Homology, Homotopy and Applications, vol. 10(1), 2008, pp.283–304

COTORSION PAIRS AND DEGREEWISE HOMOLOGICAL MODEL STRUCTURES

JAMES GILLESPIE

(*communicated by Michael A. Mandell*)

Abstract

Let $\mathcal{C}$ be an abelian category. We show that under certain hypotheses, a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $\mathcal{C}$ may induce two natural homological model structures on $\mathrm{Ch}(\mathcal{C})$. One is such that the (trivially) cofibrant objects form the class of (exact) complexes A for which each $A_n \in \mathcal{A}$. The other is such that the (trivially) fibrant objects form the class of (exact) complexes B for which each $B_n \in \mathcal{B}$. Special cases of these model structures such as Hovey's "locally free" model structure and "flasque" model structure have already appeared in the literature. The examples support the belief that any useful homological model structure comes from a single cotorsion pair on the ground category $\mathcal{C}$. Furthermore, one of the two types of model structures we consider requires surprisingly few assumptions to exist. For example, Theorem 4.7 implies that every cotorsion pair $(\mathcal{A}, \mathcal{B})$ of R-modules which is cogenerated by a set gives rise to a cofibrantly generated homological model structure on $\mathrm{Ch}(R)$.

1. Introduction

In his study of abelian groups, Salce introduced in [**Sal79**] the notion of a cotorsion pair (or cotorsion theory). The concept readily generalizes to any abelian category $\mathcal{C}$. In short, a cotorsion pair in an abelian category $\mathcal{C}$ is a pair of classes $(\mathcal{A}, \mathcal{B})$ which are orthogonal with respect to $\mathrm{Ext}^1_{\mathcal{C}}(-,-)$. As simple examples, any abelian category $\mathcal{C}$ has the cotorsion pairs $(\mathcal{P}, \mathcal{A})$ and $(\mathcal{A}, \mathcal{I})$ where $\mathcal{P}$ is the class of projectives, $\mathcal{I}$ is the class of injectives and $\mathcal{A}$ is the class of all objects in $\mathcal{C}$. In the category of modules over a ring R, we also have the standard example $(\mathcal{F}, \mathcal{C}')$ where $\mathcal{F}$ is the class of flat modules and $\mathcal{C}'$ is the class of cotorsion modules. In recent years we have seen that the study of cotorsion pairs is especially relevant to the study of $\mathcal{F}$-covers and $\mathcal{F}$-envelopes (where $\mathcal{F}$ may be any class of objects closed under isomorphisms) in abelian categories. Enochs and several coauthors had long been studying covers and envelopes when Eklof and Trlifaj's result published in [**ET01**] helped Enochs settle

Received August 6, 2007, revised February 8, 2008; published on March 31, 2008.
2000 Mathematics Subject Classification: 18G55, 55U35.
Key words and phrases: cotorsion pair, chain complexes, model structure.
This article is available at `http://intlpress.com/HHA/v10/n1/a12`

his flat cover conjecture. This result says that any cotorsion pair in R-Mod which is *cogenerated by a set* is necessarily *complete*. (See Section 2 for these definitions.)

Hovey noticed in [**Hov02**] that a Quillen model structure on any abelian category $\mathcal{C}$ is equivalent to two complete cotorsion pairs in $\mathcal{C}$ which are compatible in a precise way. This relationship is deepened by the fact that when these two cotorsion pairs are each *small* (the appropriate generalization of "cogenerated by a set" to cotorsion pairs in an arbitrary Grothendieck category $\mathcal{C}$) the corresponding model structure is cofibrantly generated.

In [**Gil04**] the author began the study of when a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in an abelian category $\mathcal{C}$ induces two compatible cotorsion pairs in the chain complex category $\mathrm{Ch}(\mathcal{C})$. In that paper, following the methods of Enochs in [**BBE01**], he showed that the flat cotorsion pair $(\mathcal{F}, \mathcal{C}')$ induces two compatible and complete cotorsion pairs on $\mathrm{Ch}(R)$ where R is a commutative ring with 1. By Hovey's correspondence this gave a flat model structure on $\mathrm{Ch}(R)$, which is *monoidal* in the sense of [**Hov99**]. The cofibrant complexes are the dg-flat complexes, the fibrant complexes are the dg-cotorsion complexes, and the trivial objects are the exact complexes. In [**Gil07**] we see that the flat model structure generalizes to the category of complexes of quasi-coherent sheaves on a quasi-compact, semi-separated scheme X.

The current paper continues the study of the interplay between cotorsion pairs in $\mathcal{C}$ and homological model structures on $\mathrm{Ch}(\mathcal{C})$. The author calls them *homological model structures* since they all are on chain complex categories and since we always require the weak equivalences to be homology isomorphisms. We see that a nice cotorsion pair $(\mathcal{A}, \mathcal{B})$ may give rise to two other "degreewise" model structures on $\mathrm{Ch}(\mathcal{C})$ besides the type used to construct the flat model structures in [**Gil04**] and [**Gil07**]. One of these model structures is such that the cofibrant objects form the class of complexes A for which each $A_n \in \mathcal{A}$. The other is such that the fibrant objects form the class of complexes B for which each $B_n \in \mathcal{B}$.

We now summarize the layout of the paper as well as give some detail as to how these model structures are constructed. Section 2 provides relevant definitions and notation which will be used throughout the paper. Since our theorems are about model categories we assume the reader is familiar with and interested in model categories. However, if one believes Hovey's correspondence Theorem 2.2 from [**Hov02**] then one really does not need to know anything about model categories to understand the paper. A nice introduction to the basic idea of a model category can be found in [**DS95**]. In Section 3 we define the two new pairs of cotorsion pairs in $\mathrm{Ch}(\mathcal{C})$. One must also show that each of these pairs is *small* to get factorizations in our model category. Since this process involves Quillen's small object argument and since all objects are small in a Grothendieck category we begin in Section 4 to assume $\mathcal{C}$ is a Grothendieck category which we then denote by $\mathcal{G}$. Here we show that if $(\mathcal{A}, \mathcal{B})$ is a small cotorsion pair with cogenerating set $\{A_i\}$, and if $\mathcal{A}$ contains a generator of finite projective dimension, then there is an induced model structure on $\mathrm{Ch}(\mathcal{G})$ where the (trivially) fibrant objects are the (exact) complexes B with $B_n \in \mathcal{B}$. The trivially cofibrant objects are all the complexes A which are retracts of transfinite extensions of the disks $\{D^n(G), D^n(A_i)\}$. The cofibrant objects are the complexes which are retracts of transfinite extensions of $\{D^n(G), D^n(A_i), S^n(G)\}$. In Section 5 we tackle the apparently dual model structure. Here we do not need the generator $G \in \mathcal{A}$ to have finite projective dimension. Any generator will do, as long as there is one in $\mathcal{A}$. However, we need to assume that $\mathcal{A}$ is a Kaplansky class and closed under direct limits. We use the definition of Kaplansky class from [**Gil07**] which is a categorical version of the definition given by Edgar Enochs in [**ELR02**]. Section 5 is a little more technical in that the reader will want to refer to [**Gil07**] for some definitions and theorems that will be used. Throughout Sections 4 and 5 we give examples of cotorsion pairs which induce such model structures. Some of the resulting model structures already appeared in [**Hov01**] before the connection between model structures and cotorsion pairs was realized.

2. Preliminaries

Definition 2.1. A pair of classes $(\mathcal{A}, \mathcal{B})$ in an abelian category $\mathcal{C}$ is a cotorsion pair if the following conditions hold:

1. $\mathrm{Ext}^1_{\mathcal{C}}(A, B) = 0$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.
2. If $\mathrm{Ext}^1_{\mathcal{C}}(A, X) = 0$ for all $A \in \mathcal{A}$, then $X \in \mathcal{B}$.
3. If $\mathrm{Ext}^1_{\mathcal{C}}(X, B) = 0$ for all $B \in \mathcal{B}$, then $X \in \mathcal{A}$.

As mentioned in the introduction, every abelian category $\mathcal{C}$ has the projective cotorsion pair $(\mathcal{P}, \mathcal{A})$ and the injective cotorsion pair $(\mathcal{A}, \mathcal{I})$. When $\mathcal{C}$ has a tensor product then we ought to have the flat cotorsion pair $(\mathcal{F}, \mathcal{C}')$ as well. For a proof that $(\mathcal{F}, \mathcal{C}')$ is in fact a cotorsion pair when $\mathcal{C}$ is the category of R-modules, see for example [**EJ01**]. In fact [**EJ01**] is a good reference for both cotorsion pairs and cotorsion modules.

The cotorsion pair is said to have *enough projectives* if for any $X \in \mathcal{C}$ there is a short exact sequence $0 \to B \to A \to X \to 0$, where $B \in \mathcal{B}$ and $A \in \mathcal{A}$. We say it has *enough injectives* if it satisfies the dual statement. If both of these hold we say the cotorsion pair is *complete*. All of the examples of cotorsion pairs in the last paragraph are complete when the category is R-Mod. The phrases "enough projective" and "enough injectives" are standard in reference to cotorsion pairs. Unfortunately, we also use the phrase "enough projectives/injectives" in reference to a category. This should not be confusing since we will always explicitly refer to either a category or a cotorsion pair. Note however that saying that the projective cotorsion pair, $(\mathcal{P}, \mathcal{A})$, has enough projecctives is equivalent to saying that the category has enough projectives. Similarly for the injective cotorsion pair and "enough injectives". So in fact the terminology applied to a cotorsion theory is just a generalization of the usual terminology. In addition however, for any class of objects $\mathcal{F}$ in an abelian category $\mathcal{C}$, the author will use the terminology *enough $\mathcal{F}$-objects* to mean for any object $X \in \mathcal{C}$ there exists an $F \in \mathcal{F}$ and an epimorphism $F \to X$. Thus if $(\mathcal{F}, \mathcal{C})$ is the "flat" cotorsion pair described above, saying we have *enough $\mathcal{F}$-objects* means we can find a surjection $F \to M$ where F is flat. But we say $(\mathcal{F}, \mathcal{C})$ *has enough projectives* to mean there exists a short exact sequence

$$0 \to C \to F \to M \to 0,$$

where $C \in \mathcal{C}$ and $F \in \mathcal{F}$.

An abelian category $\mathcal{C}$ is called a Grothendieck category if $\mathcal{C}$ has a generator $G \in \mathcal{C}$ and if direct limits are exact. The categories we are interested in, including module categories, sheaf categories and their corresponding chain complex categories are all examples of Grothendieck categories. For more information on Grothendieck categories we refer the reader to Chapter V of [**Sten75**].

Definition 2.2. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be cogenerated by a set if there is a *set* $\mathcal{S} \subset \mathcal{A}$ (not just a class) such that $\mathcal{S}^{\perp} = \mathcal{B}$. Here $\mathcal{S}^{\perp}$ is the class of all objects $X \in \mathcal{C}$ for which $\operatorname{Ext}^1_{\mathcal{C}}(S, X) = 0$ for all $S \in \mathcal{S}$. If we furthermore assume that $\mathcal{C}$ is a Grothendieck category and that $\mathcal{A}$ contains some generator G then a cotorsion pair is called *small* if for each $S \in \mathcal{S}$ there is a monomorphism i_S, with $\operatorname{cok} i_S = S$, satisfying the following: For all $X \in \mathcal{C}$, if $\mathcal{C}(i_S, X)$ is surjective for all $S \in \mathcal{S}$, then $X \in \mathcal{B}$. We denote by I the set of monomorphisms together with the monomorphism $0 \to G$, and we say I is a set of *generating monomorphisms* for the cotorsion pair $(\mathcal{A}, \mathcal{B})$.

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ in R-Mod is complete whenever it is cogenerated by a set. This important result is due to Eklof and Trlifaj and can be found in [**ET01**]. More generally, if we are in a Grothendieck category and we have a generator $G \in \mathcal{A}$, then by Theorem 6.5 of [**Hov02**] we see that any small cotorsion is complete. Small cotorsion pairs first appeared in [**Hov02**]. We also refer the reader to [**Gil07**] for more detail on small cotorsion pairs. We give the definition of a *Kaplansky class* in Section 5 but again the reader will probably want to refer to [**Gil07**] for more details. In general we will cite theorems from [**Hov02**], [**Gil07**], and [**Gil04**] throughout the paper.

