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A NEW HIGHER HOMOTOPY GROUPOID: THE FUNDAMENTAL GLOBULAR ω-GROUPOID OF A FILTERED SPACE

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Abstract

We show that the graded set of filter homotopy classes rel vertices of maps from the n-globe to a filtered space may be given the structure of (strict) globular ω-groupoid. The proofs use an analogous fundamental cubical ω-groupoid due to the author and Philip Higgins in 1981. This method also relates the construction to the fundamental crossed complex of a filtered space, and this relation allows the proof that the crossed complex associated to the free globular ω-groupoid on one element of dimension n is the fundamental crossed complex of the n-globe.

Contents

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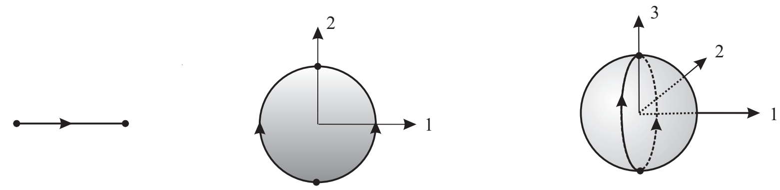
1. Introduction

By the n-globe G^n we mean the subspace of Euclidean n-space $\mathbb{R}^n$ of points x such that $\|x\| \leqslant 1$ but with the cell structure for $n \geqslant 1$

$$G^n = e^0_\pm \cup e^1_\pm \cup \cdots \cup e^{n-1}_\pm \cup e^n. \tag{1}$$

This structure will be given precisely in Section 2.



A *filtered space* is a compactly generated space X_∞ and a sequence of subspaces

$$X_* \colon X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X_\infty. \tag{2}$$

A *map of filtered spaces* $f \colon Y_* \to X_*$ is a map $f \colon Y_\infty \to X_\infty$ such that $f(Y_n) \subseteq X_n$ for all $n \geqslant 0$. This gives the category FTop of filtered spaces. A *filter homotopy* $f_t \colon f_0 \simeq f_1$ is a continuous family of filtered maps $f_t \colon Y_* \to X_*$ for $0 \leqslant t \leqslant 1$. This is to be contrasted with a *homotopy* of filtered maps which has the requirement $f_t(Y_n) \subseteq X_{n+1}$ for all t and $n \geqslant 0$.

The n-globe G^n has a skeletal filtration giving a filtered space G^n_*. If X_* is a filtered space then we have a *globular singular complex* $R^{\bigcirc} X_*$, which in dimension n is $\mathsf{FTop}(G^n_*, X_*)$. In Appendix A we will explain the structure of $R^{\bigcirc} X_*$ as a *globular set*.

We define

$$\rho^{\bigcirc} X_* = (R^{\bigcirc} X_* / \equiv), \tag{3}$$

where $\equiv$ is the relation of filter homotopy *rel vertices*. It is clear that $\rho^{\bigcirc} X_*$ inherits from $R^{\bigcirc} X_*$ the structure of globular set. Our main result is the following:

Theorem 1.1 (Main Theorem)**.** *For $n \geqslant 1$ there are compositions $\circ_i, 1 \leqslant i \leqslant n$ in dimension n giving the globular set $\rho^{\bigcirc} X_*$ the structure of strict globular ω-groupoid.*

We call $\rho^{\bigcirc} X_*$ the *fundamental globular higher homotopy groupoid of the filtered space* X_*. The proof of this theorem goes via the notion of cubical higher homotopy groupoid of a filtered space, established in [**BH81b**]. It should be useful therefore to put these results in context.

A general characterisation of work on higher homotopy groupoids in which the author has been involved may be subsumed in the following diagram of categories and functors and its properties:

$$\begin{tikzcd} \text{topological data} \arrow[rr, shift left, "\Xi"] \arrow[dr, "U"'] & & \text{algebraic data} \arrow[ll, shift left, "\mathbb{B}"] \arrow[dl, "B"] \\ & \text{topological spaces.} & \end{tikzcd} \tag{4}$$

It has been possible to give categories of 'topological data', of 'algebraic data' and functors as above, where U is the forgetful functor and $B = U \circ \mathbb{B}$, with the following properties:

(1) the functor Ξ is defined homotopically and satisfies a higher homotopy van Kampen theorem (HHvKT)[1], in that it preserves certain colimits;

(2) $\Xi \circ \mathbb{B}$ is naturally equivalent to 1;

(3) there is a natural transformation $1 \to \mathbb{B} \circ \Xi$ preserving some homotopical information.

The purpose of *(1)* is to allow some calculation of Ξ by gluing simple examples, such as convex subsets, following the use of the fundamental groupoid in [**Bro06**]. This condition *(1)* at present also rules out some widely used algebraic data, such as for example simplicial groups or groupoids, or differential graded algebras, since for those cases no such functor Ξ is known. *(2)* shows that the algebraic data faithfully captures some of the topological data. The imprecise *(3)* gives further information on the algebraic modelling. The functor B should be called a *classifying space* because it often generalises the classifying space of a group or groupoid. It has also been found useful in the homotopy classification of maps.

Here is a table of examples that have been found.

Topological data	*Algebraic data*
space with base point	groups
space with set of base points	groupoids
pointed pair of spaces	crossed modules
filtered space	crossed complexes
n-cube of pointed spaces	cat^n-groups
n-cube of pointed spaces	crossed n-cube of groups

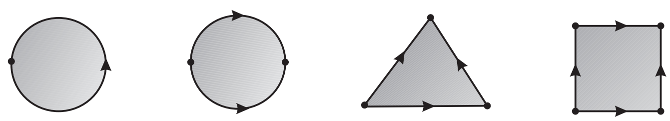
Strong results in the last two cases are shown in [**BL87, ES87**].

