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INERTIA AND DELOCALIZED TWISTED COHOMOLOGY

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Abstract

Orbispaces are the analog of orbifolds, where the category of manifolds is replaced by topological spaces. We construct the loop orbispace LX of an orbispace X in the language of stacks in topological spaces. Furthermore, to a twist given by a $U(1)$-banded gerbe $G \to X$ we associate a $U(1)^\delta$-principal bundle $\tilde{G}^\delta \to LX$. We use sheaf theory on topological stacks in order to define the delocalized twisted cohomology by

$$H^*_{\mathrm{deloc}}(X, G) := H^*(G_L, f_L^*\mathcal{L}),$$

where $f_L \colon G_L \to LX$ is the pull-back of the gerbe $G \to X$ via the natural map $LX \to X$, and $\mathcal{L} \in \mathtt{Sh_{Ab}}\mathbf{LX}$ is the sheaf of sections of the $\mathbb{C}^\delta$-bundle associated to $\tilde{G}^\delta \to LX$.

The same constructions can be applied in the case of orbifolds, and we show that the sheaf theoretic delocalized twisted cohomology is isomorphic to the twisted de Rham cohomology, where the isomorphism depends on the choice of a geometric structure on the gerbe $G \to X$.

Contents

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1. Introduction

1.1. Motivation

In the recent mathematical literature cohomological and topological properties of orbifolds became an intensively studied subject. A considerable part of the motivation comes from the mirror symmetry program where orbifolds arise naturally. Cornerstones[1] of the recent developments were the introduction of twisted orbifold K-theory [**1**] and the orbifold quantum cohomology [**14**] on the topological side, and the investigation of gerbes [**26**] and loop groupoids [**25**] on the geometric side.

Classically, orbifolds are defined like manifolds as spaces which are locally homeomorphic to a quotient of a euclidean space by a finite group. Alternatively, orbifolds are represented by proper étale smooth groupoids [**32**], [**33**]. Working with groupoid representations of orbifolds is like working with manifolds with a fixed atlas. In the modern coordinate invariant point of view an orbifold is a smooth stack in smooth manifolds which admits an orbifold atlas. By considering orbifolds as objects in the 2-category of smooth stacks one makes the notion of morphisms[2] and other constructions like fibre products transparent. The framework of stacks is most natural if one wants to include gerbes into the picture.

If one replaces smooth manifolds by topological spaces, then the corresponding analog of an orbifold is an orbispace. The goal of the present paper is to show that many geometric constructions on orbifolds are in fact topological concepts and extend to orbispaces.

The fixed point manifolds of the elements of the local automorphism groups of an orbifold X can be assembled into a new orbifold LX called the inertia or loop orbifold or the orbifold of twisted sectors. In the present paper we show that the loop orbifold can be characterized as the 2-categorical (in the 2-category of stacks) equalizer of the pair $(\mathtt{id}_X, \mathtt{id}_X)$. The same definition applies to orbispaces in the topological context. Since 2-categorical equalizers always exist in 2-categories of stacks it is clear that LX exists as a stack. But it is not *a priori* clear that LX is again an orbifold (or orbispace, respectively). In the present paper we show that taking loop stacks preserves orbispaces.

A $U(1)$-banded gerbe $G \to X$ over an orbifold gives rise to a $U(1)$-principal bundle $\tilde{G} \to LX$ over the loop orbifold of X. This bundle has a natural reduction of structure groups to the discrete $U(1)^\delta$. The traditional way to construct this reduction is to choose a connection and curving on the gerbe $G \to X$. This geometric data induces a connection on $\tilde{G} \to LX$ which turns out to be flat. The flat connection gives the

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[1] Here we mention those works which are relevant for the present paper. Note that there is a huge literature on orbifolds in algebraic geometry and mathematical physics.

[2] The right notion of a morphism between orbifolds is a representable morphism of stacks. This definition corresponds to the notion of a good morphism in the literature.

reduction of structure groups, and we can form the sheaf $\mathcal{L}$ of locally constant sections of the associated flat line bundle $L \to LX$.

In the present paper we give a topological construction of the reduction of the structure group of $\tilde{G} \to LX$ to $U(1)^\delta$ and of the sheaf $\mathcal{L}$. Furthermore, we calculate its holonomy in terms of the Dixmier-Douady class of the gerbe $G \to X$.

The third concept which we generalize to the topological case is that of twisted delocalized orbifold cohomology. The usual definition in the smooth case is based on the de Rham complex of forms on LX with coefficients in $L \to LX$. The differential of this complex involves the flat connection on L and a closed three-form on X which represents the image of the Dixmier-Douady class of the gerbe $f\colon G \to X$ in real cohomology. Let $f_L\colon G_L \to LX$ denote the pull-back of the gerbe via $LX \to X$. In the present paper we use the sheaf theory for smooth (or topological, respectively) stacks [**13**] in order to define the twisted delocalized orbifold cohomology as sheaf cohomology $H^*(LX, \mathtt{Tw}_G(\mathcal{L}))$, where $\mathtt{Tw}_G(\mathcal{L}) := R(f_L)_* f_L^*(\mathcal{L})$. Our main result is that in the smooth case the twisted delocalized cohomology according to this sheaf theoretic definition is isomorphic to the former construction using the de Rham complex. In addition to the fact that it works in the topological context our sheaf theoretic definition of twisted delocalized orbifold cohomology has the advantage that it is functorial in the gerbe $G \to X$.

In the remaining parts of the introduction we give a detailed description of the results of the present paper and explain how they are related to the existing literature.

1.2. A description of the results

In the present paper we consider stacks in smooth manifolds or stacks in topological spaces. Our basic reference for stacks in these contexts is [**19**], but see also [**34**], [**30**], and [**9**]. A stack X in smooth manifolds (topological spaces, respectively) is called a smooth stack (topological stack, respectively) if it admits an atlas $A \to X$. The atlas is called an orbifold (orbispace, respectively) atlas if the smooth (topological, respectively) groupoid $A \times_X A \Rightarrow A$ is proper étale (very proper, étale and separated (see 2.30)). An orbifold (orbispace, respectively) is a smooth (topological, respectively) stack which admits an orbifold (orbispace, respectively) atlas.

We refer to [**12**] for an introduction to orbispaces, and e.g. to [**14**, Sec. 2] for some basic information on orbifolds.

In subsection 2.1 we review the notion of 2-categorical limits. The 2-categorical equalizer of a pair of maps is a special kind of limit. We will see that equalizers exist in the 2-category of stacks on a site and in the 2-category of groupoids in topological spaces.

The goal of subsections 2.2 and 2.3 of the present paper is to place the construction of the loop orbifold LX (or orbispace, respectively) into the framework of stacks in manifolds (topological spaces, respectively).

We consider the orbifold (orbispace, respectively) X as a stack and define its inertia stack $IX \to X$ as the 2-categorical equalizer of the pair $(\mathtt{id}_X, \mathtt{id}_X)$. The loop stack LX is defined in an ad-hoc manner; see Definition 2.16 and Remark 2.24. We will see that it is canonically equivalent to IX. Though Definition 2.10 of the 2-categorical equalizer by a pull-back diagram is quite constructive we prefer to work with the simpler construction LX from now on. If X is an orbifold (orbispace, respectively), then *a priori* LX is a stack in smooth manifolds (topological spaces, respectively).

Proposition 1.1. *The loop stack of a topological stack is again a topological stack. Moreover, the loop stack of an orbispace is an orbispace, too.*

The proof of this proposition will be given in Lemmas 2.26 and 2.33.

In the smooth case, the fact that the loop stack of an orbifold is again an orbifold is well known; see [**14**, Lemma 3.1.1] or [**25**, Cor. 2.6.2].