We denote the category of chain complexes by $\operatorname{Ch}(\mathcal{C})$. The differentials d of our chain complexes lower degree. Given an object $C \in \mathcal{C}$, we denote its n-sphere by $S^n(C)$ and its n-disk by $D^n(C)$. Note that, for each n, we have a short exact sequence $0 \to S^n(C) \to D^{n+1}(C) \to S^{n+1}(C) \to 0$. Using Baer's description of Ext, there is a subgroup $\operatorname{Ext}^1_{dw}(X, Y)$ of $\operatorname{Ext}^1_{\operatorname{Ch}}(X, Y)$ consisting of all short exact sequences $0 \to Y \to Z \to X \to 0$ which are split in each degree n.

We now list two standard lemmas which will be used throughout the paper.

Lemma 2.3. *Let G be an object in an abelian category $\mathcal{C}$. G is a generator for $\mathcal{C}$ if and only if given any morphism $d \colon C \to D$, d is an epimorphism whenever d_* is an epimorphism. Here $d_* \colon \mathcal{C}(G, C) \to \mathcal{C}(G, D)$ is defined by $d_*(t) = dt$.*

Lemma 2.4. *Let $\mathcal{C}$ be an abelian category with generator G. Let X be a chain complex. If every chain map $f \colon S^n(G) \to X$ extends to $D^{n+1}(G)$, then X is exact.*

Proof. Let n be an arbitrary integer. By Lemma 2.3, showing exactness in degree n requires showing that any morphism $f \colon G \to Z_n X$ lifts over $d \colon X_{n+1} \to Z_n X$. But it is easy to see that this is the same as showing that the induced chain map $S^n(G) \to X$ extends to a morphism $D^{n+1}(G) \to X$. □

3. Cotorsion pairs of complexes

Given a class of objects $\mathcal{A}$ in an abelian category $\mathcal{C}$ the author denotes by $\widetilde{\mathcal{A}}$ the class of all exact chain complexes $X \in \operatorname{Ch}(\mathcal{C})$ such that $Z_n X \in \mathcal{A}$. From [**Gil04**] and [**Gil07**] we saw that when $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair in $\mathcal{C}$ (and with some hypotheses on the cotorsion pair and $\mathcal{C}$ itself), the class $\widetilde{\mathcal{A}}$ forms the trivially cofibrant objects and the class $\widetilde{\mathcal{B}}$ forms the trivially fibrant objects for a homological model structure on $\mathrm{Ch}(\mathcal{C})$. This model structure is intermediate to the degreewise model structures we study in this paper. We now define the classes of complexes that will be the basis for defining these model structures.

Definition 3.1. Let $\mathcal{A}$ be a class of objects in an abelian category $\mathcal{C}$. We denote by $dw\,\widetilde{\mathcal{A}}$ the class of all complexes $X \in \mathrm{Ch}(\mathcal{C})$ such that $X_n \in \mathcal{A}$ and we denote by $ex\,\widetilde{\mathcal{A}}$ the class of all exact complexes $X \in \mathrm{Ch}(\mathcal{C})$ such that $X_n \in \mathcal{A}$.

The "*dw*" is meant to be thought of as "degreewise" while the "*ex*" is meant to be thought of as "exact".

Proposition 3.2. *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in $\mathcal{C}$. Then $(dw\,\widetilde{\mathcal{A}}, (dw\,\widetilde{\mathcal{A}})^{\perp})$ is a cotorsion pair and $(dw\,\widetilde{\mathcal{A}})^{\perp}$ is the class of all complexes Y for which $Y_n \in \mathcal{B}$ and for which each map $X \to Y$ is null homotopic whenever $X \in dw\,\widetilde{\mathcal{A}}$. Similarly, the pair $({}^{\perp}(dw\,\widetilde{\mathcal{B}}), dw\,\widetilde{\mathcal{B}})$ is a cotorsion pair and ${}^{\perp}(dw\,\widetilde{\mathcal{B}})$ is the class of all complexes X for which $X_n \in \mathcal{A}$ and for which each map $X \to Y$ is null homotopic whenever $Y \in dw\,\widetilde{\mathcal{B}}$.*

Proof. We just prove the first statement since the second follows by duality. For the proof we will let $\widehat{\mathcal{B}}$ denote the class of all complexes Y for which $Y_n \in \mathcal{B}$ and for which each map $X \to Y$ is null homotopic whenever $X \in dw\,\widetilde{\mathcal{A}}$. It is easy to check that $\widehat{\mathcal{B}}$ is closed under taking suspensions and also contains all disks $D^n(B)$ whenever $B \in \mathcal{B}$. Now we simply wish to show that $(dw\,\widetilde{\mathcal{A}}, \widehat{\mathcal{B}})$ is a cotorsion pair.

First we show $\mathrm{Ext}^1_{\mathrm{Ch}}(A, B) = 0$ for any $A \in dw\,\widetilde{\mathcal{A}}$ and $B \in \widehat{\mathcal{B}}$. In this case, it is clear that any element of $\mathrm{Ext}^1_{\mathrm{Ch}}(A, B)$ (That is, a short exact sequence $B \to Z \to A$) is degreewise split and so is an element of $\mathrm{Ext}^1_{dw}(A, B)$. But using Lemma 2.1 of [**Gil04**], it is clear that $\mathrm{Ext}^1_{dw}(A, B) = 0$.

Next suppose $\mathrm{Ext}^1_{\mathrm{Ch}}(A, X) = 0$ for all $A \in dw\,\widetilde{\mathcal{A}}$. We wish to show $X \in \widehat{\mathcal{B}}$. First, given any $A \in \mathcal{A}$, $D^n(A) \in dw\,\widetilde{\mathcal{A}}$. Using Lemma 3.1 of [**Gil04**] we have the isomorphism $\mathrm{Ext}^1_{\mathrm{Ch}}(D^n(A), X) \cong \mathrm{Ext}^1_{\mathcal{C}}(A, X_n) = 0$. So $X_n \in \mathcal{B}$. Now suppose $A \to X$ is a chain map where $A \in dw\,\widetilde{\mathcal{A}}$. We would like to show that it is null homotopic. By Lemma 2.1 of [**Gil04**], this will be the case if we can show $\mathrm{Ext}^1_{dw}(A, \Sigma^{-1}X) = 0$. But clearly, $\mathrm{Ext}^1_{dw}(A, \Sigma^{-1}X) = \mathrm{Ext}^1_{dw}(\Sigma A, X)$ and this last group equals 0 since $\Sigma A \in dw\,\widetilde{\mathcal{A}}$. Therefore $X \in \widehat{\mathcal{B}}$.

Last we suppose $\mathrm{Ext}^1_{\mathrm{Ch}}(X, B) = 0$ for all $B \in \widehat{\mathcal{B}}$. We wish to show $X \in dw\,\widetilde{\mathcal{A}}$, which means we want to show $\mathrm{Ext}^1_{\mathcal{C}}(X_n, B) = 0$ for all $B \in \mathcal{B}$. But for any such B, the disk $D^{n+1}(B) \in \widehat{\mathcal{B}}$, so $\mathrm{Ext}^1_{\mathcal{C}}(X_n, B) \cong \mathrm{Ext}^1_{\mathrm{Ch}}(X, D^{n+1}(B)) = 0$. □

Recall that if A is an object in an abelian category $\mathcal{C}$, then we say A has *finite projective dimension* if there exists a positive integer n such that $\mathrm{Ext}^i_{\mathcal{C}}(A, C) = 0$ for all $C \in \mathcal{C}$ and $i \geqslant n$. Finite injective dimension is defined similarly using the other variable.

Proposition 3.3. *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in $\mathcal{C}$. If $\mathcal{B}$ contains a cogenerator of finite injective dimension then $(ex\,\widetilde{\mathcal{A}}, (ex\,\widetilde{\mathcal{A}})^{\perp})$ is a cotorsion pair. Furthermore,*

$(ex\,\widetilde{\mathcal{A}})^{\perp}$ is the class of all complexes Y for which $Y_n \in \mathcal{B}$ and for which every map $X \to Y$ is null homotopic whenever $X \in ex\,\widetilde{\mathcal{A}}$. If $\mathcal{A}$ contains a generator of finite projective dimension then we have the obvious dual statement about $({}^{\perp}(ex\,\widetilde{\mathcal{B}}), ex\,\widetilde{\mathcal{B}})$.

Proof. Again we will just prove one of the statements since the other is dual. We will show that if $\mathcal{A}$ contains a generator of finite projective dimension then $({}^{\perp}(ex\,\widetilde{\mathcal{B}}), ex\,\widetilde{\mathcal{B}})$ is a cotorsion pair. For the proof we will let $\widehat{\mathcal{A}}$ denote the class of all complexes X for which $X_n \in \mathcal{A}$ and for which every map $X \to Y$ is null homotopic whenever $Y \in ex\,\widetilde{\mathcal{B}}$. $\widehat{\mathcal{A}}$ is closed under taking suspensions and also contains all disks $D^n(A)$ whenever $A \in \mathcal{A}$. We need to show $(\widehat{\mathcal{A}}, ex\,\widetilde{\mathcal{B}})$ is a cotorsion pair.

Showing that $\operatorname{Ext}^1_{\mathrm{Ch}}(A, B) = 0$ for all $A \in \widehat{\mathcal{A}}$ and $B \in ex\,\widetilde{\mathcal{B}}$ is straightforward and does not require having a generator of finite projective dimension. Similarly, showing that $\operatorname{Ext}^1_{\mathrm{Ch}}(X, B) = 0$ for all $B \in ex\,\widetilde{\mathcal{B}}$ implies $X \in \widehat{\mathcal{A}}$ is straightforward and does not require a generator. This takes care of conditions (1) and (3) in the definition of a cotorsion pair. We now prove condition (2) using a generator from $\mathcal{A}$ with finite projective dimension.

Claim 1. For any $A \in \mathcal{A}$ with finite projective dimension and $B \in ex\,\widetilde{\mathcal{B}}$ we have $\operatorname{Ext}^1_{\mathcal{C}}(A, Z_nB) = 0$ for all n. Indeed from the short exact sequence $0 \to Z_{n+1}B \to B_{n+1} \to Z_nB \to 0$ we see that $\operatorname{Ext}^i_{\mathcal{C}}(A, Z_nB) \cong \operatorname{Ext}^{i+1}_{\mathcal{C}}(A, Z_{n+1}B)$ for all $i \geqslant 1$. But this holds for all n and so one can argue that $\operatorname{Ext}^i_{\mathcal{C}}(A, Z_nB) \cong \operatorname{Ext}^{i+j}_{\mathcal{C}}(A, Z_{n+j}B)$ for all i, j. Since A has finite projective dimension we conclude $\operatorname{Ext}^1_{\mathcal{C}}(A, Z_nB) = 0$ for all n.

Claim 2. For any $A \in \mathcal{A}$ with finite projective dimension we have $S^n(A) \in \widehat{\mathcal{A}}$. Indeed suppose $S^n(A) \to B$ is a chain map where $B \in ex\,\widetilde{\mathcal{B}}$. This corresponds to a map $A \to Z_nB$. Applying $\operatorname{Hom}(A, -)$ to the short exact sequence $Z_{n+1}B \to B_{n+1} \to Z_nB$ and using the fact that $\operatorname{Ext}^1_{\mathcal{C}}(A, Z_{n+1}B) = 0$, we see that the map $A \to Z_nB$ lifts over B_{n+1}. In other words, the map $S^n(A) \to B$ is null homotopic.