In this paper we will deal only with the case of filtered spaces, which of course includes the first three cases. There are still further choices of algebraic data as shown in the following diagram of equivalent categories, which is taken from [**Bro99**]:

[1] Jim Stasheff has suggested this term to the author, instead of the previously used Generalised van Kampen Theorem, to make clear the higher homotopy information contained in theorems of this type.

$$
\begin{tikzcd}
\text{cubical} \atop T\text{-complexes} \arrow[r, leftrightarrow, "(a)"] & \text{cubical} \atop {\omega\text{-groupoids} \atop \text{with connections}} \arrow[dr, "(f)"] \arrow[dd, leftrightarrow, "(b)"'] & \\
\text{poly-}T\text{-complexes} \arrow[d, leftrightarrow, "(e)"'] & & \text{globular} \atop \omega\text{-groupoids} \\
\text{simplicial} \atop T\text{-complexes} \arrow[r, leftrightarrow, "(d)"'] & \text{crossed} \atop \text{complexes.} \arrow[ur, leftrightarrow, "(c)"'] &
\end{tikzcd}
\tag{5}
$$

Each arrow here denotes an explicit functor which is an equivalence of categories. The equivalences (a) and (b) are in [**BH81a**]; (a) is an essential technical tool in the use of cubical ω-Gpds. The equivalence (c) is in [**BH81c**], and this with (b) implies the equivalence (f). A direct form of this equivalence is given in the much harder category case in [**AABS02**]. The equivalence (d) is due to Ashley in [**Ash88**]. The equivalence (e) is due to Jones [**Jon88**]. The different forms of algebra reflect different geometries, those of disks, globes, simplices, cubes, as shown in dimension 2 in the following diagram:



It is because the geometry of convex sets is so much more complicated in dimensions > 1 than in dimension 1 that new complications emerge for the theories of higher order group theory and of higher homotopy groupoids.

A classical homotopical functor on filtered spaces is the *fundamental crossed complex* ΠX_* of a filtered space, defined using relative homotopy groups (in the case X_0 is a singleton) by Blakers [**Bla48**]. Major achievements of the papers [**BH81a, BH81b**] were

- to define a homotopical functor, which here we call $\rho^\square$, from filtered spaces to cubical ω-groupoids with connections (and hence also to cubical T-complexes), which in dimension n is the filter homotopy classes rel vertices of filtered maps $I^n_* \to X_*$ (but see Remark 2.3);
- to prove that this functor preserved certain colimits;
- to relate $\rho^\square$ with the classical functor Π from filtered spaces to crossed complexes, and so to prove that Π preserves certain colimits.

The proofs *do not involve traditional techniques such as singular homology or simplicial approximation.* The results give nonabelian information on second relative homotopy groups using crossed module theory (see a survey in [**Bro99**]), and in higher dimensions give information on the action of the fundamental group. In particular the HHvKT has as a corollary a previously unnoticed homotopical excision theorem which has the Relative Hurewicz Theorem as a corollary [**BH81b**, Example 6, p. 34], while extra information on monoidal closed structures gives the Homotopy Addition Lemma for a simplex [**BS07**], and homotopy classification theorems [**BH91**].

Analogous methods to those of [**BH81b**] were used by Ashley in [**Ash88**] to define a functor ρ^{Δ} from filtered spaces to simplicial T-complexes. His ideas contributed to [**BH81b**], and his results are applied in [**FMa07**].

However, there has been a lack of a directly defined homotopical functor from filtered spaces to globular ω-groupoids, and this gap will be filled in this paper.

The definition of classifying space is convenient via well-developed simplicial constructions. In this way we get the classifying space of a crossed complex [**BH91**]. Its properties are further exploited in, for example, [**BGPT, FMa07, FMP06**]. However the exposition in [**BHS08**] adopts an earlier cubical approach.

The equivalence of the category of globular ω-groupoids with the category of cubical ω-groupoids with connection, and the monoidal closed structure on the latter constructed in [**BH87**], imply a monoidal closed structure on the category of globular ω-groupoids. Further, it is shown in [**BH91**] that the simple rule $[f] \otimes [g] \mapsto [f \otimes g]$ gives a natural transformation

$$\rho^{\square} X_* \otimes \rho^{\square} Y_* \to \rho^{\square}(X_* \otimes Y_*)$$

for any filtered spaces X_*, Y_*, where $X_* \otimes Y_*$ is the usual tensor product of filtered spaces given by

$$(X_* \otimes Y_*)_n = \bigcup_{p+q=n} X_p \times Y_q.$$

The induced transformation on crossed complexes is shown in [**BB93**] to be an isomorphism if X_*, Y_* are connected (Definition 4.1) and cofibred (each inclusion $X_n \to X_r$, $r > n \geqslant 0$, is a closed cofibration). It follows from the above that there is a natural transformation

$$\rho^{\bigcirc} X_* \otimes \rho^{\bigcirc} Y_* \to \rho^{\bigcirc}(X_* \otimes Y_*).$$

This natural transformation, which is difficult to construct directly, may be used to enrich the category of filtered spaces over the monoidal closed category of globular ω-groupoids.

It should be apparent from the above that it is the cubical case which gives the intuition and power in formulating and proving these theorems; the basic reason is that cubical theory is handy for: subdivision and its inverse, multiple compositions, the notion of commutative cube, and is also good for tensor products and higher homotopies. Many theorems can then, by equivalences of categories, be translated to the other cases. However the proofs for the cubical cases, particularly the properties of thin elements and T-complexes, involve also the use of crossed complexes and the equivalence of categories (a), (b) of diagram (5). Crossed complexes also have a well-developed homotopy theory [**BG89**], and they have a clear relation with chain complexes with operators [**BH90**].