The loop orbifold is also known as the orbifold of twisted sectors (compare [**14**, Sec. 3.1]) or inertia orbifold. It plays an important role in the construction of the delocalized orbifold cohomology. The twisted sectors first appeared in connection with the orbifold index theorem [**22**], [**23**]. In the framework of a topological groupoid G the corresponding object is called the inertia groupoid ΛG which has been studied in detail in [**25**]. In order to keep our notation uniform in the present paper we will denote the inertia groupoid by LG and call it the loop groupoid.[3]

Let $f\colon G \to X$ be a topological gerbe with band $U(1)$ over a topological stack X. The induced map $Lf\colon LG \to LX$ can be factored canonically as $LG \xrightarrow{p} G_L \xrightarrow{f_L} LX$, where $G_L := LX \times_X G$. Here $f_L\colon G_L \to LX$ is a topological gerbe with band $U(1)$, and $p\colon LG \to G_L$ is (the underlying map of) a $U(1)$-principal bundle.

Proposition 1.2. *The bundle $LG \to G_L$ descends canonically to a $U(1)$-principal bundle $\tilde{G} \to LX$. If X is an orbispace, then $\tilde{G} \to LX$ has a canonical reduction of the structure group $\tilde{G}^\delta \to LX$ from $U(1)$ to $U(1)^\delta$, the group $U(1)$ with the discrete topology.*

The assertions of this proposition are shown in subsection 2.5.

Let $L \to LX$ denote the line bundle associated to $\tilde{G}^\delta \to LX$. Since its structure group is discrete, we can form the sheaf $\mathcal{L}$ of its locally constant sections.

By (2.50) we have actually an extension

$$X \times U(1)^\delta \to \tilde{G}^\delta \to LX$$

of group stacks over X. The induced algebraic structures on $L \to LX$ turn this line bundle into an inner local system in the sense of [**37**, Def. 2.1], [**26**, Def. 2.2.2].

In the framework of groupoids the construction of $\tilde{G} \to LX$ has been previously given in [**25**, Thm. 6.4.2] and [**40**, Prop. 2.9]. In the smooth case a reduction of the structure group of a line bundle from $U(1)$ to $U(1)^\delta$ is equivalent to a flat unitary connection. It has been observed in [**27**, Lemma 5.0.1] and [**40**, Prop. 3.9] that a connection on the gerbe $G \to X$ induces a flat connection on $L \to LX$.

Our original contribution here is to give a construction of this reduction of the structure group in purely topological terms. In addition to simplifications this extends the previous results to the topological case.

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[3]Note that the loop groupoid LG in [**25**] is a much bigger object, and it is related to ΛG by the equation $LG^{\mathbb{R}} \cong \Lambda G$ in the notation of [**25**, Prop. 3.6.6].

A twisted torsion in the language of [**37**] is a class $\alpha \in H^2(\pi_1^{\mathrm{orbifold}}(X), U(1))$, i.e. an isomorphism class of central $U(1)$-extensions

$$1 \to U(1) \to \widehat{\pi_1^{\mathrm{orbifold}}}(X) \to \pi_1^{\mathrm{orbifold}}(X) \to 1.$$

The orbifold fundamental group $\pi_1^{\mathrm{orbifold}}(X)$ is the automorphism group of the universal orbifold covering $Y \to X$. The map

$$G_\alpha := [Y/\widehat{\pi_1^{\mathrm{orbifold}}}(X)] \to [Y/\pi_1^{\mathrm{orbifold}}(X)] = X$$

is a topological gerbe with band $U(1)$ over X. In [**37**, Sec. 4] or [**26**, Example 2.2.2] an inner local system L_α is associated directly to a twisted torsion α. In the philosophy of the present paper we would consider L_α as the bundle associated to the gerbe $G_\alpha \to X$ via the $U(1)^\delta$-bundle $\tilde{G}_\alpha^\delta \to LX$.

The sheaf of locally constant sections $\mathcal{L}$ of the line bundle L (also called inner local system) plays an important role in the definition of twisted delocalized cohomology of an orbifold [**1**], [**37**, Def. 2.2],[4] [**40**, Def. 3.10].

To a topological group G we associate the classifying stack $\mathcal{B}G := [*/G]$ (see [**19**, Example 1.5]). A G-principal bundle over a stack X is by definition a map $p\colon X \to \mathcal{B}G$.[5] Applying the loop functor and using the canonical isomorphism $L\mathcal{B}G \cong [G/G]$ (where G acts on itself by conjugations) we get a map $Lp\colon LX \to [G/G]$. If G is abelian, then this map lifts to a function $h\colon LX \to G$. We are in particular interested in the case $G = U(1)$ and give various geometric and cohomological interpretations of this function.

In the present paper, ordinary cohomology of an orbispace X is understood in the sense of [**12**, Sec. 2.2]. Let $A \to X$ be an atlas and form the simplicial space $A^\cdot$ such that $A^n := A \times_X \cdots \times_X A$ ($n+1$-factors). Here the fibre product is taken in stacks in topological spaces, but the stack A^n is in fact equivalent to a space since the map $A \to X$ is representable. The cohomology of X with integral coefficients is then defined as

$$H^*(X; \mathbb{Z}) := H^*(|A^\cdot|; \mathbb{Z}),$$

where $|A^\cdot|$ denotes the realization of the simplicial space. Independence of the choice of the atlas has been shown in [**12**, Sec. 2.2] and [**8**].[6] An alternative definition of the cohomology of X can be based on the sheaf theory for orbifolds which will be discussed below. The group $H^2(X;\mathbb{Z})$ classifies isomorphism classes of $U(1)$-principal bundles $p\colon E \to X$ (see [**12**, Sec. 4.2] for this fact).

[4]This is the cohomology of LX with coefficients in $\mathcal{L}$ with shifted grading. It is different from the gerbe-twisted delocalized cohomology.

[5]Sometimes we will use a more sloppy language and say that $E \to X$ is a G-principal bundle, where $E \to X$ is defined by the pull-back

$$\begin{tikzcd} E \arrow[r] \arrow[d] & * \arrow[d] \\ X \arrow[r, "p"] \arrow[ur, dotted] & \mathcal{B}G. \end{tikzcd}$$

[6]The result in [**8**] is more general. The only condition on the atlas $A \to X$ is that the range and source maps of the groupoid $A \times_X A \rightrightarrows A$ are topological submersions.

If Γ is a finite group, then we have

$$H^2([*/\Gamma];\mathbb{Z}) \cong H^2(\Gamma;\mathbb{Z}) \cong H^1(\Gamma;U(1)) \cong \hat{\Gamma},$$

where $\hat{\Gamma} := \mathtt{Hom}(\Gamma, U(1))$. A class $\chi \in H^2([*/\Gamma];\mathbb{Z})$ thus gives rise to a function

$$\bar{\chi}\colon L[*/\Gamma] \cong [\Gamma/\Gamma] \to U(1).$$

This construction extends to general orbispaces X and associates to each class $\chi \in H^2(X;\mathbb{Z})$ a function $\bar{\chi}\colon LX \to U(1)$. A class $\chi \in H^2(X;\mathbb{Z})$ also classifies a $U(1)$-principal bundle and therefore gives rise to a function $h_\chi\colon LX \to U(1)$.

Proposition 1.3. *We have the equality* $\bar{\chi} = h_\chi$.

This is shown in Lemma 2.43.

G-principal bundles can be defined in terms of cocycles. We will give an interpretation of the function h_χ in terms of the cocyle. A third cohomological interpretation uses the transgression $\mathtt{Tr}\colon H^2(X;U(1)) \to H^1(LX;U(1))$ introduced in [**2**], [**27**], [**39**].

