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ON THE ALGEBRAIC CLASSIFICATION OF K-LOCAL SPECTRA

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(*communicated by Brooke Shipley*)

Abstract

In 1996, Jens Franke proved the equivalence of certain triangulated categories possessing an Adams spectral sequence. One particular application of this theorem is that the $K_{(p)}$-local stable homotopy category at an odd prime can be described as the derived category of an abelian category. We explain this proof from a topologist's point of view.

1. Introduction

In 1985, Bousfield published a paper about the category of $E(1)$-local (or, equivalently, K-local) spectra at an odd prime. There, he gave an algebraic description of isomorphism classes of $E(1)$-local spectra in their homotopy category via $E(1)$-homology and a certain "k-invariant" coming from a d_2-differential in the Adams spectral sequence. However, with this setup he could only describe the morphisms in the $E(1)$-local stable homotopy category up to Adams filtration.

In 1996, Jens Franke constructed an abstract equivalence between certain triangulated categories possessing an Adams spectral sequence. Applying Franke's main theorem to the special case of $E(1)$-local spectra, one obtains an algebraic description of the homotopy category of $E(1)$-local spectra also covering the morphisms. In this paper, we give a streamlined exposition of Franke's result adapted to this special case:

Theorem (Franke)**.** There is an equivalence of categories

$$\mathcal{R}\colon \mathcal{D}^{2p-2}(\mathcal{B}) \longrightarrow \mathrm{Ho}(L_1\mathcal{S}),$$

where $\mathcal{D}^{2p-2}(\mathcal{B})$ denotes the derived category of twisted chain complexes over the abelian category $\mathcal{B}$, and $\mathrm{Ho}(L_1\mathcal{S})$ the homotopy category of $E(1)$-local spectra. Further, there are natural isomorphisms

$$E(1)_*(\mathcal{R}(C)) \cong \bigoplus_{i=0}^{2p-3} H_i(C)[i].$$

This paper is organised as follows: In the first chapter, the categories playing the main role for the construction are introduced: firstly, the category of so-called twisted

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chain complexes of $E(1)_*E(1)$-comodules and secondly, a certain diagram category of spectra with a fixed diagram shape and a model structure related to the model structure of $E(1)$-local spectra.

In the next section, a functor $\mathcal{Q}$ is constructed which gives an equivalence of twisted chain complexes and the homotopy category of above diagram spectra. In the third section this equivalence $\mathcal{Q}$ is extended to an equivalence of the derived category of twisted chain complexes and the homotopy category of $E(1)$-local spectra. Further, as Section 4 will show, this equivalence gives an "exotic model" for $E(1)$-local spectra: the homotopy categories of the chain complexes and $E(1)$-local spectra are equivalent as categories, yet there is no Quillen equivalence between them.

While we follow the overall structure of Franke's argument, we supply new and sometimes simpler proofs using methods which are more homotopy-theoretic. In particular, we show how this work is related to modern stable homotopy theory such as rigidity questions first studied by Stefan Schwede in [**Sch07**] and later by the author in [**Roi07**].

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2. The main ingredients

2.1. $E(1)_*E(1)$-comodules

We begin with describing an abelian category denoted $\mathcal{A}$, which is equivalent to the category of $E(1)_*E(1)$-comodules [**Bou85**, 10.3]. Bousfield describes $\mathcal{A}$ as follows: Let p be an odd prime and let $\mathcal{B} = \mathcal{B}(p)$ denote the category of $\mathbb{Z}_{(p)}$-modules together with Adams operations ψ^k, $k \in \mathbb{Z}^*_{(p)}$, satisfying the following:

For each $M \in \mathcal{B}(p)$,

- There is an eigenspace decomposition
$$M \otimes \mathbb{Q} \cong \bigoplus_{j \in \mathbb{Z}} W_{j(p-1)}$$
such that for all $w \in W_{j(p-1)}$ and $k \in \mathbb{Z}_{(p)}$:
$$(\psi^k \otimes id)w = k^{j(p-1)}w.$$
- For all $x \in M$ there is a finitely generated submodule $C(x)$ containing x satisfying: for all $m \geqslant 1$ there is an n such that the action of $\mathbb{Z}^*_{(p)}$ on $C(x)/p^m C(x)$ factors through the quotient of $(\mathbb{Z}/(p^{n+1}))^*$ by its subgroup of order $p-1$.

(Details can also be found in [**CCW07**].) To build the category $\mathcal{A}$ out of the above category, we additionally need the following: Let $T^{j(p-1)} \colon \mathcal{B} \longrightarrow \mathcal{B}$, $j \in \mathbb{Z}$, denote the following self-equivalence:

For all $M \in \mathcal{B}, T^{j(p-1)}(M) = M$ as a $\mathbb{Z}_{(p)}$-module, but on $T^{j(p-1)}(M)$, the Adams operation ψ^k now equals $k^{j(p-1)}\psi^k \colon M \longrightarrow M$ for all $k \in \mathbb{Z}$.

Now an object $\mathcal{M} \in \mathcal{A}$ is defined as a collection of modules $\mathcal{M} = (M_n)_{n\in\mathbb{Z}}, M_n \in \mathcal{B}$, together with isomorphisms

$$T^{p-1}(M_n) \longrightarrow M_{n+2p-2} \qquad \text{for all} \quad n \in \mathbb{Z}.$$

In this paper we will often make use of the following: Let X be a spectrum. Then the $E(1)_*E(1)$-comodule $E(1)_*(X)$ is an object of $\mathcal{A}$ in the above sense by taking $M_n := E(1)_n(X)$, and the operations ψ^k being the usual Adams operations.

From now on $\mathcal{B}$ will be viewed as the subcategory of $\mathcal{A}$ consisting of those objects $(M_n)_{n\in\mathbb{Z}}$ such that

$$M_n = \begin{cases} T^{j(p-1)}M, & \text{if } n = j(2p-2) \\ 0, & \text{else} \end{cases}$$

for some module M as above.

This describes a so-called *split* of period $2p-2$ of $\mathcal{A}$: $\mathcal{B} \subset \mathcal{A}$ is a Serre class such that

$$\begin{aligned} \bigoplus_{0\leqslant i<2p-2} \mathcal{B} &\longrightarrow \mathcal{A} \\ (B_i)_{0\leqslant i<2p-2} &\longmapsto \bigoplus_{0\leqslant i<2p-2} B_i[i] \end{aligned}$$

is an equivalence of categories, where $[i]$ denotes the i-fold internal shift in the grading; i.e., $M[i]_n = M_{n-i}$.

Remark 2.1. There exists a similar splitting of period $2p-2$ for the category of $E(n)_*E(n)$-comodules with arbitrary n and p odd. Moreover, the proof of the uniqueness theorem will not only work for the case p odd and $n = 1$ but for all p and n such that $n^2 + n < 2p - 2$, i.e., when the maximal injective dimension of $E(n)_*E(n)$-comodules is smaller than the splitting period [**Fra96**, Theorem 2.2.5].

2.2. Twisted chain complexes

In this section we describe the source of the equivalence to be constructed. Let $\mathcal{B}$ and T be as before.

Definition 2.2. The category $\mathcal{C}^{2p-2}(\mathcal{B})$ of *twisted chain complexes* is defined as follows:

The objects are chain complexes C_* with $C_i \in \mathcal{B}$ for all i together with an isomorphism of chain complexes

$$\alpha_C \colon T^{(2p-2)(p-1)}(C_*) \longrightarrow C_*[2p-2] = C_{*-2p+2}.$$

The morphisms in this category are morphisms of chain complexes $f\colon C_* \to D_*$ that are compatible with those isomorphisms; i.e., the following diagram commutes

$$\begin{array}{ccc} T^{(2p-2)(p-1)}(C_*) & \xrightarrow{\alpha_C} & C_*[2p-2] \\ \downarrow{\scriptstyle T^{(2p-2)(p-1)}(f)} & & \downarrow{\scriptstyle f[2p-2]} \\ T^{(2p-2)(p-1)}(D_*) & \xrightarrow{\alpha_D} & D_*[2p-2]. \end{array}$$

Such a chain complex C_* is called *injective* if each C_i is injective in $\mathcal{B}$. A morphism in $\mathcal{C}^{2p-2}(\mathcal{B})$ is called a *quasi-isomorphism* if it induces an isomorphim in cohomology. C_* is called *strictly injective* if it is injective and for each acyclic complex D_*, the chain complex $\mathrm{Hom}^*_{\mathcal{C}^{2p-2}(\mathcal{B})}(D_*, C_*)$ is again acyclic.

Proposition 2.3 (Franke)**.** *There is a model structure on $\mathcal{C}^{2p-2}(\mathcal{B})$ such that*

- *weak equivalences are the quasi-isomorphisms,*
- *cofibrations are the monomorphisms,*
- *fibrations are the degreewise split epimorphisms with strictly injective kernel.*

Notation. $\mathcal{D}^{2p-2}(\mathcal{B})$ denotes the derived category of $\mathcal{C}^{2p-2}(\mathcal{B})$, i.e., the homotopy category of this model category with respect to the above model structure.

2.3. Diagram categories of spectra

By a *spectrum* we mean the following: A spectrum X is a collection of simplicial sets X_n for $n \geqslant 0$ together with morphisms of simplicial sets $\sigma_n \colon \Sigma X_n \to X_{n+1}$. A morphism $f \colon X \to Y$ of spectra is a collection of morphisms $f_n \colon X_n \to Y_n$ of simplicial sets that commute with the structure maps σ_n; i.e., $\sigma_n \circ \Sigma f_n = f_{n+1} \circ \sigma_n$ [**BF78**]. Let $L_1\mathcal{S}$ denote the category of spectra together with the following model structure which is a localisation of the Bousfield-Friedlander model structure: a map $f \colon X \longrightarrow Y$ is

- a weak equivalence if $E(1)_*(f)$ is an isomorphism in $\mathcal{A}$,
- a cofibration if each $g_n \colon X_n \bigcup_{\Sigma X_{n-1}} \Sigma Y_{n-1} \longrightarrow Y_n$ is a cofibration of simplicial sets,
- a fibration if f has the right lifting property with respect to acyclic cofibrations.