Finally, suppose $G \in \mathcal{A}$ is a generator of finite projective dimension. We show that if X is a complex for which $\operatorname{Ext}^1_{\mathrm{Ch}}(A, X) = 0$ for all $A \in \widehat{\mathcal{A}}$, then $X \in ex\,\widetilde{\mathcal{B}}$. Indeed, for such an X apply the functor $\operatorname{Hom}(-, X)$ to the short exact sequence $S^n(G) \to D^{n+1}(G) \to S^{n+1}(G)$. Since $S^{n+1}(G) \in \widehat{\mathcal{A}}$ we have $\operatorname{Ext}^1_{\mathrm{Ch}}(S^{n+1}(G), X) = 0$ and so we see that any map $S^n(G) \to X$ extends over $D^{n+1}(G)$. Therefore X is exact by Lemma 2.4. It is left to see that $X_n \in \mathcal{B}$. But for all $A \in \mathcal{A}$ we have $D^n(A) \in \widehat{\mathcal{A}}$ and so $0 = \operatorname{Ext}^1_{\mathrm{Ch}}(D^n(A), X) = \operatorname{Ext}^1_{\mathcal{C}}(A, X_n)$. So $X_n \in \mathcal{B}$. □

In the above, we started with a cotorsion pair $(\mathcal{A}, \mathcal{B})$ on $\mathcal{C}$ and a suitable class of complexes and defined cotorsion pairs of complexes using the notion of null homotopy. All of the classes that arose in these cotorsion pairs are closed under suspensions and contain the appropriate disks $D^n(A)$ and $D^n(B)$. We end this section by noting that any such cotorsion pair on $\mathrm{Ch}(\mathcal{C})$ comes from a cotorsion pair on the ground category $\mathcal{C}$ in this way. To make this precise we make the following definition.

Definition 3.4. A cotorsion pair $(\mathcal{J}, \mathcal{K})$ on $\mathrm{Ch}(\mathcal{C})$ is *degreewise orthogonal* if for every pair i, j of integers we have $\operatorname{Ext}^1_{\mathcal{C}}(J_i, K_j) = 0$ whenever $J \in \mathcal{J}$ and $K \in \mathcal{K}$.

For the rest of this section we fix the following notation: If $(\mathcal{J},\mathcal{K})$ is a cotorsion pair on $\mathrm{Ch}(\mathcal{C})$ then we let $\mathcal{J}'$ be the class of all objects in $\mathcal{C}$ which appear as an entry to a complex in $\mathcal{J}$. That is, if $X \in \mathcal{J}'$, then $X = J_n$ for some $J \in \mathcal{J}$. Similarly, we let $\mathcal{K}'$ denote the class of all entries from complexes in $\mathcal{K}$.

Lemma 3.5. *The following are equivalent for a cotorsion pair $(\mathcal{J},\mathcal{K})$ of complexes:*

(1) $(\mathcal{J},\mathcal{K})$ is degreewise orthogonal.

(2) $\mathcal{J}$ and $\mathcal{K}$ each contain all disks on entries. In other words, if $J \in \mathcal{J}$, then we have $D^n(J_i) \in \mathcal{J}$ for each integer i and n. Similarly for $\mathcal{K}$.

(3) $(\mathcal{J}',\mathcal{K}')$ is a cotorsion pair on $\mathcal{C}$.

Proof. *(3)* implies *(1)* is clear. We now show *(2)* implies *(3)*. First, if $A \in \mathcal{J}'$ and $B \in \mathcal{K}'$, then $\mathrm{Ext}^1_{\mathcal{C}}(A,B) = \mathrm{Ext}^1_{\mathrm{Ch}}(D^0(A),D^1(B)) = 0$. Second, suppose $\mathrm{Ext}^1_{\mathcal{C}}(X,B) = 0$ whenever $B \in \mathcal{K}'$. We want to show $X \in \mathcal{J}'$. Now for any $K \in \mathcal{K}$, we must have $\mathrm{Ext}^1_{\mathrm{Ch}}(D^n(X),K) = \mathrm{Ext}^1_{\mathcal{C}}(X,K_n) = 0$ since $K_n \in \mathcal{K}'$. Therefore $D^n(X) \in \mathcal{J}$. So $X \in \mathcal{J}'$. Conversely, say $\mathrm{Ext}^1_{\mathcal{C}}(A,Y) = 0$ whenever $A \in \mathcal{J}'$. Then for any $J \in \mathcal{J}$, we have $\mathrm{Ext}^1_{\mathrm{Ch}}(J,D^n(Y) = \mathrm{Ext}^1_{\mathcal{C}}(J_{n-1},Y) = 0$, so $Y \in \mathcal{K}'$.

Finally *(1)* implies *(2)*. For example, suppose $J \in \mathcal{J}$. We want to show $D^n(J_i) \in \mathcal{J}$. But this is true since for any $K \in \mathcal{K}$, we have $\mathrm{Ext}^1_{\mathrm{Ch}}(D^n(J_i),K) = \mathrm{Ext}^1_{\mathcal{C}}(J_i,K_n) = 0$.

Note that *(2)* implies *(1)* can easily be proved directly using the easy fact that $\mathrm{Ext}^1_{\mathcal{C}}(A,B) = \mathrm{Ext}^1_{\mathrm{Ch}}(D^0(A),D^1(B)) = 0$. □

Note that if $\mathcal{J}$ and $\mathcal{K}$ are each closed under suspensions, then degreewise orthogonal just means $\mathrm{Ext}^1\,\mathcal{C}(J_0,K_0) = 0$ for all $J \in \mathcal{J}$ and $K \in \mathcal{K}$. One can check that $\mathcal{J}$ is closed under suspensions if and only if $\mathcal{K}$ is closed under suspensions. Similarly, $\mathcal{J}$ contains all disks on entries if and only if $\mathcal{K}$ does.

Example 3.6. Let R be a commutative ring with 1. There are cotorsion pairs in $\mathrm{Ch}(R)$ which are neither closed under suspensions nor degreewise orthogonal. For example, let $\mathcal{S} = \{S^0(F)\}$ be the singleton set where F is a flat module which cogenerates the flat cotorsion pair in R-Mod. Then $({}^{\perp}(\mathcal{S}^{\perp}),\mathcal{S}^{\perp})$ is a cotorsion pair in $\mathrm{Ch}(R)$. It can be shown that every complex $X \in {}^{\perp}(\mathcal{S}^{\perp})$ is a retract of a transfinite extension of the set $\{D^n(R),S^0(F)\}$. Since retracts and transfinite extensions are taken degreewise we see that for all $n \neq 0$, X_n is a projective R-module, while X_0 need only be a flat R-module. In particular $S^n(F) \in {}^{\perp}(\mathcal{S}^{\perp})$ only for $n = 0$. So ${}^{\perp}(\mathcal{S}^{\perp})$ is not closed under suspensions. It is also not degreewise orthogonal since it does not contain the disk $D^1(F)$.

Proposition 3.7. *Let $(\mathcal{J},\mathcal{K})$ be a degreewise orthogonal cotorsion pair of complexes where $\mathcal{J}$, and $\mathcal{K}$ are closed under suspensions. Let $(\mathcal{J}',\mathcal{K}')$ be the corresponding cotorsion pair on $\mathcal{C}$. Then $\mathcal{J}$ equals the class of all complexes X for which $X_n \in \mathcal{J}'$ and such that $X \to K$ is null homotopic whenever $K \in \mathcal{K}$. Similarly, $\mathcal{K}$ equals the class of all complexes Y for which $Y_n \in \mathcal{K}'$ and such that $J \to Y$ is null homotopic whenever $J \in \mathcal{J}$.*

Proof. We will just prove the first statement. First suppose we are given an X for which $X_n \in \mathcal{J}'$ and such that $X \to K$ is null homotopic whenever $K \in \mathcal{K}$. We show that $X \in {}^{\perp}\mathcal{K}$. So let $K \in \mathcal{K}$ be given. Since each $X_n \in \mathcal{J}'$ and each $K_n \in \mathcal{K}'$ we have that $\mathrm{Ext}^1_{\mathrm{Ch}}(X,K) = \mathrm{Ext}^1_{dw}(X,K)$. But if we let $\sim$ represent the relation of chain homotopy, then it is easy to check that

$$\mathrm{Ext}^1_{dw}(X,K) = \mathrm{Ch}(\mathcal{C})(X,\Sigma K)/\sim$$

and by hypothesis this last group is zero. Thus $X \in {}^{\perp}\mathcal{K}$.

Conversely, say $X \in {}^{\perp}\mathcal{K}$. Then for any $K \in \mathcal{K}'$, we have $D^{n+1}(K) \in \mathcal{K}$, and so $\mathrm{Ext}^1_{\mathcal{C}}(X_n, K) \cong \mathrm{Ext}^1_{\mathrm{Ch}}(X, D^{n+1}(K)) = 0$. Therefore $X_n \in \mathcal{J}'$. Also, if we are given a map $X \to K$ where K is a complex in $\mathcal{K}$, then it is null homotopic since $0 = \mathrm{Ext}^1_{\mathrm{Ch}}(X, \Sigma^{-1}K)$ implies

$$0 = \mathrm{Ext}^1_{dw}(X, \Sigma^{-1}K) = \mathrm{Ch}(\mathcal{C})(X,K)/\sim . \qquad \square$$

4. Model structures from small cotorsion pairs

Again we let $\mathcal{C}$ be an abelian category. In this section we prove that any small cotorsion pair $(\mathcal{A},\mathcal{B})$ where the class $\mathcal{A}$ contains a generator of finite projective dimension induces a homological model structure on $\mathrm{Ch}(\mathcal{C})$. The class ${}^{\perp}(ex\,\widetilde{\mathcal{B}})$ will be the class of cofibrant complexes and the class $dw\,\widetilde{\mathcal{B}}$ will be the class of fibrant complexes. We will then look at a few examples of this type of model structure, some of which already appeared in [**Hov01**].

We refer the reader to Section 6 of [**Hov02**] and Section 3 of [**Gil07**] for the definition of small cotorsion pair. In particular we use the notation of [**Gil07**]. The utility of small cotorsion pairs is that if we are in a Grothendieck category then the cotorsion pair is (functorially) complete. This was proved by Hovey in Theorem 6.5 of [**Hov02**], where the notion of a small cotorsion pair first appeared. Hovey's proof makes use of Quillen's "small object argument".

We start by providing a correction to parts (7) and (8) of Lemma 3.1 in [**Gil04**]. References to these lemmas that the author knows about are still correct by using arguments pointing to the corrected versions in Lemma 4.2 below.

Lemma 4.1. *Suppose we have a morphism of short exact sequences as shown:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{j} & B & \xrightarrow{p} & C & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \\ 0 & \longrightarrow & X & \xrightarrow{j'} & Y & \xrightarrow{p'} & Z & \longrightarrow & 0. \end{array}$$

If h is an isomorphism, then the left square is both a pullback and a pushout square. If f is an isomorphism, then the right square is both a pullback and a pushout square.

Proof. The two statements are dual and we will prove the first one. So suppose h is an isomorphism. First we will see why $gj = j'f$ is a pullback square. Suppose there is an object D and arrows α and β such that $j'\alpha = g\beta$. One can see that $p\beta = 0$ and since $j = \ker p$, there exists $\xi\colon D \to A$ such that $j\xi = \beta$. Then one argues that $j'\alpha = j'f\xi$ and since j' is left cancellable we get $\alpha = f\xi$. Therefore the left square is a pullback.

Now we show the square is a pushout. Let P be the pushout of j, f and form the diagram below (where the maps ξ and ϕ are yet to be explained):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{j} & B & \xrightarrow{p} & C & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & {\scriptstyle f''}\downarrow & & \| & & \\ 0 & \longrightarrow & X & \xrightarrow{j''} & P & \xrightarrow{p''} & C & \longrightarrow & 0 \\ & & \| & & {\scriptstyle \xi}\downarrow & & {\scriptstyle \phi}\downarrow & & \\ 0 & \longrightarrow & X & \xrightarrow{j'} & Y & \xrightarrow{p'} & Z & \longrightarrow & 0. \end{array}$$

Since $gj = j'f$, there exists a unique map $\xi\colon P \to Y$ such that $j' = \xi j''$ and $\xi f'' = g$. (The plan is to show ξ is an isomorphism.) Now since $(p'\xi)j'' = 0$ and $p'' = \operatorname{cok} j''$ there exists a unique map $\phi\colon C \to Z$ such that $\phi p'' = p'\xi$. We have $\phi p = \phi p'' f'' = p'\xi f'' = p'g = hp$. Since p is right cancellable, $\phi = h$. In particular, ϕ is an isomorphism and the snake lemma tells us that ξ is also an isomorphism. □

Lemma 4.2. *Let $\mathcal{C}$ be an abelian category. For any object $C \in \mathcal{C}$ and chain complex X, we have monomorphisms*

$$\operatorname{Ext}^1_{\mathcal{C}}(C, Z_nX) \to \operatorname{Ext}^1_{Ch(\mathcal{C})}(S^nC, X),$$
$$\operatorname{Ext}^1_{\mathcal{C}}(X_n/B_nX, C) \to \operatorname{Ext}^1_{Ch(\mathcal{C})}(X, S^nC).$$

If X is an exact complex, then these are actually isomorphisms.