It is an interesting problem to calculate the holonomy of the bundle $\tilde{G}^\delta \to LX$ in terms of the Dixmier-Douady class $d \in H^3(X;\mathbb{Z})$ of the gerbe $G \to X$. We discuss this question in a typical case in subsection 2.6. Let $\pi\colon E \to X$ be a $U(1)$-principal bundle in orbispaces and $G \to E$ be a topological gerbe with band $U(1)$ and Dixmier-Douady class $d \in H^3(E;\mathbb{Z})$. Let $\chi \in H^2(X;\mathbb{Z})$ be the first Chern class of $E \to X$. As explained above we get a function $\bar{\chi}\colon LX \to U(1)$. Let $LX_1 := \bar{\chi}^{-1}(1)$. We will see that the canonical map $LE \to LX$ factorizes over LX_1, and that $LE \to LX_1$ is again a $U(1)$-principal bundle (see Lemma 2.39). The holonomy of the bundle $\tilde{G}^\delta \to LE$ along the fibres of $LE \to LX_1$ can be considered as a function $g\colon LX_1 \to U(1)$.

Note that $\pi\colon E \to X$ is an oriented fibre bundle. We have an integration map $\pi_!\colon H^3(E;\mathbb{Z}) \to H^2(X;\mathbb{Z})$. In particular we can form $\pi_!(d) \in H^2(X;\mathbb{Z})$ and the associated function

$$\overline{\pi_!(d)}\colon LX \to U(1).$$

Proposition 1.4 (2.54)**.** *We have the following equality of functions*

$$g = \overline{\pi_!(d)}_{|LX_1}.$$

Section 3 of the present paper is devoted to twisted delocalized cohomology. We are in particular interested in a version which is the target of the Chern character from twisted K-theory. We refer to subsection 1.3 for a detailed introduction and a motivation of the particular definition of twisted delocalized cohomology. Our main original contribution in the present paper is a construction of this cohomology in the framework of sheaf theory on topological stacks. All previous definitions used the de Rham complex and are therefore tied to the orbifold case.

To a topological stack (smooth stack, respectively) X we associate a site $\mathbf{X}$. The smooth case was discussed at length in [**13**]. Details of the sheaf theory on topological stacks[7] are discussed in [**11**]. So let us fix our conventions for the topological case here.

[7]For the purpose of duality theory in [**11**] conditions of local compactness were added. The part of the theory which we use in the present paper works without this assumption.

An object of $\mathbf{X}$ is a map $(\phi\colon U \to X)$ in stacks in topological spaces, where U is a topological space (or more precisely a stack which is equivalent to a space), and ϕ is a representable map which admits local sections.[8] The morphisms in $\mathbf{X}$ are commutative diagrams

$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\Downarrow"] & V \arrow[dl] \\ & X & \end{tikzcd}$$

consisting of a morphism $U \to V$ and a 2-morphism. A family $(U_i \to U)_{i\in I}$ of morphisms in $\mathbf{X}$ is a covering family if all maps $U_i \to U$ admit local sections and the induced map $\sqcup_{i\in I} U_i \to U$ is surjective. To the site $\mathbf{X}$ we can associate the category $\mathtt{Sh}\mathbf{X}$ of sheaves of sets and the abelian category $\mathtt{Sh}_{\mathtt{Ab}}\mathbf{X}$ of sheaves of abelian groups.

A map between topological (respectively smooth) stacks $f\colon X \to Y$ induces an adjoint pair of functors

$$f^*\colon \mathtt{Sh}\mathbf{Y} \Leftrightarrow \mathtt{Sh}\mathbf{X}\colon f_*$$

relating the categories of sheaves on these sites. In the smooth case the construction of this adjoint pair was given by [**13**, Sec. 2.1]. The construction in the case of topological stacks is very similar; see [**11**].

The restriction $f_*\colon \mathtt{Sh}_{\mathtt{Ab}}\mathbf{X} \to \mathtt{Sh}_{\mathtt{Ab}}\mathbf{Y}$ of f_* to abelian sheaves is left-exact and admits a right-derived functor

$$Rf_*\colon D^+(\mathtt{Sh}_{\mathtt{Ab}}\mathbf{X}) \to D^+(\mathtt{Sh}_{\mathtt{Ab}}\mathbf{Y})$$

between the lower-bounded derived categories.

Let $G \to X$ be a topological (smooth, respectively) gerbe with band $U(1)$ on an orbispace (respectively orbifold) X. In order to define the G-twisted delocalized cohomology we need some notation.

The twist $G \to X$ gives rise to the $U(1)^\delta$-principal bundle $\tilde{G}^\delta \to LX$ (see Prop. 1.2) and an associated locally constant sheaf $\mathcal{L}$ of $\mathbb{C}$-vector spaces on the site $\mathbf{LX}$. We consider the diagram

$$\begin{tikzcd} * & G_L \arrow[l, "p"'] \arrow[r] \arrow[d, "f_L"] & G \arrow[d, "f"] \\ & LX \arrow[r] \arrow[ur, phantom, "\Nearrow"] & X, \end{tikzcd}$$

where the square is 2-cartesian and the map $p\colon G_L \to *$ is the canonical projection to the point. Since $\mathtt{Site}(*)$ is the big site of the point, i.e. the category of all topological spaces, we need the evaluation functor $\mathtt{ev}\colon D^+(\mathtt{Sh}_{\mathtt{Ab}}\mathtt{Site}(*)) \to D^+(\mathtt{Ab})$ at the object $(* \to *) \in \mathtt{Site}(*)$.

Definition 1.5. The G-twisted delocalized cohomology of X is defined as

$$H^*_{\mathrm{deloc}}(X;G) := H^*(\mathtt{ev} \circ Rp_* \circ f_L^*(\mathcal{L})). \tag{1.6}$$

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[8]Note that $\mathbf{X}$ must be small. A precise definition would either involve universes or a cardinality restriction.

The G-twisted delocalized cohomology of X is functorial in the data $G \to X$ (see Lemma 3.7). For further details we refer to subsection 3.5.

Our main result is the comparison of this sheaf-theoretic definition of G-twisted delocalized cohomology with the previous de Rham model [**40**, Def. 3.10] in the case of orbifolds.

We now explain the de Rham model for the twisted delocalized cohomology. Let X be an orbifold and $G \to X$ be a smooth gerbe with band $U(1)$. In this case we can define three versions of twisted delocalized de Rham cohomology. The 2-periodic twisted delocalized cohomology is the correct target of the Chern character and will be defined in (1.14) below. The sheaf theoretic cohomology (1.6) is not 2-periodic. In the following we describe its appropriate de Rham model. We choose a closed three-form $\lambda \in \Omega^3(LX)$ which represents the image of the Dixmier-Douady class of $G_L \to LX$ in real cohomology. Then we define a sheaf $\Omega_{LX}[[z]]_\lambda \in C^+(\mathtt{Sh_{Ab}}\mathbf{LX})$ of complexes which associates to each object $(\phi\colon U \to LX) \in \mathbf{LX}$ the complex $(\Omega(U)[[z]], d_\lambda)$, where $(\Omega(U), d_{dR})$ is the de Rham complex of the smooth manifold U, z is a formal variable of degree 2, and $d_\lambda = d_{dR} + \frac{d}{dz}\phi^*\lambda$. Let $\Omega(LX;\mathcal{L})[[z]]_\lambda := \Gamma_{LX}(\Omega_{LX}[[z]]_\lambda \otimes \mathcal{L})$ denote the complex of global sections (see (3.15) for the definition of global sections) of the tensor product of sheaves $\Omega_{LX}[[z]]_\lambda \otimes \mathcal{L}$. Its cohomology is the twisted delocalized de Rham cohomology

$$H^*_{dR,\mathrm{deloc}}(X,(G,\lambda)) := H^*(\Omega(LX;\mathcal{L})[[z]]_\lambda) \tag{1.7}$$

(see 3.23).