(For a reference for this model structure, see e.g. the introduction of [**Bou79**].) Note that $\mathrm{Ho}(L_1\mathcal{S})$ is equivalent to the homotopy category of $E(1)$-local spectra.

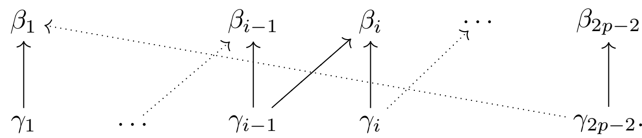
By a *poset* we mean a partially ordered finite set. For a poset C, $L_1\mathcal{S}^C$ denotes the category of C-shaped diagrams with values in $L_1\mathcal{S}$. For each $c \in C$ and $X \in L_1\mathcal{S}^C$, let X_c denote the value of X at the vertex c. For example, taking the poset $\underline{1} = (0 \to 1)$, an object of $L_1\mathcal{S}^{\underline{1}}$ is determined by a morphism $X_0 \longrightarrow X_1$ in $L_1\mathcal{S}$.

For fixed C, there is a model structure on $L_1\mathcal{S}^C$: A morphism $f \colon X \longrightarrow Y$ of diagrams is

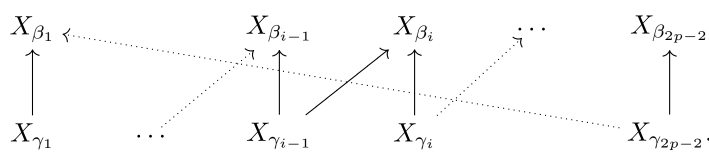
- a weak equivalence if it is a vertexwise weak equivalence in $L_1\mathcal{S}$ (i.e., $f_c \colon X_c \to Y_c$ induces an isomorphism in $E(1)$-homology for each $c \in C$),
- a fibration if it is vertexwise a fibration in $L_1\mathcal{S}$,
- a cofibration if for all $c \in C$, $X_c \coprod_{\mathrm{colim}_{c'<c} X_{c'}} Y_{c'} \longrightarrow Y_c$ is a cofibration.

This gives $L_1\mathcal{S}^C$ the structure of a stable model category [**DS95**, 10.13]; thus $\mathrm{Ho}(L_1\mathcal{S}^C)$ is a triangulated category (see e.g. [**Hov99**]).

From now on, C will be the poset consisting of elements β_i and γ_i for $i \in \mathbb{Z}/(2p-2)$ such that $\beta_i > \gamma_i$ and $\beta_i > \gamma_{i-1}$ for $i \in \mathbb{Z}/(2p-2)$; i.e.,



So an object X of $\mathrm{Ho}(L_1\mathcal{S}^C)$ is a diagram of spectra



N.B. It should be pointed out that we work in the homotopy category of a diagram category of spectra and not with diagrams taking values in the homotopy category of spectra.

In this particular case it is not too hard to characterise the fibrant and cofibrant objects of $L_1\mathcal{S}^C$:

- $X \in L_1\mathcal{S}^C$ is fibrant if and only if each $X_{\beta_i}, X_{\gamma_i}$ is fibrant in $L_1\mathcal{S}$;
- $X \in L_1\mathcal{S}^C$ is cofibrant if and only if each $X_{\beta_i}, X_{\gamma_i}$ is cofibrant in $L_1\mathcal{S}$ and for all $i \in \mathbb{Z}/(2p-2)$,

$$X_{\gamma_{i-1}} \vee X_{\gamma_i} \longrightarrow X_{\beta_i}$$

  is a cofibration in $L_1\mathcal{S}$.

2.4. Generalised $E(1)_*$-Moore spectra

Let us briefly recover the notion of $E(1)_*$-Moore spectra and their properties, as established in [**Bou85**, 8.7 and 9.5]. They will provide a useful tool for the proofs in the following sections.

Definition 2.4. Let $i \in \mathbb{Z}$ and $M \in \mathcal{B}[i]$. A spectrum $X \in \mathrm{Ho}(L_1\mathcal{S})$ is called an *$E(1)_*$-Moore spectrum of type* (M, i) if $E(1)_*(X) \cong M$.

The $E(1)_*$-Moore spectra satisfy the following properties:

- For each $M \in \mathcal{B}[i]$, an $E(1)_*$-Moore spectrum of type (M, i) exists and is unique up to weak equivalence.
- For two Moore spectra X of type (M, i) and Y of type (N, i), $E(1)_*$-homology induces an isomorphism

$$E(1)_*\colon [X, Y]_*^{L_1\mathcal{S}} \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{B}}(M, N).$$

(Here, $[A, B]_k^{E(1)}$ denotes $\mathrm{Hom}_{\mathrm{Ho}(L_1\mathcal{S})}(\Sigma^k A, B)$.) Further, one can use the $E(1)$-Adams spectral sequence to compute homotopy classes of morphisms between sums of $E(1)_*$-Moore spectra.As the splitting index of $\mathcal{B}$ in $\mathcal{A}$ is larger than the injective dimension of $\mathcal{B}$ (i.e., $p > 2$), the Adams spectral sequence collapses and one obtains

Proposition 2.5 (Bousfield). *Let $k \in \mathbb{Z}$, $M_i, N_i \in \mathcal{B}[i]$, X_i and Y_i Moore spectra of type (M_i, i) and (N_i, i), respectively,*

$$X = \bigvee_{i=0}^{2p-3} X_i \quad \textit{and} \quad Y = \bigvee_{i=0}^{2p-3} Y_i.$$

Then

$$[X, Y]_k^{L_1\mathcal{S}} \cong \bigoplus_{s=0}^{2} \mathrm{Ext}_{\mathcal{B}}^s(E(1)_*(X), E(1)_{*+s+k}(Y)).$$

In particular,

$$[X_i, Y_i]_0^{L_1\mathcal{S}} \cong \mathrm{Hom}_{\mathcal{B}}(M_i, N_i), \qquad [X_i, Y_i]_1^{L_1\mathcal{S}} = 0,$$
$$[X_{i-1}, Y_i]_0^{L_1\mathcal{S}} \cong \mathrm{Ext}^1_{\mathcal{B}}(M_{i-1}, N_i), \qquad [X_{i-1}, Y_i]_1^{L_1\mathcal{S}} \cong \mathrm{Hom}_{\mathcal{B}}(M_{i-1}, N_i).$$

3. The functor $\mathcal{Q}$

3.1. Defining $\mathcal{Q}$

We would now like to build twisted chain complexes out of diagrams of spectra. Let X be an object of $\mathrm{Ho}(L_1\mathcal{S}^C)$. The given morphism

$$p_i\colon X_{\gamma_i} \longrightarrow X_{\beta_i}$$

as a part of the diagram X induces a morphism in $\mathcal{A}$

$$\pi_i := E(1)_*(p_i)[-i]\colon E(1)_*(X_{\gamma_i})[-i] \longrightarrow E(1)_*(X_{\beta_i})[-i].$$

Notation. $G_i(X) := E(1)_*(X_{\gamma_i})[-i]$ and $B_i(X) := E(1)_*(X_{\beta_i})[-i]$.

The objects $B_i(X)$ will play the role of the boundaries in the chain complex $C_*(X)$ to be built, and the $G_i(X)$'s will play the role of the quotient of the chains by the boundaries.

Now we would like to assign to each $k_i\colon X_{\gamma_{i-1}} \longrightarrow X_{\beta_i} \in \mathrm{Ho}(L_1\mathcal{S}^{\underline{1}})$ (see Section 1.5) an exact triangle

$$X_{\gamma_{i-1}} \xrightarrow{k_i} X_{\beta_i} \longrightarrow \mathrm{cone}(k_i) \longrightarrow \Sigma X_{\gamma_{i-1}}$$

in a functorial(!) way. This is done by using Franke's cone functor

$$\mathrm{cone}\colon \mathrm{Ho}(L_1\mathcal{S}^{\underline{1}}) \longrightarrow \mathrm{Ho}(L_1\mathcal{S}), (f\colon A \to B) \mapsto \mathrm{Hocolim}(* \leftarrow A \xrightarrow{f} B).$$

Notation. Define $C_i(X) := E(1)_*(\mathrm{cone}(k_i))[-i] \in \mathcal{A}$.

Applying $E(1)_*$ to the above exact triangle we obtain a long exact sequence

$$\cdots \to G_{i-1}(X)[-1] \to B_i(X) \to C_i(X) \to G_{i-1}(X) \to B_i(X)[1] \to \dots \quad (1)$$

Now let $\mathcal{L}$ be the full subcategory of $\mathrm{Ho}(L_1\mathcal{S}^C)$ consisting of all objects X such that:

- $G_i(X)$ and $B_i(X)$ are not just objects in $\mathcal{A}$ but actually objects in the splitting $\mathcal{B}$ of $\mathcal{A}$ (see Section 1.1). This implies that X_{γ_i} and X_{β_i} are $E(1)_*$-Moore spectra of type $(G_i(X), i)$ and $(B_i(X), i)$, respectively.
- $\pi_i\colon G_i(X) \longrightarrow B_i(X)$ is surjective for all i.