The relation with simplicial theory is useful because of the wide development of simplicial theory. Finally, the relation with the globular theory could be useful because of the wide familiarity of uses of weak structures and lax functors and natural transformations. For example, compare the discussion of Schreier theory using crossed complexes in [**BH82, BP96**] with the use of 2-groupoids in [**BBF05**]. Calculational applications are usually made using crossed complexes. For example, the paper [**BP96**] uses the notion of small free crossed resolution to give calculations which yield small parametrisations of some nonabelian extensions of groups, whereas the notion of *free globular ω-groupoid* is lacking, or undeveloped.

2. Disks, globes, and cubes

Our results follow from an analysis of the relations between globes and cubes. These results are probably well known but need to be done carefully for our purposes.

We give real space $\mathbb{R}^n$ the Euclidean norm $\|x\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$. We embed $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$ as usual by $x \mapsto (x, 0)$. The n-cube I^n will be the subset of $\mathbb{R}^n$ of points x such that $|x_i| \leqslant 1$ for all i. Thus $I = I^1$ is identified with $[-1, 1]$ and we also identify I^n with the n-fold product of I with itself.

The *n-disk* is the subspace D^n of $\mathbb{R}^n$ of points x with $\|x\| \leqslant 1$. The $(n-1)$-sphere S^{n-1} is the subspace of D^n of points x with $\|x\| = 1$.

We define the *n-globe* G^n to be D^n as a space, but with the cell structure

$$G^n = e^0_\pm \cup e^1_\pm \cup \cdots \cup e^{n-1}_\pm \cup e^n.$$

Here for $i < n$ the closed cell $\bar{e}^i_\pm$ is the set of points $x = (x_1, \ldots, x_n) \in G^n$ such that $\|x\| = 1, x_j = 0$ for $j < n - i$ and $\pm x_{n-i} \geqslant 0$. This convention is in keeping with the relationship with cubes which we find convenient. Note that the $(n-1)$-skeleton of G^n is contained in S^{n-1}.

For each of $Q = \Delta, \Box, \circ$ we have a singular complex $S^Q X$ of a topological space X, giving the well-known simplicial and cubical singular complex, and also a 'globular' singular complex consisting of maps $G^n \to X$. We will later describe this as a 'globular set'.

Definition 2.1. We now define by induction the maps $\phi_n \colon I^n \to G^n, n \geqslant 1$, with the following properties, for $x = (x_1, \ldots, x_n) \in I^n$:

(i) $\phi_1(x_1) = x_1$;
(ii) $|x_i| = 1$ for some $i = 1, \ldots, n$ if and only if $\|\phi_n(x)\| = 1$;
(iii) $|x_i| = 1$ for some $i = 1, \ldots, n$ implies $(\phi_n(x))_j = 0$ for $j < i$.

We set for $x = (t, y) \in I \times I^{n-1}$:

$$\phi_n(t, y) = (t\sqrt{1 - \|\phi_{n-1}(y)\|^2}, \phi_{n-1}(y)). \tag{6}$$

First note that if $x = (t, y)$, then

$$\|\phi_n(x)\|^2 = t^2 + (1 - t^2)\|\phi_{n-1}(y)\|^2.$$

This easily proves (ii) and (iii) by induction.

The maps $\phi_n\colon I^n \to G^n$ induce a map $\bar{\phi}\colon S^{\bigcirc}X \to S^{\square}X$.

We define the *globular subset* γK of a cubical set K to agree with K in dimensions 0, 1 and to be in dimension $n \geqslant 2$ the set of k such that $\partial_i^{\pm} k \in \operatorname{Im}\varepsilon_1^{i-1}, i = 2, \dots, n$.

Proposition 2.2. *The image of $\bar{\phi}\colon S^{\bigcirc}X \to S^{\square}X$ is exactly the globular subset of $S^{\square}X$.*

Proof. We prove by induction from the formula for ϕ_n that the image is globular. Let $p_1^i\colon \mathbb{R}^n \to \mathbb{R}^{n-i}$ be the projection omitting the first i coordinates. Suppose that $\phi_{n-1}\bar{\partial}_i^{\pm} = f_{n-1}p_1^{i-1}$. Then $\phi_n\bar{\partial}_{i+1}^{\pm} = f'_{n-1}p_1^i$ where $f'_{n-1}(x) = (0, f_{n-1}(x))$.

For the converse, we prove by induction that these are the only identifications that ϕ_n makes. Suppose $\phi_n(t,y) = \phi_n(t',y')$. Then $\phi_{n-1}(y) = \phi_{n-1}(y')$ and

$$t\sqrt{1 - \|\phi_{n-1}(y)\|^2} = t'\sqrt{1 - \|\phi_{n-1}(y')\|^2}.$$

Thus if $\|\phi_{n-1}(y)\| \neq 1$ then $t = t'$. But $\|\phi_{n-1}(y)\| = 1$ implies some $|y_i| = 1$, by the inductive hypothesis. $\square$

Let X_* be a filtered space. Then we obtain three *filtered singular complexes* $R^Q X_*$ for $Q = D, \circ, \square$ defined as graded sets by

$$(R^Q X_*)_n = \mathsf{FTop}(Q_*^n, X_*).$$

There are also associated graded homotopy sets $\rho^Q X_*$, which in dimension n are given by the quotient maps

$$p^Q\colon R^Q, X_* \to \rho^Q X_* = R^Q X_* / \equiv,$$

where $\equiv$ is the relation of homotopy rel vertices through filtered maps.