The twisted delocalized de Rham cohomology defined in [**40**, Def. 3.10] is related to the definition of the present paper by a duality. For simplicity we assume that LX is oriented. Otherwise one must plug in orientation bundles. Let us first recall the definition [**40**, Def. 3.10]. Let u be a formal variable of degree -2 and define the complex of sheaves $\Omega_{LX}((u))$ which associates to $(\phi\colon U \to LX)$ the space of formal Laurent series of forms $\Omega(U)((u))_\lambda$ with the differential $d'_\lambda := d_{dR} - ui\phi^*\lambda$. The twisted cohomology in [**40**, Def. 3.10] is the cohomology of the complex of compactly supported global sections $\Omega(LX;\mathcal{L})_{\mathrm{comp}}((u))_\lambda$[9] of $\Omega_{LX}((u))_\lambda \otimes \mathcal{L}$. Note that the multiplication by u induces an isomorphism of complexes which makes the cohomology of [**40**, Def. 3.10] two-periodic.

We define the pairing (using the hermitean structure of $\mathcal{L}$)

$$\langle \dots, \dots \rangle \colon \Omega(LX;\mathcal{L})_{\mathrm{comp}}((u))_\lambda \otimes \Omega(LX;\mathcal{L})[[z]]_\lambda \to \mathbb{C} \tag{1.8}$$

by

$$\langle u^n\omega, z^m\alpha\rangle = \delta_{m,n} m! \int_{LX} \omega \wedge \alpha,$$

where $\omega \in \Omega(LX;\mathcal{L})_{\mathrm{comp}}$ and $\alpha \in \Omega(LX;\mathcal{L})$. One easily checks that

$$\langle d'_\lambda\omega, \alpha\rangle = (-1)^{|\omega|+1}\langle \omega, d_\lambda\alpha\rangle.$$

The pairing (1.8) induces an embedding of $\Omega(LX;\mathcal{L})[[z]]_\lambda$ into the dual complex of $\Omega(LX;\mathcal{L})_{\mathrm{comp}}((u))_\lambda$.

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[9] Here we use the freedom of rescaling λ by non-zero factors as explained in [**40**, Rem. 3.11(1)].

Let us now explain the relation between (1.7) and the 2-periodic version (1.14). Note that the complex of sheaves $\Omega_{LX}[[z]]_\lambda$ admits an action of the operation $T := \frac{d}{dz}$ of degree -2. We consider the system

$$\mathcal{S}\colon \Omega_{LX}[[z]]_\lambda \xleftarrow{T} \Omega_{LX}[[z]]_\lambda[2] \xleftarrow{T} \Omega_{LX}[[z]]_\lambda[4] \xleftarrow{T} \dots$$

in the category $C(\mathtt{Sh_{Ab}}\mathbf{LX})$ of unbounded complexes. The discussion of [**13**, 1.3.23] can be subsumed in the assertion that $\Gamma_{LX}(\lim \mathcal{S} \otimes \mathcal{L})$ is exactly the periodic complex (1.14).

Our basic result is an extension of [**13**, Thm. 1.1] from smooth manifolds to orbifolds.

Theorem 1.9 (Theorem 3.24)**.** *If $G \to X$ is a $U(1)$-banded gerbe over an orbifold, then there exists an isomorphism*

$$R(f_L)_*(\mathbb{R}_{\mathbf{G_L}}) \cong \Omega_{LX}[[z]]_\lambda \tag{1.10}$$

in the derived category $D^+(\mathtt{Sh_{Ab}}\mathbf{LX})$.

This isomorphism is not canonical and depends on the choice of a connection on the gerbe $G \to X$. As a consequence of (1.10) we get

Theorem 1.11 (Theorem 3.25)**.** *If $G \to X$ is a $U(1)$-banded gerbe over an orbifold, then there exists an isomorphism*

$$H^*_{dR,\mathrm{deloc}}(X;(G,\lambda)) \cong H^*_{\mathrm{deloc}}(X;G).$$

This isomorphism of $\mathbb{C}$-vector spaces is again not canonical and depends on the choice of a connection on the gerbe $G \to X$.

The main goal of the forthcoming paper [**11**] will be a sheaf theoretic construction of 2-periodic twisted delocalized cohomology. The idea is to define an analog T of the operation $\frac{d}{dz}$ on the left-hand side of the derived category isomorphism (1.10). In analogy with the de Rham model we then will consider the system

$$\mathcal{T}\colon R(f_L)_*(\mathbb{R}_{\mathbf{G_L}}) \xleftarrow{T} R(f_L)_*(\mathbb{R}_{\mathbf{G_L}})[2] \xleftarrow{T} R(f_L)_*(\mathbb{R}_{\mathbf{G_L}})[4] \xleftarrow{T} \dots$$

in $D(\mathtt{Sh_{Ab}}\mathbf{LX})$. The sheaf-theoretic version of periodic delocalized twisted cohomology will be defined as

$$H^*(\mathtt{ev} \circ Rp_*(\mathtt{holim}\mathcal{T} \otimes \mathcal{L})).$$

In order to make this rough idea precise we must solve various problems, in particular

(1) The homotopy limit $\mathtt{holim}\mathcal{T}$ of the diagram $\mathcal{T}$ in the derived category is only well-defined up to non-canonical isomorphism. In order to define a functorial periodic cohomology we must work hard to construct a much more concrete version of the system $\mathcal{T}$.

(2) The push-forward $Rp_*(\mathrm{colim}\mathcal{T} \otimes \mathcal{L})$ is not a standard derived functor since it acts between unbounded derived categories. We use a model category approach in order to construct functors like Rp_*.

The main application and technical tool in [**11**] will be T-duality. The results of subsections 2.4 and 2.6 of the present paper will be needed in [**11**] in a crucial way.

1.3. Motivation of the definition of twisted delocalized cohomology

In the present subsection we motivate the definition of twisted delocalized cohomology as the correct target for the Chern character from twisted K-theory.

It is a well-known fact that the Chern character $\mathbf{ch}\colon K(X) \to H(X;\mathbb{Q})$ from the complex K-theory of a finite CW-complex X to the rational cohomology of X induces an isomorphism $K(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} H(X;\mathbb{Q})$ (we consider both sides as $\mathbb{Z}/2\mathbb{Z}$-graded groups).

Complex K-theory and rational cohomology both have equivariant generalizations. Every generalized cohomology E theory has the Borel extension. If X is a G-space, then the Borel extension of E to G-spaces associates to X the group $E_G^{\mathrm{Borel}}(X) := E(EG \times_G X)$. Here EG is a universal space for G, i.e. a contractible space on which G acts freely and properly. The Chern character induces an equivariant Chern character $\mathbf{ch}_G\colon K_G^{\mathrm{Borel}}(X) \to H_G^{\mathrm{Borel}}(X;\mathbb{Q})$ which gives again a rational isomorphism.

The interesting equivariant extension of K-theory is not the Borel extension but the extension due to Atiyah-Segal based on equivariant vector bundles [**4**]. It will be denoted by $K_G(X)$. In order to see the difference between K_G^{Borel} and K_G consider the simple example of finite group G acting trivially on the point $*$. The equivariant Atiyah-Segal K-theory is isomorphic to the representation ring $R(G)$ of G. In [**3**] is was shown that $K_G^{\mathrm{Borel}}(*)$ is isomorphic to the completion $\widehat{R(G)}_I$ of the representation ring at the dimension ideal $I := \ker(\dim\colon R(G) \to \mathbb{Z})$.