Proof. The two statements are dual. We will prove the first statement. Suppose $\mathcal{S}$ is the short exact sequence $0 \to Z_nX \xrightarrow{f} D \xrightarrow{g} C \to 0$ in $\operatorname{Ext}^1_{\mathcal{C}}(C, Z_nX)$. Taking the pushout of f and the inclusion $i\colon Z_nX \to X_n$, we get a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & Z_nX & \xrightarrow{f} & D & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & {\scriptstyle i}\downarrow & & {\scriptstyle i'}\downarrow & & \| & & \\ 0 & \longrightarrow & X_n & \xrightarrow{f'} & P & \xrightarrow{g'} & C & \longrightarrow & 0 \\ & & {\scriptstyle d_n}\downarrow & & \downarrow{\scriptstyle h} & & & & \\ 0 & \longrightarrow & B_{n-1}X & = & B_{n-1}X & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0. & & & & \end{array}$$

Now form the short exact sequence indicated below. (The middle complex is indeed a complex since $\operatorname{Im} h \subseteq B_{n-1}X$.)

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & X_{n+1} & = & X_{n+1} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle d} & & \Big\downarrow{\scriptstyle f'd} & & \Big\downarrow & & \\
0 & \longrightarrow & X_n & \xrightarrow{f'} & P & \xrightarrow{g'} & C & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle d_n} & & \Big\downarrow{\scriptstyle h} & & \Big\downarrow & & \\
0 & \longrightarrow & X_{n-1} & = & X_{n-1} & \longrightarrow & 0 & \longrightarrow & 0. \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

This short exact sequence is an element of $\operatorname{Ext}^1_{\operatorname{Ch}(\mathcal{C})}(S^nC, X)$ which we denote by $\phi(\mathcal{S})$. This gives us a morphism $\phi\colon \operatorname{Ext}^1_{\mathcal{C}}(C, Z_nX) \to \operatorname{Ext}^1_{\operatorname{Ch}(\mathcal{C})}(S^nC, X)$. The reader can check that ϕ is a well-defined homomorphism of abelian groups. Now we argue that ϕ is a monomorphism. Suppose the above complex splits and $s = \{s_n\}$ is a section. By commutativity of the diagram we have $hs_n = 0$. Since $i' = \ker h$, there is a unique morphism $s'\colon C \to D$ such that $i's' = s_n$. Now s' is a section for g since $gs' = g'i's' = g's_n = 1_C$. So the sequence $0 \to Z_nX \xrightarrow{f} D \xrightarrow{g} C \to 0$ also splits.

Now suppose X is an exact complex. In this case we can construct an inverse ψ for ϕ. Let $0 \to X \to Y \to S^nC \to 0$ be a short exact sequence in $\operatorname{Ext}^1_{\operatorname{Ch}(\mathcal{C})}(S^nC, X)$ and denote it by $\mathcal{T}$. Define $\psi\colon \operatorname{Ext}^1_{\operatorname{Ch}(\mathcal{C})}(S^nC, X) \to \operatorname{Ext}^1_{\mathcal{C}}(C, Z_nX)$ by sending $\mathcal{T}$ to $0 \to Z_nX \to Z_nY \to C \to 0$. Note that this sequence is exact since X is an exact complex. (Truncate $\mathcal{T}$ above degree n and apply the fundamental lemma of homological algebra. We actually only need $H_{n-1}X = 0$ here.) It is clear that if $\mathcal{S}$ is the short exact sequence $0 \to Z_nX \xrightarrow{f} D \xrightarrow{g} C \to 0$ in $\operatorname{Ext}^1_{\mathcal{C}}(C, Z_nX)$, then $\psi(\phi(\mathcal{S})) = \mathcal{S}$.

On the other hand, let $\mathcal{T}$ be $0 \to X \to Y \to S^nC \to 0$ in $\operatorname{Ext}^1_{\operatorname{Ch}(\mathcal{C})}(S^nC, X)$. Then we may assume it has the form:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & X_{n+1} & = & Y_{n+1} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle d} & & \Big\downarrow{\scriptstyle \alpha d} & & \Big\downarrow & & \\
0 & \longrightarrow & X_n & \xrightarrow{\alpha} & Y_n & \xrightarrow{\beta} & C & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle d_n} & & \Big\downarrow{\scriptstyle h} & & \Big\downarrow & & \\
0 & \longrightarrow & X_{n-1} & = & Y_{n-1} & \longrightarrow & 0 & \longrightarrow & 0. \\
 & & \Big\downarrow{\scriptstyle d_{n-1}} & & \Big\downarrow{\scriptstyle d_{n-1}} & & \vdots & &
\end{array}
$$

Since Y is a complex we have $\operatorname{Im} h \subseteq Z_{n-1}Y = Z_{n-1}X = B_{n-1}X$. But since $h\alpha = d_n$ we have $B_{n-1}X \subseteq \operatorname{Im} h$. Therefore $B_{n-1}X = \operatorname{Im} h$. So $\mathcal{T}$ gives rise to the diagram below with exact rows and columns:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & Z_nX & \xrightarrow{\alpha|_{Z_nX}} & Z_nY & \longrightarrow & C & \longrightarrow & 0 \\ & & {\scriptstyle i}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & X_n & \xrightarrow{\alpha} & Y_n & \xrightarrow{\beta} & C & \longrightarrow & 0\\ & & {\scriptstyle d_n}\downarrow & & \downarrow{\scriptstyle h} & & & & \\ 0 & \longrightarrow & B_{n-1}X & = & B_{n-1}X & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0. & & & & \end{array}$$

By Lemma 4.1 the upper-left square is a pushout. Therefore we have $\phi(\psi(\mathcal{T})) = \mathcal{T}$. So ϕ and ψ are inverse isomorphisms. □

In [**Gil07**] the author proved the following proposition which appeared as Proposition 3.8.

Proposition 4.3. *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in an abelian category $\mathcal{C}$ which has a generator $G \in \mathcal{A}$. If $(\mathcal{A}, \mathcal{B})$ is cogenerated by a set $\{A_i\}_{i \in I_0}$, then the induced cotorsion pair $({}^\perp\widetilde{\mathcal{B}}, \widetilde{\mathcal{B}})$ is cogenerated by the set*

$$\mathcal{S} = \{S^n(G) | n \in \mathbb{Z}\} \cup \{S^n(A_i) | n \in \mathbb{Z}, i \in I_0\}.$$

Furthermore, suppose $(\mathcal{A}, \mathcal{B})$ is small with generating monomorphisms the map $0 \to G$ together with monomorphisms k_i as below (one for each $i \in I_0$):

$$0 \to Y_i \xrightarrow{k_i} Z_i \to A_i \to 0.$$

Then $({}^\perp\widetilde{\mathcal{B}}, \widetilde{\mathcal{B}})$ is small with generating monomorphisms the set

$$I = \{0 \to D^n(G)\} \cup \{S^{n-1}(G) \to D^n(G)\} \cup \{S^n(Y_i) \xrightarrow{S^n(k_i)} S^n(Z_i)\}.$$

Using an approach similar to the one used in [**Gil07**], we will find sets which cogenerate $({}^\perp(dw\,\widetilde{\mathcal{B}}), dw\,\widetilde{\mathcal{B}})$ and $({}^\perp(ex\,\widetilde{\mathcal{B}}), ex\,\widetilde{\mathcal{B}})$. We included the statement of Proposition 4.3 here to compare the statement to the ones in Propositions 4.4 and 4.6.

Proposition 4.4. *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in a abelian category $\mathcal{C}$. If $(\mathcal{A}, \mathcal{B})$ is cogenerated by a set $\{A_i\}_{i \in I_0}$, then the induced cotorsion pair $({}^\perp(dw\,\widetilde{\mathcal{B}}), dw\,\widetilde{\mathcal{B}})$ is cogenerated by the set*

$$\mathcal{S} = \{\, D^n(A_i) | n \in \mathbb{Z}, i \in I_0\}.$$

Furthermore, suppose $\mathcal{A}$ contains a generator G and $(\mathcal{A}, \mathcal{B})$ is small with generating monomorphisms the map $0 \to G$ together with monomorphisms k_i as below (one for *each $i \in I_0$):*

$$0 \to Y_i \xrightarrow{k_i} Z_i \to A_i \to 0.$$

Then $({}^{\perp}(dw\,\widetilde{\mathcal{B}}), dw\,\widetilde{\mathcal{B}})$ is small with generating monomorphisms the set

$$J = \{0 \to D^n(G)\} \cup \{D^n(Y_i) \xrightarrow{D^n(k_i)} D^n(Z_i)\}.$$

Proof. We have already noted that ${}^{\perp}(dw\,\widetilde{\mathcal{B}})$ contains all disks on objects in $\mathcal{A}$ and so $\mathcal{S} \subseteq {}^{\perp}(dw\,\widetilde{\mathcal{B}})$. It follows that $\mathcal{S}^{\perp} \supseteq dw\,\widetilde{\mathcal{B}}$. Conversely if $X \in \mathcal{S}^{\perp}$, then we have $0 = \mathrm{Ext}^1_{\mathrm{Ch}(\mathcal{C})}(D^n(A_i), X)$ for all $i \in I_0$. But $\mathrm{Ext}^1_{\mathrm{Ch}(\mathcal{C})}(D^n(A_i), X) \cong \mathrm{Ext}^1_{\mathcal{C}}(A_i, X_n)$ (by Lemma 3.1 in [**Gil04**]). So $\mathrm{Ext}^1_{\mathcal{C}}(A_i, X_n) = 0$ which implies $X_n \in \mathcal{B}$ since the set $\{A_i\}$ cogenerates $(\mathcal{A}, \mathcal{B})$. Therefore $\mathcal{S}^{\perp} = dw\,\widetilde{\mathcal{B}}$.

Next we prove the statement about smallness. First note that since G generates $\mathcal{C}$, the complexes $D^n(G)$ generate $\mathrm{Ch}(\mathcal{C})$. Also since $G \in \mathcal{A}$, we have ${}^{\perp}(dw\,\widetilde{\mathcal{B}})$ contains the generators $\{D^n(G)\}$. Now let X be any chain complex. We wish to show that "extending through monomorphisms in J" implies $X \in dw\,\widetilde{\mathcal{B}}$. But any map $Y_i \to X_n$ determines a morphism $D^n(Y_i) \to X$, which we assume extends over $D^n(k_i)$ to a map $D^n(Z_i) \to X$. In particular, any map $Y_i \to X_n$ extends over k_i to a map $Z_i \to X_n$. By the smallness hypothesis this implies $X_n \in \mathcal{B}$. So $X \in dw\,\widetilde{\mathcal{B}}$. ☐

Next we prove a similar statement for the cotorsion pair $({}^{\perp}(ex\,\widetilde{\mathcal{B}}), ex\,\widetilde{\mathcal{B}})$.