In the cases $Q = D, \square$ it is known that these graded sets obtain additional structure giving us, for $Q = D$, the fundamental crossed complex ΠX_*, and, for $Q = \square$, what is called in [**BH81b**] the *fundamental (cubical) ω-groupoid (with connections)* of X_*. However the proof that the standard compositions on $R^{\square}X_*$ are inherited by $\rho^{\square}X_*$ is nontrivial, as is the crucial result that $p^{\square}$ is a Kan fibration of cubical sets.

Remark 2.3. In [**BH81b**], the homotopies are not taken rel vertices and a condition J_0 is imposed, so that each map $\dot{I}^2 \to X_0$, where $\dot{I}^2$ is the boundary of I^2, may be extended to a map $I^2 \to X_1$. This condition is in many ways inconvenient. The filling processes used in the proofs can all be started by assuming instead that the homotopies are rel vertices so that the maps $\dot{I}^2 \to X_0$ required to be extended are in fact all constant. The details will be available in [**BHS08**].

Our first main result is:

Theorem 2.4. *The induced map*

$$\phi^*\colon \rho^{\bigcirc}X_* \to \rho^{\square}X_*$$

is injective.

Proof. Let $[\alpha], [\beta] \in (\rho^{\bigcirc} X_*)_n$ be such that $\phi^*[\alpha] = \phi^*[\beta]$; that is

$$[\alpha\phi] = [\beta\phi]$$

in $(\rho^{\square} X_*)_n$. Let $H\colon \alpha\phi \equiv \beta\phi$ be such a homotopy. Then H is a map $I_*^{n+1} \to X_*$ such that writing $I^{n+1} = I^n \times I$, each $H_t\colon I_*^n \to X_*$ is a filtered map.

We use a folding map $\Phi\colon I^n \to I^n$ given by Definition 3.1 of [**AABS02**] (see Definition B.2) which has the property that Φ factors through ϕ.

We now define a new homotopy $K_t = \Phi H_t\colon I_*^n \to X_*$. Then K_t is a globular homotopy $\Phi\alpha\phi \equiv \Phi\beta\phi$. But, by assumption, $\alpha\phi, \beta\phi$ are already globular maps. So the proof is completed with the following lemma.

Lemma 2.5. *If $a\colon I_*^n \to X_*$ is a globular map, then Φa is globularly equivalent to a.*

Proof. Since Φ is a composition of the folding operations ψ_i, it is sufficient to prove that $\psi_i a \equiv_{\bigcirc} a$. We follow the proof of [**AABS02**, Proposition 3.4]. By the definition of ψ_i:

$$\psi_i a = \Gamma_i^+ \partial_{i+1}^- a \circ_{i+1} a \circ_{i+1} \Gamma_i^- \partial_{i+1}^+ a.$$

But $\partial_{i+1}^- a$ and $\partial_{i+1}^+ a$ are globular. From the laws in Appendix B, we obtain, since a is globular,

$$\Gamma_i^{\mp} \partial_{i+1}^{\pm} a \in \operatorname{Im} \Gamma_i^{\mp} \varepsilon_i = \operatorname{Im} \varepsilon_i^2 = \operatorname{Im} \varepsilon_{1+1} \varepsilon_i.$$

So standard contractions of the two cubes $\Gamma_i^{\mp} \partial_{i+1}^{\pm} a$ yield a homotopy of $\psi_i a \equiv_{\bigcirc} a$ through globular maps. $\square$

It now follows that $\alpha, \beta\colon G_*^n \to X_*$ are globularly equivalent. $\square$

This proof is a higher-dimensional version of an argument in [**BHKP02**, Sec. 6].

Corollary 2.6. *The compositions in $\rho^{\square} X_*$ are inherited by $\rho^{\bigcirc} X_*$ to give the latter the structure of globular ω-groupoid.*

We do not know how to prove directly that $\rho^{\bigcirc} X_*$ may be given this structure of globular ω-groupoid.

3. The free globular ω-groupoid on one generator

Let X_* be a filtered space. Then we have a diagram of maps of homotopy sets

$$(\Pi X_*)_n \xrightarrow{i} (\rho^{\bigcirc} X_*)_n \xrightarrow{j} (\rho^{\square} X_*)_n. \tag{7}$$

We know from [**BH81b**] that the composition $j \circ i$ is injective. We already know that j is injective. It follows that i is injective. Thus the globular ω-groupoid $\rho^{\bigcirc} X_*$ contains the crossed complex ΠX_*, and the results of [**BH81c**] show that the latter generates the former as ω-Gpd.

We need the following result.

Theorem 3.1. *If G is a globular ω-groupoid, then there is a filtered space X_* such that $\rho^{\bigcirc} X_* \cong G$.*

Proof. Let C be the crossed complex associated with the ω-groupoid G under the equivalence (c) of diagram (5). By Corollary 9.3 of [**BH81b**], there is a filtered space X_* such that $\Pi X_* \cong C$. (Here X is the classifying space BC filtered by $X_n = BC^{(n)}$, where $C^{(n)}$ is the nth truncation of C.) It follows that $\rho^{\bigcirc} X_* \cong G$. □

Theorem 3.2. *The globular ω-groupoid $\rho^{\bigcirc} G^n_*$ is the free globular ω-groupoid on the class of the identity map, and its associated crossed complex is isomorphic to ΠG^n_*.*

Proof. Let $\iota\colon G^n_* \to G^n_*$ denote the identity map, and $[\iota]$ its class in $\rho^{\bigcirc} G^n_*$. Let H be a globular ω-groupoid and let $x \in H_n$. We have to show there is a unique morphism $\alpha\colon \rho^{\bigcirc} G^n_* \to H$ such that $\alpha[\iota] = x$. By Theorem 3.1 we may assume that H is of the form $\rho^{\bigcirc} X_*$ for some filtered space X_*; then x has a representative $g\colon G^n_* \to X_*$. It follows that $\rho^{\bigcirc}(g)([\iota]) = x$. This proves existence of such a morphism.

