $x \in E^*(X)$ and $y \in E^*(Y)$, their *cross product* $x \times y \in E^*(X \wedge Y)$ is

$$x \times y\colon X \wedge Y \xrightarrow{x\wedge y} E_x \wedge E_y \xrightarrow{\phi} E_{xy}.$$

It is clearly associative and natural in X and Y.

Similarly, given homology classes $a \in E_*(X)$ and $b \in E_*(Y)$, their *cross product* $a \times b \in E_*(X \wedge Y)$ is

$$\begin{aligned} a \times b\colon S \simeq S \wedge S &\xrightarrow{a\wedge b} E_a \wedge X \wedge E_b \wedge Y \\ &\simeq E_a \wedge E_b \wedge X \wedge Y \xrightarrow{\phi\wedge X\wedge Y} E_{ab} \wedge X \wedge Y. \end{aligned}$$

It, too, is associative and natural.

Bimodule structure

We next make $E^*(X)$ and $E_*(X)$ into E_*-bimodules.

We define the *left* action of E_* on $E^*(X)$ as usual: given $x \in E^*(X)$ and $v \in E_*$, the element $vx \in E^*(X)$ corresponds to $v \times x \in E^*(S \wedge X)$ under the isomorphism $E^*(X) \cong E^*(S \wedge X)$. We similarly define the *right* action of E_* by making $xv \in E^*(X)$ correspond to $x \times v \in E^*(X \wedge S)$. In particular, $1x = x = x1$, as $1 \in E_*$ is $\eta\colon S \to E$. By associativity of E, $(vx)v' = v(xv')$, and the two actions make $E^*(X)$ an *E_*-bimodule*; further,

$$\times\colon E^*(X) \otimes E^*(Y) \longrightarrow E^*(X \wedge Y) \tag{11}$$

becomes a homomorphism of E_*-bimodules (in the usual sense). Moreover, it remains continuous in the profinite topology on cohomology (see [**Bo95**, Defn. 4.9]) defined by filtering $E^*(X)$ by the subbimodules

$$F^\alpha E^*(X) = \mathrm{Ker}[i_\alpha^*\colon E^*(X) \longrightarrow E^*(X_\alpha)],$$

where X_α runs over the finite subspectra of X, included by $i_\alpha\colon X_\alpha \subset X$.

Similarly, we make $E_*(X)$ an E_*-bimodule and

$$\times\colon E_*(X) \otimes E_*(Y) \longrightarrow E_*(X \wedge Y) \tag{12}$$

a homomorphism of E_*-bimodules (except that homology is always discrete).

The Künneth Theorem

The proof of Theorem 4.2 in [**Bo95**] extends without change to this general context in the following form.

Theorem 3.1. *Suppose that $E_*(X)$ is free or flat as a right E_*-module or that $E_*(Y)$ is free or flat as a left E_*-module. Then diagram* (12) *is an isomorphism of E_*-bimodules.* □

Twisted cross products

Unfortunately, cross products are not enough. As we already saw in [**BW07**] and §2, we need also the *twisted* cross product $a \mathbin{\overline{\times}} b \in E_*(X \wedge Y)$, defined as

$$\begin{aligned} a \mathbin{\overline{\times}} b \colon S \simeq S \wedge S &\xrightarrow{a \wedge b} E_a \wedge X \wedge E_b \wedge Y \\ &\simeq E_a \wedge E_b \wedge X \wedge Y \xrightarrow{\overline{\phi} \wedge X \wedge Y} E_{ba} \wedge X \wedge Y. \end{aligned}$$

It is no longer a homomorphism of E_*-bimodules; instead, given $v \in E_*$, we find that

(i) $(va) \mathbin{\overline{\times}} b = a \mathbin{\overline{\times}} (bv) \colon S \longrightarrow E_{bva} \wedge X \wedge Y$;

(ii) $(av) \mathbin{\overline{\times}} b = (a \mathbin{\overline{\times}} b)v \colon S \longrightarrow E_{bav} \wedge X \wedge Y$;

(iii) $a \mathbin{\overline{\times}} (vb) = v(a \mathbin{\overline{\times}} b) \colon S \longrightarrow E_{vba} \wedge X \wedge Y$.

(There are also twisted cross products in cohomology that we do not need here.)

Scalar products

Given $x \in E^*(X)$ and $a \in E_*(X)$, Adams [**Ad69**, p. 72] in effect defined their *scalar* (or *Kronecker*) *product* $\langle x, a \rangle \in E_*$ as

$$\langle x, a \rangle \colon S \xrightarrow{a} E_a \wedge X \xrightarrow{\mathrm{id} \wedge x} E_a \wedge E_x \xrightarrow{\overline{\phi}} E_{xa},$$

being careful to end up in E_{xa} in order to preserve the lexical order of x and a. In our context, this ensures that, given $v \in E_*$,

(i) $\langle vx, a \rangle = v\langle x, a \rangle \colon S \longrightarrow E_{vxa}$;

(ii) $\langle xv, a \rangle = \langle x, va \rangle \colon S \longrightarrow E_{xva}$;

(iii) $\langle x, av \rangle = \langle x, a \rangle v \colon S \longrightarrow E_{xav}$.

Equivalently,

$$\langle -, - \rangle \colon E^*(X) \otimes E_*(X) \longrightarrow E_* \tag{13}$$

is another homomorphism of E_*-bimodules. In particular, taking $X = S$, we have $\langle v, w \rangle = vw$ for $v, w \in E_*$.

There is one surprise, caused by the shuffling of factors E. Given also $y \in E^*(Y)$ and $b \in E_*(Y)$, it is essential for our purposes in [**BW07**, Lemma 6.1] to expand $\langle x \times y, a \times b \rangle$. However, $\langle x \times y, a \times b \rangle \colon S \longrightarrow E_{xyab}$ *does not simplify* in this generality, and is not the same as $\langle x, a \rangle \langle y, b \rangle \colon S \longrightarrow E_{xayb}$. But if we mix the cross products, we can show that

$$\langle x \times y, a \mathbin{\overline{\times}} b \rangle = \langle x, \langle y, b \rangle a \rangle \colon S \longrightarrow E_{xyba}. \tag{14}$$

Both sides easily reduce to the same map

$$\begin{aligned} S \simeq S \wedge S &\xrightarrow{a \wedge b} E_a \wedge X \wedge E_b \wedge Y \xrightarrow{\mathrm{id} \wedge x \wedge \mathrm{id} \wedge y} \\ &E_a \wedge E_x \wedge E_b \wedge E_y \simeq E_x \wedge E_y \wedge E_b \wedge E_a \xrightarrow{\phi_4} E_{xyba}. \end{aligned}$$

There is also the *twisted* scalar product $\overline{\langle x, a \rangle} \in E_*$, defined as

$$\overline{\langle x, a \rangle} \colon S \xrightarrow{a} E_a \wedge X \xrightarrow{\mathrm{id} \wedge x} E_a \wedge E_x \xrightarrow{\phi} E_{ax},$$

with ϕ instead of $\bar{\phi}$. Its linearity properties are:

$$\begin{aligned} &\text{(i)} \quad \overline{\langle vx, a\rangle} = \overline{\langle x, av\rangle}\colon S \longrightarrow E_{avx};\\ &\text{(ii)} \quad \overline{\langle xv, a\rangle} = \overline{\langle x, a\rangle}v\colon S \longrightarrow E_{axv};\\ &\text{(iii)} \quad \overline{\langle x, va\rangle} = v\overline{\langle x, a\rangle}\colon S \longrightarrow E_{vax}; \end{aligned} \tag{15}$$

and for $w \in E_*$, $\overline{\langle v, w\rangle} = wv$.

Duality

The advantage of the twisted scalar product is that given $x \in E^*(X)$, (iii) shows that $\overline{\langle x, -\rangle}\colon E_*(X) \to E_*$ is a homomorphism of left E_*-modules, whereas $\langle x, -\rangle$ is not. Denote by Mod_* the graded category of *left* E_*-modules; we define the *duality homomorphism*

$$d = d_X\colon E^*(X) \longrightarrow DE_*(X) = \mathit{Mod}_*(E_*(X), E_*) \tag{16}$$

by $dx = d_X x = \overline{\langle x, -\rangle}\colon E_*(X) \to E_*$.

Equations (15) actually show more, that d is a homomorphism of E_*-bimodules, provided we make the dual $DM = \mathit{Mod}_*(M, E_*)$ of any E_*-bimodule M into an E_*-bimodule by endowing it with the non-obvious actions defined on $f \in DM$ by

$$(vf)m = f(mv), \qquad (fv)m = (fm)v, \tag{17}$$

for any $v \in E_*$ and $m \in M$.

As in [**Bo95**, Defn. 4.8], we define the *dual-finite* topology on the dual DM of any (discrete) left E_*-module M by filtering DM by the submodules

$$F^K DM = \operatorname{Ker}[-|K\colon DM \longrightarrow DK],$$

where K runs over all finitely generated left submodules of M. This topology is useful mainly when M is a free module, in which case DM may be viewed as a cartesian product of copies of E_* (with degree shifts), equipped with the product topology.

Note that the definition of d_X is asymmetric, and that in this context, equation (17) makes DM a *right* E_*-module, filtered by right submodules. Even if M is a bimodule, $F^K DM$ will in general not be a subbimodule; in fact, we do not use the right E_*-action on $E_*(X)$ at all.

Theorem 3.2. *Suppose $E_*(X)$ is a free left E_*-module. Then the homomorphism d_X in* (16) *is a homeomorphism of filtered right E_*-modules. In particular, $E^*(X)$ is complete Hausdorff.*

Proof. We apply the proof of Theorem 9.25 in [**Bo95**], specifically the commutative diagram

$$\begin{array}{ccccc} E^*(X) & \xrightarrow[\cong]{} & E\text{-}\mathit{Mod}_*(E \wedge X, E) & \xrightarrow[\pi_*^S(-)]{} & DE_*(X) \\ & & \cong \downarrow \mathrm{Mor}(g,E) & & \cong \downarrow Dg_* \\ E^*(W) & \xrightarrow{\cong} & E\text{-}\mathit{Mod}_*(E \wedge W, E) & \xrightarrow{\pi_*^S(-)} & DE_*(W) \end{array}$$

where $E\text{-}\mathit{Mod}_*$ denotes the graded category of left E-module spectra, W is a wedge of spheres, the rows are d_X and d_W, and the vertical arrows are induced by an isomorphism $g\colon E\wedge W\to E\wedge X$ of left E-module spectra constructed from a suitable map $f\colon W\to E_f\wedge X$ of spectra. The proof given there still delivers a diagram of homeomorphisms.

All the horizontal arrows are isomorphisms of E_*-bimodules, using the E_*-actions on $DE_*(X)$ supplied by (17) and on $E\text{-}\mathit{Mod}_*(E\wedge X,E)$ by transferring in the obvious way from $E^*(X)$. However, the resulting bijection $g^*\colon E^*(X)\cong E^*(W)$ is *not* an isomorphism of E_*-bimodules, and neither is Dg_*. Given $x\in E^*(X)$, or map $x\colon X\to E_x$, g^*x is the composite

$$g^*x\colon W\xrightarrow{f}E_f\wedge X\xrightarrow{\mathrm{id}\wedge x}E_f\wedge E_x\xrightarrow{\phi}E_{fx}.$$

Given also $v\in E_*$, or map $v\colon S\to E_v$, we compute that $g^*(xv)=(g^*x)v\colon W\to E_{fxv}$, showing that g^* preserves the right E_*-action. In contrast, $v(g^*x)$ is the composite

$$v(g^*x)\colon W\simeq S\wedge W\xrightarrow{\mathrm{id}\wedge f}S\wedge E_f\wedge X\xrightarrow{v\wedge\mathrm{id}\wedge x}E_v\wedge E_f\wedge E_x\xrightarrow{\phi_3}E_{vfx},$$

while $g^*(vx)$ reduces to the same, but with ϕ_3 replaced by

$$E_v\wedge E_f\wedge E_x\xrightarrow{\overline{\phi}\wedge\mathrm{id}}E_{fv}\wedge E_x\xrightarrow{\phi}E_{fvx},$$

which is different in general. Equations (17) show the same for Dg_*, where the homomorphism $g_*\colon E_*(W)\to E_*(X)$ of left E_*-modules fails to preserve the right E_*-action. □

4. Products in $P(n)$-(co)homology

In this section, we specialize the results of the previous section by taking $E=P(n)$. From now on, we work mainly in the homotopy category Ho of *unbased* spaces and use *absolute* homology $P(n)_*(X)$ and cohomology $P(n)^*(X)$ (which may be viewed stably as the *relative* (co)homology of the pair (X^+,o), where X^+ denotes the disjoint union of X and a (new) basepoint o).