Lemma 4.5. *Let G be a generator for $\mathcal{C}$ and let $\mathcal{S}$ be any set containing $\{S^n(G)\}$. Then every complex in $\mathcal{S}^{\perp}$ is exact.*

Proof. Let $X \in \mathcal{S}^{\perp}$. Consider the short exact sequence

$$0 \to S^{n-1}(G) \to D^n(G) \to S^n(G) \to 0.$$

It induces an exact sequence of abelian groups

$$\mathrm{Hom}_{\mathrm{Ch}(\mathcal{C})}(D^n(G), X) \to \mathrm{Hom}_{\mathrm{Ch}(\mathcal{C})}(S^{n-1}(G), X) \to \mathrm{Ext}^1_{\mathrm{Ch}(\mathcal{C})}(S^n(G), X) = 0.$$

It now follows from Lemma 2.4 that X is exact. ☐

Proposition 4.6. *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in an abelian category $\mathcal{C}$ and let $G \in \mathcal{A}$ be a generator with finite projective dimension. If $(\mathcal{A}, \mathcal{B})$ is cogenerated by a set $\{A_i\}_{i \in I_0}$, then the induced cotorsion pair $({}^{\perp}(ex\,\widetilde{\mathcal{B}}), ex\,\widetilde{\mathcal{B}})$ is cogenerated by the set*

$$\mathcal{S} = \{S^n(G) | n \in \mathbb{Z}\} \cup \{D^n(A_i) | n \in \mathbb{Z}, i \in I_0\}.$$

Furthermore, if $(\mathcal{A}, \mathcal{B})$ is small with generating monomorphisms, then the map $0 \to G$ together with monomorphisms k_i as below (one for each $i \in I_0$):

$$0 \to Y_i \xrightarrow{k_i} Z_i \to A_i \to 0.$$

Then $({}^{\perp}(ex\,\widetilde{\mathcal{B}}), ex\,\widetilde{\mathcal{B}})$ is small with generating monomorphisms the set

$$I = \{0 \to D^n(G)\} \cup \{S^{n-1}(G) \to D^n(G)\} \cup \{D^n(Y_i) \xrightarrow{D^n(k_i)} D^n(Z_i)\}.$$

Proof. We already know that each $D^n(A_i) \in {}^{\perp}(ex\,\widetilde{\mathcal{B}})$. Also, we showed in the proof of Proposition 3.3 that $S^n(G) \in {}^{\perp}(ex\,\widetilde{\mathcal{B}})$. Therefore $\mathcal{S} \subseteq {}^{\perp}(ex\,\widetilde{\mathcal{B}})$ and so $\mathcal{S}^{\perp} \supseteq ex\,\widetilde{\mathcal{B}}$.

Conversely, we will show $\mathcal{S}^{\perp} \subseteq ex\,\widetilde{\mathcal{B}}$. If $X \in \mathcal{S}^{\perp}$ we know from Lemma 4.5 that X is exact. So it is left to show that $X_n \in \mathcal{B}$. But $0 = \mathrm{Ext}^1_{\mathrm{Ch}(\mathcal{C})}(D^n(A_i), X)$ for all $i \in I_0$ and $\mathrm{Ext}^1_{\mathrm{Ch}(\mathcal{C})}(D^n(A_i), X) \cong \mathrm{Ext}^1_{\mathcal{C}}(A_i, X_n)$ by Lemma 3.1 in [**Gil04**]. So $\mathrm{Ext}^1_{\mathcal{C}}(A_i, X_n) = 0$ which implies $X_n \in \mathcal{B}$ since the set $\{A_i\}$ cogenerates the cotorsion theory.

Next we prove the statement about smallness. First note that since $G \in \mathcal{A}$, the class ${}^{\perp}(ex\,\widetilde{\mathcal{B}})$ contains the generators $\{D^n(G)\}$. Now let X be any chain complex. We wish to show that "extending through monomorphisms in I" implies $X \in ex\,\widetilde{\mathcal{B}}$. But any map $Y_i \to X_n$ determines a morphism $D^n(Y_i) \to X$, which we assume extends over $D^n(k_i)$ to a map $D^n(Z_i) \to X$. In particular, any map $Y_i \to X_n$ extends over k_i to a map $Z_i \to X_n$. By the smallness hypothesis this implies $X_n \in \mathcal{B}$. Also, as we have already seen from Lemma 2.4, extending over the morphisms $S^{n-1}(G) \to D^n(G)$ forces X to be exact. So $X \in ex\,\widetilde{\mathcal{B}}$. □

Theorem 4.7. *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in a Grothendieck category $\mathcal{G}$ and suppose $G \in \mathcal{A}$ is a generator with finite projective dimension. Furthermore, suppose $(\mathcal{A}, \mathcal{B})$ is small with generating monomorphisms the map $0 \to G$ together with monomorphisms k_i as below (for each $i \in I_0$, where $\{A_i\}_{i \in I_0}$ cogenerates $(\mathcal{A}, \mathcal{B})$):*

$$0 \to Y_i \xrightarrow{k_i} Z_i \to A_i \to 0.$$

Then we have a cofibrantly generated model structure on $\mathrm{Ch}(\mathcal{G})$ described as follows: The weak equivalences are the homology isomorphisms. The cofibrations (respectively, trivial cofibrations) are the monomorphisms whose cokernels are in ${}^{\perp}(ex\,\widetilde{\mathcal{B}})$ (respectively, ${}^{\perp}(dw\,\widetilde{\mathcal{B}})$). The fibrations (respectively, trivial fibrations) are the epimorphisms whose kernels are in $dw\,\widetilde{\mathcal{B}}$ (respectively, $ex\,\widetilde{\mathcal{B}}$). Furthermore, we have ${}^{\perp}(dw\,\widetilde{\mathcal{B}}) = {}^{\perp}(ex\,\widetilde{\mathcal{B}}) \cap \mathcal{E}$ and $ex\,\widetilde{\mathcal{B}} = dw\,\widetilde{\mathcal{B}} \cap \mathcal{E}$, where $\mathcal{E}$ is the class of exact complexes. The set I given in the statement of Proposition 4.6 form the generating cofibrations. The set J given in the statement of Proposition 4.4 form the generating trivial cofibrations.

Proof. This all follows from work in [**Hov02**] and we just give an outline of the proof. First, the Grothendieck category hypothesis ensures that every object in $\mathcal{G}$ is small, and so we can apply the small object argument to argue (as in the proof of Theorem 6.5 of [**Hov02**]) that $({}^{\perp}(dw\,\widetilde{\mathcal{B}}), dw\,\widetilde{\mathcal{B}})$ and $({}^{\perp}(ex\,\widetilde{\mathcal{B}}), ex\,\widetilde{\mathcal{B}})$ are functorially complete cotorsion pairs.

It is clear from the definitions that $ex\,\widetilde{\mathcal{B}} = dw\,\widetilde{\mathcal{B}} \cap \mathcal{E}$. We now show that ${}^{\perp}(dw\,\widetilde{\mathcal{B}}) = {}^{\perp}(ex\,\widetilde{\mathcal{B}}) \cap \mathcal{E}$. First, if $A \in {}^{\perp}(dw\,\widetilde{\mathcal{B}})$, then clearly $A \in {}^{\perp}(ex\,\widetilde{\mathcal{B}})$. Also A must be exact. Indeed by Lemma 3.5 of [**Gil07**], A must be a retract of a transfinite extension of complexes appearing as cokernels of some map in the set J. But all of these complexes are in $\mathcal{E}$, and $\mathcal{E}$ is closed under transfinite extensions and retracts. So A must be exact. Therefore ${}^{\perp}(dw\,\widetilde{\mathcal{B}}) \subseteq {}^{\perp}(ex\,\widetilde{\mathcal{B}}) \cap \mathcal{E}$. On the other hand, say $X \in {}^{\perp}(ex\,\widetilde{\mathcal{B}}) \cap \mathcal{E}$. Since $({}^{\perp}(dw\,\widetilde{\mathcal{B}}), dw\,\widetilde{\mathcal{B}})$ is complete we can find a short exact sequence $0 \to B \to A \to X \to 0$ where $B \in dw\,\widetilde{\mathcal{B}}$ and $A \in {}^{\perp}(dw\,\widetilde{\mathcal{B}})$. Since A and X are exact it follows (from the induced long exact sequence of homology objects) that B is exact. So $B \in dw\,\widetilde{\mathcal{B}} \cap \mathcal{E} = ex\,\widetilde{\mathcal{B}}$. Since $({}^{\perp}(ex\,\widetilde{\mathcal{B}}), ex\,\widetilde{\mathcal{B}})$ is a cotorsion pair the sequence

$0 \to B \to A \to X \to 0$ must split, making X a direct summand of A. But then X must belong to ${}^{\perp}(dw\,\widetilde{\mathcal{B}})$ since the left side of a cotorsion pair is always closed under retracts. This completes the proof that ${}^{\perp}(dw\,\widetilde{\mathcal{B}}) = {}^{\perp}(ex\,\widetilde{\mathcal{B}}) \cap \mathcal{E}$.

The existence of the model structure comes from Theorem 2.2 of [**Hov02**] and one can see that the cofibrations, fibrations, weak equivalences, and generating (trivial) cofibrations are how we describe by looking at Sections 5 and 6 of [**Hov02**]. In particular, see Definition 5.1 and the proof of Lemma 6.7 in [**Hov02**]. □

4.1. Examples concerning modules over a ring.

Suppose R is a commutative ring with 1. If $\mathcal{G}$ is the category of R-modules, then every cotorsion pair $(\mathcal{A}, \mathcal{B})$ which is cogenerated by a set $\mathcal{S}$ is small. Indeed we can find a set of generating monomorphisms by taking, for each $S \in \mathcal{S}$, a monomorphism $K \hookrightarrow P$ where $P/K \cong S$. Furthermore, the class $\mathcal{A}$ will always contain the projective generator R. Therefore *any* cotorsion pair of R-modules which is cogenerated by a set gives rise to a homological model structure as in Theorem 4.7.

In particular, the injective cotorsion pair $(\mathcal{A}, \mathcal{I})$ is small. Indeed Baer's criterion implies that it is cogenerated by the set $\mathcal{S} = \{R/I\}$ where I ranges through all possible ideals of R. Thus Theorem 4.7 gives an alternate injective model structure on $\mathrm{Ch}(R)$ where the (trivially) fibrant objects are the (exact) complexes I for which each I_n is injective. The cofibrant complexes in this model structure are all of the complexes X which have the property that any chain map $X \to I$, where I is trivially fibrant, is null homotopic. Note that in general, for the class $dg\widetilde{\mathcal{I}}$ of dg-injective complexes, we have $dg\widetilde{\mathcal{I}} \subseteq dw\,\widetilde{\mathcal{I}}$. But also bounded above complexes in $dw\,\widetilde{\mathcal{I}}$ are in $dg\widetilde{\mathcal{I}}$ by Lemma 3.4 of [**Gil04**].

If we start with the projective cotorsion pair $(\mathcal{P}, \mathcal{A})$, then since this pair is small (take $\mathcal{S} = \{R\}$) we have an induced model structure where the fibrant objects are in $dw\,\widetilde{\mathcal{A}}$. However, it is clear that $dw\,\widetilde{\mathcal{A}}$ is the class of all complexes. Therefore ${}^{\perp}(dw\,\widetilde{\mathcal{A}})$ is the class of all projective complexes. So in the notation of [**Gil04**], $(\widetilde{\mathcal{P}}, dg\widetilde{\mathcal{A}}) = ({}^{\perp}(dw\,\widetilde{\mathcal{A}}), dw\,\widetilde{\mathcal{A}})$ and $(dg\widetilde{\mathcal{P}}, \widetilde{\mathcal{A}}) = ({}^{\perp}(ex\,\widetilde{\mathcal{A}}), ex\,\widetilde{\mathcal{A}})$. In short, Theorem 4.7 just recovers the usual projective model structure on $\mathrm{Ch}(R)$.

Of course there are many other cotorsion pairs of R-modules, each inducing a model structure on $\mathrm{Ch}(R)$. For example, it was shown in [**BBE01**] that the cotorsion pair $(\mathcal{F}, \mathcal{C})$ where $\mathcal{F}$ is the class of flat R-modules and $\mathcal{C}$ is the class of cotorsion R-modules is cogenerated by the set $\mathcal{S}$ of all flat modules with cardinality less than or equal to $\max\{\omega, |R|\}$. Therefore we get a homological model structure where the cofibrant objects are in ${}^{\perp}(ex\,\widetilde{\mathcal{C}})$ and the fibrant objects are in $dw\,\widetilde{\mathcal{C}}$.