Suppose $\beta\colon \rho^{\bigcirc} G^n_* \to H$ is another morphism such that $\beta([\iota]) = x$. Then

$$\gamma(\alpha), \gamma(\beta)\colon \Pi G^n_*$$

agree on the generating element $c^n \in \pi_n(G^n, G^n_{n-1}, 1)$ of that group. However, ΠG^n_* is generated as crossed complex by all elements $\Phi dc^n \in \pi_r(G^n_r, G^n_{r-1}, 1)$ for all globular face operators d from dimension n to dimension r for $0 \leqslant r \leqslant n$. Since α, β are morphisms of ω-groupoids, $\alpha(\Phi dc^n) = d\alpha\Phi c^n = d\beta\Phi c^n = \beta(d\Phi c^n)$. Therefore α, β agree on ΠG^n_*, but the latter generates $\rho^{\bigcirc} G^n_*$ as ω-groupoid. So $\alpha = \beta$. □

The form of this crossed complex may be deduced from the cubical Homotopy Addition Lemma [**BH81a**, Lemma 7.1].

$$\delta x = \begin{cases} -x_1^+ - x_2^- + x_1^- + x_2^+ & \text{if } n = 2, \\ -x_3^+ - (x_2^-)^{u_2\mathbf{x}} - x_1^+ + (x_3^-)^{u_3\mathbf{x}} + x_2^+ + (x_1^-)^{u_1\mathbf{x}} & \text{if } n = 3, \\ \sum_{i=1}^n (-1)^i \{x_i^+ - (x_i^-)^{u_i\mathbf{x}}\} & \text{if } n \geqslant 4 \end{cases}$$

(where $u_i = \partial_1^+ \partial_2^+ \cdots \widehat{\imath} \cdots \partial_{n+1}^+$). In the case when x is globular, this reduces to

$$\delta x = -x_1^+ + x_1^- \text{ if } n \geqslant 2.$$

Notice that this is a groupoid formula if $n = 2$.

It would be interesting to have a purely algebraic proof of this result.

4. The higher homotopy van Kampen theorem

Suppose for the rest of this section that X_* is a filtered space. We suppose given a cover $\mathcal{U} = \{U^\lambda\}_{\lambda \in \Lambda}$ of X_∞ such that the interiors of the sets of $\mathcal{U}$ cover X_∞. For each $\zeta \in \Lambda^n$ we set $U^\zeta = U^{\zeta_1} \cap \cdots \cap U^{\zeta_n}$, $U^\zeta_i = U^\zeta \cap X_i$. Then $U^\zeta_0 \subseteq U^\zeta_1 \subseteq \cdots$ is called the *induced filtration* U^ζ_* of U^ζ. So the globular homotopy ω-groupoids in the following

$\varrho^{\bigcirc}$-*diagram* of the cover are well defined:

$$\bigsqcup_{\zeta\in\Lambda^2}\varrho^{\bigcirc}U^{\zeta}_* \overset{a}{\underset{b}{\rightrightarrows}} \bigsqcup_{\lambda\in\Lambda}\varrho^{\bigcirc}U^{\lambda}_* \xrightarrow{c} \varrho^{\bigcirc}X_*. \tag{8}$$

Here $\bigsqcup$ denotes disjoint union (which is the same as a coproduct in the category of globular ω-groupoids); a, b are determined by the inclusions $a_\zeta\colon U^\lambda\cap U^\mu \to U^\lambda$ and $b_\zeta\colon U^\lambda\cap U^\mu\to U^\mu$ for each $\zeta=(\lambda,\mu)\in\Lambda^2$; and c is determined by the inclusions $c_\lambda\colon U^\lambda\to X_\infty$.

Definition 4.1. A filtered space X_* is said to be *connected* if the following conditions hold for each $n\geqslant 0$:

- If $r>0$, the map $\pi_0X_0\to\pi_0X_r$, induced by inclusion, is surjective; i.e. X_0 meets all path connected components of all stages of the filtration X_r.
- (for $n\geqslant 1$): If $r>n$ and $x\in X_0$, then $\pi_n(X_r,X_n,x)=0$.

Theorem 4.2. *Suppose that for every finite intersection U^ζ of elements of $\mathcal{U}$, the induced filtration U^ζ_* is connected. Then*

(Conn) *X_* is connected;*

(Iso) *c in the above $\varrho^{\bigcirc}$-diagram is the coequaliser of a, b in the category of globular ω-groupoids.*

Proof. This follows from Theorem B of [**BH81b**], i.e. the analogous theorem for $\rho^{\square}$, and the fact that the equivalence from the category of globular ω-groupoids to that of cubical ω-groupoids with connections takes $\rho^{\bigcirc}X_*$ to $\rho^{\square}X_*$. □

Remark 4.3. If one could find convenient 'globular filtrations' of spaces, analogous to the cell decompositions of CW-complexes, then one should be able to use these results to show that $\rho^{\bigcirc}$ of such a filtration was a free globular ω-groupoid.

5. Monoidal closed structures

The category of cubical ω-groupoids with connection is monoidal closed [**BH87**]. We recall from that paper how the tensor product is defined.