So if X is an object of $\mathcal{L}$, what does this mean for the long exact sequence (1)? If $X \in \mathcal{L}$, then by definition

$$G_{i-1}(X)[-1] \in \mathcal{B}[-1] \quad \text{and} \quad B_i(X) \in \mathcal{B}.$$

Therefore, by definition of $\mathcal{B}$, the maps

$$G_{i-1}(X)[-1] \longrightarrow B_i(X) \quad \text{and} \quad G_{i-1}(X) \longrightarrow B_i(X)[1]$$

in the long exact sequence (1) are zero. Thus, (1) splits into short exact sequences

$$0 \longrightarrow B_i(X) \xrightarrow{\iota_i} C_i(X) \xrightarrow{\rho_i} G_{i-1}(X) \longrightarrow 0. \tag{2}$$

To make a chain complex out of the objects $C_i(X)$, we need a differential $d\colon C_i(X) \longrightarrow C_{i-1}(X)$ which we define as the composition

$$C_i(X) \xrightarrow{\rho_i} G_{i-1}(X) \xrightarrow{\pi_{i-1}} B_{i_1}(X) \xrightarrow{\iota_{i-1}} C_{i-1}(X). \tag{3}$$

Then d^2 is indeed zero since it factors over $\rho_{i-1} \circ \iota_{i-1}$, which is part of the short exact sequence (2) and thus zero itself. The morphisms ρ_i and π_i are surjective since $X \in \mathcal{L}$, so $\mathrm{im}(d) = B_*(X)$. Also, because of the shape of the underlying poset we work with, $C_*(X)$ is $2p-2$-twistperiodic. So this construction gives a functor

$$\mathcal{Q}\colon \mathcal{L} \longrightarrow \mathcal{C}^{2p-2}(\mathcal{B}), \quad X \longmapsto C_*(X).$$

Theorem 3.1. *The functor $\mathcal{Q}$ is an equivalence of categories.*

The proof will follow in the next two subsections.

3.2. $\mathcal{Q}$ is full and faithful

Proposition 3.2. *The functor $\mathcal{Q}$ is full and faithful.*

Proof. We have to prove that for objects X and $\tilde{X}$ of $\mathcal{L}$, the map

$$M := \mathrm{Hom}_{\mathrm{Ho}(L_1\mathcal{S}^C)}(X, \tilde{X}) \xrightarrow{q} N \tag{4}$$

with

$$N := \bigoplus_i \mathrm{Hom}_{\mathcal{B}^{\underline{1}}}\big((B_i(X) \to C_i(\tilde{X})), (B_i(\tilde{X}) \to C_i(\tilde{X}))\big)$$

induced by $\mathcal{Q}$ is injective and its image consists of those morphisms that are morphisms of chain complexes. A morphism $f = (f_i)_i \in N$ is also a morphism of chain complexes if and only if it is compatible with the differentials, i.e. (remembering the definition of d) if and only if it makes the outer square in the following diagram commute.

$$\begin{array}{ccccccc} C_i(X) & \xrightarrow{\rho_i} & G_{i-1}(X) & \xrightarrow{\pi_{i-1}} & B_{i-1}(X) & \xrightarrow{\iota_{i-1}} & C_{i-1}(X) \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle \overline{f}_i} & & \downarrow{\scriptstyle f_{i-1}} & & \downarrow{\scriptstyle f_{i-1}} \\ C_i(\tilde{X}) & \xrightarrow{\rho_i} & G_{i-1}(\tilde{X}) & \xrightarrow{\pi_{i-1}} & B_{i-1}(\tilde{X}) & \xrightarrow{\iota_{i-1}} & C_{i-1}(\tilde{X}). \end{array}$$

Since $f \in N$ and $G_{i-1} \cong C_i/B_i$, we know that the first and the third small square commute. So, f is a morphism of chain complexes if and only if the middle small square commutes, i.e. if and only if f lies in the kernel of the map

$$D\colon N \longrightarrow \bigoplus_i \operatorname{Hom}_{\mathcal{A}}\big(G_i(X), B_i(\tilde{X})\big),$$

where D sends $f = (f_i)_i \in N$ to $f_i \circ \pi_i - \pi_i \circ \overline{f}_{i+1}$, with $\overline{f}_i\colon G_{i-1}(X) \to G_{i-1}(\tilde{X})$ induced by f_i.

So, showing that $\mathcal{Q}$ is full and faithful is equivalent to showing that

$$0 \longrightarrow M \xrightarrow{q} N \xrightarrow{D} \bigoplus_i \operatorname{Hom}_{\mathcal{A}}\big(G_i(X), B_i(\tilde{X})\big) \tag{5}$$

is exact. To show the exactness of (5), we would first like to get a description of M and N in terms of some other exact sequences.

We start with M. A morphism of $\operatorname{Hom}_{\operatorname{Ho}(L_1\mathcal{S}^C)}(X, \tilde{X})$ consists of the following data: the morphisms at each vertex plus commutativity conditions coming from the shape of C. To be more precise, the mapping space $\operatorname{map}_{L_1\mathcal{S}^C}(X, \tilde{X})$ (see Section 5) is the upper left corner of the following pullback square of mapping spaces

$$\begin{array}{ccc} \operatorname{map}_{L_1\mathcal{S}^C}(X, \tilde{X}) & \longrightarrow & \prod_i \operatorname{map}_{L_1\mathcal{S}}(X_{\beta_i}, \tilde{X}_{\beta_i}) \\ \downarrow & & \downarrow \\ \prod_i \operatorname{map}_{L_1\mathcal{S}}(X_{\gamma_i}, \tilde{X}_{\gamma_i}) & \longrightarrow & \prod_i \operatorname{map}_{L_1\mathcal{S}}(X_{\gamma_{i-1}}, \tilde{X}_{\beta_i}) \times \prod_i \operatorname{map}_{L_1\mathcal{S}}(X_{\gamma_i}, \tilde{X}_{\beta_i}), \end{array}$$

where the lower left and upper right corner contain the information about the maps at each vertex and the lower right corner plus the maps into it give the commutativity conditions. The right vertical map is the precomposition with the maps

$$X_{\gamma_{i-1}} \vee X_{\gamma_i} \longrightarrow X_{\beta_i}, \tag{6}$$

and the lower horizontal map is the composition with the maps

$$\tilde{X}_{\gamma_i} \longrightarrow \tilde{X}_{\beta_i}, \qquad \text{resp.} \qquad \tilde{X}_{\gamma_i} \longrightarrow \tilde{X}_{\beta_{i+1}}.$$

Without loss of generality one can assume X to be cofibrant and $\tilde{X}$ to be fibrant (see Section 1.3). Since (6) is then a cofibration for each i and $L_1\mathcal{S}$ is a simplicial model category [**GJ99**, II.3], the right vertical map in the pullback square is a fibration. Therefore, the pullback square is a homotopy pullback square, and the left vertical map is a fibration as well.

From a homotopy pullback square one obtains a long exact homotopy sequence. Since X is cofibrant and $\tilde{X}$ fibrant, we have as homotopy groups

$$\pi_k \operatorname{map}_{L_1\mathcal{S}}(X_{\gamma_i}, \tilde{X}_{\gamma_i}) \cong [X_{\gamma_i}, \tilde{X}_{\gamma_i}]_k^{E(1)}$$

(analogously for the other indices), and

$$\pi_0 \operatorname{map}_{L_1\mathcal{S}^C}(X, \tilde{X}) = M \cong \operatorname{Hom}_{\operatorname{Ho}(L_1\mathcal{S}^C)}(X, \tilde{X}).$$

Writing down the first five terms of the long exact homotopy sequence we obtain

$$\begin{array}{c} \bigoplus_i [X_{\gamma_i}, \tilde{X}_{\gamma_i}]_1^{E(1)} \oplus \bigoplus_i [X_{\beta_i}, \tilde{X}_{\beta_i}]_1^{E(1)} \\ \downarrow \\ \bigoplus_i [X_{\gamma_{i-1}}, \tilde{X}_{\beta_i}]_1^{E(1)} \oplus \bigoplus_i [X_{\gamma_i}, \tilde{X}_{\beta_i}]_1^{E(1)} \\ \downarrow \\ M \\ \downarrow \\ \bigoplus_i [X_{\gamma_i}, \tilde{X}_{\gamma_i}]_0^{E(1)} \oplus \bigoplus_i [X_{\beta_i}, \tilde{X}_{\beta_i}]_0^{E(1)} \\ \downarrow \\ \bigoplus_i [X_{\gamma_{i-1}}, \tilde{X}_{\beta_i}]_0^{E(1)} \oplus \bigoplus_i [X_{\gamma_i}, \tilde{X}_{\beta_i}]_0^{E(1)}. \end{array} \tag{7}$$

Next, we would like to simplify the terms of this sequence using the results of Subsection 2.4 for $E(1)_*$-Moore spectra. By definition of $\mathcal{L}$, the spectra X_{γ_i} and X_{β_i} are $E(1)_*$-Moore spectra of type $(G_i(X), i)$ and $(B_i(X), i)$, respectively. Using Proposition 2.5, we have

$$[X_{\beta_i}, \tilde{X}_{\beta_i}]_1^{E(1)} = 0 = [X_{\gamma_i}, \tilde{X}_{\gamma_i}]_1^{E(1)} = [X_{\gamma_i}, \tilde{X}_{\beta_i}]_1^{E(1)}$$

and

$$\begin{aligned} [X_{\beta_i}, \tilde{X}_{\beta_i}]_0^{E(1)} &\cong \operatorname{Hom}_{\mathcal{B}}(B_i(X), B_i(\tilde{X})), \\ [X_{\gamma_i}, \tilde{X}_{\gamma_i}]_0^{E(1)} &\cong \operatorname{Hom}_{\mathcal{B}}(G_i(X), G_i(\tilde{X})), \\ [X_{\gamma_i}, \tilde{X}_{\beta_i}]_0^{E(1)} &\cong \operatorname{Hom}_{\mathcal{B}}(G_i(X), B_i(\tilde{X})), \\ [X_{\gamma_{i-1}}, \tilde{X}_{\beta_i}]_1^{E(1)} &\cong \operatorname{Hom}_{\mathcal{B}}(G_{i-1}(X), B_i(\tilde{X})), \\ [X_{\gamma_{i-1}}, \tilde{X}_{\beta_i}]_0^{E(1)} &\cong \operatorname{Ext}^1_{\mathcal{B}}(G_{i-1}(X), B_i(\tilde{X})). \end{aligned}$$