4.2. The dimensionwise injective model structure in Grothendieck categories.

The dimensionwise injective model structure on $\mathrm{Ch}(R)$ from the above example generalizes to any Grothendieck category $\mathcal{G}$ which has a generator G of finite projective dimension. This is because Proposition V.2.9 of [**Sten75**] generalizes Baer's criterion. From this proposition, the set of all monomorphisms $V \hookrightarrow G$ is a set of generating monomorphisms for $(\mathcal{A}, \mathcal{I})$. In particular we recover Hovey's "locally free model structure" (Proposition 2.3 of [**Hov01**]) on the category of chain complexes of quasi-coherent sheaves on a nice enough Noetherian scheme X.

4.3. A flasque model structure on complexes of sheaves

As another example, we will construct a "flasque" model structure on Ch($\mathcal{O}_X$-Mod) when X is a finite- dimensional and Noetherian topological space. The existence of this model structure is interesting since it is a formalization of the fact that sheaf cohomology is computable using flasque coresolutions (which need not be injective). This model structure also first appeared in [**Hov01**], although there it was referred to as a "flat" model structure rather than a "flasque" model structure. One advantage of the cotorsion pair approach to defining this model structure is that the cofibrant objects are more easily understood. They form the class ${}^{\perp}(ex\,\widetilde{\mathcal{F}})$ where $({}^{\perp}\mathcal{F}, \mathcal{F})$ is the flasque cotorsion pair discussed below.

Henceforth, we let $(X, \mathcal{O}_X)$ denote any ringed space, and we will specify explicitly when we want X to be finite- dimensional and Noetherian. We let $\mathcal{F}$ denote the class of all flasque sheaves in $\mathcal{O}_X$-Mod. That is, $F \in \mathcal{F}$ if for all open $U \subseteq X$, the restriction map $F(X) \to F(U)$ is a surjection. Although our language of *small* cotorsion pairs was not used, one can see by examining the proof Proposition 2.2 in [**EO01**], that $({}^{\perp}\mathcal{F}, \mathcal{F})$ is a small cotorsion pair. We will now show this in detail.

First let us recall the standard set of generators for $\mathcal{O}_X$-Mod. For each open $U \subseteq X$, extend $\mathcal{O}_{|U}$ by 0 outside of U to get a presheaf, we denote as $\mathcal{O}_U$. Now sheafify to get an $\mathcal{O}_X$-module, which we will denote $j!(\mathcal{O}_U)$. One can prove without difficulty that we have isomorphisms $\mathrm{Hom}(\mathcal{O}_U, G) \cong G(U)$. So by the universal property of sheafification we get $\mathrm{Hom}(j!(\mathcal{O}_U), G) \cong G(U)$. It follows at once that the set $\{j!(\mathcal{O}_U)\}$ forms a generating set since the modules $j!(\mathcal{O}_U)$ "pick out points". Hence the direct sum $\bigoplus_{U \subseteq X} j!(\mathcal{O}_U)$ is a generator. Note that each $j!(\mathcal{O}_U)$ is a flat $\mathcal{O}_X$-module since $[j!(\mathcal{O}_U)]_p \cong (\mathcal{O}_U)_p$, which equals $\mathcal{O}_p$ if $p \in U$ and 0 if $p \in X \backslash U$.

For each open $U \subseteq X$, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_U & \xrightarrow{k_U} & \mathcal{O}_X & \xrightarrow{p_U} & \mathcal{O}_X/\mathcal{O}_U & \longrightarrow & 0 \\ & & {\scriptstyle \xi_{\mathcal{O}_U}}\big\downarrow & & {\scriptstyle \xi_{\mathcal{O}_X}}\big\downarrow & & {\scriptstyle \xi_U}\big\downarrow & & \\ 0 & \longrightarrow & j!(\mathcal{O}_U) & \xrightarrow{k_U^+} & (\mathcal{O}_X)^+ & \xrightarrow{p_U^+} & (\mathcal{O}_X/\mathcal{O}_U)^+ & \longrightarrow & 0, \end{array}$$

where k_U is the obvious inclusion. This makes the top row presheaf exact. The bottom row is obtained by sheafification and is therefore sheaf exact. In fact, using basic sheaf isomorphisms, we can replace this diagram with

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_U & \xrightarrow{k_U} & \mathcal{O}_X & \xrightarrow{p_U} & \mathcal{O}_X/\mathcal{O}_U & \longrightarrow & 0 \\ & & {\scriptstyle \xi_{\mathcal{O}_U}}\big\downarrow & & \big\| & & {\scriptstyle \xi_U}\big\downarrow & & \\ 0 & \longrightarrow & j!(\mathcal{O}_U) & \xrightarrow{k_U^+} & \mathcal{O}_X & \xrightarrow{p_U^+} & \mathcal{O}_X/j!(\mathcal{O}_U) & \longrightarrow & 0. \end{array}$$

In light of the above discussion, the following lemma is easy to prove.

Lemma 4.8. *An $\mathcal{O}_X$-module F is flange if and only if for every open $U \subseteq X$, the map $(k_U^+)^*\colon \mathrm{Hom}(\mathcal{O}_X, F) \to \mathrm{Hom}(j!(\mathcal{O}_U), F)$ is surjective.*

Lemma 4.9. *The generators $\{j!(\mathcal{O}_U)\}$ are in ${}^{\perp}\mathcal{F}$.*

Proof. For an open set U, the functor $\mathrm{Hom}(j!(\mathcal{O}_U), -)$ is isomorphic to $\Gamma(U, (-)|_U)$. Also, the restriction functor $(-)|_U$ preserves injectives, so we have isomorphisms

$$\mathrm{Ext}^n(j!(\mathcal{O}_U), -) \cong H^n(U, (-)|_U).$$

Now given $F \in \mathcal{F}$, the restriction $F|_U$ is also flasque. So by Proposition III.2.5 of [**Har77**] we have $\mathrm{Ext}^1(j!(\mathcal{O}_U), F) \cong H^1(U, F|_U) = 0$. □

Lemma 4.10. *For any flasque sheaf F, we have* $\mathrm{Ext}^1(\mathcal{O}_X/j!(\mathcal{O}_U), F) = 0$.

Proof. Suppose F is flasque. Given $U \subseteq X$, we have the short exact sequence

$$0 \to j!(\mathcal{O}_U) \xrightarrow{k_U^+} \mathcal{O}_X \xrightarrow{p_U^+} \mathcal{O}_X/j!(\mathcal{O}_U) \to 0.$$

By applying $\mathrm{Hom}(-, F)$ and using Lemma 4.9 we get the exact sequence

$$\mathrm{Hom}(\mathcal{O}_X, F) \xrightarrow{(k_U^+)^*} \mathrm{Hom}(j!(\mathcal{O}_U), F) \to \mathrm{Ext}^1(\mathcal{O}_X/j!(\mathcal{O}_U), F) \to 0.$$

But $(k_U^+)^*$ is surjective by Lemma 4.8. Therefore $\mathrm{Ext}^1(\mathcal{O}_X/j!(\mathcal{O}_U), F) = 0$. □

Proposition 4.11. *Let $(X, \mathcal{O}_X)$ be any ringed space and $\mathcal{F}$ be the class of flasque $\mathcal{O}_X$-modules. Then $({}^\perp\mathcal{F}, \mathcal{F})$ is a small cotorsion pair. The generating monomorphisms are the maps $0 \to j!(\mathcal{O}_U)$, along with the maps $k_U^+ \colon j!(\mathcal{O}_U) \to \mathcal{O}_X$.*

Proof. Given *any* set $\mathcal{S}$ in an abelian category, $({}^\perp(\mathcal{S}^\perp), \mathcal{S}^\perp)$ is a cotorsion pair, cogenerated by $\mathcal{S}$. If we take $classS = \{classO_X/j!(\mathcal{O}_U) \colon U \subseteq X\}$, then by Lemmas 4.8 and 4.10 we have that $\mathcal{F} = \mathcal{S}^\perp$ and so $({}^\perp\mathcal{F}, \mathcal{F})$ is a cotorsion pair cogenerated by $\{\mathcal{O}_X/j!(\mathcal{O}_U) \colon U \subseteq X\}$. Finally, it is clear from Lemmas 4.9 and 4.8 that the set

$$I_0 = \{0 \to j!(\mathcal{O}_U)\} \cup \{j!(\mathcal{O}_U) \to \mathcal{O}_X\}$$

is a set of generating monomorphisms for $({}^\perp\mathcal{F}, \mathcal{F})$. □

Of course to get the model structure of Theorem 4.7 induced by $({}^\perp\mathcal{F}, \mathcal{F})$ we need to know that the generators $j!(\mathcal{O}_U)$ each have finite projective dimension. However, as in the proof of Lemma 4.9, we know that for any open set $U \subseteq X$, we have

$$\mathrm{Ext}^i(j!(\mathcal{O}_U), -) \cong H^i(U, (-)|_U).$$

This suggests that the generators $j!(\mathcal{O}_U)$ each have finite projective dimension if and only if $(X, \mathcal{O}_X)$ has finite hereditary global dimension which we define below.

Definition 4.12. A ringed space $(X, \mathcal{O}_X)$ has *finite global dimension* if there is a positive integer n for which the sheaf cohomology $H^i(X, G) = 0$ for all $\mathcal{O}_X$-modules G and $i \geqslant n$. We say $(X, \mathcal{O}_X)$ has *finite hereditary global dimension* if for every open $U \subseteq X$, the ringed space $(U, \mathcal{O}|_U)$ has finite global dimension.

It follows from Grothendieck's vanishing theorem (Theorem III.2.7 of [**Har77**]) and exercises I.1.7(c) and I.1.10(a) of [**Har77**] that $(X, \mathcal{O}_X)$ has finite hereditary global dimension whenever X is a finite-dimensional Noetherian topological space, in particular when $(X, \mathcal{O}_X)$ is a finite-dimensional Noetherian scheme. Also, as pointed out by Hovey in [**Hov01**], a finite-dimensional compact manifold has finite hereditary global dimension. The next theorem shows we have re-arrived at a model structure constructed by Hovey in [**Hov01**].

Theorem 4.13. *Let $(X, \mathcal{O}_X)$ be a ringed space with finite hereditary global dimension. Let $\mathcal{G}$ denote the category of sheaves of $\mathcal{O}_X$-modules and let $\mathcal{F}$ denote the class of flasque $\mathcal{O}_X$-modules in $\mathcal{G}$. Then the small cotorsion pair $({}^{\perp}\mathcal{F}, \mathcal{F})$ induces a cofibrantly generated model structure on $\mathrm{Ch}(\mathcal{G})$ described as follows: The weak equivalences are the homology isomorphisms. The cofibrations (respectively, trivial cofibrations) are the monomorphisms whose cokernels are in ${}^{\perp}(ex\,\widetilde{\mathcal{F}})$ (respectively, ${}^{\perp}(dw\,\widetilde{\mathcal{F}})$). The fibrations (respectively, trivial fibrations) are the epimorphisms whose kernels are in $dw\,\widetilde{\mathcal{F}}$ (respectively, $ex\,\widetilde{\mathcal{F}}$). Furthermore, the set I of generating cofibrations is the set*

$$\{0 \to D^n(j!(\mathcal{O}|_U))\} \cup \{S^{n-1}(j!(\mathcal{O}|_U)) \to D^n(j!(\mathcal{O}|_U))\} \cup \{D^n(k_U^+)\}$$

while the set J of generating trivial cofibrations is the set

$$\{0 \to D^n(j!(\mathcal{O}|_U))\} \cup \{D^n(k_U^+)\}.$$

Proof. This follows from our comments above and by applying Theorem 4.7 to the small cotorsion pair $({}^{\perp}\mathcal{F}, \mathcal{F})$ from Proposition 4.11. □

We will call this model structure the "dimensionwise flasque" model structure on $\mathrm{Ch}(\mathcal{O}_X\text{-Mod})$. It is interesting to note that there is also a "dimensionwise cotorsion" model structure on $\mathrm{Ch}(\mathcal{O}_X\text{-Mod})$ whenever $(X, \mathcal{O}_X)$ is a ringed space with finite hereditary global dimension. This model structure is induced by the flat cotorsion pair and is a direct generalization of the model structure we described in Section 4.1. Since one can argue that the set $\mathcal{S} = \{\mathcal{O}_X/j!(\mathcal{O}_U) : U \subseteq X\}$ is contained in the class of flat $\mathcal{O}_X$-modules we see that all cotorsion $\mathcal{O}_X$-modules are flasque. Therefore the class of "dimensionwise cotorsion" complexes is contained in the class of "dimensionwise flasque" complexes.