As before, we identify the stable map $Q\colon P(n)\to P(n)$ with the stable operation Q on $P(n)$-cohomology. Unstably, the space $\underline{P(n)}_s$ represents the cohomology theory $P(n)^s(-)$ on Ho, so that the operation Q is represented by maps $Q\colon \underline{P(n)}_s\to\underline{P(n)}_{s+N}$.

The companion homology operation Q is defined (stably) on $a\in P(n)_*(X)$ as

$$Qa\colon S\xrightarrow{a}P(n)\wedge X\xrightarrow{Q\wedge\mathrm{id}}P(n)\wedge X.$$

For the twisted cross product, equation (3) immediately yields

$$T_*(a\times b)=b\mathbin{\overline{\times}}a=b\times a+v_n\,Qb\times Qa\qquad(\text{in }P(n)_*(Y\times X))\tag{18}$$

for any $a\in P(n)_*(X)$ and $b\in P(n)_*(Y)$, or equivalently,

$$T_*(b\mathbin{\overline{\times}}a)=a\times b=a\mathbin{\overline{\times}}b+v_n\,Qa\mathbin{\overline{\times}}Qb.\tag{19}$$

For the cup product in $P(n)^*(X)$, we have the commutation rule

$$yx=xy+v_n\,(Qx)(Qy).\tag{20}$$

Modules

As the ring $P(n)_*$ is commutative, there is no algebraic distinction between left $P(n)_*$-modules and right $P(n)_*$-modules. What we actually need is a stronger statement.

Proposition 4.1. *The right action of $P(n)_*$ on $P(n)^*(X)$ and $P(n)_*(X)$ coincides with the left action: $xv = vx$ and $av = va$ for $v \in P(n)_*$, $x \in P(n)^*(X)$ and $a \in P(n)_*(X)$. Moreover, the action of $P(n)_*$ is independent of the choice of the multiplication ϕ or $\bar{\phi}$ on $P(n)$.*

Proof. If we take Y to be a point in (18), we see that

$$T_*(a \times v) = v \times a + v_n\, Qv \times Qa = v \times a,$$

as $Qv = 0$ trivially in $P(n)_*(\text{point}) = P(n)_*$. Hence $av = va$. The analogous formula for cohomology (see [**BW07**, §2.2]) yields $xv = vx$.

By (18), changing ϕ to $\bar{\phi}$ adds the zero term $v_n(Qv)(Qa)$ to va, and similarly for cohomology. □

Products

Proposition 4.1 allows us to dispense with the right action of $P(n)_*$. The bimodule homomorphisms (11) and (12) simply reduce to the usual $P(n)_*$-bilinear homomorphisms in cohomology,

$$\times\colon P(n)^*(X) \otimes P(n)^*(Y) \longrightarrow P(n)^*(X \times Y),$$

and homology,

$$\times\colon P(n)_*(X) \otimes P(n)_*(Y) \longrightarrow P(n)_*(X \times Y).$$

Thus the Künneth formula, Theorem 3.1, reverts to its traditional form, as stated in [**Bo95**, Thm. 4.2].

Theorem 4.2. *If $P(n)_*(X)$ or $P(n)_*(Y)$ is a free or flat $P(n)_*$-module, the cross product induces an isomorphism*

$$\times\colon P(n)_*(X) \otimes P(n)_*(Y) \xrightarrow{\cong} P(n)_*(X \times Y)$$

of $P(n)_*$-modules, even for $p = 2$. □

Proposition 4.3. *The (co)homology operation Q behaves as expected on products:*

(a) *Given $a \in P(n)_*(X)$ and $b \in P(n)_*(Y)$, we have*

$$Q(a \times b) = Qa \times b + a \times Qb. \tag{21}$$

We may replace $\times$ throughout by the twisted cross product $\overline{\times}$.

(b) *Given $x \in P(n)^*(X)$ and $y \in P(n)^*(Y)$, we have*

$$Q(x \times y) = Qx \times y + x \times Qy.$$

Again, we may replace $\times$ throughout by $\overline{\times}$.

Proof. Both parts follow directly from (2) and (4). □

The stable operation Q is automatically additive. More is true.

Corollary 4.4. *The homology operation* $Q\colon P(n)_*(Y) \to P(n)_*(Y)$ *and cohomology operation* $Q\colon P(n)^*(Y) \to P(n)^*(Y)$ *are* $P(n)_*$*-linear homomorphisms.*

Proof. We take X as a point in Proposition 4.3 and $a = v \in P(n)_*(\text{point}) = P(n)_*$. Since $Qv = 0$ trivially, (21) reduces to $Q(v \times b) = v \times Qb$ and hence $Q(vb) = vQb$.
The proof for cohomology is algebraically the same. □

Scalar products also simplify in the same way.

Proposition 4.5. *For* $p = 2$*, given* $x \in P(n)^*(X)$ *and* $a \in P(n)_*(X)$*:*

(a) *The scalar product* $\langle x, a\rangle$ *is* $P(n)_*$*-bilinear;*
(b) *The scalar product* $\langle x, a\rangle$ *is independent of the choice of multiplication* ϕ *or* $\bar{\phi}$ *on* $P(n)$*, and coincides with the twisted scalar product* $\overline{\langle x, a\rangle}$ *defined in §3;*
(c) *The operation* Q *is self-adjoint, in the sense that* $\langle Qx, a\rangle = \langle x, Qa\rangle$*.*

Proof. For (a), Proposition 4.1 allows us to treat (13) as being bilinear in the ordinary sense. Because Q acts trivially on $P(n)_*$, (4) yields $0 = Q\langle x, a\rangle = \langle Qx, a\rangle + \langle x, Qa\rangle$, which gives (c). Then in (b), by (3), the two candidates for $\langle x, a\rangle$ differ by $v_n\langle Qx, Qa\rangle = v_n\langle x, QQa\rangle = 0$. □

We can now deduce the following result, which is Proposition 2.4 in [**BW07**].

Proposition 4.6. *Given elements* $x \in P(n)^*(X)$*,* $y \in P(n)^*(Y)$*,* $a \in P(n)_*(X)$ *and* $b \in P(n)_*(Y)$*, we have*

$$\langle x \times y, a \times b\rangle = \langle x, a\rangle\langle y, b\rangle + v_n\langle x, Qa\rangle\langle Qy, b\rangle. \tag{22}$$

If instead we mix the products, we find

$$\langle x \times y, a \mathbin{\overline{\times}} b\rangle = \langle x, a\rangle\langle y, b\rangle. \tag{23}$$

Proof. Proposition 4.1 allows us to extricate $\langle y, b\rangle$ from its enclosing scalar product in (14) and rewrite that equation as (23). If we replace $a \mathbin{\overline{\times}} b$ by $a \times b$, (19) provides the extra term $v_n\langle x, Qa\rangle\langle y, Qb\rangle$. Then Proposition 4.5(c) allows us to replace $\langle y, Qb\rangle$ by $\langle Qy, b\rangle$ in order to obtain the more natural-looking (22). □

Duality

In view of Propositions 4.5 and 4.1, Theorem 3.2 for $P(n)$ also reverts to its standard form, as stated in [**Bo95**, Thm. 4.14].

Theorem 4.7. *Even for* $p = 2$*, if* $P(n)_*(X)$ *is a free* $P(n)_*$*-module,*

$$d_X\colon P(n)^*(X) \cong DP(n)_*(X) = \mathsf{Mod}_*(P(n)_*(X), P(n)_*)$$

is a homeomorphism of filtered $P(n)_*$*-modules.* □

The Künneth formula in cohomology is a direct consequence of Theorems 4.7 and 4.2, just as in [**Bo95**, Thm. 4.19].

Theorem 4.8. *Even for* $p = 2$*, if* $P(n)_*(X)$ *and* $P(n)_*(Y)$ *are free* $P(n)_*$*-modules, the completed cross product*

$$\times\colon P(n)^*(X) \mathbin{\widehat{\otimes}} P(n)^*(Y) \xrightarrow{\cong} P(n)^*(X \times Y)$$

is a homeomorphism of filtered $P(n)_*$*-modules.* □

5. Three environments

Three flavors of operations in $P(n)$-cohomology were encoded in [**BW07**] as the $P(n)_*$-duals of the three algebraic objects in the diagram of homomorphisms of left $P(n)_*$-modules (among other structures),

$$\overline{P(n)}_*\Big(\underline{P(n)}_*\Big) \xrightarrow{\ q\ } A^*_* = Q\overline{P(n)}_*\Big(\underline{P(n)}_*\Big) \xrightarrow{\ \sigma\ } \Gamma = \overline{P(n)}_*(P(n)). \tag{24}$$

Stable operations are dual to the Hopf algebroid Γ, which has $P(n)_*$-generators $a_{(i)}$ and $b_{(j)}$, described in detail in [ibid., Thm. 6.4]. Part of the structure is the right unit ring homomorphism $\eta_R\colon P(n)_* \to \Gamma$, which makes Γ a right $P(n)_*$-module.

Additive unstable operations are dual to the bigraded Hopf algebroid A^*_*, where A^s_i denotes the indecomposables in degree i of the Hopf algebra $\overline{P(n)}_*\Big(\underline{P(n)}_s\Big)$ and is assigned the *total* degree $i-s$. (It was named Q^*_* in [**BW07**], but we wish to avoid any confusion with the homology homomorphisms Q_* induced by the maps Q in Ho. Worse, the notation $Q\overline{P(n)}_*\Big(\underline{P(n)}_s\Big)$ is ambiguous; here, Q denotes the indecomposables in the Hopf algebra, not the homology operation Q.) It has generators e, $a_{(i)}$, $b_{(j)}$ and $w_k = \eta_R v_k$, described in detail in [ibid., Thm. 7.2]. The stabilization $\sigma\colon A^s_i \to \Gamma_{i-s}$ has (total) degree zero and maps $a_{(i)}$ and $b_{(j)}$ to their namesakes, also $\sigma e = 1$ and $\sigma w_k = w_k = \eta_R v_k \in \Gamma$.

Unstable operations are dual to the Hopf ring $\overline{P(n)}_*\Big(\underline{P(n)}_*\Big)$ (not a Hopf ring in quite the ordinary sense; see §6). It has generators e, $a_{(i)}$, $b_{(j)}$ and $[v_k]$, described in detail in [ibid., Thms. 11.1 and 11.3]; there are also useful elements f_i (see [**BW07**, (10.9)]), b_j, and $[v]$ for any $v \in P(n)_*$. The map $q_s\colon \overline{P(n)}_*\Big(\underline{P(n)}_s\Big) \to A^s_*$ simply quotients out by the $*$-decomposable elements and $P(n)_*$-multiples of 1_s in the Hopf algebra (with a degree shift of $-s$) and maps each generator to its namesake, except that $q[v_k] = w_k$.

Because the homology operation Q and the homology homomorphisms Q_* appear in so many formulae in §6, we record how they act on all three objects in diagram (24). It is also useful to include the linear functional $\langle Q, -\rangle$ that corresponds to the cohomology operation Q. As q and σ respect Q, Q_* and $\langle Q,-\rangle$, we actually work mostly in the Hopf ring. A few complicated proofs have to be deferred to §6.

The stable environment

By Proposition 4.1, $\Gamma = \overline{P(n)}_*(P(n))$ is exactly the same set and $P(n)_*$-module as $P(n)_*(P(n))$; only the multiplication is different. The homology homomorphism Q_* is automatically $P(n)_*$-linear. By Proposition 4.3(a), the homology operation Q is a derivation.

Proposition 5.1. *In the Hopf algebroid* $\Gamma = \overline{P(n)}_*(P(n))$*:*

(a) $Q = Q_*$ *is the* $P(n)_*$*-linear derivation defined on the generators by* $Qa_{(i)} = 0$ *for* $0 \leqslant i < n-1$*,* $Qa_{(n-1)} = 1$*, and* $Qb_{(j)} = 0$*; also* $Qw_k = 0$*.*

(b) *The linear functional* $\langle Q, -\rangle$ *is given on the generators by* $\langle Q, a_{(i)}\rangle = 0$ *for* $0 \leqslant i < n-1$*,* $\langle Q, a_{(n-1)}\rangle = 1$*, and* $\langle Q, b_{(j)}\rangle = 0$*.*

(c) *We have the relation*

$$a_{(i)}^2 = b_{(i+1)} \qquad (\text{for } 0 \leqslant i \leqslant n-1). \tag{25}$$

(d) *The multiplication on* Γ *is commutative and is independent of the choice of multiplication* ϕ *or* $\bar{\phi}$ *on* $P(n)$.

It is *not* obvious that the ring Γ is commutative when $p = 2$.

The additive environment

Here, Q remains a $P(n)_*$-linear derivation, but this is far less obvious for Q_*. Now that $Q\colon A_i^s \to A_{i-N}^s$ and Q_* have different bidegrees, they no longer coincide.