Putting this into the sequence (7), we obtain the exact sequence

$$\begin{array}{c} 0 \\ \downarrow \\ \bigoplus_i \mathrm{Hom}_{\mathcal{B}}(G_{i-1}(X), B_i(\tilde{X})) \\ \downarrow \\ M \\ \downarrow \\ \bigoplus_i \mathrm{Hom}_{\mathcal{B}}(G_i(X), G_i(\tilde{X})) \oplus \bigoplus_i \mathrm{Hom}_{\mathcal{B}}(B_i(X), B_i(\tilde{X})) \\ \downarrow \\ \bigoplus_i \mathrm{Ext}^1_{\mathcal{B}}(G_{i-1}(X), B_i(\tilde{X})) \oplus \bigoplus_i \mathrm{Hom}_{\mathcal{B}}(G_i(X), B_i(\tilde{X})). \end{array} \tag{8}$$

Now we would like to find a similar description of

$$N = \bigoplus_i \mathrm{Hom}_{\mathcal{B}^{\perp}} \big((B_i(X) \to C_i(X)), (B_i(\tilde{X}) \to C_i(\tilde{X}))\big).$$

As mentioned before, morphisms in N can be viewed as morphisms of the short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_i(X) & \longrightarrow & C_i(X) & \longrightarrow & G_{i-1}(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle \overline{f}_i} & & \\ 0 & \longrightarrow & B_i(\tilde{X}) & \longrightarrow & C_i(\tilde{X}) & \longrightarrow & G_{i-1}(\tilde{X}) & \longrightarrow & 0. \end{array}$$

Thus, we get a canonical map

$$N \longrightarrow N' := \bigoplus_i \mathrm{Hom}_{\mathcal{B}}(B_i(X), B_i(\tilde{X})) \oplus \bigoplus_i \mathrm{Hom}_{\mathcal{B}}(G_i(X), G_i(\tilde{X})) \tag{9}$$

by sending $f \in N$ to $(f_i, \overline{f}_i)_i$. The kernel of this map consists of morphisms of the same exact sequences of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_i(X) & \longrightarrow & C_i(X) & \longrightarrow & G_{i-1}(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & B_i(\tilde{X}) & \longrightarrow & C_i(\tilde{X}) & \longrightarrow & G_{i-1}(\tilde{X}) & \longrightarrow & 0. \end{array}$$

Every Φ of the form

$$C_i(X) \longrightarrow G_{i-1}(X) \xrightarrow{\phi} B_i(\tilde{X}) \longrightarrow C_i(\tilde{X})$$

lies in the kernel of (10). From applying the snake lemma to the above diagram it also follows that every Φ in the kernel looks exactly like this. Therefore, the kernel of (10) is isomorphic to $\bigoplus_i \mathrm{Hom}_{\mathcal{B}}(G_{i-1}(X), B_i(\tilde{X}))$. Consequently,

$$0 \longrightarrow \bigoplus_i \mathrm{Hom}_{\mathcal{B}}(G_{i-1}(X), B_i(\tilde{X})) \longrightarrow N \longrightarrow N' \tag{10}$$

is exact.

The next question is: when is an element of N' hit by an element of N? In other words, given $f_B\colon B_i(X) \to B_i(\tilde{X})$ and $f_G\colon G_{i-1}(X) \to G_{i-1}(\tilde{X})$, when is there a map $f_C\colon C_i(X) \to C_i(\tilde{X})$ making the following diagram commute?

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_i(X) & \longrightarrow & C_i(X) & \longrightarrow & G_{i-1}(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_B} & & \downarrow{\scriptstyle f_C} & & \downarrow{\scriptstyle f_G} & & \\ 0 & \longrightarrow & B_i(\tilde{X}) & \longrightarrow & C_i(\tilde{X}) & \longrightarrow & G_{i-1}(\tilde{X}) & \longrightarrow & 0. \end{array}$$

The upper sequence corresponds to an element $S \in \mathrm{Ext}^1_{\mathcal{B}}(G_{i-1}(X), B_i(X))$, the lower one to an element $\tilde{S} \in \mathrm{Ext}^1_{\mathcal{B}}(G_{i-1}(\tilde{X}), B_i(\tilde{X}))$. The maps f_B and f_G give rise to maps

$$\begin{aligned} (f_B)_*\colon\ & \mathrm{Ext}^1_{\mathcal{B}}(G_{i-1}(X), B_i(X)) \longrightarrow \mathrm{Ext}^1_{\mathcal{B}}(G_{i-1}(X), B_i(\tilde{X})), \\ (f_G)^*\colon\ & \mathrm{Ext}^1_{\mathcal{B}}(G_{i-1}(\tilde{X}), B_i(\tilde{X})) \longrightarrow \mathrm{Ext}^1_{\mathcal{B}}(G_{i-1}(X), B_i(\tilde{X})). \end{aligned}$$

So for given f_B and f_G there is a morphism f_C making the above diagram commute if and only if $(f_B)_*(S) = (f_G)^*(\tilde{S})$. It follows that

$$\begin{array}{c} 0 \\ \downarrow \\ \bigoplus_i \mathrm{Hom}_{\mathcal{B}}(G_{i-1}(X), B_i(\tilde{X})) \\ \downarrow \\ N \\ \downarrow \\ N' = \bigoplus_i \mathrm{Hom}_{\mathcal{B}}(B_i(X), B_i(\tilde{X})) \oplus \bigoplus_i \mathrm{Hom}_{\mathcal{B}}(G_i(X), G_i(\tilde{X})) \\ \downarrow \\ \bigoplus_i \mathrm{Ext}^1_{\mathcal{B}}(G_{i-1}(X), B_i(\tilde{X}))k \end{array} \tag{11}$$

is exact, where the last map sends a pair (f_B, f_G) to $(f_B)_*(S) - (f_G)^*(\tilde{S})$. Putting this sequence together with the sequence (9), we obtain

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\bigoplus_i \mathrm{Hom}_{\mathcal{B}}(G_{i-1}(X), B_i(\tilde{X})) & = & \bigoplus_i \mathrm{Hom}_{\mathcal{B}}(G_{i-1}(X), B_i(\tilde{X})) \\
\downarrow a & & \downarrow b \\
M & \xrightarrow{\quad q \quad} & N \\
\downarrow & & \downarrow \\
\begin{array}{c} \bigoplus_i \mathrm{Hom}_{\mathcal{B}}(B_i(X), B_i(\tilde{X})) \\ \oplus \\ \bigoplus_i \mathrm{Hom}_{\mathcal{B}}(G_i(X), G_i(\tilde{X})) \end{array} & = & \begin{array}{c} \bigoplus_i \mathrm{Hom}_{\mathcal{B}}(B_i(X), B_i(\tilde{X})) \\ \oplus \\ \bigoplus_i \mathrm{Hom}_{\mathcal{B}}(G_i(X), G_i(\tilde{X})) \end{array} \\
\downarrow & & \downarrow \\
\begin{array}{c} \bigoplus_i \mathrm{Ext}^1_{\mathcal{B}}(G_{i-1}(X), B_i(\tilde{X})) \\ \oplus \\ \bigoplus_i \mathrm{Hom}_{\mathcal{B}}(G_i(X), B_i(\tilde{X})) \end{array} & \xrightarrow{pr} & \bigoplus_i \mathrm{Ext}^1_{\mathcal{B}}(G_{i-1}(X), B_i(\tilde{X})),
\end{array}
$$

where the second horizontal arrow is the morphism induced by the functor $\mathcal{Q}$ and the last one is the projection onto the first summand. One has to check that all the squares actually commute, which they do.

Then, a small diagram chase shows that q is injective. Also, by construction of q, in

$$0 \longrightarrow M \xrightarrow{q} N \xrightarrow{D} \bigoplus_i \mathrm{Hom}_{\mathcal{A}}\big(G^i(X), B^i(\tilde{X})\big), \tag{12}$$

the image of q lies in the kernel of D. With a slightly bigger diagram chase, it follows that the image of q is the entire kernel of D.

This completes the proof that $\mathcal{Q}$ is full and faithful. □

3.3. $\mathcal{Q}$ is essentially surjective

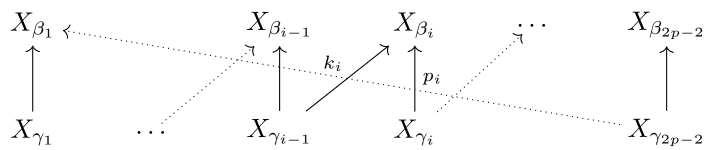
To complete the proof of Theorem 3.1, i.e. that

$$\mathcal{Q}\colon \mathcal{L} \longrightarrow \mathcal{C}^{2p-2}(\mathcal{B})$$

is an equivalence of categories, it is left to show that $\mathcal{Q}$ is essentially surjective; i.e., each $C_* \in \mathcal{C}^{2p-2}(\mathcal{B})$ is isomorphic to an object in the image of $\mathcal{Q}$.