5. Model structures from Kaplansky classes

We now turn to the problem of finding conditions on $(\mathcal{A}, \mathcal{B})$ which will guarantee that $(ex\,\widetilde{\mathcal{A}}, (ex\,\widetilde{\mathcal{A}})^{\perp})$ and $(dw\,\widetilde{\mathcal{A}}, (dw\,\widetilde{\mathcal{A}})^{\perp})$ induce a model structure. We will need the notion of a Kaplansky class that was given in [**Gil07**]. This definition was inspired by the definition for R-Mod given in [**ELR02**]. The reader may want to consult [**Gil07**] if the definition appears abstruse.

Definition 5.1. Let $\mathcal{F}$ be a class of objects in an abelian category and let κ be a regular cardinal. We say $\mathcal{F}$ is a *κ-Kaplansky class* if the following property holds: Given $X \subseteq F \neq 0$ where $F \in \mathcal{F}$ and X is κ-generated, there exists a κ-presentable object $S \neq 0$ such that $X \subseteq S \subseteq F$ and $S, F/S \in \mathcal{F}$. We say $\mathcal{F}$ is a *Kaplansky class* if it is a κ-Kaplansky class for some regular cardinal κ.

In [**Gil07**] the author showed that if we start with a nice enough Kaplansky class $\mathcal{F}$, in a Grothendieck category $\mathcal{G}$, then $(\widetilde{\mathcal{F}}, dg\widetilde{\mathcal{C}})$ is a small cotorsion pair, where $\mathcal{C} = \mathcal{F}^{\perp}$. It follows that the cotorsion pairs $(dg\widetilde{\mathcal{F}}, \widetilde{\mathcal{C}})$ and $(\widetilde{\mathcal{F}}, dg\widetilde{\mathcal{C}})$ give rise to a model structure on $\mathrm{Ch}(\mathcal{G})$. We use a similar technique here to show that $(ex\,\widetilde{\mathcal{F}}, (ex\,\widetilde{\mathcal{F}})^{\perp})$ and $(dw\,\widetilde{\mathcal{F}}, (dw\,\widetilde{\mathcal{F}})^{\perp})$ are each small cotorsion pairs whenever $\mathcal{F}$ is a nice enough Kaplansky class. Since we clearly have $ex\,\widetilde{\mathcal{F}} = dw\,\widetilde{\mathcal{F}} \cap \mathcal{E}$, where $\mathcal{E}$ is the class of exact complexes, it will be automatic that $(ex\,\widetilde{\mathcal{F}}, (ex\,\widetilde{\mathcal{F}})^{\perp})$ and $(dw\,\widetilde{\mathcal{F}}, (dw\,\widetilde{\mathcal{F}})^{\perp})$ induce another model structure on $\mathrm{Ch}(\mathcal{G})$. Our "nice enough" assumption will include $\mathcal{F}$ being closed under direct limits, so our methods work well when $\mathcal{F}$ is some class of flat objects in a Grothendieck category.

Proposition 5.2. *Let $\mathcal{F}$ be a κ-Kaplansky class where $\kappa > \omega$ is a regular cardinal. Then the class $dw\,\widetilde{\mathcal{F}}$ is a locally κ-cogenerated class.*

Proof. $dw\,\widetilde{\mathcal{F}}$ is the class of all chain complexes F with $F_n \in \mathcal{F}$. Suppose $0 \neq F \in dw\,\widetilde{\mathcal{F}}$ is given. We wish to construct a nonzero subcomplex $S \subseteq F$ in such a way that *(i)* S_n is κ-generated, *(ii)* $S_n \in \mathcal{F}$, and *(iii)* $F_n/S_n \in \mathcal{F}$.

Since $0 \neq F$, there must exist an n such that $F_n \neq 0$. Using the κ-Kaplansky class hypothesis we can find $0 \neq S_n \subseteq F_n$ such that *(i)* S_n is κ-presentable (hence κ-generated too), *(ii)* $S_n \in \mathcal{F}$, and *(iii)* $F_n/S_n \in \mathcal{F}$. Now $d_n(S_n)$ is κ-generated, and is contained in $F_{n-1} \in \mathcal{F}$. If $F_{n-1} = 0$ we can stop. Otherwise, use the κ-Kaplansky hypothesis again to find $d_n(S_n) \subseteq S_{n-1} \subseteq F_{n-1}$ such that *(i)* S_{n-1} is κ-presentable, *(ii)* $S_{n-1} \in \mathcal{F}$, and *(iii)* $F_{n-1}/S_{n-1} \in \mathcal{F}$. Now it is clear that we can continue downward to construct a bounded above subcomplex $S \subseteq F$ such that $S, F/S \in dw\,\widetilde{\mathcal{F}}$ and each S_n is κ-generated. Since $\kappa > \omega$ it follows from Lemma 4.10 of [**Gil07**] that S is a κ-generated complex. □

From Appendix B of [**Gil07**] we know that in any Grothendieck category we can find a regular cardinal κ large enough so that the κ-generated objects coincide with the κ-presentable objects. As a result all subobjects and quotient objects of a κ-generated object are also κ-generated.

Lemma 5.3. *Suppose $\mathcal{G}$ is a locally κ-generated Grothendieck category. We assume that $\kappa > \omega$ is a regular cardinal chosen so that the κ-generated and κ-presentable objects coincide. Let $\mathcal{F}$ be a κ-Kaplansky class, and let $S \subseteq F$ where $F \in ex\,\widetilde{\mathcal{F}}$ and S is an exact κ-generated subcomplex. Given any integer n for which $F_n \neq 0$, we can find an exact κ-generated subcomplex $T \subseteq F$ containing S and such that $T_n \neq 0$, and $T_n, F_n/T_n \in \mathcal{F}$.*

Proof. Without loss of generality we let $n = 0$ and suppose $F_n \neq 0$. By Lemma 4.10 of [**Gil07**] S is κ-generated if and only if each S_n is κ-generated. Using the Kaplansky class condition, we can find a κ-generated T_0 containing S_0 for which T_0 and F_0/T_0 are both in $\mathcal{F}$. So all we need to do is extend T_0 into an exact κ-generated subcomplex containing S. We build down by setting $T_{-1} = S_{-1} + d(T_0)$ and $T_i = S_i$ for all $i < -1$. One can check that

$$T_0 \to S_{-1} + d(T_0) \to S_{-2} \to \cdots$$

is exact. In particular, we have exactness in degree -1 since $d(S_0) \subseteq d(T_0)$.

Next we build up from T_0. To do this we use Lemma 4.4 of [**Gil07**]. For example, take the kernel of $T_0 \to T_{-1}$ and find a κ-generated $T_1' \subseteq F_1$ such that T_1' maps surjectively onto this kernel. Then take $T_1 = S_1 + T_1'$. Now T_1 also maps surjectively onto this kernel. We continue upward to build $T_2, T_3, \cdots$ in the same way. Since each T_n is κ-generated we are done. □

Proposition 5.4. *Suppose $\mathcal{G}$ is a locally κ-generated Grothendieck category. Assume that $\kappa > \omega$ is a regular cardinal chosen so that the κ-generated and κ-presentable objects coincide. If $\mathcal{F}$ is a κ-Kaplansky class and is closed under direct limits, then the class $ex\,\widetilde{\mathcal{F}}$ is locally κ-cogenerated.*

Proof. $ex\,\widetilde{\mathcal{F}}$ is the class of all exact chain complexes F with $F_n \in \mathcal{F}$. Suppose $0 \neq F \in ex\,\widetilde{\mathcal{F}}$ is given. We wish to construct a nonzero exact complex $S \subseteq F$ in such a way that *(i)* S_n is κ-generated, *(ii)* $S_n \in \mathcal{F}$, and *(iii)* $F_n/S_n \in \mathcal{F}$.

We start by choosing an n for which $F_n \neq 0$. Without loss of generality suppose that $n = 0$. Using Lemma 5.3 (with $S = 0$), find a κ-generated subcomplex $T^1 \subseteq F$ such that $0 \neq (T^1)_0 \in \mathcal{F}$ and $F_0/(T^1)_0 \in \mathcal{F}$. Now use the lemma again, with $S = T^1$, to get a κ-generated subcomplex T^2 containing T^1 and such that $(T^2)_{-1}$ and $F_{-1}/(T^2)_{-1}$ are each in $\mathcal{F}$. Lets say that T^1 was constructed using a "degree 0 operation" and T^2 was constructed using a "degree -1 operation". Then we can continue to build an increasing union of subcomplexes, $\{T^k\}$, using the following "back and forth" pattern and using "degree n operations":

$$0, \quad -1, 0, 1, \quad -2, -1, 0, 1, 2, \quad -3, -2, -1, 0, 1, 2, 3 \quad \cdots .$$

Finally set $S = \bigcup_{k\in\mathbb{N}} T^k$. Since $\mathcal{F}$ is closed under direct limits, we see (by a cofinality argument) that $S_n \in \mathcal{F}$ for each n. By a similar cofinality argument we see that $F_n/S_n \in \mathcal{F}$ for each n. Also each S_n is κ-generated since it is only a countable union of κ-generated objects and $\kappa > \omega$. Therefore S is the desired subcomplex. □

Theorem 5.5. *Suppose $\mathcal{G}$ is a locally κ-generated Grothendieck category where $\kappa > \omega$ is a regular cardinal chosen large enough so that the κ-generated and κ-presentable objects coincide. Let $\mathcal{F}$ be a κ-Kaplansky class which contains a κ-presentable generator G. Furthermore, suppose $\mathcal{F}$ is closed under extensions, retracts and direct limits and let $\mathcal{C} = \mathcal{F}^\perp$. Then $(dw\,\widetilde{\mathcal{F}}, (dw\,\widetilde{\mathcal{F}})^\perp)$ is a small cotorsion pair with generating monomorphisms the set*

$$I = \{0 \to D^n(G)\} \cup \{X \to Y\},$$

where $X \to Y$ ranges over all (representatives from isomorphism classes of) monomorphisms where Y is a κ-generated complex and $Y/X \in dw\,\widetilde{\mathcal{F}}$. Similarly, the pair $(ex\,\widetilde{\mathcal{F}}, (ex\,\widetilde{\mathcal{F}})^\perp)$ is a small cotorsion pair with generating monomorphisms the set

$$J = \{0 \to D^n(G)\} \cup \{X \to Y\},$$

where $X \to Y$ ranges over the monomorphisms where Y is a κ-generated complex and $Y/X \in ex\,\widetilde{\mathcal{F}}$. Furthermore these cotorsion pairs are compatible in the sense that $ex\,\widetilde{\mathcal{F}} = dw\,\widetilde{\mathcal{F}} \cap \mathcal{E}$ and $(dw\,\widetilde{\mathcal{F}})^\perp = (ex\,\widetilde{\mathcal{F}})^\perp \cap \mathcal{E}$ where $\mathcal{E}$ is the class of exact complexes.

Proof. Since $\mathcal{F}$ is a κ-Kaplansky class it is also a locally κ-cogenerated class. Since it also contains a κ-presentable generator G and is closed under extensions, retracts, and direct limits we see by Propositions 4.8 of [**Gil07**] that $(\mathcal{F}, \mathcal{F}^\perp)$ is a small, and hence complete, cotorsion pair. From Proposition 3.2 we get that $(dw\,\widetilde{\mathcal{F}}, (dw\,\widetilde{\mathcal{F}})^\perp)$ is a cotorsion pair. Since any Grothendieck category contains an injective cogenerator, and since such a cogenerator is necessarily in $\mathcal{F}^\perp$, we get from Proposition 3.3 that $(ex\,\widetilde{\mathcal{F}}, (ex\,\widetilde{\mathcal{F}})^\perp)$ is a cotorsion pair.