It is not true that the Atiyah-Segal equivariant K-theory is rationally isomorphic to the Borel extension of rational cohomology. In the case of discrete groups and proper actions the appropriate target of the Chern character was found in [**7**]. It will be called the delocalized cohomology in this paper. Let G be a discrete group which acts properly on a space X. Then we define a new proper G-space

$$\Lambda X := \bigsqcup_{g\in G} X^g,$$

where $X^g \subset X$ is the subspace of fixed points of g. The action of $h \in G$ on ΛX maps $x \in X^g$ to $hx \in X^{hgh^{-1}}$. The delocalized cohomology of the G-space X is the cohomology of the quotient $\Lambda X/G$.

A G-space X gives rise to a topological quotient stack $[X/G]$. If G is a discrete group which acts properly on X, then the quotient $[X/G]$ is an example of an orbispace. But not every orbispace can be represented in this form. We refer to [**12**] for the description of the category of orbispaces. The stack $[\Lambda X/G]$ has a description in the language of topological stacks. If Z is a topological stack, then we define its loop stack LZ (see 2.16 and 2.24)[10] such that $L[X/G] = [\Lambda X/G]$ for a discrete group acting properly on a space X.

If G is a discrete group which acts properly on a space X, then the quotient X/G is a reasonable topological space. It is the coarse moduli space of the orbispace $[X/G]$. The definition of the coarse moduli space extends to arbitrary orbispaces. The coarse moduli space of the orbispace Z will be denoted by $|Z|$. If $Z^1 \Rightarrow Z^0$ is a presentation of the orbispace by a proper étale groupoid, then $|Z| = Z^0/Z^1$.

[10]In the present paper we use the name loop stack. In the literature it is also known under the name inertia stack.

The rational cohomology of an orbispace Z is the cohomology of its coarse moduli space $|Z|$. Therefore we can define the delocalized cohomology of an orbispace as the cohomology of $|LZ|$. This generalizes the definition of the delocalized cohomology from global quotient orbispaces to general orbispaces.

Note that this is not quite the definition of delocalized cohomology which we are going to use in the main part of the paper but sufficient for the present discussion. Later we prefer a sheaf-theoretic definition of the delocalized cohomology.

Delocalized cohomology for orbifolds appeared in connection with the index theorem for orbifolds [**23**]. In a completely different context of quantum cohomology for orbifolds it was constructed in [**14**], [**36**]. Note that the grading used in [**14**] is different from the grading in the present paper.

A different generalization of K-theory is twisted K-theory (see [**5**]). The search for the target of an appropriate Chern character lead to the definition of 2-periodic twisted de Rham cohomology[11]. Usually it is defined on smooth manifolds X. Given a closed three-form $\lambda \in \Omega^3(X)$, twisted de Rham cohomology is the cohomology of the complex

$$\cdots \stackrel{d_\lambda}{\to} \Omega^{\mathrm{even}}(X) \stackrel{d_\lambda}{\to} \Omega^{\mathrm{odd}}(X) \stackrel{d_\lambda}{\to} \Omega^{\mathrm{even}}(X) \stackrel{d_\lambda}{\to} \dots, \tag{1.12}$$

where $d_\lambda := d_{dR} + \lambda$.

A Chern character for twisted K-theory with values in λ-twisted de Rham cohomology was constructed in [**10**], [**29**], and [**6**]. The twist of K-theory is classified by a class $\lambda_{\mathbb{Z}} \in H^3(X;\mathbb{Z})$. The closed form $\lambda \in \Omega^3(X)$ should represent the image of $\lambda_{\mathbb{Z}}$ in real cohomology. It was shown that this $\mathbb{Z}/2\mathbb{Z}$-graded cohomology theory is again isomorphic to twisted K-theory tensored with $\mathbb{R}$.

Twisted K-theory on orbifolds has first been considered in [**1**]. In this paper the twist was given by a so-called inner local system of twisted torsion. The natural object to be used to twist complex K-theory is a gerbe $G \to X$ with band S^1 (see [**17**] for details and more general twists). Gerbe twisted K-theory for orbifolds was discussed in [**26**]. For general local quotient stacks it was defined in [**18**], [**17**]. Using topological groupoids in order to represent stacks a very general definition of twisted K-theory in terms of the groupoid C^*-algebra was given in [**41**].

The result of [**7**] in the case of global quotient orbispaces obtained from proper actions of discrete groups shows that the correct target of the Chern character has to take the topology of the fixed point sets into account. Thus the target of the Chern character from twisted K-theory of an orbifold should be a delocalized version of twisted de Rham cohomology. If X is an orbifold, then LX is again an orbifold. In particular we can consider differential forms on LX. Given a three-form $\lambda \in \Omega^3(LX)$ we can define the twisted delocalized de Rham cohomology as the cohomology of the complex

$$\cdots \stackrel{d_\lambda}{\to} \Omega^{\mathrm{even}}(LX) \stackrel{d_\lambda}{\to} \Omega^{\mathrm{odd}}(LX) \stackrel{d_\lambda}{\to} \Omega^{\mathrm{even}}(LX) \stackrel{d_\lambda}{\to} \dots . \tag{1.13}$$

[11]This could also be reversed. The equations for fields associated to D-branes in string theory with B-field backgroup can be expressed in terms of the twisted de Rham differential. In this history twisted K-theory was found as a cohomology theory with a (Chern character) map to twisted de Rham cohomology giving the integrality lattice of D-brane charges [**31**], [**42**].

It turned out that this cohomology is not the correct target of the Chern character. This has already been observed in [**1**].

Let (L,∇^L) be the flat complex line bundle associated to $\tilde{G}^\delta \to LX$. We let $\Omega(LX;L)$ denote the differential forms with values in L, and d^L be the differential induced by d_{dR} and the flat connection ∇^L. We let $\lambda \in \Omega^3(LX)$ be a closed three-form which represents the image of the Dixmier-Douady class $\lambda_{\mathbb{Z}} \in H^3(LX;\mathbb{Z})$ of the gerbe $G_L \to LX$ in real cohomology. We set $d^L_\lambda := d^L + \lambda$. The correct target of the Chern character on G-twisted K-theory of the orbifold X is the 2-periodic cohomology of the complex

$$\cdots \to \Omega^{ev}(LX;L) \stackrel{d^L_\lambda}{\to} \Omega^{\mathrm{odd}}(LX;L) \stackrel{d^L_\lambda}{\to} \Omega^{ev}(LX;L) \to \dots . \tag{1.14}$$

This Chern character was constructed in [**40**].

In the context of equivariant cohomology theories a canonical target for the Chern character can be constructed as Bredon cohomology; see [**24**]. Bredon cohomology has been constructed for orbifolds (see [**35**]), and it should be possible to extend its definition to sufficiently nice orbispaces.

It would be very interesting to incorporate twists into the definition of orbispace Bredon cohomology, and furthermore to construct a Chern character in this context. In a very special situation (torsion twist and Γ-CW complex for a discrete group Γ) this has been carried out in [**16**] by a reduction to the non-twisted equivariant situation of [**24**]. The discussion in [**1**, Sec. 8] suggests to do this more generally.

The present paper follows a different route placing the relevant cohomology theories in the general framework of sheaf theory on stacks. We leave it to future work to develop the relevant Bredon cohomology and to compare it to the construction of the present paper.

2. Inertia

2.1. 2-limits in 2-categories

In the present paper we consider stacks on some site or groupoids in some ambient category like topological spaces or manifolds. A common feature of these constructs is that they are objects in a 2-category. Of particular importance for the present paper is the notion of a 2-limit. The goal of this subsection is to explain this notion.