Proposition 3.3. *The functor $\mathcal{Q}$ is essentially surjective.*

Proof. So let C_* be an object of $\mathcal{C}^{2p-2}(\mathcal{B})$, and let $B_*(C)$ denote the boundaries of C_* and $G_{*-1}(C) = C_*/B_*$ the quotient of C_* by its boundaries. We know that in a potential preimage of C_*



in $\mathcal{L}$ the vertices X_{γ_i} and X_{β_i} have to be $E(1)_*$-Moore spectra of type $(G_i(C), i)$ and $(B_i(C), i)$, respectively. (Remember that such $E(1)_*$-Moore spectra exist and are unique up to weak equivalence.) Now we have to construct the correct morphisms

$$p_i\colon X_{\gamma_i} \longrightarrow X_{\beta_i} \quad \text{and} \quad k_i\colon X_{\gamma_{i-1}} \longrightarrow X_{\beta_i}.$$

The differential d of C_* can be factored as

$$d^i\colon C_i \xrightarrow{\partial_i} B_{i-1}(C) \xrightarrow{\iota_{i-1}} C_{i-1},$$

where ∂_i is a surjection and ι_{i-1} the inclusion of the boundaries into the chains. Since $\partial_i(B_{i+1}(C)) = 0$, we have a morphism $\pi_i\colon G_i(C) \longrightarrow B_i(C)$. From Subsection 2.4 we know that

$$E(1)_*\colon [X_{\gamma_i}, X_{\beta_i}]_0^{L_1\mathcal{S}} \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{B}}(G_i(C), B_i(C))$$

is an isomorphism, so let $p_i\colon X_{\gamma_i} \longrightarrow X_{\beta_i}$ be the morphism $p_i = E(1)_*^{-1}(\pi_i)$. We also know from Proposition 2.5 that

$$[X_{\gamma_{i-1}}, X_{\beta_i}]_0^{L_1\mathcal{S}} \cong \mathrm{Ext}^1_{\mathcal{B}}(G_{i-1}(C), B_i(C)). \tag{13}$$

Let κ_i denote the class in Ext of the exact sequence

$$0 \longrightarrow B_i(C) \xrightarrow{\iota_i} C_i \xrightarrow{\rho_i} G_{i-1}(C) \longrightarrow 0$$

and k_i the corresponding morphism $X_{\gamma_{i-1}} \longrightarrow X_{\beta_i}$ under the isomorphism (13).

With the above choices, the diagram X satisfies $\mathcal{Q}(X) \cong C_*$. This completes the proof that $\mathcal{Q}$ is essentially surjective and consequently the proof of Theorem 3.1 that $\mathcal{Q}$ is an equivalence of categories. □

4. The reconstruction functor $\mathcal{R}$

4.1. Defining $\mathcal{R}$

In the last section we showed that

$$\mathcal{Q}\colon \mathcal{L} \longrightarrow \mathcal{C}^{2p-2}(\mathcal{B})$$

is an equivalence of categories. To prove the main theorem, we would like to build an equivalence of categories

$$\mathcal{R}\colon \mathcal{D}^{2p-2}(\mathcal{B}) = \mathrm{Ho}(\mathcal{C}^{2p-2}(\mathcal{B})) \longrightarrow \mathrm{Ho}(L_1\mathcal{S})$$

with the help of $\mathcal{Q}$. Define

$$\mathcal{R}' := \mathrm{Hocolim} \circ \mathcal{Q}^{-1}\colon \mathcal{C}^{2p-2}(\mathcal{B}) \longrightarrow \mathrm{Ho}(L_1\mathcal{S}^{C}) \longrightarrow \mathrm{Ho}(L_1\mathcal{S}).$$

We would like to show that $\mathcal{R}'$ factors over the derived category of $\mathcal{C}^{2p-2}(\mathcal{B})$. This will give us the desired reconstruction functor $\mathcal{R}$. We will show in this section that it is an equivalence of categories.

However, we first look at some properties of

$$E(1)_* \circ \mathcal{R}' \colon \mathcal{C}^{2p-2}(\mathcal{B}) \longrightarrow \mathcal{A}.$$

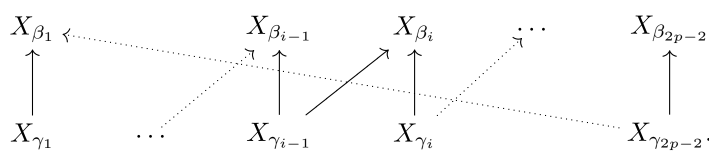
Lemma 4.1.

$$E(1)_*(\mathrm{Hocolim}_C X) \cong \bigoplus_i H_i(\mathcal{Q}(X))[i].$$

Proof. By definition,

$$\mathrm{Hocolim}_C X = \mathrm{colim}_C X^{cof},$$

where X^{cof} denotes a cofibrant replacement of $X \in L_1\mathcal{S}^C$. Now let us look at the colimit of a diagram



We have morphisms

$$X_{\gamma_i} \vee X_{\gamma_{i-1}} \longrightarrow X_{\beta_i}$$

for each i. Taking the wedge sum of those morphisms for even i, one obtains a morphism

$$\bigvee_{i=1}^{2p-2} X_{\gamma_i} \longrightarrow \bigvee_{i \text{ even}} X_{\beta_i},$$

and simultaneously, for odd i,

$$\bigvee_{i=1}^{2p-2} X_{\gamma_i} \longrightarrow \bigvee_{i \text{ odd}} X_{\beta_i}.$$

The colimit of the diagram X is the same as the colimit of the following diagram

$$\bigvee_{i \text{ odd}} X_{\beta_i} \longleftarrow \bigvee_{i=1}^{2p-2} X_{\gamma_i} \longrightarrow \bigvee_{i \text{ even}} X_{\beta_i};$$

i.e., the colimit of X is the pushout of the upper left corner in

$$\begin{array}{ccc} \bigvee_{i=1}^{2p-2} X_{\gamma_i} & \longrightarrow & \bigvee_{i \text{ even}} X_{\beta_i} \\ \downarrow & & \downarrow \\ \bigvee_{i \text{ odd}} X_{\beta_i} & \longrightarrow & \mathrm{colim}_C X. \end{array}$$

Without loss of generality, let X be cofibrant, so that the colimit of X models the homotopy colimit. Then the left vertical and upper horizontal maps in the square are cofibrations, and the pushout diagram is also a homotopy pushout diagram. Therefore,

$$\bigvee_{i=1}^{2p-2} X_{\gamma_i} \to \bigvee_{i \text{ odd}} X_{\beta_i} \vee \bigvee_{i \text{ even}} X_{\beta_i} \cong \bigvee_{i=1}^{2p-2} X_{\beta_i} \to \mathrm{Hocolim}_C X \to \Sigma\Big(\bigvee_{i=1}^{2p-2} X_{\gamma_i}\Big)$$

is an exact triangle in $\mathrm{Ho}(L_1\mathcal{S})$. Applying $E(1)$-homology, one obtains a long exact sequence

$$\begin{aligned}\cdots \bigoplus_i E(1)_n(X_{\gamma_i}) \to \bigoplus_i E(1)_n(X_{\beta_i}) \to E(1)_n(\mathrm{Hocolim}_C X)\\ \to \bigoplus_i E(1)_{n-1}(X_{\gamma_i}) \to \bigoplus_i E(1)_{n-1}(X_{\beta_i}) \dots .\end{aligned} \tag{14}$$

The map

$$\oplus\pi_i[i+1]\colon \bigoplus_i E(1)_{n-1}(X_{\gamma_i}) \to \bigoplus_i E(1)_{n-1}(X_{\beta_i})$$

is surjective for all n by assumption since $X \in \mathcal{L}$, so

$$\bigoplus_i E(1)_n(X_{\gamma_i}) \longrightarrow E(1)_n(\mathrm{Hocolim}_C X)$$

is the zero map. So we get a short exact sequence in $\mathcal{A}$

$$0 \to E(1)_*(\mathrm{Hocolim}_C X) \longrightarrow \bigoplus_i E(1)_{*-1}(X_{\gamma_i}) \xrightarrow{\oplus\pi_i[i+1]} E(1)_{*-1}(X_{\beta_i}) \to 0.$$

Therefore,

$$E(1)_*(\mathrm{Hocolim}_C X) \cong \bigoplus_i \ker(\pi_i)[i+1].$$

Now we prove that $\ker(\pi_i)$ is isomorphic to $H_{i+1}(\mathcal{Q}(X))$. Let us remember how the differential d of $C_*(X) = \mathcal{Q}(X)$ had been defined (see Section 2.1). Here is d^2:

$$\begin{aligned}C_i(X) \xrightarrow{\rho_i} G_{i-1}(X) \xrightarrow{\pi_{i-1}} B_{i-1}(X) \xrightarrow{\iota_{i-1}} C_{i-1}(X)\\ \xrightarrow{\rho_{i-1}} G_{i-2}(X) \xrightarrow{\pi_{i-2}} B_{i-2}(X) \xrightarrow{\iota_{i-2}} C_{i-2}(X).\end{aligned}$$

We have $\mathrm{im}(\iota_{i-1}) = \ker(\rho_{i-1})$ since they are part of the short exact sequence (2). Since ρ_i and π_{i-1} are surjective, $\mathrm{im}(d) = \mathrm{im}(\iota_{i-1})$. We also have

$$\ker(d) = \ker(\pi_{i-2} \circ \rho_{i-1}).$$

By basic algebra,

$$\ker(\pi_{i-2}) \cong \frac{\ker(\pi_{i-2} \circ \rho_{i-1})}{\ker(\rho_{i-1})} \cong \frac{\ker(d)}{\mathrm{im}(\iota_{i-1})} \cong \frac{\ker(d)}{\mathrm{im}(d)} \cong H_{i-1}(\mathcal{Q}(X)).$$