Lemma 5.2. *The homology homomorphism* $Q_*\colon A_i^s \to A_i^{s+N}$ *is a derivation.*

The proof will follow immediately from Theorem 6.4 or Lemma 7.4, by neglecting decomposables.

Proposition 5.3. *In the bigraded Hopf algebroid* A_*^**:*

(a) Q *is the* $P(n)_*$*-linear derivation defined on the generators by the equations* $Qe = 0$, $Qa_{(i)} = 0$ *for* $0 \leqslant i < n-1$, $Qa_{(n-1)} = e$, $Qb_{(j)} = 0$, *and* $Qw_k = 0$.

(b) Q_* *is the* $P(n)_*$*-linear derivation defined on the generators by the equations* $Q_*e = 0$, $Q_*a_{(i)} = 0$ *for* $0 \leqslant i < n-1$, $Q_*a_{(n-1)} = b_{(0)}^{2^{n-1}}$, $Q_*b_{(j)} = 0$, *and* $Q_*w_k = 0$.

(c) *The linear functional* $\langle Q, -\rangle$ *takes the following values:* $\langle Q, e\rangle = 0$, $\langle Q, a_{(i)}\rangle = 0$ *for* $0 \leqslant i < n-1$, $\langle Q, a_{(n-1)}\rangle = 1$, $\langle Q, b_{(j)}\rangle = 0$, *and* $\langle Q, w_k\rangle = 0$.

(d) *We have the relation*

$$a_{(i)}^2 = b_{(i+1)} \qquad (\text{for } 0 \leqslant i \leqslant n-1).$$

(e) *The multiplication on* A_*^* *is commutative and is independent of the choice of multiplication* ϕ *or* $\bar{\phi}$ *on* $P(n)$.

Proof of Proposition 5.1, assuming Proposition 5.3. We copy (c) from (7). For the other parts, we simply apply the stabilization $\sigma\colon A_*^* \to \Gamma$ to Proposition 5.3, noting that $\sigma e = \sigma b_{(0)} = 1$. $\square$

It is even less obvious that the ring A_*^* is commutative for $p = 2$. Although Proposition 5.3 bears a strong formal resemblance to Proposition 5.1, stable proofs do not apply. Instead, we must use Hopf ring methods.

The unstable environment

The Hopf ring $\overline{P(n)}_*\left(\underline{P(n)}_*\right)$ has two multiplications, $c * d = \mu_*(c \mathbin{\overline{\times}} d)$ and $c \circ d = \phi_*(c \mathbin{\overline{\times}} d)$, where the maps $\mu\colon \underline{P(n)}_s \times \underline{P(n)}_s \to \underline{P(n)}_s$ and $\phi\colon \underline{P(n)}_s \times \underline{P(n)}_m \to \underline{P(n)}_{s+m}$ represent, in *Ho*, addition and multiplication in $P(n)$-cohomology. As discussed in [**BW07**, §10.2], the two Cartan formulae (10.23) and (10.36) in [**BJW95**] for $r(x+y)$ and $r(xy)$ continue to hold in this context.

Here, Q remains a derivation, but Q_* does not. It will be convenient to replace the Hopf ring generator $[v_k]$ by $[v_k] - 1$.

Proposition 5.4. *In the Hopf ring* $\overline{P(n)}_*\left(\underline{P(n)}_*\right)$*:*

(a) *Q is a $P(n)_*$-linear derivation for both multiplications,*

$$Q(c * d) = Qc * d + c * Qd, \quad Q(c \circ d) = Qc \circ d + c \circ Qd,$$

and is given on the generators by $Qe = 0$, $Qa_{(i)} = 0$ for $0 \leqslant i < n-1$, $Qa_{(n-1)} = e$, $Qb_{(j)} = 0$, and $Q([v_k] - 1) = 0$; further, $Qf_i = 0$ for $i < 2^n$ and $Qb_j = 0$.

(b) *Q_* is a $*$-homomorphism,*

$$Q_*(c * d) = (Q_*c) * (Q_*d), \tag{26}$$

and is given on the generators by $Q_*e = 0$, $Q_*a_{(i)} = 0$ for $0 \leqslant i < n-1$, $Q_*a_{(n-1)} = b_{(0)}^{\circ 2^{n-1}}$, $Q_*b_{(j)} = 0$, and $Q_*([v_k] - 1) = 0$; further, $Q_*f_i = 0$ for $0 < i < 2^n$, $Q_*b_j = 0$ for $j > 0$, $Q_*[v] = 1_{-s+N}$ for any $v \in P(n)_s$, and $Q_*1_s = 1_{s+N}$.

(c) *The linear functional $\langle Q, - \rangle$ is given on generators by $\langle Q, e \rangle = 0$, $\langle Q, a_{(i)} \rangle = 0$ for $0 \leqslant i < n-1$, $\langle Q, a_{(n-1)} \rangle = 1$, $\langle Q, b_{(j)} \rangle = 0$, and $\langle Q, [v_k] - 1 \rangle = 0$.*

(d) *We have the relation*

$$a_{(i)}^{\circ 2} = b_{(i+1)} \qquad (\text{for } 0 \leqslant i \leqslant n-1).$$

(e) *The effect of changing the choice of multiplication on $P(n)$ from ϕ to $\bar{\phi}$ is to reverse both multiplications: $c * d$ is replaced by $d * c$ and $c \circ d$ by $d \circ c$.*

(f) *Denote by $\overline{\psi}$ the comultiplication on the Hopf ring defined using $\bar{\phi}$ instead of ϕ as the multiplication on $P(n)$. Then the comultiplication is also reversed: if*

$$\psi c = \sum_\alpha c'_\alpha \otimes c''_\alpha, \tag{27}$$

then $\overline{\psi} c = \sum_\alpha c''_\alpha \otimes c'_\alpha$.

Theorem 6.9 and its accompanying examples will show that the Hopf ring is definitely *not* commutative or cocommutative.

Conspicuous by its absence is the formula for $Q_*(c \circ d)$, which is complicated; it will be the subject of Theorem 6.4. The proof of (d) is also deferred to §6. Before we prove the other parts, we need to review the coalgebra structure.

The unstable coalgebra structure

We recall the coalgebra structure (ψ, ϵ) on the Hopf ring (which we warn is unrelated to the coalgebra structures on Γ and A^*_*). We need to know how it relates to Q, Q_* and the two multiplications.

The counit $\epsilon\colon \overline{P(n)}_*\left(\underline{P(n)}_*\right) \to P(n)_*$ is simply induced by the maps $\omega_s\colon \underline{P(n)}_s \to$ point. There are no surprises, $\epsilon(c * d) = (\epsilon c)(\epsilon d)$ and $\epsilon(c \circ d) = (\epsilon c)(\epsilon d)$. Further, $\epsilon Qc = 0$ and $\epsilon Q_* c = \epsilon c$. For the first of these, $\epsilon Qc = \omega_{s*} Qc = Q(\omega_{s*} c) = 0$ by naturality. The second follows from $\omega_{s+N} \circ Q = \omega_s\colon \underline{P(n)}_s \to$ point.

The comultiplication ψ is defined in terms of the diagonal map $\Delta\colon \underline{P(n)}_s \to \underline{P(n)}_s \times \underline{P(n)}_s$ as

$$\psi\colon \overline{P(n)}_*\Big(\underline{P(n)}_s\Big) \xrightarrow{\Delta_*} \overline{P(n)}_*\Big(\underline{P(n)}_s \times \underline{P(n)}_s\Big) \xleftarrow{\cong} \overline{P(n)}_*\Big(\underline{P(n)}_s\Big) \otimes \overline{P(n)}_*\Big(\underline{P(n)}_s\Big),$$

using the Künneth isomorphism in $\overline{P(n)}$-homology. Algebraically, equation (27) is equivalent to

$$\Delta_* c = \sum_\alpha c'_\alpha \,\overline{\times}\, c''_\alpha. \tag{28}$$

To find ψQc, we apply Q to get, using naturality,

$$\Delta_* Qc = Q\Delta_* c = \sum_\alpha Q(c'_\alpha \,\overline{\times}\, c''_\alpha) = \sum_\alpha Qc'_\alpha \,\overline{\times}\, c''_\alpha + \sum_\alpha c'_\alpha \,\overline{\times}\, Qc''_\alpha.$$

The analogue of (28) for Qc now implies that

$$\psi Qc = \sum_\alpha Qc'_\alpha \otimes c''_\alpha + \sum_\alpha c'_\alpha \otimes Qc''_\alpha.$$

For $\psi Q_* c$, we use the trivial equation $\Delta \circ Q = (Q \times Q) \circ \Delta$ to write

$$\Delta_* Q_* c = (Q \times Q)_* \Delta_* c = \sum_\alpha (Q \times Q)_*(c'_\alpha \,\overline{\times}\, c''_\alpha) = \sum_\alpha Q_* c'_\alpha \,\overline{\times}\, Q_* c''_\alpha,$$

which we similarly convert to

$$\psi Q_* c = \sum_\alpha Q_* c'_\alpha \otimes Q_* c''_\alpha.$$

Preliminaries for Proposition 5.4

We begin with results on $a_{(n-1)}$ and f_i.

Lemma 5.5. *We have $Qa_{(n-1)} = e$, $Q_* a_{(n-1)} = b_{(0)}^{\circ 2^{n-1}}$, and $\langle Q, a_{(n-1)}\rangle = 1$. Further, $Q_* f_i = 0$ for $0 < i < 2^n$.*

Proof. We know from [**BW07**, (6.15)] that $P(n)^*(\mathbb{R}P^{2N}) = P(n)_*[t]/(t^{2N+1})$. The Atiyah–Hirzebruch spectral sequence in homology also collapses; $P(n)_*(\mathbb{R}P^{2N})$ is a free $P(n)_*$-module with basis elements y_i for $0 \leqslant i \leqslant 2N$, where y_i is dual to t^i. We may view the cohomology class t as a map $t\colon \mathbb{R}P^{2N} \to \underline{P(n)}_1$; in $\overline{P(n)}$-homology, it induces $t_* y_i = f_i$, and in particular, $t_* y_1 = f_1 = e$ and $t_* y_{2^n} = f_{2^n} = a_{(n-1)}$. (Indeed, this is how $a_{(i)}$ was originally defined in [**Wi84**], or compare [**BJW95**, Prop. 10.5].)

We start from the equation $Qt = t^{2^n}$ in cohomology. (The only alternative is $Qt = 0$, which soon implies that Q is identically zero.) Then dually, by Proposition 4.5(c), in homology we have $Qy_{2^n} = y_1$, and by naturality, $Qa_{(n-1)} = e$.

For any cohomology operation r, we have, from [**BW07**, (10.9)],

$$r(t) = \langle r, 1_1\rangle 1 + \langle r, e\rangle t + \sum_{i=2}^{2N} \langle r, f_i\rangle t^i. \tag{29}$$

If we take $r = Q$, we see that $\langle Q, a_{(n-1)}\rangle = 1$. If we replace r by $r \circ Q$, we get

$$r(t^{2^n}) = r(Qt) = (r \circ Q)(t) = \sum_{i=0}^{2N} \langle r \circ Q, f_i\rangle t^i = \sum_{i=0}^{2N} \langle r, Q_* f_i\rangle t^i,$$

using $\langle r \circ Q, f_i\rangle = \langle Q^* r, f_i\rangle = \langle r, Q_* f_i\rangle$. Since $1 \circ e = 0$, the Cartan formula for $r(t^{2^n})$ simplifies to

$$r(t^{2^n}) = \langle r, 1\rangle + \langle r, e^{\circ 2^n}\rangle t^{2^n} + \text{higher terms.}$$

Since this holds for all r, comparison of the coefficients of t^i yields $Q_* f_i = 0$ for $0 < i < 2^n$ and $Q_* a_{(n-1)} = Q_* f_{2^n} = e^{\circ 2^n} = b_{(0)}^{\circ 2^{n-1}}$. □

The same techniques apply to b_j and $[v]$, with simplifications.

Lemma 5.6. *We have $Qb_j = 0$ for all j and $Q_* b_j = 0$ for $j > 0$. In particular, for $b_{(j)} = b_{2^j}$ we have $Qb_{(j)} = 0$ and $Q_* b_{(j)} = 0$.*

Proof. In this case, $P(n)_*(\mathbb{C}P^\infty)$ is a free $P(n)_*$-module on basis elements $z_j \in P(n)_{2j}(\mathbb{C}P^\infty)$ for $j \geqslant 0$, and $P(n)^*(\mathbb{C}P^\infty) = P(n)_*[[x]]$, with x^j dual to z_j. The map $x\colon \mathbb{C}P^\infty \to \underline{P(n)}_2$ induces $x_* z_j = b_j$. Trivially, $Qz_j = 0$, hence $Qb_j = 0$.