By a 2-*category* we always mean a strict 2-category. In our main examples of 2-categories have the property that all 2-morphisms are isomorphisms, but in the present subsection we do not assume this. For objects a and b of a 2-category we denote by $\mathtt{Hom}_{\mathcal{C}}(a,b)$ the Hom-category from a to b (we will often omit the subscript and write $\mathtt{Hom}(a,b)$). We will write the objects as straight arrows $a \to b$, and the morphisms between two arrows $f,g\colon a \to b$ as $f \rightsquigarrow g$.

By a 2-functor we always mean a pseudo-2-functor, as explained for example in [**21**, Def. 1.4.2]. By a *strict* 2-functor we mean such a functor where all unit and composition 2-isomorphisms are identities.

Let $\mathcal{C}$ be a 2-category. For any $X \in \mathtt{Ob}\mathcal{C}$ we denote by $\mathcal{C}/X$ the over-2-category

- with objects the 1-arrows $A \to X$,

- whose 1-morphisms are triangles filled in with a 2-morphism

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & \arrow[d, Rightarrow, dotted] & B \arrow[dl] \\ & X & \end{tikzcd},$$

- and where 2-morphisms are the ones of $\mathcal{C}$ making the natural diagram commutative.

There is a version of this construction for a 2-functor $\mathcal{D} \to \mathcal{C}$ and an object X of $\mathcal{C}$, denoted $\mathcal{D}/X$. Note that if $\mathcal{D}$ is a 1-category then so is $\mathcal{D}/X$.

Let $\mathcal{C}$ be a 2-category and D a small category. Let $F, G\colon D \to \mathcal{C}$ be two 2-functors. A *natural* 2-*transformation* φ from F to G is an assignment of a 1-morphism $\varphi(a)\colon F(a) \to G(a)$ for any object a of D and a 2-isomorphism $\varphi(f)$ for any $f\colon a \to b$ in D filling in the square

$$\begin{tikzcd} F(a) \arrow[r, "\varphi(a)"] \arrow[d, "F(f)"'] & G(a) \arrow[d, "G(f)"] \\ F(b) \arrow[r, "\varphi(b)"] \arrow[ur, Rightarrow, dotted, "\phi(f)"] & G(b), \end{tikzcd}$$

satisfying the obvious compatibility for compositions of maps in D.

Let $\varphi, \psi\colon F \to G$ be two natural 2-transformations. A *modification* t from φ to ψ consists of a 2-morphism $t(a)\colon \varphi(a) \rightsquigarrow \psi(a)$ for any object a of D satisfying an again obvious compatibility with $\varphi(f)$ and $\psi(f)$ for any map f in D.

With these definitions the 2-functors, the natural 2-transformations and the modifications form a 2-category.

For F, G as above we denote by $\mathtt{Hom}_{\mathcal{C}^{\mathcal{D}}}(F, G)$ the corresponding category of natural transformations from F to G.

For an object c of $\mathcal{C}$ we denote by $\underline{D}_c$ the constant diagram on c, i.e. the (strict) 2-functor from D to $\mathcal{C}$ sending all objects to c and all morphisms to $\mathtt{id}_c$.

Definition 2.1. Let $F\colon D \to \mathcal{C}$ be a 2-functor. A 2-*limit* of F is an object c of $\mathcal{C}$ together with a natural 2-transformation $\varphi\colon \underline{D}_c \to F$ such that for any object T of C the functor $\mathtt{Hom}_{\mathcal{C}}(T, c) \to \mathtt{Hom}_{\mathcal{C}^{\mathcal{D}}}(\underline{D}_T, F)$ given by composition with φ is an equivalence of categories.

The constant diagram functor $c \mapsto \underline{D}_c$ is a 2-functor $\mathcal{C} \to \mathcal{C}^D$. Note that $F \in \mathcal{C}^D$. Using $\underline{D}$ we form the over-2-category $\mathcal{C}/F$. By definition a 2-limit (c, ϕ) of F is an object of $\mathcal{C}/F$.

For example, a 2-final object of $\mathcal{C}$ is an object c such that for all objects T of $\mathcal{C}$ the projection from $\mathtt{Hom}(T, c)$ to the point category is an equivalence.

Lemma 2.2. *Let $u\colon \mathcal{C} \to \mathcal{D}$ be a 2-functor between 2-categories, X an object of $\mathcal{D}$. Let c, $f\colon u(c) \to X$ be an object of $\mathcal{C}/X$. Then if the functor*

$$\mathtt{Hom}_{\mathcal{C}}(T, c) \to \mathtt{Hom}_{\mathcal{D}}(u(T), X)$$

is an equivalence for all objects T of $\mathcal{C}$ the object (c, f) is 2-final in $\mathcal{C}/X$. If the 2-morphisms in $\mathcal{D}$ and $\mathcal{C}$ are all 2-isomorphisms the converse holds.

Proof. Let $(c', f') \in \mathcal{C}/X$ be another object. Then there is a canonical 2-cartesian square

$$\begin{tikzcd} \mathtt{Hom}_{\mathcal{C}/X}((c',f'),(c,f)) \arrow[r] \arrow[d] & \mathtt{Hom}_{\mathcal{C}}(c',c) \arrow[d] \\ \mathrm{pt} \arrow[r, "f'"] & \mathtt{Hom}_{\mathcal{D}}(u(c'),X) \end{tikzcd}$$

in $\mathtt{Cat}$. Hence the first statement follows. The second statement follows from the fact that a map $\varphi\colon A \to B$ between groupoids is an equivalence if and only if all (2-categorical) fibres over objects of B are contractible. $\square$

An equivalence between two objects c and d of $\mathcal{C}$ are 1-arrows $f\colon c \to d$ and $g\colon d \to c$ together with 2-isomorphisms $\varphi\colon \mathtt{id}_c \rightsquigarrow g \circ f$ and $\psi\colon \mathtt{id}_d \rightsquigarrow f \circ g$ satisfying the triangular identities as for units and counits of adjunctions.

As a particular case consider two 2-final objects c, c' in a 2-category $\mathcal{D}$. Then there is an equivalence between c and c' which is unique up to a unique 2-isomorphism.

Lemma 2.3. *If an object $(c, \varphi) \in \mathcal{C}/F$ is a 2-limit of F then it is 2-final in $\mathcal{C}/F$. If all 2-morphisms in $\mathcal{C}$ are 2-isomorphisms or if $\mathcal{C}$ has all small 2-limits, then the converse is true. Any two choices of 2-limits are equivalent in $\mathcal{C}/F$, unique up to unique 2-isomorphism, in particular the underlying objects in $\mathcal{C}$ are (canonically) equivalent.*

Proof. The first statement follows from Lemma 2.2. The second statement under the assumption on the 2-morphisms also follows from that lemma, and under the completeness assumption it follows from the first statement and the uniqueness (up to unique isomorphism) of 2-final objects. The third statement also follows from the properties of 2-final objects stated above. $\square$

Lemma 2.4. *In $\mathtt{Cat}$, the 2-category of small categories, small 2-limits exist.*

Proof. The usual construction gives a *preferred model*: For a 2-functor $F\colon D \to \mathtt{Cat}$ define c to be the category whose objects are collections of objects $x_a \in F(a)$ for any object a of D together with isomorphisms $\varphi_f\colon (Ff)(x_a) \to x_b$ for any map $f\colon a \to b$ in D satisfying a compatibility condition for compositions of maps in D, and whose morphisms from (x_a) to (y_a) are compatible systems of morphisms $x_a \to y_a$. The transformation $\underline{D}_c \to F$ induced by projections exhibits c as a 2-limit of F. $\square$

Let us consider for example the category

$$D := \begin{tikzcd} & b \arrow[d] \\ a \arrow[r] & c. \end{tikzcd} \tag{2.5}$$

A functor $F\colon D \to \mathcal{C}$ is a diagram

$$\begin{tikzcd} & B \arrow[d, "v"] \\ A \arrow[r, "u"] & C. \end{tikzcd} \tag{2.6}$$

A 2-categorical fibre product of F is a diagram

$$\begin{tikzcd} A \times_C B \arrow[r] \arrow[d] & B \arrow[d, "v"] \\ A \arrow[r, "u"] \arrow[ur, Rightarrow, dotted, "\psi"] & C \end{tikzcd} \tag{2.7}$$

fulfilling some natural properties. Such a diagram gives in two natural ways an object in $\mathcal{C}/F$ (by requiring the map $A \times_C B \to C$ be one of the two possible compositions), and it is easily checked that the usual properties are equivalent to this object being a 2-limit. If these properties are fulfilled we call a diagram as above 2*-cartesian*.