For cubical ω-Gpds F, G, H, we define a *bimorphism*

$$b\colon F,G\to H \tag{9}$$

to be a family of functions $b=b_{p,q}\colon F_p\times G_q\to H_{p+q}$ such that if $x\in F_p$, $y\in G_q$ and $p+q=n$, then:

(i) $\partial_i^\alpha b(x,y)=\begin{cases} b(\partial_i^\alpha x,y) & \text{if } 1\leqslant i\leqslant p,\\ b(x,\partial_{i-p}^\alpha y) & \text{if } p+1\leqslant i\leqslant n;\end{cases}$

(ii) $\varepsilon_i b(x,y)=\begin{cases} b(\varepsilon_i x,y) & \text{if } 1\leqslant i\leqslant p+1,\\ b(x,\varepsilon_{i-p} y) & \text{if } p+1\leqslant i\leqslant n+1;\end{cases}$

(iii) $\Gamma_i b(x,y) = \begin{cases} b(\Gamma_i x, y) & \text{if } 1 \leqslant i \leqslant p, \\ b(x, \Gamma_{i-p} y) & \text{if } p+1 \leqslant i \leqslant n; \end{cases}$

(iv) $b(x \circ_i x', y) = b(x,y) \circ_i b(x',y)$ if $1 \leqslant i \leqslant p$ and $x \circ_i x'$ is defined in F;

(v) $b(x, y \circ_j y') = b(x,y) \circ_{p+j} b(x,y')$ if $1 \leqslant j \leqslant q$ and $y \circ_j y'$ is defined in G.

The *tensor product* of cubical ω-groupoids F, G is given by the the universal bimorphism $F, G \to F \otimes G$: that is any bimorphism $F, G \to H$ uniquely factors through a morphism $F \otimes G \to H$.

We next recall a result from [**BH91**].

Proposition 5.1. *Let X_*, Y_* be filtered spaces. Then there is a natural transformation*

$$\eta \colon \rho^{\square} X_* \otimes \rho^{\square} Y_* \to \rho^{\square}(X_* \otimes Y_*).$$

Proof. This natural transformation is determined by the bimorphism

$$([f],[g]) \mapsto [f \otimes g],$$

where $f \colon I_*^p \to X_*$, $g \colon I_*^q \to Y_*$. The proof that this is well defined and gives a bimorphism is routine, given the geometry of the cubes, that $I_*^p \otimes I_*^q \cong I_*^{p+q}$, and the well definedness of compositions on filter homotopy classes, as proved in [**BH81b**]. $\square$

It is proved in [**BH91**], by considering the corresponding free crossed complexes, that this morphism is an isomorphism if X_*, Y_* are skeletal filtrations of CW-complexes, and in [**BB93**] that this is an isomorphism if X_*, Y_* are connected and cofibred.

Because the categories of cubical and of globular ω-groupoids are equivalent, and the former has a monoidal closed structure, this is inherited by the latter.

So we deduce from the above results:

Theorem 5.2. *Let X_*,Y_* be filtered spaces. Then there is a natural transformation*

$$\eta \colon \rho^{\bigcirc} X_* \otimes \rho^{\bigcirc} Y_* \to \rho^{\bigcirc}(X_* \otimes Y_*)$$

which is an isomorphism if X_*, Y_* are connected and cofibred.

6. The nerve and classifying space functors on globular ω-groupoids

Here we just show how to define a simplicial nerve $N^{\Delta} G$ of a globular ω-groupoid G, by the standard procedure:

$$(N^{\Delta} G)_n = \omega\text{-}\mathsf{Gpd}(\rho^{\bigcirc} \Delta_*^n, G). \tag{10}$$

The geometric realisation of this simplicial set then defines the *classifying space BG* of G. However it is not so easy to see how to exploit this. The classifying space of a crossed complex is applied in, for example, [**BH91, BGPT, FMa07, FMP06**].

Appendix A. The globular site

We now recall from [**BH81c**] a definition, which in [**Str87**] and later work, is termed a *globular set*. This is a sequence $(S_n)_{n\geqslant 0}$ of sets with two families of functions

$$\begin{aligned} d_i^{\pm} &: S_n \to S_i, i = 0, \ldots, n-1, \\ s_i &: S_i \to S_n, i = 0, \ldots, n-1, \end{aligned}$$

satisfying the following laws, where $\alpha, \beta = \pm$:

(i) $d_i^{\alpha} d_j^{\beta} = d_i^{\alpha}$ for $i < j, \alpha, \beta = \pm$;

(ii) $s_j s_i = s_i$ for $i < j$;

(iii) $d_j^{\beta} s_i = \begin{cases} s_j^{\beta} & \text{for } j < i, \\ 1 & \text{for } j = i, \\ s_i & \text{for } j > i. \end{cases}$

A *globular site* GS is a small category such that globular sets can be identified with contravariant functors $GS \to \mathsf{Set}$. We want to identify such a site whose objects are the globes G^n of Section 2. We therefore define the maps

$$\bar{d}_i^{\pm} : G^i \to G^n, \qquad \bar{s}_i : G^n \to G^i \tag{11}$$

$$x \mapsto (0_{n-i}, \pm\sqrt{1 - \|x\|^2}, x), \quad (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_i), \tag{12}$$

for $i < n$, where $0_j = \underbrace{(0, \ldots, 0)}_{j}$.