For any cohomology operation r, since $Qx = 0$ trivially, [**BW07**, (10.4)] gives

$$r(0) = r(Qx) = (r \circ Q)(x) = \sum_{j=0}^{\infty} \langle r \circ Q, b_j\rangle x^j = \sum_{j=0}^{\infty} \langle r, Q_* b_j\rangle x^j.$$

We compare with $r(0) = \langle r, 1\rangle 1$, to deduce that $Q_* b_j = 0$ for $j > 0$. □

Lemma 5.7. *For any $v \in P(n)_s$, we have $Q_*[v] = 1_{-s+N}$. In particular, $Q_* 1_{-s} = Q_*[0_s] = 1_{-s+N}$ and $Q_*([v_k] - 1) = 0$.*

Proof. Given $v \in P(n)^{-s}(\text{point}) = P(n)_s$, we have $[v] \in \overline{P(n)}_0\left(\underline{P(n)}_{-s}\right)$. Trivially, $Qv = 0$ in $P(n)^{-s+N}(\text{point})$. Then by [**BW07**, (10.2)],

$$r(0) = r(Qv) = (r \circ Q)(v) = \langle r \circ Q, [v]\rangle = \langle r, Q_*[v]\rangle$$

for any r. Comparing with $r(0) = \langle r, 1\rangle$, we deduce that $Q_*[v] = 1_{-s+N}$. □

Proof of Proposition 5.4, except (d). For (a), from (21) we have

$$\begin{aligned} Q(c * d) &= Q\mu_*(c \mathbin{\overline{\times}} d) = \mu_* Q(c \mathbin{\overline{\times}} d) \\ &= \mu_*(Qc \mathbin{\overline{\times}} d + c \mathbin{\overline{\times}} Qd) = Qc * d + c * Qd, \end{aligned}$$

and similarly for $Q(c \circ d)$.

In (b), (26) depends on the fact that the cohomology operation Q is additive, $Q(x+y) = Qx + Qy$. For the representing map $Q\colon \underline{P(n)}_s \to \underline{P(n)}_{s+N}$, it follows that $Q \circ \mu = \mu \circ (Q \times Q)$; then

$$Q_*(c * d) = Q_*\mu_*(c \mathbin{\overline{\times}} d) = \mu_*(Q \times Q)_*(c \mathbin{\overline{\times}} d) = \mu_*(Q_*c \mathbin{\overline{\times}} Q_*d) = Q_*c * Q_*d.$$

Lemmas 5.5, 5.6 and 5.7 take care of many of the statements. For Qe, we may write $Qe = QQa_{(n-1)} = 0$ (which is trivial except when $n = 1$). The remaining values are trivial, as they lie in groups that are zero.

Part (e) was proved in [**BW07**, §10.1].

For (f), we use $T \circ \Delta = \Delta\colon \underline{P(n)}_k \to \underline{P(n)}_k \times \underline{P(n)}_k$. Equation (28) yields

$$\Delta_*c = T_*\Delta_*c = \sum_\alpha T_*(c'_\alpha \mathbin{\overline{\times}} c''_\alpha) = \sum_\alpha c''_\alpha \times c'_\alpha, \tag{30}$$

with the help of (19). The analogue of (28) for $\overline{\psi}$ translates this into $\overline{\psi}c = \sum_\alpha c''_\alpha \otimes c'_\alpha$, as required. $\square$

Proof of Proposition 5.3 except (e), assuming Lemma 5.2. We apply the quotient map $q\colon \overline{P(n)}_*\left(\underline{P(n)}_*\right) \to A^*_*$ to Proposition 5.4 and use $q([v_k] - 1) = w_k$. $\square$

The two missing items will be proved in §6. The proofs of Propositions 5.3(e) and 5.4(d) will be applications following Proposition 7.2.

6. Structure of the Hopf Ring

Here, we explain in detail what kind of object the Hopf ring $\overline{P(n)}_*\left(\underline{P(n)}_*\right)$ is when $p = 2$. Because the multiplication on $P(n)$ is noncommutative, several of the Hopf ring axioms require modification. Whenever two spaces are shuffled, equation (19) introduces extra terms. We exhibit only the four affected axioms and the three rules for (co)commutation. All the other axioms listed in [**BJW95**, §10] survive unaltered. Several of the more complicated proofs are deferred to the next section.

From now on, as we are almost exclusively concerned with the Hopf ring, we simplify the notation. We write $H(s)$ for the Hopf algebra $\overline{P(n)}_*\left(\underline{P(n)}_s\right)$, $H(s)_i$ for the group of elements of degree i in $H(s)$, and $H(*)$ for the whole Hopf ring $\overline{P(n)}_*\left(\underline{P(n)}_*\right)$.

In this section, we need only limited information from [**BW07**, §11] on the Hopf ring. As noted earlier, we replace each generator $[v_k]$ by $[v_k] - 1$, to make all our $\circ$-generators lie in the augmentation ideal $\operatorname{Ker}\epsilon$.

(31)
- (i) We start with the $\circ$-*generators* e, $a_{(i)}$, $b_{(j)}$ and $[v_k] - 1$;
- (ii) A $\circ$-*monomial* is any $\circ$-product of $\circ$-generators (including the empty product $[1] - 1_0 \in H(0)$);
- (iii) A $*\circ$-*monomial* is any $*$-product of $\circ$-monomials (including the empty product $1_s \in H(s)$ for each s).

To stay inside $\operatorname{Ker}\epsilon$, we follow the convention of [**RW77**], that for any element

$d \in \operatorname{Ker}\epsilon$, we define $d^{\circ 0} = [1] - 1_0$, with the result that $d^{\circ 0} \circ d = d$ still holds. Corollary 6.8 will verify that the $*\circ$-monomials do indeed span the Hopf ring $H(*)$ as a $P(n)_*$-module.

The coalgebra structure

Theorem 6.1. *For any elements $c, d \in H(s) = \overline{P(n)}_*\big(\underline{P(n)}_s\big)$, where ψc is given by equation* (27) *and similarly $\psi d = \sum_\beta d'_\beta \otimes d''_\beta$, we have*

$$\psi(c * d) = \sum_\alpha \sum_\beta c'_\alpha * d'_\beta \otimes c''_\alpha * d''_\beta + \sum_\alpha \sum_\beta v_n\, c'_\alpha * Qd'_\beta \otimes Qc''_\alpha * d''_\beta. \tag{32}$$

Remark 6.2. We could arrange the formula to read simply $\psi(c * d) = (\psi c) * (\psi d)$, which states that ψ preserves $*$-products, by endowing the tensor product $H(s) \otimes H(s)$ with the nonstandard $*$-multiplication suggested by [**BW07**, (2.6)],

$$(c \otimes c') * (d \otimes d') = c * d \otimes c' * d' + v_n c * Qd \otimes Qc' * d'.$$

It will be useful to note that we do not need formulae involving the elements 1_j, because for any $c \in H(s)$ we have, as usual,

$$1_s * c = c = c * 1_s, \quad 1_m \circ c = (\epsilon c)1_{s+m} = c \circ 1_m, \quad \psi 1_s = 1_s \otimes 1_s, \quad \epsilon 1_s = 1.$$

We therefore concentrate attention on elements of the augmentation ideal $\operatorname{Ker}\epsilon$. Because ϵ is the counit, we may rewrite ψc for $c \in \operatorname{Ker}\epsilon$ in the more useful form

$$\psi c = c \otimes 1 + \sum_i c'_i \otimes c''_i + 1 \otimes c, \tag{33}$$

breaking out the two end terms, where $\epsilon c'_i = 0$ and $\epsilon c''_i = 0$ for all i, and similarly

$$\psi d = d \otimes 1 + \sum_j d'_j \otimes d''_j + 1 \otimes d. \tag{34}$$

Then equation (28) is replaced by

$$\Delta_* c = c \mathbin{\overline{\times}} 1 + \sum_i c'_i \mathbin{\overline{\times}} c''_i + 1 \mathbin{\overline{\times}} c. \tag{35}$$

Theorem 6.3. *Given elements c and d of the Hopf ring that satisfy $\epsilon c = 0$ and $\epsilon d = 0$, with ψc and ψd given by* (33) *and and* (34)*, we have*

$$\begin{aligned} \psi(c \circ d) = c \circ d \otimes 1 &+ \sum_i \sum_j c'_i \circ d'_j \otimes c''_i \circ d''_j + 1 \otimes c \circ d \\ &+ \sum_i \sum_j v_n\, c'_i \circ Qd'_j \otimes Qc''_i \circ d''_j. \end{aligned} \tag{36}$$

The previous remark applies equally well here.

Now we can state the formula for $Q_*(c \circ d)$, deferred from §5.

Theorem 6.4. *Given elements c and d of the Hopf ring that satisfy $\epsilon c = 0$ and $\epsilon d = 0$, with ψc and ψd given by* (33) *and* (34)*, the action of Q_* on $\circ$-products is given by*

$$Q_*(c \circ d) = Q_* c \circ d + \sum_i \sum_j (c'_i \circ Q_* d'_j) * (Q_* c''_i \circ d''_j) + c \circ Q_* d. \tag{37}$$

The distributive laws

Because the distributive laws involve some shuffling of factors, they have to be modified.

Theorem 6.5. *For any elements a, b, c in $\overline{P(n)}_*\left(\underline{P(n)}_*\right)$ that satisfy $\epsilon a = 0$, $\epsilon b = 0$ and $\epsilon c = 0$, with ψc given by* (33)*, the distributive laws are*

$$(a * b) \circ c = \sum_i (a \circ c_i') * (b \circ c_i'') + \sum_i v_n \, (a \circ Qc_i') * (Qb \circ c_i'') \tag{38}$$

and

$$c \circ (a * b) = \sum_i (c_i' \circ a) * (c_i'' \circ b) + \sum_i v_n \, (c_i' \circ Qa) * (Qc_i'' \circ b). \tag{39}$$

It is clear that the Hopf ring element $a_{(n-1)}$ causes most of our difficulties.

Definition 6.6. Let us call a $\circ$-monomial or other expression in our $\circ$-generators *wild* if it explicitly contains the element $a_{(n-1)}$, or *tame* if it does not.

Proposition 6.7. *Let c and d be $*\circ$-monomials as in in* (31)(iii)*.*

(a) *$c \circ d$ is a $P(n)_*$-linear combination of $*\circ$-monomials. If c and d are tame, so is $c \circ d$.*

(b) *If $c \neq 1$, ψc has the form* (33)*, where every c_i' and c_i'' is a $P(n)_*$-linear combination of $*\circ$-monomials. If c is tame, so are every c_i' and c_i''.*

(c) *Qc is a sum of $*\circ$-monomials, and is zero if c is tame.*

(d) *$Q_* c$ is a $P(n)_*$-linear combination of $*\circ$-monomials, and is zero if c is tame and $c \neq 1$.*

By (a), the set of linear combinations of $*\circ$-monomials is closed under $\circ$-multiplication as well as $*$-multiplication. We know from [**BW07**] that the $\circ$-generators generate $H(*)$ as a Hopf ring over $P(n)_*$.

Corollary 6.8. *The $*\circ$-monomials* (31)(iii) *span the Hopf ring $H(*)$ as a $P(n)_*$-module.* □

Proof of Proposition 6.7. We assume the previous four theorems (of which none is yet proved). None of the formulae we use introduces an $a_{(n-1)}$ where there was not one before, so tameness is preserved. Part (c) follows directly from Proposition 5.4(a) by induction.

We prove the other three parts together, by induction on degree. In degree zero, we have only the $\circ$-generators $[v_k] - 1$, which are easily handled by the standard formulae $[v] * [v'] = [v + v']$ and $[v] \circ [v'] = [vv']$ for $v, v' \in P(n)_*$, and the formula for $\psi([v]-1)$, below. For (b), we start from the formulae in [**BW07**],

$$\begin{aligned}\psi e &= e\otimes 1 + 1\otimes e,\\ \psi a_{(i)} &= \sum_{j+k=2^{i+1}} f_j\otimes f_k,\\ \psi b_{(i)} &= \sum_{j+k=2^i} b_j\otimes b_k,\\ \psi([v]-1) &= ([v]-1)\otimes 1 + ([v]-1)\otimes([v]-1)\\ &\qquad + 1\otimes([v]-1)\quad (\text{for any } v\in P(n)_*),\end{aligned}\tag{40}$$

where f_j is a $*$-product of selected generators $a_{(k)}$, also (if j is odd) e, and b_j is a (complicated) linear combination of $*\circ$-monomials in the generators $b_{(k)}$ and $[v_k]-1$. Then we use Theorem 6.3 for $\psi(c\circ d)$ and Theorem 6.1 for $\psi(c*d)$. For (a), we use the distributive laws in Theorem 6.5 to expand $(z*c)\circ d$ and $c\circ(z*d)$, where z is a $\circ$-monomial. For (d), we use Theorem 6.4 and Proposition 5.4(b). □

The commutation rules

Neither multiplication is commutative in this Hopf ring environment, nor is the comultiplication cocommutative.