Assume that $\mathcal{C} = \mathtt{Cat}$. A model of $A \times_C B$ is then the category whose objects are triples (a, b, γ), where $a \in \mathtt{Ob}(A)$, $b \in \mathtt{Ob}(b)$ and $\gamma\colon u(a) \to v(b)$. A morphism $(a, b, \gamma) \to (a', b', \gamma')$ is a pair $(f\colon a \to a', g\colon b \to b')$ such that $\gamma' \circ u(f) = v(f) \circ \gamma$. The 2-morphism in (2.7) is given by $\psi(a, b, \gamma) := \gamma$.

We see in particular that 2-categorical fibre products in $\mathtt{Cat}$ are 2-limits.

We call a 2-cartesian diagram (2.7) a *standard model* of the fibre product in a 2-category $\mathcal{C}$ if for any object T the functor $\mathtt{Hom}(T, \_)$ produces a diagram which is *isomorphic* (with respect to an obvious map) to the model in $\mathtt{Cat}$ from above. Note that this is not the preferred model introduced above.

Like ordinary limits 2-categorical limits are characterized by a universal property for Hom-categories.

Lemma 2.8. *Let $F\colon D \to \mathcal{C}$ be a 2-functor, $(c, \varphi) \in \mathcal{C}/F$ a 2-limit of F and T an object of $\mathcal{C}$. Consider the 2-functor $H\colon D \to \mathtt{Cat}$ given by $a \mapsto \mathtt{Hom}_{\mathcal{C}}(T, F(a))$. Then the natural map $\underline{D}_{\mathtt{Hom}_{\mathcal{C}}(T,c)} \to H$ is a 2-limit of the functor H.*

Proof. In fact $\mathtt{Hom}_{\mathcal{C}^{\mathcal{D}}}(\underline{D}_c, F)$ is naturally isomorphic to the preferred model of the 2-limit of the diagram $a \mapsto \mathtt{Hom}_{\mathcal{C}}(T, F(a))$. □

Lemma 2.8 implies an equivalence of categories

$$\mathtt{Hom}_{\mathcal{C}}(T, 2\text{-}\lim_{a \in D} F(a)) \cong 2\text{-}\lim_{a \in D} \mathtt{Hom}_{\mathcal{C}}(T, F(a)),$$

where the left 2-limit is taken in $\mathcal{C}$, and the right 2-limit is taken in $\mathtt{Cat}$.

Let C be another small category and suppose given a 2-functor $F\colon C \times D \to \mathcal{C}$. For simplicity suppose that $\mathcal{C}$ has all small 2-limits.

Proposition 2.9. *Let the notation be as above. The assignment*

$$a \mapsto 2\text{-}\lim_{b \in D} F(a, b)$$

can be made into a 2-functor $K\colon C \to \mathcal{C}$, and two such choices are canonically equivalent. Moreover the 2-limit of K is canonically equivalent to the 2-limit of F.

Proof. The first assertion is a consequence of Lemma 2.3. We sketch the proof of the second statement. By Lemma 2.8 we are reduced to prove the statement in $\mathtt{Cat}$. But taking everywhere preferred models produces isomorphic models of the two 2-limits in question. □

We will assume that $\mathcal{C}$ has a final object and admits standard models of all 2-categorical fibre products. The absolute product $\times$ is understood as a standard model of the fibre product over the final object. Consider a pair of maps

$$\begin{tikzcd} X \arrow[dr, "f"] & & X \arrow[dl, "g"'] \\ & Y. & \end{tikzcd}$$

Definition 2.10. The equalizer $E(f,g)$ of the pair of maps $f,g\colon X \to Y$ is defined as a standard model of the 2-categorical fibre product

$$\begin{tikzcd} E(f,g) \arrow[r] \arrow[d] & Y \arrow[d, "\mathtt{diag}"] \\ X \arrow[r, "{(f,g)}"] \arrow[ur, Rightarrow, dotted] & Y \times Y. \end{tikzcd}$$

Note that on Hom-categories this definition yields in fact the preferred model of the equalizer diagram.

Definition 2.11. We define the inertia object of X as the equalizer

$$IX := E(\mathtt{id}_X, \mathtt{id}_X).$$

We say that a 2-category is 2-complete if it admits all small 2-limits. There is an analogous notion of a 2-colimit, and the category is called 2-cocomplete if all small 2-colimits exist. The category is called 2-bicomplete if it is 2-complete and 2-cocomplete.

The 2-category of small groupoids $\mathtt{gpd}$ is 2-bicomplete as well as bicomplete as a category. The same holds for the 2-category $\mathtt{PSt}I$ of prestacks on a small category I, which is by definition the 2-category of 2-functors $\mathtt{gpd}^{I^{\mathrm{op}}}$. The 2-category of stacks $\mathtt{St}\mathbf{S}$ on a small site $\mathbf{S}$ is 2-bicomplete.

We consider the 2-category $\mathtt{gpd}(\mathcal{U})$ of groupoids in a category $\mathcal{U}$ which has finite limits. Our basic example for $\mathcal{U}$ is the category $\mathtt{Top}$ of topological spaces.

Lemma 2.12. *The category $\mathtt{gpd}(\mathcal{U})$ admits standard models of all 2-categorical fibre products.*

Proof. The objects and morphisms of the standard model of a fibre product in $\mathtt{gpd}(\mathcal{U})$ can be expressed in terms of fibre products in $\mathcal{U}$. □

Lemma 2.13. *In $\mathtt{gpd}(\mathcal{U})$ equalizers exist for any pair of maps.*

Proof. We observe that $\mathtt{gpd}(\mathcal{U})$ has a final object and admits 2-categorical fibre products (Lemma 2.12). In fact, the limit of the empty diagram in $\mathcal{U}$ is the final object $*$ of $\mathcal{U}$. The groupoid $* \Rightarrow *$ is the final object in $\mathtt{gpd}(\mathcal{U})$. □

We assume that $\mathcal{C}$ has a final object and admits standard models of all 2-categorical fibre products, and we consider a diagram (2.6).

Lemma 2.14. *We have a natural isomorphism $I(A \times_C B) \cong IA \times_{IC} IB$, where we use standard models for the fibre products.*

Proof. We only have to check this for $\mathcal{C} = \mathtt{Cat}$ since everything can be stated in terms of Hom-categories. We let $\tilde{\mathcal{D}}$ be the category freely generated by two objects 0, 1, and two isomorphisms from 0 to 1. Then we have an isomorphism $IA \cong \mathtt{Hom}(\tilde{\mathcal{D}}, A)$; see also Lemma 2.15 in the case of groupoids. Since standard fibre products commute with the cotensor structure the claim follows. □

2.2. Loops

In a 2-category of groupoids $\mathtt{gpd}(\mathcal{U})$ or stacks $\mathtt{St}(\mathcal{S})$ the preferred model of the inertia IX (see Def. 2.11) of X is quite complicated. The goal of the present subsection is the construction of a simpler model of IX which we call the loop object LX.