It follows that

$$E(1)_*(\mathrm{Hocolim}_C X) \cong \bigoplus_i H_i(\mathcal{Q}(X))[i]. \qquad \square$$

Because of the lemma we now see that the functor $E(1)_* \circ \mathcal{R}'$ sends weak equivalences (i.e. quasi-isomorphisms) in $\mathcal{C}^{2p-2}(\mathcal{B})$ to isomorphisms in $\mathcal{A}$ and thus factors over $\mathcal{D}^{2p-2}(\mathcal{B}) = \mathrm{Ho}(\mathcal{C}^{2p-2}(\mathcal{B}))$. In other words, for C_*, D_* quasi-isomorphic chain complexes we get

$$E(1)_*(\mathcal{R}'(C_*)) \cong \bigoplus_i H_i(C_*)[i] \cong \bigoplus_i H_i(D_*)[i] \cong E(1)_*(\mathcal{R}'(D_*)).$$

However, two objects of $\mathrm{Ho}(L_1\mathcal{S})$ are isomorphic if and only if there is a morphism of spectra inducing an isomorphism in $E(1)$-homology, so $\mathcal{R}'(C_*) \cong \mathcal{R}'(D_*)$ for quasi-isomorphic C_* and D_*, and consequently $\mathcal{R}'$ itself factors over the derived category $\mathcal{D}^{2p-2}(\mathcal{B})$. So we have obtained a functor

$$\mathcal{R}\colon \mathcal{D}^{2p-2}(\mathcal{B}) \longrightarrow \mathrm{Ho}(L_1\mathcal{S}).$$

4.2. The main theorem

Theorem 4.2. *$\mathcal{R}$ is an equivalence of categories.*

Proof. First again, we prove that $\mathcal{R}$ is full and faithful; i.e., for

$$C^1_*, C^2_* \in \mathcal{D}^{2p-2}(\mathcal{B}),$$

the map

$$r\colon \mathrm{Hom}_{\mathcal{D}^{2p-2}(\mathcal{B})}(C^1_*, C^2_*) \longrightarrow [\mathcal{R}(C^1_*), \mathcal{R}(C^2_*)]^{E(1)}$$

induced by $\mathcal{R}$ is an isomorphism.

To show this, we once more make use of the Adams spectral sequence [**Fra96**, 2.1.1]

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{A}}\Big(\bigoplus_i H_i(C^1_*)[i+t], \bigoplus_i H_i(C^2_*)[i]\Big) \Rightarrow \mathrm{Hom}^{t-s}_{\mathcal{D}^{2p-2}(\mathcal{B})}(C^1_*, C^2_*), \qquad (15)$$

where $C^1_*, C^2_* \in \mathcal{D}^{2p-2}(\mathcal{B})$. This spectral sequence arises as follows: We begin with an injective resolution of $\bigoplus_i H_i(C^2_*)[i]$:

$$\begin{array}{ccccccccc} \bigoplus H_i(C^2_*)[i] & \hookrightarrow & I^0 & \xrightarrow{d_1} & I^1 & \xrightarrow{d_2} & I^2 & \longrightarrow & 0. \\ & & \downarrow\!\!\downarrow & \nearrow & \downarrow\!\!\downarrow & \nearrow & & & \\ & & \mathrm{im}(d_1) & & \mathrm{im}(d_2) & & & & \end{array} \qquad (16)$$

(This resolution stops at I^2 since the injective dimension of an object in $\mathcal{A}$ is at most 2.)

This resolution gives rise to an Adams resolution

$$\begin{array}{ccccccc} C^2_* = C^2_{(0)} & \longleftarrow & C^2_{(1)} & \longleftarrow & C^2_{(2)} & \longleftarrow & 0. \\ \downarrow & \nearrow_{+} & \downarrow & \nearrow_{+} & \downarrow & \nearrow_{+} & \\ E_{I^0} & & E_{I^1} & & E_{I^2} & & \end{array} \qquad (17)$$

The Adams resolution is characterised by the following: First, by applying

$$\bigoplus_i H_i(-)[i]$$

to the diagram

$$\begin{array}{ccccccccc} C^2_* = C^2_{(0)} & \longrightarrow & E_{I^0} & \longrightarrow & E_{I^1} & \longrightarrow & E_{I^2} & \longrightarrow & 0, \\ & & \downarrow & \nearrow & \downarrow & \nearrow & & & \\ & & C_2^{(1)} & & C_2^{(2)} & & & & \end{array} \tag{18}$$

one obtains exactly the diagram (16). Besides, each triangle in (17) is an exact triangle in $\mathcal{D}^{2p-2}(\mathcal{B})$ (the diagonal maps are maps raising the degree by one). Further, E_I denotes the Eilenberg-MacLane object for $I \in \mathcal{A}$; i.e.,

$$\mathrm{Hom}_{\mathcal{A}}(\bigoplus_i H_i(C_*)[i], I) \cong \mathrm{Hom}_{\mathcal{D}^{2p-2}(\mathcal{B})}(C_*, E_I) \text{ for all } C_* \in \mathcal{D}^{2p-2}(\mathcal{B}),$$

and for $C_* = E_I$, the image of the identity in

$$\mathrm{Hom}_{\mathcal{A}}(\bigoplus_i H_i(E_I)[i], I)$$

is an isomorphism. (Note that by Lemma 2.1.1 of [**Fra96**], $C^2_{(2)}$ is indeed an Eilenberg-MacLane object for I^2!) Applying $\mathrm{Hom}_{\mathcal{D}^{2p-2}(\mathcal{B})}(C^1_*, -)$ to the resolution (17) gives an exact couple, and with it the desired spectral sequence.

We now apply the reconstruction functor $\mathcal{R}$ to (17) and claim that the result

$$\begin{array}{ccccccc} \mathcal{R}(C^2) = \mathcal{R}(C^2_{(0)}) & \longleftarrow & \mathcal{R}(C^2_{(1)}) & \longleftarrow & \mathcal{R}(C^2_{(2)}) & \longleftarrow & 0 \\ \downarrow & \overset{+}{\nearrow} & \downarrow & \overset{+}{\nearrow} & \cong\downarrow & \overset{+}{\nearrow} & \\ \mathcal{R}(E_{I^0}) & & \mathcal{R}(E_{I^1}) & & \mathcal{R}(E_{I^2}) & & \end{array} \tag{19}$$

is an Adams resolution for $\mathcal{R}(C^*_2)$ with respect to $E(1)$-homology.

We have to prove the following:

- applying $E(1)_*$ to (19) gives an injective resolution of $E(1)_*(\mathcal{R}(C^2_*))$,
- each triangle in (19) is exact,
- $\mathcal{R}(E_I)$ is again an Eilenberg-MacLane object in $\mathrm{Ho}(L_1\mathcal{S})$.

The first point is clear after Lemma 4.1, which says that

$$E(1)_*(\mathcal{R}(C^*)) \cong \bigoplus_i H_i(C_*)[i].$$

To prove the second point we make use of the following fact without giving the details of its proof:

Let $C^0_* \to C^1_* \to C^2_* \to C^0_*[1]$ be an exact triangle in $\mathcal{D}^{2p-2}(\mathcal{B})$ with $H^*(C^0_*) \to H^*(C^1_*)$ a monomorphism. Then

$$\mathcal{R}(C^0_*) \to \mathcal{R}(C^1_*) \to \mathcal{R}(C^2_*) \to \mathcal{R}(C^0_*[1])$$

is an exact triangle in $\mathrm{Ho}(L_1\mathcal{S})$.

Using Lemma 4.1 again, we see that the vertical arrows in (17) give monomorphisms in cohomology. So, applying the above fact, we have that the triangles in (17) are indeed exact.

To show that $\mathcal{R}(E_I)$ is again an Eilenberg-MacLane object in $\mathrm{Ho}(L_1\mathcal{S})$ for injective $I \in \mathcal{A}$, we have to show that

$$\mathrm{Hom}_{\mathcal{A}}(E(1)_*(X), I) \cong [X, \mathcal{R}(E_I)]^{E(1)} \quad \text{for all} \quad X \in \mathrm{Ho}(L_1\mathcal{S}).$$

We know that

$$E(1)_*(\mathcal{R}(E_I)) \cong \bigoplus_i H_i(E_I)[i] \cong I,$$

so $\mathcal{R}(E_I)$ has injective $E(1)$-homology.

Now we look at the classical Adams spectral sequence

$$\begin{aligned} E_2^{s,t} &= \mathrm{Ext}^s_{\mathcal{A}}(E(1)_*(X)[t], E(1)_*(\mathcal{R}(E_I))) = \mathrm{Ext}^s_{\mathcal{A}}(E(1)_*(X)[t], I) \\ &\Rightarrow [X, \mathcal{R}(E_I)]^{E(1)}_{t-s} \end{aligned}$$

for $X \in \mathrm{Ho}(L_1\mathcal{S})$. Since I is injective in A, the Ext-term vanishes unless $s = 0$, so the spectral sequence collapses, and

$$\mathrm{Ext}^0_{\mathcal{A}}(E(1)_*(X)[t], I) = \mathrm{Hom}_{\mathcal{A}}(E(1)_*(X)[t], I) \cong [X, \mathcal{R}(E_I)]^{E(1)}_t$$

as desired.