Theorem 6.9. *Given elements of the Hopf ring* $c,d\in\overline{P(n)}_*\left(\underline{P(n)}_*\right)$ *that satisfy* $\epsilon c=0$ *and* $\epsilon d=0$, *with* ψc *and* ψd *given by* (33) *and* (34), *we have the following (co)commutation rules:*

(a) *For the comultiplication* ψ,

$$\sum_i c'_i\otimes c''_i = \sum_i c''_i\otimes c'_i + \sum_i v_n\, Qc''_i\otimes Qc'_i;$$

(b) *For the* $*$-*multiplication,*

$$d*c = c*d + v_n\, Qc*Qd;\tag{41}$$

(c) *For the* $\circ$-*multiplication,*

$$\begin{aligned}d\circ c &= c\circ d + v_n\, Qc\circ Qd + Q_*c\circ Q_*d\circ[v_n]\\ &\quad + \sum_i\sum_j (c'_i\circ d'_j)*(Q_*c''_i\circ Q_*d''_j\circ[v_n])\\ &\quad + \sum_i\sum_j v_n(Qc'_i\circ Qd'_j)*(Q_*c''_i\circ Q_*d''_j\circ[v_n]).\end{aligned}\tag{42}$$

Nevertheless, we shall see in Proposition 7.2 that all tame elements in the Hopf ring are $\circ$-central.

Remark 6.10. Of course $a_{(n-1)}$ commutes with itself, but equation (42) yields

$$a_{(n-1)}\circ a_{(n-1)} = a_{(n-1)}\circ a_{(n-1)} + v_n\, e\circ e + b_{(0)}^{\circ 2^{n-1}}\circ b_{(0)}^{\circ 2^{n-1}}\circ[v_n].$$

This is consistent, because by [**BW07**, §10.5], the main relation $(\mathcal{R}_n)$ is $v_n b_{(0)} = b_{(0)}^{\circ 2^n}\circ[v_n]$ and $e\circ e = b_{(0)}$.

Remark 6.11. We present concrete examples of non(co)commutativity.

It is easy to see that $*$ is not commutative in general. If we take $c = a_{(n-1)} \circ b_{(0)}$ and $d = a_{(n-1)} \circ b_{(1)}$, equation (41) shows directly that

$$d * c = c * d + v_n\,(e \circ b_{(0)}) * (e \circ b_{(1)}).$$

It is less obvious that $\circ$-multiplication is not commutative. If we set $c = (e \circ a_{(n-1)}) * b_{(0)}$ and $d = (e \circ a_{(n-1)}) * b_{(1)}$, equation (42) reduces to

$$d \circ c = c \circ d + v_n(b_{(0)} \circ b_{(1)}) * b_{(0)}^{\circ 2}.$$

Alternatively, we may reduce $c \circ d$ and $d \circ c$ to standard form as in [**BW07**, §11.3]. Since $e \circ a_{(n-1)}$ and $b_{(0)} = e \circ e$ are both suspensions and therefore primitive, equation (32) yields

$$\psi c = c \otimes 1 + e \circ a_{(n-1)} \otimes b_{(0)} + b_{(0)} \otimes e \circ a_{(n-1)} + 1 \otimes c.$$

Then the distributive law (38) yields, after some simplification,

$$d \circ c = (b_{(0)} \circ b_{(n)}) * (b_{(0)} \circ b_{(1)}) + (e \circ a_{(n-1)} \circ b_{(0)}) * (e \circ a_{(n-1)} \circ b_{(1)}),$$

with no extra term, while for $c \circ d$, (39) yields the same, plus the extra term $v_n\, b_{(0)}^{\circ 2} * (b_{(0)} \circ b_{(1)})$.

Neither is the comultiplication cocommutative. If we take (as above) $c = (e \circ a_{(n-1)}) * b_{(0)}$ and $d = (e \circ a_{(n-1)}) * b_{(1)}$ and use equation (32) to compute $\psi(c * d)$ and $\psi(d * c)$, we find that both contain all the same terms one would normally expect, which are symmetric. In addition, $\psi(c * d)$ has two extra terms $v_n\, b_{(0)}^{*3} \otimes b_{(0)}^{*2} + v_n\, b_{(0)}^{*2} \otimes b_{(0)} * b_{(1)}$ that are not symmetric, whereas $\psi(d * c)$ has $v_n\, b_{(0)} * b_{(1)} \otimes b_{(0)}^{*2} + v_n\, b_{(0)}^{*2} \otimes b_{(0)}^{*3}$.

7. Proofs of the Hopf ring structure

We establish the remaining theorems in §6. All our proofs follow the same pattern (already seen in §5): we represent some relation in $P(n)$-cohomology in the category *Ho*, apply $\overline{P(n)}$-homology, and evaluate on a generic twisted cross product.

A diagonal map Δ introduces the comultiplication by way of equation (35). Whenever two factors are switched, equation (19) adds extra terms involving Q. (Some of these extra terms will in due course go away, but we do not know which in advance.)

Proof of Theorems 6.1 and 6.3. To abbreviate, we write $W = \underline{P(n)}_s$. The maps

$$W \times W \xrightarrow{\mu} W \xrightarrow{\Delta} W \times W$$

and

$$W \times W \xrightarrow{\Delta\times\Delta} W \times W \times W \times W$$
$$\xrightarrow{\mathrm{id}\times T\times\mathrm{id}} W \times W \times W \times W \xrightarrow{\mu\times\mu} W \times W$$

trivially coincide. We apply $\overline{P(n)}$-homology and evaluate on the element $c \,\overline{\times}\, d \in \overline{P(n)}_*\Big(\underline{P(n)}_s \times \underline{P(n)}_s\Big)$, using (35) and (19) to obtain (32).

The proof of (36) is completely analogous, with ϕ replacing μ. □

Proof of Theorem 6.5. We represent the distributive law $(x+y)z = xz + yz$ in $P(n)$-cohomology by the equality of the maps

$$\underline{P(n)}_s \times \underline{P(n)}_s \times \underline{P(n)}_m \xrightarrow{\mu\times\mathrm{id}} \underline{P(n)}_s \times \underline{P(n)}_m \xrightarrow{\phi} \underline{P(n)}_{s+m}$$

and

$$\begin{aligned}\underline{P(n)}_s \times \underline{P(n)}_s \times \underline{P(n)}_m &\xrightarrow{\mathrm{id}\times\mathrm{id}\times\Delta} \underline{P(n)}_s \times \underline{P(n)}_s \times \underline{P(n)}_m \times \underline{P(n)}_m \\ &\xrightarrow{\mathrm{id}\times T\times\mathrm{id}} \underline{P(n)}_s \times \underline{P(n)}_m \times \underline{P(n)}_s \times \underline{P(n)}_m \\ &\xrightarrow{\phi\times\phi} \underline{P(n)}_{s+m} \times \underline{P(n)}_{s+m} \xrightarrow{\mu} \underline{P(n)}_{s+m}\end{aligned}$$

We apply $\overline{P(n)}$-homology to $a \overline{\times} b \overline{\times} c \in \overline{P(n)}_*\Big(\underline{P(n)}_s \times \underline{P(n)}_s \times \underline{P(n)}_m\Big)$ to obtain the distributive law (38). The other distributive law is analogous. □

We next establish a commutation rule for $d \circ c$ that will in due course reduce to equation (42).

Lemma 7.1. *Assuming that $\epsilon c = 0$ and $\epsilon d = 0$, with ψc and ψd given by equations* (33) *and* (34), *we have*

$$\begin{aligned}d \circ c = c \circ d &+ v_n\, Qc \circ Qd + Q_*c \circ Q_*d \circ [v_n] + v_n\, Q_*Qc \circ Q_*Qd \circ [v_n] \\ &+ \sum_{i,j} (c_i' \circ d_j') * (Q_*c_i'' \circ Q_*d_j'' \circ [v_n]) \\ &+ \sum_{i,j} v_n\, (Qc_i' \circ Qd_j') * (Q_*c_i'' \circ Q_*d_j'' \circ [v_n]) \\ &+ \sum_{i,j} v_n\, (c_i' \circ d_j') * (Q_*Qc_i'' \circ Q_*Qd_j'' \circ [v_n]) \\ &+ \sum_{i,j} v_n^2\, (Qc_i' \circ Qd_j') * (Q_*Qc_i'' \circ Q_*Qd_j'' \circ [v_n]) \\ &+ \sum_{i,j} v_n\, (Qc_i' \circ d_j') * (Q_*c_i'' \circ Q_*Qd_j'' \circ [v_n]).\end{aligned} \tag{43}$$

Proof. We rewrite (20) as $yx = xy + (Qx)(Qy)v_n$, and represent it in *Ho* by the maps $\phi\colon \underline{P(n)}_m \times \underline{P(n)}_s \to \underline{P(n)}_{s+m}$ and

$$\begin{aligned}\underline{P(n)}_m \times \underline{P(n)}_s &\xrightarrow{\Delta\times\Delta} \underline{P(n)}_m \times \underline{P(n)}_m \times \underline{P(n)}_s \times \underline{P(n)}_s \\ &\xrightarrow{\mathrm{id}\times T\times\mathrm{id}} \underline{P(n)}_m \times \underline{P(n)}_s \times \underline{P(n)}_m \times \underline{P(n)}_s \\ &\xrightarrow{T\times T} \underline{P(n)}_s \times \underline{P(n)}_m \times \underline{P(n)}_s \times \underline{P(n)}_m \\ &= \underline{P(n)}_s \times \underline{P(n)}_m \times \underline{P(n)}_s \times \underline{P(n)}_m \times \text{point} \\ &\xrightarrow{\mathrm{id}\times\mathrm{id}\times Q\times Q\times v_n} \\ &\qquad \underline{P(n)}_s \times \underline{P(n)}_m \times \underline{P(n)}_{s+N} \times \underline{P(n)}_{m+N} \times \underline{P(n)}_{-2N} \\ &\xrightarrow{\phi\times\phi_3} \underline{P(n)}_{s+m} \times \underline{P(n)}_{s+m} \xrightarrow{\mu} \underline{P(n)}_{s+m}.\end{aligned}$$

We apply $\overline{P(n)}$-homology and evaluate on $d \overline{\times} c \in \overline{P(n)}_*\Big(\underline{P(n)}_m \times \underline{P(n)}_s\Big)$, using $Q \circ Q = 0$, $Q_*1 = 1$, $\epsilon Qc = 0$ and $\epsilon Q_* c = \epsilon c$. The result is (43). □

Proposition 7.2. *Every tame element of the Hopf ring $H(*)$ is $\circ$-central.*

Proof. We take d in (43) to be any of our $\circ$-generators other than $a_{(n-1)}$. By (40) and Proposition 5.4, Qd, Q_*d and each $Q_*d''_j$ all vanish, so that (43) simplifies to $d \circ c = c \circ d$.

If c_1 and c_2 are both tame and $\circ$-central, so is $c_1 \circ c_2$, trivially. To see that $c_1 * c_2$ is also $\circ$-central, we compare the two distributive laws (39) and (38) (which have now been proved) for any $d \in \operatorname{Ker} \epsilon$, with ψd expressed as in (34),

$$d \circ (c_1 * c_2) = \sum_j (d'_j \circ c_1) * (d''_j \circ c_2) = \sum_j (c_1 \circ d'_j) * (c_2 \circ d''_j) = (c_1 * c_2) \circ d.$$

It follows that any tame expression is $\circ$-central. □

Proof of Proposition 5.3(e). All our $\circ$-generators (31)(i) commute with each other, as $a_{(n-1)}$ is the only wild one. When we pass to the indecomposables A^*_* of the Hopf ring, the $\circ$-generators map to e, $a_{(i)}$, $b_{(j)}$ and w_k, which therefore still commute and generate the $P(n)_*$-algebra A^*_*.

From Proposition 5.4(e), switching the multiplication on $P(n)$ from ϕ to $\bar{\phi}$ reverses the $\circ$-multiplication, which has no effect on A^*_*. □

Remark 7.3. As another example of noncommutativity, we note that Proposition 7.2 does not extend to the elements f_i for $i > 2^n$, which are wild. Take $c = f_{2^n+1} = e * a_{(n-1)}$ and $d = f_{2^n+2} = a_{(0)} * a_{(n-1)}$, where $n \geqslant 2$. (If $n = 1$, $a_{(0)} * a_{(n-1)} = 0$ and f_4 does not exist.) We compute as before that

$$f_{2^n+2} \circ f_{2^n+1} = f_{2^n+1} \circ f_{2^n+2} + v_n\, b_{(0)} * (e \circ a_{(0)}).$$

Proof of Proposition 5.4(d). This is similar to the stable proof of (25) in [**BW07**, (6.17)], and in fact relies on it.