Appendix B. The cubical site

Let K be a cubical set, that is, a family of sets $\{K_n; n \geqslant 0\}$ with face maps

$$\partial_i^{\alpha} : K_n \to K_{n-1} (i = 1, 2, \ldots, n; \alpha = +, -)$$

and degeneracy maps

$$\varepsilon_i : K_{n-1} \to K_n (i = 1, 2, \ldots, n)$$

satisfying the usual cubical relations:

$$\partial_i^{\alpha} \partial_j^{\beta} = \partial_{j-1}^{\beta} \partial_i^{\alpha} \qquad (i < j), \tag{B.1)(i}$$

$$\varepsilon_i \varepsilon_j = \varepsilon_{j+1} \varepsilon_i \qquad (i \leqslant j), \tag{B.1)(ii}$$

$$\partial_i^{\alpha} \varepsilon_j = \begin{cases} \varepsilon_{j-1} \partial_i^{\alpha} & (i < j) \\ \varepsilon_j \partial_{i-1}^{\alpha} & (i > j) \\ \mathrm{id} & (i = j), \end{cases} \tag{B.1)(iii}$$

We say that K is a *cubical set with connections* if it has additional structure maps (called *connections*) $\Gamma_i^+, \Gamma_i^- : K_{n-1} \to K_n (i = 1, 2, \dots, n-1)$ satisfying the relations:

$$\Gamma_i^\alpha \Gamma_j^\beta = \Gamma_{j+1}^\beta \Gamma_i^\alpha \qquad (i < j), \tag{B.2)(i}$$

$$\Gamma_i^\alpha \Gamma_i^\alpha = \Gamma_{i+1}^\alpha \Gamma_i^\alpha, \tag{B.2)(ii}$$

$$\Gamma_i^\alpha \varepsilon_j = \begin{cases} \varepsilon_{j-1}\Gamma_i^\alpha & (i < j) \\ \varepsilon_j \Gamma_{i-1}^\alpha & (i > j), \end{cases} \tag{B.2)(iii}$$

$$\Gamma_j^\alpha \varepsilon_j = \varepsilon_j^2 = \varepsilon_{j+1}\varepsilon_j, \tag{B.2)(iv}$$

$$\partial_i^\alpha \Gamma_j^\beta = \begin{cases} \Gamma_{j-1}^\beta \partial_i^\alpha & (i < j) \\ \Gamma_j^\beta \partial_{i-1}^\alpha & (i > j+1), \end{cases} \tag{B.2)(v}$$

$$\partial_j^\alpha \Gamma_j^\alpha = \partial_{j+1}^\alpha \Gamma_j^\alpha = \mathrm{id}, \tag{B.2)(vi}$$

$$\partial_j^\alpha \Gamma_j^{-\alpha} = \partial_{j+1}^\alpha \Gamma_j^{-\alpha} = \varepsilon_j \partial_j^\alpha. \tag{B.2)(vii}$$

The connections are to be thought of as extra 'degeneracies'. (A degenerate cube of type $\varepsilon_j x$ has a pair of opposite faces equal and all other faces degenerate. A cube of type $\Gamma_i^\alpha x$ has a pair of adjacent faces equal and all other faces of type $\Gamma_j^\alpha y$ or $\varepsilon_j y$.)

The prime example of a cubical set with connections is the singular cubical complex $K = S^\square X$ of a space X. Here K_n is the set of singular n-cubes in X (i.e. continuous maps $I^n \to X$). The face maps are induced as usual by maps $\bar{\partial}_i^\pm : I^{n-1} \to I^n$ and the degeneracies by the projections $p_i : I^n \to I^{n-1}$. The connections $\Gamma_i^\alpha : K_{n-1} \to K_n$ are induced by the maps $\gamma_i^\alpha : I^n \to I^{n-1}$ defined by

$$\gamma_i^\alpha(t_1, t_2, \dots, t_n) = (t_1, t_2, \dots, t_{i-1}, A(t_i, t_{i+1}), t_{i+2}, \dots, t_n),$$

where $A(s,t) = \max(s,t), \min(s,t)$ as $\alpha = -, +$, respectively.

The complex $S^\square X$ has some further relevant structure, namely the composition of n-cubes in the n different directions. Accordingly, we define a *cubical set with connections and compositions* to be a cubical set K with connections in which each K_n has n partial compositions $\circ_j (j = 1, 2, \dots, n)$ satisfying the following axioms.

If $a, b \in K_n$, then $a \circ_j b$ is defined if and only if $\partial_j^- b = \partial_j^+ a$, and then

$$\begin{cases} \partial_j^-(a \circ_j b) = \partial_j^- a \\ \partial_j^+(a \circ_j b) = \partial_j^+ b, \end{cases} \qquad \partial_i^\alpha(a \circ_j b) = \begin{cases} \partial_j^\alpha a \circ_{j-1} \partial_i^\alpha b & (i < j) \\ \partial_i^\alpha a \circ_j \partial_i^\alpha b & (i > j). \end{cases} \tag{B.3}$$

The interchange laws. If $i \neq j$, then

$$(a \circ_i b) \circ_j (c \circ_i d) = (a \circ_j c) \circ_i (b \circ_j d) \tag{B.4}$$

whenever both sides are defined. (The diagram

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{tikzcd}[row sep=small] \cdot \arrow[r, "i"] \arrow[d, "j"'] & {} \\ {} & \end{tikzcd}$$

will be used to indicate that both sides of the above equation are defined and also to denote the unique composite of the four elements.)

If $i \neq j$, then

$$\varepsilon_i(a \circ_j b) = \begin{cases} \varepsilon_i a \circ_{j+1} \varepsilon_i b & (i \leqslant j) \\ \varepsilon_i a \circ_j \varepsilon_i b & (i > j), \end{cases} \tag{B.5}$$

$$\Gamma_i^\alpha(a \circ_j b) = \begin{cases} \Gamma_i^\alpha a \circ_{j+1} \Gamma_i^\alpha b & (i < j) \\ \Gamma_i^\alpha a \circ_j \Gamma_i^\alpha b & (i > j), \end{cases} \tag{B.6)(i}$$

$$\Gamma_j^+(a \circ_j b) = \begin{bmatrix} \Gamma_j^+ a & \varepsilon_j a \\ \varepsilon_{j+1} a & \Gamma_j^+ b \end{bmatrix} \quad \begin{array}{l} \longrightarrow j \\ \big\downarrow \\ j+1 \end{array}, \tag{B.6)(ii}$$

$$\Gamma_j^-(a \circ_j b) = \begin{bmatrix} \Gamma_j^- a & \varepsilon_{j+1} b \\ \varepsilon_j b & \Gamma_j^- b \end{bmatrix} \quad \begin{array}{l} \longrightarrow j \\ \big\downarrow \\ j+1 \end{array}. \tag{B.6)(iii}$$

These last two equations are the *transport laws*[2].