Applying $[\mathcal{R}(C^1_*), -]^{E(1)}$ to (19) gives an exact couple leading to the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{A}}(E(1)_*(\mathcal{R}(C^1_*))[t], E(1)_*(\mathcal{R}(C^2_*))) \Rightarrow [\mathcal{R}(C^1_*), \mathcal{R}(C^2_*)]^{E(1)}_{t-s}.$$

So $\mathcal{R}$ induces a morphism of exact couples that is also an isomorphism on the E_1-terms

$$r\colon\ \mathrm{Hom}^t_{\mathcal{D}^{2p-2}(\mathcal{B})}(C^1_*, E_{I^s}) \longrightarrow [\mathcal{R}(C^1_*), \mathcal{R}(E_{I^s})]^{E(1)}_t.$$

By definition of an Eilenberg-MacLane object, the left side is isomorphic to

$$\mathrm{Hom}^t_{\mathcal{A}}(\bigoplus_i H_i(C^1_*)[i], I^s).$$

Since $\mathcal{R}(E_{I^s})$ is an Eilenberg-MacLane object with respect to $E(1)_*$, the right side is isomorphic to

$$\mathrm{Hom}^t_{\mathcal{A}}(E(1)_*(\mathcal{R}(C^*_1)), I^s).$$

So because of Lemma Lemma 4.1, the two sides are isomorphic. It follows that r is an isomorphism on the targets of the spectral sequences, and thus, $\mathcal{R}$ is full and faithful.

Now it remains to show that $\mathcal{R}$ is essentially surjective. Let Y be an object of $\mathrm{Ho}(L_1\mathcal{S})$ and let

$$\begin{array}{ccccccc} Y = Y^{(0)} & \longleftarrow & Y^{(1)} & \longleftarrow & Y^{(2)} & \longleftarrow & 0 \\ \downarrow & \nearrow_{+} & \downarrow & \nearrow_{+} & \downarrow{\scriptstyle\cong} & \nearrow_{+} & \\ \mathcal{E}_{I^0} & & \mathcal{E}_{I^1} & & \mathcal{E}_{I^2} & & \end{array} \tag{20}$$

be an Adams resolution for Y. First, we show that all Eilenberg-MacLane objects

$\mathcal{E}_I \in \mathrm{Ho}(L_1\mathcal{S})$ lie in the essential image of $\mathcal{R}$: Let E_I be the Eilenberg-MacLane object for I in $\mathcal{D}^{2p-2}(\mathcal{B})$. We already showed that $\mathcal{R}(E_I)$ is an Eilenberg-MacLane object for I in $\mathrm{Ho}(L_1\mathcal{S})$, and thus, $\mathcal{E}_I \cong \mathcal{R}(E_I)$, so $\mathcal{E}_I$ lies in the essential image of $\mathcal{R}$.

Next, we would like to show that Y lies in the essential image. We start with showing this for $Y^{(1)}$. We know that there are Eilenberg-MacLane objects $E_{I^1}, E_{I^2} \in \mathcal{D}^{2p-2}(\mathcal{B})$ such that $\mathcal{R}(E_{I^1}) \cong \mathcal{E}_{I^1}$ and $\mathcal{R}(E_{I^2}) \cong \mathcal{E}_{I^2}$. We started with an injective resolution

$$E(1)_*(Y) \to I^0 \to I^1 \to I^2 \to 0$$

for $E(1)_*(Y) \in \mathcal{A}$. Using Lemma 4.1, this resolution equals

$$E(1)_*(Y) \to \bigoplus_i H_i(E_{I^0})[i] \xrightarrow{d^1} \bigoplus_i H_i(E_{I^1})[i] \xrightarrow{d^2} \bigoplus_i H_i(E_{I^2})[i] \to 0 \tag{21}$$

with Eilenberg-MacLane objects in $\mathcal{D}^{2p-2}(\mathcal{B})$ as above. We take those Eilenberg-MacLane objects and complete them to an exact triangle

$$E_{I^2} \to D_* \to E_{I^1} \to E_{I^2}[1] \tag{22}$$

in $\mathcal{D}^{2p-2}(\mathcal{B})$. Applying

$$\bigoplus_i H_i(-)[i]$$

to this triangle gives rise to a long exact sequence in $\mathcal{A}$. Since d^2 in (21) is a surjection, the third morphism in this triangle induces a surjection in cohomology as well. Consequently, the second morphism $D \to E_{I^1}$ must give an injection in cohomology. So we can apply the formerly stated fact that in this case,

$$\mathcal{R}(E_{I^2}) \to \mathcal{R}(D_*) \to \mathcal{R}(E_{I^1}) \to \mathcal{R}(E_{I^2}[1])$$

is an exact triangle in $\mathrm{Ho}(L_1\mathcal{S})$.

Consider

$$\begin{array}{ccccccc}
\mathcal{E}_{I^2} & \longrightarrow & Y^{(1)} & \longrightarrow & \mathcal{E}_{I^1} & \longrightarrow & \Sigma\mathcal{E}_{I^2} \\
\cong \downarrow \mathcal{R} & & \vdots & & \cong \downarrow \mathcal{R} & & \cong \downarrow \mathcal{R} \\
\mathcal{R}(E_{I^2}) & \longrightarrow & \mathcal{R}(D_*) & \longrightarrow & \mathcal{R}(E_{I^1}) & \longrightarrow & \mathcal{R}(E_{I^2}[1])
\end{array}$$

with the upper row being an exact triangle coming from (20). The third square commutes since $\mathcal{R}$ is full. By the axioms of a triangulated category there exist a morphism $Y^{(1)} \to \mathcal{R}(D_*)$ making the whole diagram commute. By the 5-lemma, this is an isomorphism; thus $Y^{(1)} \cong \mathcal{R}(D_*)$, and so $Y^{(1)}$ lies in the essential image of $\mathcal{R}$. Similarly, this also follows for Y, which completes the proof that $\mathcal{R}$ is an equivalence of categories. □

Corollary 4.3. *$\mathcal{R}$ preserves the Adams filtration.*

Remark 4.4. Nora Ganter recently proved in [**Gan07**] that, for the case of $E(1)$-local spectra, $\mathcal{R}$ is not just an equivalence of categories but $\mathcal{R}$ also carries tensor products of chain complexes into smash products of spectra. (This is not known to be true for arbitrary n with $n^2 + n < 2p - 2$.)

5. A further application

As proved, $\mathcal{R}$ provides an equivalence of triangulated categories which also happen to be homotopy categories of model categories. The next question now is if $\mathcal{D}^{2p-2}(\mathcal{B})$ and $\mathrm{Ho}(L_1\mathcal{S})$ are equivalent as categories, can their model structures also be positively compared; i.e., is there a Quillen equivalence between them? The answer to that is remarkable:

Proposition 5.1. *The categories $\mathcal{D}^{2p-2}(\mathcal{B})$ and $\mathrm{Ho}(L_1\mathcal{S})$ are not Quillen equivalent. In particular, $\mathcal{R}$ is not derived from a Quillen equivalence.*

Proof. To prove this, we compare the homotopy types of certain mapping spaces for each category. Let us first collect the necessary definitions. For a pointed simplicial model category $\mathcal{C}$, there is a mapping space functor

$$\mathrm{map}_{\mathcal{C}}(-,-)\colon \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathrm{sSet}_*$$

to the category of pointed simplicial sets satisfying

$$\mathrm{map}_{\mathcal{C}}(X,Y)_0 = \mathrm{Hom}_{\mathcal{C}}(X,Y)$$

for all $X, Y \in \mathcal{C}$ and certain adjointness properties [**GJ99**, II.2.1]. However, $\mathcal{C}^{2p-2}(\mathcal{B})$ is not a simplicial category. The next best thing we can achieve for an arbitrary pointed model category $\mathcal{C}$ is a notion of a mapping space that is well-defined up to homotopy, which will do for our purposes.

To achieve this, we look at the category $\mathcal{C}^{\Delta}$ of cosimplicial objects in $\mathcal{C}$ and view X as a constant object in $\mathcal{C}^{\Delta}$. The category $\mathcal{C}^{\Delta}$ of cosimplicial objects in a model category $\mathcal{C}$ can be given a model structure, the so-called Reedy model structure. For details of this, see [**Hov99**, Section 5.2]. We now define a special replacement of X in $\mathcal{C}^{\Delta}$, a so-called *frame*. To do this, we first need the following:

Definition 5.2. Via the methods of [**Hov99**, Remark 5.2.3 and Example 5.2.4], there are functors $\mathbf{l}^\bullet, \mathbf{r}^\bullet\colon \mathcal{C} \longrightarrow \mathcal{C}^{\Delta}$ with the following properties:

Let $X \in \mathcal{C}$:

- the n^{th} level space of the object $\mathbf{l}^\bullet X$ is the $n+1$-fold coproduct of A;
- $\mathbf{l}^\bullet\colon \mathcal{C} \longrightarrow \mathcal{C}^{\Delta}$ is a left adjoint to the evaluation functor $ev_0\colon \mathcal{C}^{\Delta} \longrightarrow \mathcal{C}$ that sends $A^\bullet$ to $A^\bullet[0]$;
- the n^{th} level space of the object $\mathbf{r}^\bullet X$ is X itself;
- $\mathbf{r}^\bullet\colon \mathcal{C} \longrightarrow \mathcal{C}^{\Delta}$ is a right adjoint to $ev_0\colon \mathcal{C}^{\Delta} \longrightarrow \mathcal{C}$.

Remark 5.3. One can prove that $\mathbf{r}^\bullet$ is the constant cosimplicial functor. There is a natural transformation $\mathbf{l}^\bullet \longrightarrow \mathbf{r}^\bullet$ that is the identity in degree zero and the fold map in higher degrees.