We apply the Cartan formula to (29) to obtain

$$r(t^2) = \sum_{i=0}^{2N} \sum_{j=0}^{2N} \langle r, f_i \circ f_j \rangle t^{i+j}$$

for all operations r. Now t^2 is the Chern class of the complexified real Hopf line bundle over $\mathbb{R}P^{2N}$, so [**BW07**, (10.4)] gives

$$r(t^2) = \sum_{k=0}^{N} \langle r, b_k \rangle t^{2k}.$$

Comparing coefficients of t^{2k}, we deduce that $b_k = \sum_{i+j=2k} f_i \circ f_j$. As the elements f_i for $i < 2^n$ are tame and so $\circ$-commute by Proposition 7.2, this simplifies to $b_k = f_k^{\circ 2}$, provided $2k \leqslant 2^n$; in particular, $a_{(i)}^{\circ 2} = b_{(i+1)}$ for $0 \leqslant i \leqslant n-2$.

This proof is not available for $a_{(n-1)}^{\circ 2}$ in $\overline{P(n)}_*\Big(\underline{P(n)}_*\Big)$, as $t^{2^{n+1}} = 0$, but it does show that $a_{(n-1)}^{\circ 2} = b_{(n)}$ in $\overline{P(n+1)}_*\Big(\underline{P(n+1)}_2\Big)$. The homomorphism of Hopf rings

$H(*) \to \overline{P(n+1)}_*\Big(\underline{P(n+1)}_*\Big)$ induced by the canonical map $\rho\colon P(n) \to P(n+1)$ carries each generator of $H(*)$ to its namesake in $\overline{P(n+1)}_*\Big(\underline{P(n+1)}_*\Big)$. It also adds one new generator, $a_{(n)}$, and kills the ideal (v_n) in $P(n)_*$. The only nonzero element of $H(2)_{2^{n+1}}$ that lies in $\operatorname{Ker}\rho_*$ is $v_n b_{(0)}$. We must have $a_{(n-1)}^{\circ 2} = b_{(n)}$ in $H(2)$, as the only other candidate, $a_{(n-1)}^{\circ 2} = b_{(n)} + v_n b_{(0)}$, does not stabilize correctly to (25). □

We next prove a similarly extended version of (37).

Lemma 7.4. *Given $c, d \in H(*)$ that satisfy $\epsilon c = 0$ and $\epsilon d = 0$, with ψc and ψd given by* (33) *and* (34)*, we have*

$$\begin{aligned} Q_*(c \circ d) = {} & Q_*c \circ d + c \circ Q_*d \\ & + \sum_{i,j} (c'_i \circ Q_*d'_j) * (Q_*c''_i \circ d''_j) \\ & + \sum_{i,j} v_n\, (c'_i \circ Q_*Qd'_j) * (Q_*Qc''_i \circ d''_j) \end{aligned} \tag{44}$$

Proof. We represent the derivation formula $Q(xy) = x(Qy) + (Qx)y$ in $P(n)$-cohomology unstably by equality of the two maps

$$\underline{P(n)}_s \times \underline{P(n)}_m \xrightarrow{\phi} \underline{P(n)}_{s+m} \xrightarrow{Q} \underline{P(n)}_{s+m+N}$$

and

$$\begin{aligned} \underline{P(n)}_s \times \underline{P(n)}_m & \xrightarrow{\Delta\times\Delta} \underline{P(n)}_s \times \underline{P(n)}_s \times \underline{P(n)}_m \times \underline{P(n)}_m \\ & \xrightarrow{\mathrm{id}\times T\times\mathrm{id}} \underline{P(n)}_s \times \underline{P(n)}_m \times \underline{P(n)}_s \times \underline{P(n)}_m \\ & \xrightarrow{\mathrm{id}\times Q\times Q\times\mathrm{id}} \underline{P(n)}_s \times \underline{P(n)}_{m+N} \times \underline{P(n)}_{s+N} \times \underline{P(n)}_m \\ & \xrightarrow{\phi\times\phi} \underline{P(n)}_{s+m+N} \times \underline{P(n)}_{s+m+N} \xrightarrow{\mu} \underline{P(n)}_{s+m+N}. \end{aligned}$$

Again, we apply $\overline{P(n)}$-homology and evaluate on $c \,\overline{\times}\, d \in \overline{P(n)}_*\Big(\underline{P(n)}_s \times \underline{P(n)}_m\Big)$, to obtain (44). □

Lemma 7.5. *Take any element c in the Hopf ring.*

(a) *If g is any of our $\circ$-generators* (31)(i) *other than $a_{(n-1)}$, we have $Q_*(c \circ g) = (Q_*c) \circ g$.*

(b) *For $g = a_{(n-1)}$, we have*

$$Q_*(c \circ a_{(n-1)}) = (Q_*c) \circ a_{(n-1)} + c \circ b_{(0)}^{\circ 2^{n-1}}.$$

Proof. We may assume $c \in \operatorname{Ker}\epsilon$, as everything is zero if $c = 1$. We put $d = g$ in (44). Since all the $Q_*d'_j$ and Qd'_j vanish by (40) and Proposition 5.4, also Q_*d in case (a), equation (44) simplifies as stated. □

Corollary 7.6. *Let z be a $\circ$-monomial as in* (31)(ii)*. Then Q_*z is tame if z is wild, and $Q_*z = 0$ if z is tame.*

Proof. If z is tame, $Q_*z = 0$ from Lemma 7.5(a), by induction on the number of $\circ$-factors in z. If z is wild, now that we know from Proposition 7.2 that all our $\circ$-generators $\circ$-commute ($a_{(n-1)}$ is the only wild generator), we may shuffle all factors $a_{(n-1)}$ to the end. We use the relation (now proved) $a_{(n-1)}^{\circ 2} = b_{(n)}$ to reduce the number of factors $a_{(n-1)}$ until we have at most one. If now $z = z' \circ a_{(n-1)}$, with z' tame, Lemma 7.5(b) shows that $Q_*z = z' \circ b_{(0)}^{\circ 2^{n-1}}$, which is tame. □

Lemma 7.7. *For all $*\circ$-monomials c we have $Q_*Qc = QQ_*c = 0$.*

Proof. By naturality of the homology operation Q, we have $Q_*Qc = QQ_*c$. We show that $QQ_*c = 0$.

The result is trivial for $c = 1$. Otherwise, we may write $c = z_1 * z_2 * \cdots * z_r$, where each z_i is a $\circ$-monomial and $r > 0$. By Corollary 7.6, each Q_*z_i is tame (if not zero). Then Proposition 5.4 shows that (i) each $QQ_*z_i = 0$, (ii) $Q_*c = Q_*z_1 * Q_*z_2 * \cdots * Q_*z_r$, and hence (iii) $QQ_*c = 0$. □

Proof of Theorem 6.4. Lemma 7.7 allows us to omit the unwanted terms in (44) to obtain (37). □

Proof of Theorem 6.9. For (a), we deduce from (30) that

$$\Delta_*c = 1 \times c + \sum_i c_i'' \times c_i' + c \times 1,$$

after replacing (27) by (33). By (18), we can rewrite $c_i'' \times c_i'$ as $c_i'' \overline{\times} c_i' + v_n\, Qc_i'' \overline{\times} Qc_i'$. (For the end terms we have simply $1 \times c = 1 \overline{\times} c$ and $c \times 1 = c \overline{\times} 1$.) We compare with (35).

For (b), we represent the relation $y + x = x + y$ in $P(n)$-cohomology by the equality of maps $\mu = \mu \circ T \colon \underline{P(n)}_s \times \underline{P(n)}_s \to \underline{P(n)}_s$. We apply $\overline{P(n)}$-homology to these maps and evaluate on $d \overline{\times} c \in \overline{P(n)}_*\big(\underline{P(n)}_s \times \underline{P(n)}_s\big)$ to obtain (41), with the help of (19).

For (c), Lemma 7.7 allows us to omit all the unwanted terms in Lemma 7.1 to obtain (42). □

8. A nonadditive splitting

To complete the proof of Lemma 5.1 in [**BW07**], given $m \geqslant n$, we need a *non-additive* idempotent operation $\theta(m)$ on the ungraded cohomology theory $P(n)^M(-)$, where $M = g(n,m)$ is the numerical function given by

$$g(n,m) = 2(2^n + 2^{n+1} + \cdots + 2^m). \tag{45}$$

(We know there is no relevant additive idempotent in this dimension.) We shall use $\theta(m)$ to produce what we actually want, a natural splitting $\overline{\theta(m)} \colon P(n,m)^M(-) \to P(n)^M(-)$ of the canonical projection $\rho(m) \colon P(n)^*(-) \to P(n,m)^*(-)$; we recall from [**BW07**] that $P(n,m)$ is the spectrum constructed from $P(n)$ to have the homotopy groups $P(n,m)_* = P(n)_*/J_m$, where J_m denotes the ideal $(v_{m+1}, v_{m+2}, \dots) \subset P(n)_*$.

We shall define $\theta(m)$ by its linear functional $\langle\theta(m), -\rangle\colon H(M) \to P(n)_*$ on the Hopf algebra $H(M) = \overline{P(n)}_*\Big(\underline{P(n)}_M\Big)$. In this section, we give axioms for $\langle\theta(m), -\rangle$ that ensure that $\theta(m)$ has the desired properties; in the following section, Corollary 9.4 actually constructs the linear functional.

A Hopf ring ideal

Definition 8.1. Denote by $\mathfrak{J}_m$ the $*$-ideal in the Hopf ring $H(*)$ generated by all elements of the form $c \circ ([v_k] - 1)$ with $c \in H(*)$ and $k > m$.

As in [**BW07**, Lemma 19.35], this is a Hopf ring ideal. It is closely related to the ideal J_m. Following [**BJW95**, (23.6)], the axioms we need on $\langle\theta(m), -\rangle$ are:

$$\begin{aligned} &\text{(i)}\quad \langle\theta(m), c\rangle = 0 \text{ for all } c \in \mathfrak{J}_m \cap H(M);\\ &\text{(ii)}\quad \langle\theta(m), c\rangle \equiv \epsilon_A q_M c \bmod J_m \text{ for all } c \in H(M); \end{aligned} \tag{46}$$

where we recall the projection $q_M\colon \overline{P(n)}_i\Big(\underline{P(n)}_M\Big) \to A_i^M$ and the additive counit $\epsilon_A\colon A_i^M \to P(n)_{i-M}$. These axioms imply that $\theta(m)$ has the desired properties.

First, $\theta(m)$ behaves correctly on the homotopy groups $\pi_*\Big(\underline{P(n)}_M\Big) = \Sigma^M P(n)_*$.

Lemma 8.2. *If $\theta(m)$ satisfies the axioms (46), then on the homotopy groups we have*

(i) $\theta(m)_*\Sigma^M v = 0$ *for all* $v \in J_m$;

(ii) $\theta(m)_*\Sigma^M v \equiv \Sigma^M v \bmod J_m$ *for all* $v \in P(n)_*$.

Proof. By [**BW07**, (12.1)], given $v \in P(n)_s$,

$$\theta(m)_*\Sigma^M v = \Sigma^M \langle\theta(m), e^{\circ M+s} \circ ([v] - 1)\rangle,$$

where we make use of $e \circ 1 = 0$. As $v \in J_m$ implies that $[v] - 1$ lies in $\mathfrak{J}_m$, we can read off the results from (46). □

Next, we study the homology homomorphism $\theta(m)_*\colon H(M) \to H(M)$ induced by $\theta(m)\colon \underline{P(n)}_M \to \underline{P(n)}_M$.

Lemma 8.3. *If $\theta(m)$ satisfies the axioms (46), then on the $\overline{P(n)}$-homology groups $H(M) = \overline{P(n)}_*\Big(\underline{P(n)}_M\Big)$ we have*

(i) $\theta(m)_*c = 0$ *for all* $c \in \mathfrak{J}_m \cap H(M)$;

(ii) $\theta(m)_*c \equiv c \bmod \mathfrak{J}_m$ *for all* $c \in H(M)$.

Proof. The proof of the similar result Lemma 23.4 in [**BJW95**] applies, with one minor modification. An additional case is needed to handle the extra generators $a_{(i)}$. Proposition 10.3(vii) of [**BW07**] allows us to treat these generators the same way as the others. □

It is now easy to deduce that $\theta(m)_*$ and hence $\theta(m)$ are idempotent. However, in the nonadditive context, this is not enough; a more sophisticated relation is needed, from Lemma 3.10 in [**Bo95**].