Now we collect some facts on $dw\,\widetilde{\mathcal{F}}$ and $ex\,\widetilde{\mathcal{F}}$. Being the left side of a cotorsion pair, $dw\,\widetilde{\mathcal{F}}$ and $ex\,\widetilde{\mathcal{F}}$ are themselves closed under extensions and retracts. Each class clearly contains the generators $D^n(G)$. Since direct limits are exact and since they are taken degreewise for chain complexes, it is clear that $dw\,\widetilde{\mathcal{F}}$ and $ex\,\widetilde{\mathcal{F}}$ are both closed under direct limits. Finally each class is locally κ-cogenerated by Propositions 5.2 and 5.4. It now follows from Proposition 4.8 of [**Gil07**] that $(dw\,\widetilde{\mathcal{F}}, (dw\,\widetilde{\mathcal{F}})^{\perp})$ and $(ex\,\widetilde{\mathcal{F}}, (ex\,\widetilde{\mathcal{F}})^{\perp})$ are each small cotorsion pairs with generating monomorphisms as we describe.

It is left to show that $ex\,\widetilde{\mathcal{F}} = dw\,\widetilde{\mathcal{F}} \cap \mathcal{E}$ and $(dw\,\widetilde{\mathcal{F}})^{\perp} = (ex\,\widetilde{\mathcal{F}})^{\perp} \cap \mathcal{E}$ where $\mathcal{E}$ is the class of exact complexes. Obviously $ex\,\widetilde{\mathcal{F}} = dw\,\widetilde{\mathcal{F}} \cap \mathcal{E}$, and so we can argue as in the proof of Theorem 4.7 that $(dw\,\widetilde{\mathcal{F}})^{\perp} = (ex\,\widetilde{\mathcal{F}})^{\perp} \cap \mathcal{E}$. □

Corollary 5.6. *Suppose $\mathcal{G}$ is a locally κ-generated Grothendieck category where $\kappa > \omega$ is a regular cardinal chosen large enough so that the κ-generated and κ-presentable objects coincide. Let $\mathcal{F}$ be a κ-Kaplansky class which contains a κ-presentable generator G. Furthermore, suppose $\mathcal{F}$ is closed under extensions, retracts and direct limits. Then we have a cofibrantly generated model structure on* $\mathrm{Ch}(\mathcal{G})$ *described as follows: The weak equivalences are the homology isomorphisms. The cofibrations (respectively, trivial cofibrations) are the monomorphisms whose cokernels are in $dw\,\widetilde{\mathcal{F}}$ (respectively, $ex\,\widetilde{\mathcal{F}}$). The fibrations (respectively, trivial fibrations) are the epimorphisms whose kernels are in $(ex\,\widetilde{\mathcal{F}})^{\perp}$ (respectively, $(dw\,\widetilde{\mathcal{F}})^{\perp}$). The set I in the statement of Theorem 5.5 are the generating cofibrations while J is the set of generating trivial cofibrations.*

Proof. This is immediate from Hovey's Theorem 2.2 of [**Hov02**]. □

We now look at several examples of homological model structures that come from Kaplansky classes using Corollary 5.6.

5.1. The injective model structure.

Let $\mathcal{G}$ be any Grothendieck category and let $(\mathcal{A}, \mathcal{I})$ be the injective cotorsion pair. Then as explained in Corollary 7.1 of [**Gil07**], we can always find a regular $\kappa > \omega$ large enough that *(1)* $\mathcal{G}$ is locally κ-generated, *(2)* the the κ-generated objects coincide with the κ-presentable objects, and *(3)* $\mathcal{G}$ contains a κ-presentable generator G. It is trivial that $\mathcal{A}$ (the class of all objects in $\mathcal{G}$) satisfies the rest of the hypotheses of Corollary 5.6. So we have an induced model structure on $\mathrm{Ch}(\mathcal{G})$ where the cofibrant (resp. trivially cofibrant) objects are complexes in $dw\,\widetilde{\mathcal{A}}$ (resp. $ex\,\widetilde{\mathcal{A}}$) and the fibrant (resp. trivially fibrant) complexes are those in $(ex\,\widetilde{\mathcal{A}})^{\perp}$ (resp. $(dw\,\widetilde{\mathcal{A}})^{\perp}$). However, it is clear that $dw\,\widetilde{\mathcal{A}}$ is the class of all complexes and so $(dw\,\widetilde{\mathcal{A}})^{\perp}$ is the class of all injective complexes. Similarly, it is clear that $ex\,\widetilde{\mathcal{A}}$ is the class of all exact complexes and so $(ex\,\widetilde{\mathcal{A}})^{\perp}$ must be the class of all dg-injective complexes. So the model structure on $\mathrm{Ch}(\mathcal{G})$ induced by Corollary 5.6 is just the usual injective model structure.

5.2. Examples concerning modules over a ring.

Now suppose that R is a commutative ring with 1 and $\mathcal{G}$ is the category of R-modules. Then $\mathcal{G}$ is locally finitely presentable. (For a proof of this fact, see the footnote to Theorem 4.34 of [**Lam99**].) So $\mathcal{G}$ is locally κ-presentable (and therefore locally κ-generated) for every regular cardinal κ by facts in Appendix A of [**Gil07**]. Let κ be a regular cardinal with $\kappa > \max\{|R|, \omega\}$. Then by Lemma B.2 of [**Gil07**], the κ-generated modules coincide with the κ-presentable modules and such modules M are characterized by the condition $|M| < \kappa$. So Corollary 5.6 can be applied to any such κ-Kaplansky class $\mathcal{F}$ which is closed under retracts, extensions, and direct limits, and which contains the generator R. Since it was shown in [**BBE01**] that the class $\mathcal{F}$ of flat R-modules is such a κ-Kaplansky class, there is a homological model structure on $\mathrm{Ch}(\mathcal{G})$ where the cofibrant objects are the complexes which are flat in each degree. The trivially cofibrant objects are the exact complexes which are flat in each degree. The model structure is cofibrantly generated but it is NOT monoidal in the sense defined in Chapter 4 of [**Hov99**]. If this were true, then $X \otimes_{\mathrm{Ch}(\mathcal{G})} Y$ would be trivially cofibrant for all cofibrant X and trivially cofibrant Y. However, we give a counterexample to show that in general $X \otimes_{\mathrm{Ch}(\mathcal{G})} Y$ need not be trivially cofibrant even when both X and Y are trivially cofibrant. For this example, take R to be the ring of integers mod 4. Let Y be the chain complex where each $Y_n = R$ and each differential is $R \xrightarrow{\times 2} R$. Then $Y \in ex\,\widetilde{\mathcal{F}}$, but $Y \otimes_{\mathrm{Ch}(R)} Y$ is not even exact.

The author is convinced that there is also an analogous model structure on $\mathrm{Ch}(\mathcal{G})$ coming from the projective cotorsion pair $(\mathcal{P}, \mathcal{A})$. We already know that we have induced cotorsion pairs $(dw\,\widetilde{\mathcal{P}}, (dw\,\widetilde{\mathcal{P}})^{\perp})$ and $(ex\,\widetilde{\mathcal{P}}, (ex\,\widetilde{\mathcal{P}})^{\perp})$. These can be shown to be cogenerated by a set using Kaplansky's Theorem 1 of [**Kap58**] and arguments similar to those in the proofs of Propositions 5.2 and 5.4. So there is a model structure on $\mathrm{Ch}(\mathcal{G})$ where the cofibrant objects are the complexes with a projective module in each degree.

5.3. Examples concerning sheaf categories.

The flat model structure above can be generalized to sheaf categories. First, as in Subsection 4.3 above, take $\mathcal{G} = \mathcal{O}_X$-Mod where $\mathcal{O}_X$ is a sheaf of rings on a topological space X. Here X may be any space. Let $\mathcal{F}$ be the class of flat $\mathcal{O}_X$-modules. We define the cardinality of an $\mathcal{O}_X$-modules (or *any* presheaf), F, to be $|F| = |\coprod_{U \subseteq X} F(U)|$, where $U \subseteq X$ ranges over all the open sets in X. From Section 7 of [**Gil07**] one can see that we have the following facts: Let β be an infinite cardinal such that $\beta > \max\{|X|, |\mathcal{O}_X|\}$. Now let $\kappa = 2^{\beta}$. We may also assume, and do assume, that κ is large enough that each $j!(\mathcal{O}_U)$ is κ-generated. Then *(1)* the κ-generated objects coincide with the κ-presentable objects, *(2)* $\mathcal{G}$ is a locally κ-presentable (and locally κ-generated) category, and *(3)* $\mathcal{F}$ is a κ-Kaplansky class. Also, $\mathcal{F}$ is closed under direct limits, retracts, and extensions, and contains the generators $j!(\mathcal{O}_U)$. Therefore, we have a homological model structure on $\mathrm{Ch}(\mathcal{O}_X\text{-Mod})$ where the cofibrant objects are the degreewise flat complexes.

Finally, let $\mathcal{G}$ be the category $\mathrm{Qco}(X)$ of quasi-coherent $\mathcal{O}_X$-modules, where X is any scheme in which $\mathrm{Qco}(X)$ has a flat generator. Then $\mathrm{Ch}(\mathrm{Qco}(X))$ has an analogous model structure. To obtain the model structure from Corollary 5.6, we let $\mathcal{F}$ be the class of flat quasi-coherent $\mathcal{O}_X$-modules and take $\kappa > \max\{\omega, |\mathcal{O}_X|\}$. We can assume that κ is large enough so that $\mathrm{Qco}(X)$ is locally κ-generated and also so that the flat generator is κ-presentable. Then all the hypotheses of Corollary 5.6 hold with this choice of κ. We refer the reader to [**Gil07**] and [**EE05**] for more details.

References

[BBE01] L. Bican, R. El Bashir, and E. Enochs, All modules have flat covers, *Bull. London Math Soc.* **33** (2001), 385–390.

[DS95] W.G. Dwyer and J. Spalinski, Homotopy theories and model categories, *Handbook of algebraic topology*, 73–126, North-Holland, Amsterdam, 1995.

[ET01] P. C. Eklof and J. Trlifaj, How to make Ext vanish, *Bull. London Math Soc.* **33** (2001), 41–51.

[EE05] E. Enochs and S. Estrada, Relative homological algebra in the category of quasi-coherent sheaves, *Adv. in Math.* **194** (2005), 284–295.

[ELR02] E. Enochs and J. A. López-Ramos, Kaplansky classes, *Rend. Sem. Mat. Univ. Padova* **107** (2002), 67–79.

[EJ01] E. Enochs and O. Jenda, *Relative homological algebra*, de Gruyter Expositions in Mathematics **30**, Walter De Gruyter & Co., New York, 2000.

[EO01] E. Enochs and L. Oyonarte, Flabby envelopes of sheaves, *Communications in Algebra* **29** (2001), 3449-3457.

[Gil04] J. Gillespie, The flat model structure on $\mathrm{Ch}(R)$, *Trans. Amer. Math. Soc.* **356** (2004), 3369–3390.

[Gil07] J. Gillespie, Kaplansky classes and derived categories, *Math. Zeit.* **257** (2007), 811–843.

[Har77] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics **52** Springer-Verlag, New York, 1977.

[Hov99] M. Hovey, *Model categories*, Mathematical Surveys and Monographs **63**, Amer. Math. Soc., Providence, RI, 1999.

[Hov01] M. Hovey, Model category structures on chain complexes of sheaves, *Trans. Amer. Math. Soc.* **353** (2001), 2441–2457.

[Hov02] M. Hovey, Cotorsion pairs, model category structures, and representation theory, *Math. Zeit.* **241** (2002), 553–592.

[Kap58] I. Kaplansky, Projective modules, *Ann. of Math.* **68** (1958), 372–377.

[Lam99] T. Y. Lam, *Lectures on modules and rings*, Graduate Texts in Math. **189**, Springer-Verlag, New York, 1999.

[Sal79] L. Salce, Cotorsion theories for abelian groups, *Symposia Math.* **23**, 11–32, Academic Press, New York, 1979.

[Sten75] B. Stenström, *Rings of quotients*, Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen **217**, Springer-Verlag, New York 1975.

James Gillespie `jrg21@psu.edu`

4000 University Drive, Penn State Greater Allegheny, McKeesport, PA 15132-7698