With these functors, we can now define cosimplicial frames:

Definition 5.4. Let $\mathcal{C}$ be a model category, X an object of $\mathcal{C}$. A *cosimplicial frame* for X is a cosimplicial object $X^\bullet \in \mathcal{C}^{\Delta}$ together with a factorisation of the map

$\mathbf{l}^\bullet X \longrightarrow \mathbf{r}^\bullet X$ in $\mathcal{C}^\Delta$

$$\mathbf{l}^\bullet X \rightarrowtail X^\bullet \xrightarrow{\sim} \mathbf{r}^\bullet X,$$

where the weak equivalence $X^\bullet \xrightarrow{\sim} \mathbf{r}^\bullet X$ in degree zero induces a weak equivalence in $\mathcal{C}$.

For the existence of such framings, see [**Hov99**, Theorem 5.2.8].

We now use this definition to define mapping spaces:

Definition 5.5. Let X, Y be objects of $\mathcal{C}$, $X^\bullet$ a cosimplicial frame for X and

$$Y \overset{\sim}{\rightarrowtail} Y^{\text{fib}} \twoheadrightarrow *$$

a factorisation of $Y \to *$. Then the *(left) mapping space* for X and Y is defined via

$$\operatorname{map}_{\mathcal{C}}(X,Y) := \mathcal{C}(X^\bullet, Y^{\text{fib}}) \in \text{sSet}_*,$$

where $\mathcal{C}(X^\bullet, Y^{\text{fib}})$ is the simplicial set with

$$\mathcal{C}(X^\bullet, Y^{\text{fib}})_n := \operatorname{Hom}_{\mathcal{C}}(X^\bullet[n], Y^{\text{fib}}).$$

However, it is not clear whether this definition actually deserves to be called a definition since it depends on two choices: firstly, the cosimplicial frame for X and secondly, the fibrant replacement for Y. So, for this definition to make sense we need the following:

Lemma 5.6. *Let $X_1^\bullet, X_2^\bullet$ be two cosimplicial frames for cofibrant X in $\mathcal{C}$, and let $Y_1^{\text{fib}}, Y_2^{\text{fib}}$ be two fibrant replacements for Y. Then*

$$\mathcal{C}(X_1^\bullet, Y_1^{\text{fib}}) \simeq \mathcal{C}(X_2^\bullet, Y_2^{\text{fib}})$$

in sSet$_*$.

Proof. First, let $X_1^\bullet$ and $X_2^\bullet$ be two cosimplicial frames for X. By definition, the frames $X_1^\bullet$ and $X_2^\bullet$ are linked by a zig-zag of weak equivalences

$$X_1^\bullet \xrightarrow{\sim} \mathbf{r}^\bullet X \xleftarrow{\sim} X_2^\bullet.$$

For fibrant Y, the functor $\mathcal{C}(-, Y)$ preserves weak equivalences [**SS02**, Lemma 6.3], so for fibrant Y and $X_1^\bullet$, $X_2^\bullet$ as above, we have

$$\mathcal{C}(X_1^\bullet, Y) \simeq \mathcal{C}(X_2^\bullet, Y).$$

For the second part we quote [**Hov99**, Corollary 5.4.4], which says that for cofibrant X in $\mathcal{C}$, the functor

$$\mathcal{C}(X^\bullet, -) \colon \mathcal{C} \longrightarrow \text{sSet}_*$$

preserves fibrations and acyclic fibrations, in particular between fibrant objects. So Ken Brown's lemma applies [**Hov99**, Lemma 1.1.12], and it follows that $\mathcal{C}(X^\bullet, -)$ takes weak equivalences between fibrant objects in $\mathcal{C}$ to weak equivalences in sSet$_*$, which proves the claim of our lemma. □

Now we look at the behaviour of mapping spaces under Quillen functors and Quillen equivalences.

Lemma 5.7. *Let* $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$ *be a Quillen equivalence,* $X, X' \in \mathcal{C}$ *both cofibrant. Then*

$$\mathrm{map}_{\mathcal{C}}(X, X') \cong \mathrm{map}_{\mathcal{D}}(LX, LX')$$

in $\mathrm{Ho}(\mathrm{sSet}_*)$.

Proof. First of all, let $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$ be a Quillen adjoint functor pair, $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. Then

$$\mathrm{map}_{\mathcal{D}}(LX, Y) = \mathcal{D}((LX)^{\bullet}, Y^{\mathrm{fib}})$$

by definition. Since L is a left Quillen functor, $L(X^{\bullet}) \in \mathcal{D}^{\Delta}$ is also a cosimplicial frame for LX [**Hov99**, Lemma 5.6.1], so

$$\mathcal{D}((LX)^{\bullet}, Y^{\mathrm{fib}}) \cong \mathcal{D}(L(X^{\bullet}), Y^{\mathrm{fib}})$$

by Lemma 5.6. By adjointness,

$$\mathrm{Hom}_{\mathcal{D}}(L(X^{\bullet})[n], Y^{\mathrm{fib}}) \cong \mathrm{Hom}_{\mathcal{C}}(X^{\bullet}[n], R(Y^{\mathrm{fib}})),$$

so

$$\mathcal{D}(L(X^{\bullet}), Y^{\mathrm{fib}}) \cong \mathcal{C}(X^{\bullet}, R(Y^{\mathrm{fib}})).$$

Since R is a right Quillen functor, $R(Y^{\mathrm{fib}})$ is a fibrant replacement for RY. Consequently by Lemma 5.6,

$$\mathcal{C}(X^{\bullet}, R(Y^{\mathrm{fib}})) \simeq \mathcal{C}(X^{\bullet}, (RY)^{\mathrm{fib}}) = \mathrm{map}_{\mathcal{C}}(X, RY).$$

Thus, altogether we have

$$\mathrm{map}_{\mathcal{C}}(X, RY) \simeq \mathrm{map}_{\mathcal{D}}(LX, Y). \tag{23}$$

Next, let $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$ be a Quillen equivalence and $X' \in \mathcal{C}$ cofibrant. Then

$$LX' \xrightarrow{\sim} (LX')^{\mathrm{fib}}$$

is a weak equivalence in $\mathcal{D}$ with cofibrant source and fibrant target, so by definition of a Quillen equivalence, the adjoint map

$$X' \xrightarrow{\sim} R((LX')^{\mathrm{fib}})$$

is a weak equivalence in $\mathcal{C}$. Since R is a right Quillen functor, $R((LX')^{\mathrm{fib}})$ is fibrant in $\mathcal{C}$. Consequently, $R((LX')^{\mathrm{fib}})$ is a fibrant replacement for X' in $\mathcal{C}$. By Lemma 5.6 and the above adjointness result for mapping spaces (23), it follows that

$$\mathrm{map}_{\mathcal{C}}(X, X') \simeq \mathrm{map}_{\mathcal{C}}(X, R((LX')^{\mathrm{fib}})) \simeq \mathrm{map}_{\mathcal{D}}(LX, LX')$$

in sSet_* which proves the lemma. □

Now we return to our special case: We will see that for all $C, D \in \mathcal{C}^{2p-2}(\mathcal{B})$, $\mathrm{map}_{\mathcal{C}^{2p-2}(\mathcal{B})}(C, D)$ is weakly equivalent to a product of Eilenberg-MacLane spaces. However, the mapping space $\mathrm{map}_{L_1\mathcal{S}}(S^0, S^0)$ is not a product of Eilenberg-MacLane spaces, so as a consequence of Lemma 5.7, there is no Quillen equivalence between those two model categories which was the claim of the proposition.

The category $\mathcal{C}^{2p-2}(\mathcal{B})$ is abelian, so for all $C, D \in \mathcal{C}^{2p-2}(\mathcal{B})$, the n-simplices of $\mathrm{map}_{\mathcal{C}^{2p-2}(\mathcal{B})}(C, D)$,

$$\mathcal{C}(C^\bullet, D^{\mathrm{fib}})_n = \mathrm{Hom}(C^\bullet[n], D),$$

form an abelian group, and the simplicial structure maps are group homomorphisms; thus

$$\mathcal{C}(C^\bullet, D^{\mathrm{fib}}) = \mathrm{map}_{\mathcal{C}^{2p-2}(\mathcal{B})}(C, D)$$

is not just a simplicial set but a simplicial abelian group. From Proposition III.2.20 of [**GJ99**], it follows that

$$\mathrm{map}_{\mathcal{C}^{2p-2}(\mathcal{B})}(C, D) \cong \prod_{n \geqslant 0} K(\pi_n \, \mathrm{map}_{\mathcal{C}^{2p-2}(\mathcal{B})}(C, D)_n, n),$$

where $K(G, n)$ denotes the n^{th} Eilenberg-MacLane space for the abelian group G.

However, there are spectra for which the mapping spaces over $L_1\mathcal{S}$ are not products of Eilenberg-MacLane spaces, for example,

$$\mathrm{map}_{L_1\mathcal{S}}(S^0, S^0) \cong QL_1S^0 = \mathrm{colim}_n \, \Omega^n L_1 S^n.$$

Thus, $\mathcal{C}^{2p-2}(\mathcal{B})$ and $L_1\mathcal{S}$ cannot be Quillen equivalent and $\mathcal{C}^{2p-2}(\mathcal{B})$ provides an exotic model for $L_1\mathcal{S}$. □

So we have seen that $\mathcal{C}^{2p-2}(\mathcal{B})$ provides an exotic model for $\mathrm{Ho}(L_1\mathcal{S})$. For the stable homotopy category itself such exotic models do not exist, as proved by Schwede in [**Sch07**]. At the prime 2, the author has shown in that the $E(1)$-local stable homotopy category cannot have an exotic model, either [**Roi07**]. However, this is not true for the chromatic localisations of the stable homotopy category in the cases $n^2 + n < 2p - 2$ (shown here explicitly for $n = 1$). It is not yet known how many such exotic models exist and what can be said about the other chromatic localisations.

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