Lemma 8.4. *If $\theta = \theta(m)$ satisfies the axioms (46), it satisfies the identity*

$$\theta(x + y - \theta(y)) = \theta(x) \qquad (\textit{in } P(n)^M(-)). \tag{47}$$

Proof. We repeat a technique from §6, but in reverse: we use a Hopf algebra calculation to establish the relation (47) in cohomology. The right side, considered as a function of (x, y), is represented by the map

$$W \times W \xrightarrow{\mathrm{id}\times\omega_M} W \times \text{point} = W \xrightarrow{\theta} W,$$

where we abbreviate by writing $W = \underline{P(n)}_M$. We apply $\overline{P(n)}$-homology and evaluate on $c \mathbin{\overline{\times}} d$ to get the result $\theta_*((\epsilon d)c)$.

The left side is represented by the composite

$$\begin{aligned} W \times W &\xrightarrow{\mathrm{id}\times\Delta} W \times W \times W \xrightarrow{\mathrm{id}\times\mathrm{id}\times\theta} W \times W \times W \\ &\xrightarrow{\mathrm{id}\times\mathrm{id}\times\nu} W \times W \times W \xrightarrow{\mu_3} W \xrightarrow{\theta} W \end{aligned}$$

where ν represents the operation "$-$" and induces the conjugation $\nu_* = \chi$ on $H(M)$. Again, we apply $\overline{P(n)}$-homology and evaluate on $c \mathbin{\overline{\times}} d$. If we omit the final θ, we find

$$\sum_\beta c * d'_\beta * \chi\theta_* d''_\beta \equiv c * \sum_\beta d'_\beta * \chi d''_\beta = c * (\epsilon d)1 = (\epsilon d)c \bmod \mathfrak{J}_m,$$

using Lemma 8.3(ii). When we apply θ_*, which kills $\mathfrak{J}_m \cap H(M)$ by Lemma 8.3(i), we get $\theta_*((\epsilon d)c)$. □

The other needed relation, $\theta(0) = 0$, is automatic here, as $P(n)^M(\text{point}) = 0$. As in [**Bo95**], the two relations have several immediate consequences:

$$\begin{aligned} &\text{(i)} \quad \theta(\theta(y)) = \theta(y); \\ &\text{(ii)} \quad \theta(x + y) = \theta(x) \text{ if } \theta(y) = 0; \\ &\text{(iii)} \quad \theta(y - \theta(y)) = 0; \\ &\text{(iv)} \quad \operatorname{Ker}\theta = \{x : \theta(x) = 0\} \text{ is a subgroup of } P(n)^M(-). \end{aligned} \tag{48}$$

For (i), we put $x = \theta(y)$ in (47), to show that θ is idempotent in the ordinary sense. If $\theta(y) = 0$, (47) simplifies to (ii). For (iii) we put $x = 0$. For (iv), we have $0 \in \operatorname{Ker}\theta$ by assumption. If x and y lie in $\operatorname{Ker}\theta$, so does $x + y$ by (ii). Given $\theta(y) = 0$, we put $x = -y$ in (47) to see that $\theta(-y) = 0$, to finish (iv).

By (iv), we can legitimately form the quotient group $\operatorname{Coim}\theta = P(n)^M(-)/\operatorname{Ker}\theta$ (not to be confused with the image of θ, which is *not* a subgroup of $P(n)^M(-)$); its elements are cosets $x + \operatorname{Ker}\theta$. It is easy to show as in [**Bo95**, §3], using equations (48), that $\operatorname{Ker}\theta$ and $\operatorname{Coim}\theta$ are ungraded cohomology theories (in the sense of [**Bo95**, Lemma 3.10]), representable in *Ho* by H-spaces. We build the commutative diagram of ungraded cohomology theories and natural transformations

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker}\theta & \xrightarrow{\subset} & P(n)^M(-) & \xrightarrow{q} & \operatorname{Coim}\theta & \longrightarrow & 0 \\
 & & & & \big\downarrow{\scriptstyle\theta} & \swarrow{\scriptstyle\theta'} & \big\downarrow{\scriptstyle=} & & \\
 & & & & P(n)^M(-) & \xrightarrow{q} & \operatorname{Coim}\theta & & \\
 & & & & & \searrow{\scriptstyle\rho(m)} & & & \\
 & & & & & & P(n,m)^M(-). & &
\end{array}$$

It features a short exact sequence. The non-additive operation θ' is defined by $\theta'(x + \operatorname{Ker}\theta) = \theta(x)$ to make the upper triangle commute; by (48)(ii), it is well defined. The lower triangle commutes because $\theta(x) + \operatorname{Ker}\theta = x + \operatorname{Ker}\theta$ by (iii).

We really want a splitting of $\rho(m)$, not q. By Lemma 8.2, the natural transformation $\rho(m) \circ \theta'$ induces an isomorphism on spheres and hence generally. (Equivalently, the map that represents $\rho(m) \circ \theta'$ induces an isomorphism of homotopy groups and is therefore a homotopy equivalence.) We define

$$\overline{\theta(m)} = \theta' \circ (\rho(m) \circ \theta')^{-1} \colon P(n,m)^M(-) \longrightarrow P(n)^M(-).$$

Trivially, we have $\rho(m) \circ \overline{\theta(m)} = \mathrm{id}$ and $\overline{\theta(m)} \circ \rho(m)$ is an idempotent operation on $P(n)^M(-)$. (Lemma 8.2 strongly suggests that $\rho(m) \circ \theta = \rho(m)$, which readily implies that $\overline{\theta(m)} \circ \rho(m) = \theta$; however, we do not need this, and a complete proof appears to require properties of the spectrum $P(n,m)$ beyond the scope of this paper.)

9. Reduction to standard form

Before we produce the linear functional $\langle\theta(m), -\rangle$ for §8 in Corollary 9.4, we need much more information on the Hopf ring $H(*)$. Specifically, we need a precise description of how the ideal $\mathfrak{J}_m$ sits inside $H(*)$. Our strategy is quite different and significantly simpler than the proof of the analogous result for BP given in [**BJW95**, §23], and more closely resembles the proof of the additive case [**BW07**, Lemma 9.3].

We first need to refine Corollary 6.8. As all our $\circ$-generators $\circ$-commute by Proposition 7.2 and we have the relations $e^{\circ 2} = b_{(0)}$ and $a_{(i)}^{\circ 2} = b_{(i+1)}$, any $\circ$-monomial may be written

$$\begin{aligned} z &= e^{\circ h} \circ a^{\circ I} \circ b^{\circ J} \circ ([v^K] - 1) \\ &= e^{\circ h} \circ a_{(0)}^{\circ i_0} \circ a_{(1)}^{\circ i_1} \circ \dots \circ a_{(n-1)}^{\circ i_{n-1}} \circ b_{(0)}^{\circ j_0} \circ b_{(1)}^{\circ j_1} \circ b_{(2)}^{\circ j_2} \circ \dots \\ &\qquad \circ ([v_n] - 1)^{\circ k_n} \circ ([v_{n+1}] - 1)^{\circ k_{n+1}} \circ ([v_{n+2}] - 1)^{\circ k_{n+2}} \circ \dots, \end{aligned} \tag{49}$$

as in [**BW07**, (11.1)], with multi-indices $I = (i_0, i_1, \dots, i_{n-1})$, $J = (j_0, j_1, j_2, \dots)$ and $K = (k_n, k_{n+1}, k_{n+2}, \dots)$, where h and every i_r is 0 or 1. (We used the identity $([v] - 1) \circ ([v'] - 1) = [vv'] - 1$ to make the notation less cumbersome. Of course, we could also replace $[v^K] - 1$ by simply $[v^K]$, as long as e or some $a_{(i)}$ or $b_{(j)}$ is present, but we prefer to stay within the augmentation ideal $\operatorname{Ker}\epsilon$ as much as possible.) We need to know how Q acts; by Proposition 5.4,

$$\begin{aligned} &Q(e^{\circ h} \circ a^{\circ I,0} \circ b^{\circ J} \circ ([v^K] - 1)) = 0, \\ &Q(e^{\circ h} \circ a^{\circ I,1} \circ b^{\circ J} \circ ([v^K] - 1)) = e \circ e^{\circ h} \circ a^{\circ I,0} \circ b^{\circ J} \circ ([v^K] - 1), \end{aligned} \tag{50}$$

where $I = (i_0, i_1, \dots, i_{n-2})$ and $a^{\circ I,s}$ denotes $a_{(0)}^{\circ i_0} \circ a_{(1)}^{\circ i_1} \circ \dots \circ a_{(n-2)}^{\circ i_{n-2}} \circ a_{(n-1)}^{\circ s}$.

We note that each of our $\circ$-generators lies in a group $H(s)_i$ for which $2i - s > 0$; it follows that each $H(s)_i$ contains only finitely many $\circ$-monomials, and hence only finitely many $*\circ$-monomials.

In [**BW07**, Defn. 8.1], we defined certain $\circ$-monomials as being *allowable*; the others are not (in the end) needed. (The empty $\circ$-monomial $e^{\circ 0} \circ a^{\circ 0} \circ b^{\circ 0} \circ ([1] - 1) = [1] - 1_0 \in H(0)$ is allowable.) Given s, we choose a total ordering of all the allowable $\circ$-monomials that lie in $H(s)$; then we call the $*\circ$-monomial (in $H(s)$)

$$c = z_1 * z_2 * \cdots * z_r \tag{51}$$

an *ordered* $*\circ$-monomial if each z_i is allowable and $z_1 < z_2 < \cdots < z_r$. This allows us to rephrase Theorem 11.3 of [**BW07**] in the following form.

Theorem 9.1. *For each s, the Hopf algebra $H(s) = \overline{P(n)}_*\left(\underline{P(n)}_s\right)$ is a free $P(n)_*$-module, with a basis consisting of all the ordered $*\circ$-monomials* (51) *that lie in $H(s)$* (*including the empty $*$-product 1_s*).

The rest of the section will be concerned with reducing a general $*\circ$-monomial to this standard form. (This is not as straightforward as [**BW07**] suggests.) Our proofs will make it clear that the choice of ordering is not significant.

We show that the ideal $\mathfrak{J}_m$ reduces to what one would like, but only in a certain range. Let us call the $\circ$-monomial (49) a $\mathfrak{J}_m$*-monomial* if it visibly lies in $\mathfrak{J}_m$, i.e. $v^K \in J_m$ (has a factor v_k with $k > m$).

Theorem 9.2. *If $s \leqslant M = g(n,m)$, the $*$-ideal $\mathfrak{J}_m \cap H(s)$ is the free $P(n)_*$-submodule of $H(s)$ with a basis consisting of all the ordered $*\circ$-monomials* (51) *in which at least one $*$-factor z_i is a $\mathfrak{J}_m$-monomial.*

Remark 9.3. What distinguishes this from [**BW07**, Lemma 9.3] when $p = 2$ is that if $s = M$, some allowable $\mathfrak{J}_m$-monomials fail to be Q-allowable and so do not appear there, namely those of the form

$$b_{(0)}^{\circ N} \circ b_{(0)}^{\circ 2^{n+1}} \circ b_{(d_{n+1})}^{\circ 2^{n+2}} \circ d_{(d_{n+2})}^{\circ 2^{n+3}} \circ \cdots \circ b_{(d_m)}^{\circ 2^{m+1}} \circ [v_{m+1}],$$

where $0 \leqslant d_{n+1} \leqslant d_{n+2} \leqslant \cdots \leqslant d_m$.

Corollary 9.4. *There exists a $P(n)_*$-linear functional $\langle \theta(m), - \rangle \colon H(M) \to P(n)_*$ that satisfies the axioms* (46).

Proof. We simply choose $\langle \theta(m), c \rangle \in P(n)_*$ for each ordered $*\circ$-monomial c in the form (51). If some z_i is a $\mathfrak{J}_m$-monomial, we must take $\langle \theta(m), c \rangle = 0$ to satisfy (46)(i); then (ii) is automatic, because $\epsilon_A q_M c$ lies in J_m if $r = 1$, or is zero if $r > 1$. Otherwise, we simply choose $\langle \theta(m), c \rangle$ to satisfy axiom (ii). □

Reduction of $\circ$-monomials

We show how to reduce a general $\circ$-monomial (49) to allowable $\circ$-monomials. Our starting point is [**BW07**, Thm. 8.2], except that we temporarily exclude the $\circ$-generators $a_{(i)}$. We adapt it to the Hopf ring environment by including the decomposables.

Lemma 9.5. *Any $\circ$-monomial $z = e^{\circ h} \circ b^{\circ J} \circ ([v^K] - 1)$ can be expressed as a $P(n)_*$-linear combination of:*

(i) *Allowable $\circ$-monomials z';*

(ii) *Decomposable $*$-products of two or more $\circ$-monomials.*

None of the resulting expressions contains an $a_{(i)}$.