It is easily verified that the singular cubical complex $S^\square X$ of a space X satisfies these axioms if $\circ_j$ is defined by

$$(a \circ_j b)(t_1, t_2, \ldots, t_n) = \begin{cases} a(t_1, \ldots, t_{j-1}, 2t_j, t_{j+1}, \ldots, t_n) & (t_j \leqslant \frac{1}{2}) \\ b(t_1, \ldots, t_{j-1}, 2t_j - 1, t_{j+1}, \ldots, t_n) & (t_j \geqslant \frac{1}{2}), \end{cases}$$

whenever $\partial_j^- b = \partial_j^+ a$.

We will now describe two graded subsets of a cubical set K. The *globular subset* $K^{\bigcirc}$ consists in dimension n of the elements a such that $\partial_i^\alpha a \in \operatorname{Im} \varepsilon_1^{i-1}, i = 1, \ldots, n$. The *diskal subset* K^D consists in dimension n of the elements a such that $\partial_i^\alpha a \in \operatorname{Im} \varepsilon_1^{n-1}$ for $(\alpha, i) \neq (-, 1)$. Clearly $K^D \subseteq K^{\bigcirc} \subseteq K$.

Proposition B.1. *If K is a cubical set with compositions, then the compositions $\circ_i$ are inherited by $K^{\bigcirc}$ so that if $d_i^\alpha \colon K_n^{\bigcirc} \to K_{n-i}^{\bigcirc}$ is defined by $a \mapsto (\partial_1^\alpha)^i(a)$, then $K^{\bigcirc}$ becomes a globular set with compositions. If, further, K is a cubical ω-category (-groupoid), then $K^{\bigcirc}$ is a globular ω-category (-groupoid).*

It is proved in [**BH81a**] that if K is a cubical ω-groupoid, then K^D inherits the structure of crossed complex, and in [**BH81c**] (see also [**AABS02**]), that $K^{\bigcirc}$ inherits the structure of globular ω-groupoid.

A *globular ω-category* is a globular set as above with category structures $\circ_i$ on S_n $0 \leqslant i \leqslant n-1$ for each $n \geqslant 0$ such that $\circ_i$ has S_i as its set of objects and D_i^-, D_i^+, E_i as its initial, final, and identity maps. These category structures must be compatible; that is:

(i) if $i > j$ and $\alpha = \pm$, then

$$D_i^\alpha(x \circ_j y) = D_i^\alpha x \circ_j D_i^\alpha y,$$

whenever the left-hand side is defined;

[2] From [**BS76**] recall that the term *connection* was chosen because of an analogy with path-connections in differential geometry. In particular, the transport law is a variation or a special case of the transport law for a path-connection.

(ii) $E_i(x \circ_j y) = E_i x \circ_j E_i y$ in S_n whenever the left-hand side is defined;

(iii) (The interchange law) if $i \neq j$, then

$$(x \circ_j y) \circ_i (z \circ_j w) = (x \circ_i z) \circ_j (y \circ_j w)$$

whenever both sides are defined.

It is standard to write both sides of the interchange law (when defined) as

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} \qquad \begin{array}{l} \to j \\ \downarrow i \end{array} .$$

Definition B.2. Let K be a cubical set with connections and compositions. The *folding operations* are the operations

$$\psi_i, \Psi_r, \Phi_m \colon K_n \to K_n$$

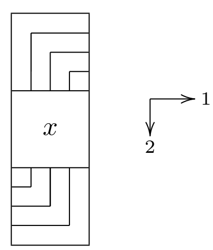
defined for $1 \leqslant i \leqslant n-1$, $1 \leqslant r \leqslant n$ and $0 \leqslant m \leqslant n$ by

$$\begin{aligned} \psi_i x &= \Gamma_i^+ \partial_{i+1}^- x \circ_{i+1} x \circ_{i+1} \Gamma_i^- \partial_{i+1}^+ x, \\ \Psi_r &= \psi_{r-1}\psi_{r-2} \dots \psi_1, \\ \Phi_m &= \Psi_1 \Psi_2 \dots \Psi_m = \psi_1(\psi_2\psi_1) \dots (\psi_{m-1} \dots \psi_1). \end{aligned}$$

Note in particular that Ψ_1, Φ_0 and Φ_1 are identity operations.

Here is a picture of $\psi_1 \colon K_2 \to K_2$:

$\psi_1(x) =$



Proposition B.3. *Let K be a cubical set with connections and compositions. The 'folding' operator $\Phi_n \colon K_n \to K_n$ satisfies $\partial_i^\pm \Phi_n x \in \operatorname{Im} \varepsilon_1^{i-1}$ for $1 \leqslant i \leqslant n$ and $x \in K_n$. That is,* $\operatorname{Im}\Phi$ *is contained in the globular subset of K.*

This is part of Proposition 3.3(iii) of [**AABS02**]. Note that the compositions are needed to define Φ_n but this property of Φ_n requires only the properties (B1), (B2) giving the relations between cubical operations and connections, and does not require any axioms on the compositions.

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