Proof. This result is actually inherited from BP. It uses only the main relations $(\mathcal{R}_k)$ and $(\mathcal{R}'_n)$ in [**BW07**], so does not introduce any $a_{(i)}$. We do not need to include 1 in (ii), as $z \in \operatorname{Ker} \epsilon$. □

In (i), we could restrict z' to be Q-allowable, but this would not be compatible with Theorem 9.1. We next reinstate the $a^{\circ I}$. We note that the definition of allowable (unlike Q-allowable) makes no reference to the factor $a^{\circ I}$. We recall from Definition 6.6 that a Hopf ring expression in our $\circ$-generators is wild if it contains $a_{(n-1)}$ or tame if it does not. For tame elements, Proposition 6.7 simplifies as follows.

Lemma 9.6. *Suppose c and d are tame $*\circ$-monomials, where $c \neq 1$. Then:*
- (a) $Q_* c = 0$;
- (b) *$c \circ d$ is a $P(n)_*$-linear combination of tame $*\circ$-monomials;*
- (c) *$\psi c = c \otimes 1 + \sum_i c'_i \otimes c''_i + 1 \otimes c$, where every c'_i and c''_i is a $P(n)_*$-linear combination of tame $*\circ$-monomials.* □

Lemma 9.7. *Any $\circ$-monomial $z = e^{\circ h} \circ a^{\circ I} \circ b^{\circ J} \circ ([v^K] - 1)$ can be expressed as a $P(n)_*$-linear combination of:*
- (i) *Allowable $\circ$-monomials z', which are wild if and only if z is wild;*
- (ii) *$*$-products of two or more tame $\circ$-monomials.*

We can recursively apply the lemma to the monomials appearing in (ii).

Corollary 9.8. *Any tame $\circ$-monomial can be expressed as a $P(n)_*$-linear combination of $*$-products of one or more tame allowable $\circ$-monomials.* □

Proof of Lemma 9.7. We apply Lemma 9.5 to $e^{\circ h} \circ b^{\circ J} \circ ([v^K] - 1)$ and then $\circ$-multiply by $a^{\circ I}$. For (i), this is obvious, as the extra $\circ$-factor $a^{\circ I}$ does not affect allowability. For (ii), we put back the $\circ$-factors $a_{(i)}$ one at a time, using the distributive law

$$a_{(i)} \circ (c_1 * c_2) = \sum_{j=1}^{2^{i+1}-1} (f_j \circ c_1) * (f_{2^{i+1}-j} \circ c_2).$$

We observe that $f_{2^n} = a_{(n-1)}$ never appears, even if $i = n - 1$. We apply Lemma 9.6 to both $*$-factors on the right. □

Reduction of $\mathfrak{J}_m$-monomials

We particularly need to know what happens when we reduce a $\mathfrak{J}_m$-monomial.

As in [**BW07**], we define the *b-length* of the $\circ$-monomial (49) as $\sum_r j_r$, the total number of $\circ$-factors of the form $b_{(j)}$ (including repetitions). We note that in Definition 8.1 of [**BW07**], the disallowed monomials of type (i) have b-length at least $\frac{1}{2} g(n, q)$, while those of type (ii) have b-length at least $\frac{1}{2} g(n, q) - 1$, with $g(n, q)$ given by equation (45). Because we are dealing with allowable rather than Q-allowable monomials, the other three types are irrelevant. This prepares us for the main lemma.

Lemma 9.9. *Provided $s \leqslant g(n, m)$, any $\mathfrak{J}_m$-monomial $z \in H(s)$ can be expressed as a $P(n)_*$-linear combination of:*
- (i) *Allowable $\mathfrak{J}_m$-monomials z', which are wild if and only if z is wild;*

(ii) *$*$-products of two or more tame $\circ$-monomials, of which at least one is a $\mathfrak{J}_m$-monomial.*

Just as we did for Corollary 9.8, we can recursively apply Corollary 9.8 and Lemma 9.9 to the $\circ$-monomials appearing in (ii).

Corollary 9.10. *If $s \leqslant g(n,m)$, any tame $\mathfrak{J}_m$-monomial in $H(s)$ can be expressed as a $P(n)_*$-linear combination of:*

(i) *Tame allowable $\mathfrak{J}_m$-monomials;*

(ii) *$*$-products of two or more tame allowable $\circ$-monomials, of which at least one is a $\mathfrak{J}_m$-monomial.* □

Proof of Lemma 9.9. We do not apply Lemma 9.7 directly. Any $\mathfrak{J}_m$-monomial has the form $z = y \circ ([v_q] - 1)$ for some $q > m$ and we apply that lemma to y instead, replacing y by an allowable $\circ$-monomial y' or a $*$-product $c = z_1 * z_2 * \cdots * z_r$ of tame $\circ$-monomials z_i.

In the second case, we write $c = d * z_r$, where $d = z_1 * z_2 * \cdots * z_{r-1}$. Then by equations (38) and (40),

$$\begin{aligned} c \circ ([v_q] - 1) = (d \circ ([v_q] - 1)) * z_r + (d \circ ([v_q] - 1)) * (z_r \circ ([v_q] - 1)) \\ + d * (z_r \circ ([v_q] - 1)), \end{aligned} \tag{52}$$

which shows by induction on r that $c \circ ([v_q] - 1)$ has the required form.

In the first case, we fix s and an arbitrarily large degree ceiling i_0, and show by *downward* induction on h that the lemma holds for all $z' = y' \circ ([v_h] - 1)$ with y' allowable that lie in $H(s)$ and have degree at most i_0. This statement is true vacuously for large enough h (depending on s and i_0), as there are only finitely many $\circ$-monomials in each $H(s)_i$. We assume it holds for all $h > q$ and prove it holds for $h = q$, assuming that $q > m$.

Case 1: y' has no factor e or $[v_k] - 1$.
Since $y' \in H(s + 2(2^q - 1))$, the b-length of z' is at most

$$\tfrac{1}{2}(s + 2(2^q - 1)) \leqslant \tfrac{1}{2}g(n,m) + 2^q - 1 \leqslant \tfrac{1}{2}g(n,q) - 1,$$

allowing for possible factors $a_{(i)}$. Thus z' is not a disallowed monomial of type (i); nor is it of type (ii), as it has no factor e. Therefore z' is allowable.

Case 2: y' has a factor e, but no factor $[v_k] - 1$.
This time, the b-length of z' is at most

$$\tfrac{1}{2}(s - 1 + 2(2^q - 1)) < \tfrac{1}{2}g(n,q) - 1$$

and is therefore allowable.

Case 3: $y' = u \circ ([v_k] - 1)$, where $k \leqslant q$.
Since y' is allowable, $z' = u \circ ([v_k] - 1) \circ ([v_q] - 1)$ remains allowable, by the form of [**BW07**, Defn. 8.1].

Case 4: $y' = u \circ ([v_k] - 1)$, *where* $k > q$.
Then $z' = (u \circ ([v_q] - 1)) \circ ([v_k] - 1)$, to which we apply the induction hypothesis. □

Lemma 9.11. *If* $s \leqslant g(n,m)$, *the* $*$-*ideal* $\mathfrak{J}_m \cap H(s)$ *in* $H(s)$ *is generated by the allowable* $\mathfrak{J}_m$-*monomials that lie in* $H(s)$.

Proof. Take a typical generator $c \circ ([v_q] - 1)$ of $\mathfrak{J}_m$. Since $1 \circ ([v_q] - 1) = 0$, we may assume $c \in \operatorname{Ker} \epsilon$. By Theorem 9.1, it is enough to consider the case $c = z_1 * z_2 * \cdots * z_r$, with $r > 0$ and each z_i a $\circ$-monomial. By (52) and induction on r, the ideal $\mathfrak{J}_m \cap H(s)$ is generated by elements of the form $z \circ ([v_q] - 1)$ with z a $\circ$-monomial and $q > m$. We apply Lemma 9.9 and Corollary 9.10 to $z \circ ([v_q] - 1)$. □

Reduction of $*\circ$-monomials

We assume from now on that $s \leqslant g(n,m)$. We take any $*$-product c of wild allowable $\circ$-monomials and tame $\circ$-monomials (whether allowable or not) that lie in $H(s)$; by Theorem 9.1 and Lemma 9.7, these generate $H(s)$ as a $P(n)_*$-module. Since any tame element is $*$-central by Proposition 6.7(c) and Theorem 6.9(b), we may write

$$c = z_1 * z_2 * \cdots * z_r * T, \tag{53}$$

where each z_i is a wild allowable $\circ$-monomial and T is a $P(n)_*$-linear combination of tame $*\circ$-monomials.

Our object is to reduce such c to the basis in Theorem 9.1. There are three steps: we must shuffle the factors z_i into their chosen order, deal with repeated wild factors, and finally expand T (which does not affect anything else).

Lemma 9.12. *Given* c *as in equation* (53), *suppose* $z_i > z_{i+1}$. *Then*

$$\begin{aligned} c &= z_1 * \dots z_{i-1} * z_{i+1} * z_i * z_{i+2} * \cdots * z_r * T \\ &\quad + z_1 * \cdots * z_{i-1} * z_{i+2} * \cdots * z_r * T', \end{aligned}$$

where $T' = v_n\, Qz_{i+1} * Qz_i * T$ *is tame.*

If z_i (*or* z_{i+1}) *is a* $\mathfrak{J}_m$-*monomial, so is* Qz_i (*or* Qz_{i+1}).

Proof. We plug in (41), $z_i * z_{i+1} = z_{i+1} * z_i + v_n Qz_{i+1} * Qz_i$. By (50), Qz_i and Qz_{i+1} are always tame $\circ$-monomials (though the extra e could disallow them). Moreover, if z is a $\mathfrak{J}_m$-monomial, so is Qz. □

Lemma 9.13. *Given* c *as in equation* (53), *suppose that* $z_i = z_{i+1} = e^{\circ h} \circ a^{\circ I,1} \circ b^{\circ J} \circ ([v^K] - 1)$ *as in* (49). *Then:*

(a) *If* $i_{n-2} = 0$, $c = z_1 * \cdots * z_{i-1} * z_{i+2} \cdots * z_r * T'$, *where* T' *has the form* $(z' + v_n z'') * T$ *and is tame. If* z_i *is a* $\mathfrak{J}_m$-*monomial, so are* z' *and* z''.

(b) *If* $i_{n-2} = 1$, c *is a* $P(n)_*$-*linear combination of terms of the forms*

$$\begin{gathered} z_1 * \cdots * z_{i-1} * z_i' * z_{i+2} * \cdots * z_r * T, \\ z_1 * \cdots * z_{i-1} * z_{i+2} * \cdots * z_r * T', \\ z_1 * \cdots * z_{i-1} * z'' * z_{i+2} * \cdots * z_r * T, \end{gathered}$$

where z_i' *and* z'' *are wild allowable and* $T' = t * T$ *is tame. If* z_i *is a* $\mathfrak{J}_m$-*monomial, so are* z_i', z'' *and* t.

Proof. We plug in the formula (11.6) or (11.7) from [**BW07**] for $Fz_i = z_i * z_i$, which (in either case) has the form $Fz_i = z' + v_n z''$, where z'' happens to be allowable. If z_i is a $\mathfrak{J}_m$-monomial, so are z' and z''.

If $i_{n-2} = 0$, z' and z'' are both tame, and we have (a).

If $i_{n-2} = 1$, z' and z'' are both wild. By Lemma 9.7, z' is a linear combination of wild allowable $\circ$-monomials z'_i and a linear combination t of tame $*\circ$-monomials. By Lemma 9.9, if z' is a $\mathfrak{J}_m$-monomial, so are z'_i and each term of t. This completes (b). □

We apply Lemmas 9.12 and 9.13 to (53) as often as possible; the process must terminate after a finite number of steps, as each application reduces the count r of wild $*$-factors z_i or improves the ordering of the wild $*$-factors (without changing r). The result is a linear combination of terms of the form (53) in which $z_1 * z_2 * \cdots * z_r$ is an ordered $*\circ$-monomial.

The final step is to expand the tame factor T by Corollary 9.8 as a linear combination of $*$-products $T' = t_1 * t_2 * \cdots * t_k$ of tame allowable $\circ$-monomials. As each t_i is $*$-central, we can shuffle the factors t_i in the product $z_1 * z_2 * \cdots * z_r * T'$ anywhere, to place all the $*$-factors in non-decreasing order, without introducing any extra terms. If any t_i is repeated in a term, [**BW07**, §11.4] shows that $Ft_i = t_i * t_i = 0$, which kills that term.

Proof of Theorem 9.2. By Lemma 9.11 and Theorem 9.1, every element of $\mathfrak{J}_m \cap H(s)$ is a linear combination of $*$-products (53) in which at least one $*$-factor is a $\mathfrak{J}_m$-monomial. This property is preserved by each application of Lemma 9.12 or Lemma 9.13; thus each c is transformed into a linear combination of ordered $*\circ$-monomials c' having the same property, as required. □

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