We start with the case of $\mathtt{gpd}(\mathcal{U})$. Let us assume that $\mathcal{U}$ is tensored and cotensored over $\mathtt{Sets}$. The cotensor functor will be denoted by

$$\underline{\mathtt{Hom}}\colon \mathtt{Sets}^{\mathrm{op}} \times \mathcal{U} \to \mathcal{U}.$$

Using the existence of finite limits in $\mathcal{U}$ we extend this functor to a bifunctor

$$\underline{\mathtt{Hom}}_{\mathtt{Cat}}\colon (\mathtt{Sets}^{\mathrm{fin}} - \mathtt{Cat}) \times (\mathcal{U} - \mathtt{Cat}) \to (\mathcal{U} - \mathtt{Cat}),$$

where for a category $\mathcal{A}$ with finite limits we write $(\mathcal{A} - \mathtt{Cat})$ for the 2-category of category objects in $\mathcal{A}$, and $\mathtt{Sets}^{\mathrm{fin}}$ is the category of finite sets.

Let $X \in \mathtt{gpd}(\mathcal{U}) \subset (\mathcal{U} - \mathtt{Cat})$ be a groupoid in $\mathcal{U}$. We consider the category

$$\mathcal{D} := \begin{tikzcd} \bullet_0 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & \bullet_1 \end{tikzcd} \in (\mathtt{Sets}^{\mathrm{fin}} - \mathtt{Cat}).$$

Since X is a groupoid, $\underline{\mathtt{Hom}}_{\mathtt{Cat}}(\mathcal{D}, X) \in (\mathcal{U} - \mathtt{Cat})$ is again a groupoid in $\mathcal{U}$.

Lemma 2.15. *We have a natural isomorphism*

$$IX \cong \underline{\mathtt{Hom}}_{\mathtt{Cat}}(\mathcal{D}, X).$$

Proof. We insert the standard model of the 2-categorical fibre product of $\mathtt{gpd}(\mathcal{U})$ into the definition of the equalizer in the special case that $f = g = \mathtt{id}_X$. Then the assertion becomes obvious. □

Later we will have the freedom to replace groupoids by equivalent groupoids. We let $\tilde{\mathcal{D}}$ be the category obtained from $\mathcal{D}$ by adjoining inverses. Since X is a groupoid we have

$$\underline{\mathtt{Hom}}_{\mathtt{Cat}}(\mathcal{D}, X) \cong \underline{\mathtt{Hom}}_{\mathtt{Cat}}(\tilde{\mathcal{D}}, X).$$

We now consider the category $\mathcal{L}$ with one object $*$ and infinite cyclic automorphism group generated by σ

$$\begin{tikzcd} * \arrow[loop left] \end{tikzcd} .$$

Then we have a natural functor $i\colon \mathcal{L} \to \tilde{\mathcal{D}}$ which maps $*$ to $\bullet_0$ and σ to $\beta^{-1} \circ \alpha$. This is an equivalence of categories. It induces an equivalence of groupoids

$$\underline{\mathtt{Hom}}_{\mathtt{Cat}}(\mathcal{D}, X) \cong \underline{\mathtt{Hom}}_{\mathtt{Cat}}(\tilde{\mathcal{D}}, X) \xrightarrow{i^*} \underline{\mathtt{Hom}}_{\mathtt{Cat}}(\mathcal{L}, X) \ .$$

Definition 2.16. The groupoid $LX := \underline{\mathtt{Hom}}_{\mathtt{Cat}}(\mathcal{L}, X)$ will be called the loop groupoid of X.

Note that we have an equivalence of groupoids

$$IX \to LX. \tag{2.17}$$

If $f\colon X \to Y$ is a morphism in $\mathtt{gpd}(\mathcal{U})$, then composition with f functorially induces a morphism $Lf\colon LX \to LY$.

It is easy to describe the objects and morphisms of the loop groupoid LX explicitly.

Lemma 2.18. *The objects LX^0 and morphisms LX^1 of LX are given by the following fibre products in $\mathcal{U}$:*

$$\begin{tikzcd} LX^0 \arrow[r] \arrow[d, "\delta"] & X^1 \arrow[d, "(s,r)"] \\ X^0 \arrow[r, "\mathtt{diag}"] & X^0 \times X^0, \end{tikzcd} \tag{2.19}$$

$$\begin{tikzcd} LX^1 \arrow[r] \arrow[d, "s"] & X^1 \arrow[d, "s"] \\ LX^0 \arrow[r, "\delta"] & X^0. \end{tikzcd} \tag{2.20}$$

The range map is given (in the language of elements) by the map

$$r((x,\gamma),\mu) := (r(\mu), \mu \circ \gamma \circ \mu^{-1}).$$

We will give another description of LX^1 which turns out to be useful later. We define P by the cartesian diagram

$$\begin{tikzcd} P \arrow[r, "k"] \arrow[d, "(p,q)"] & X^1 \arrow[d, "s,r"] \\ LX^0 \times LX^0 \arrow[r, "\delta,\delta"] & X^0 \times X^0. \end{tikzcd} \tag{2.21}$$

The composition of X induces a map $m\colon P \to X^1$ defined in the language of objects by

$$((x_0,\gamma_0),(x_1,\gamma_1),\mu) \mapsto \gamma_1^{-1} \circ \mu \circ \gamma_0 \circ \mu^{-1}.$$

Lemma 2.22. *We have a cartesian diagram*

$$\begin{tikzcd} LX^1 \arrow[r, "i"] \arrow[d, "j"] & X^0 \arrow[d, "1"] \\ P \arrow[r, "m"] & X^1, \end{tikzcd}$$

where $j := (s,r)$ and $i := \delta \circ s$.

Proof. Consider an object $T \in \mathcal{U}$. A map $f\colon T \to LX^1$ is uniquely determined by a pair (u,v), $u\colon T \to LX^0$ and $v\colon T \to X^1$ such that $\delta \circ u = s \circ v\colon T \to X^0$. The map

u is given by a pair (a, b) of maps with $a\colon T \to X^0$ and $b\colon T \to X^1$ such that $s \circ b = r \circ b = a$. We see that u is completely determined by b. Note that $\delta \circ u = s \circ b = s \circ v$. We have $j \circ f = ((s \circ b, b), (r \circ v, v \circ b \circ v^{-1}), v)$ and observe that $m \circ j \circ f = 1 \circ i \circ f$. This construction is natural in $T \to LX^0$ and therefore determines a map $LX^1 \to P \times_{X^1} X^0$.

Consider now a map $g\colon T \to LX^1 \to P \times_{X^1} X^0$ given by a pair (x, y) of maps $x\colon T \to P$ and $y\colon T \to X^0$ such that $m \circ x = 1 \circ y$. The pair $(p \circ x, k \circ x)$ satisfies $\delta \circ p \circ x = s \circ k \circ x$ and therefore defines a map $f\colon T \to LX^1$. Again, the construction is functorial in g and defines a map $P \times_{X^1} X^0 \to LX^1$.

We leave it to the reader to check that these maps are inverse to each other. □

Let $X \in \mathtt{gpd}(\mathcal{U})$ and LX be its loop groupoid. Evaluation at the unique object $*$ of $\mathcal{L}$ induces a functor $LX \to X$. Therefore LX can naturally be considered as an object of $\mathtt{gpd}(\mathcal{U})/X$. Note that a morphism in this category is a diagram

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\Downarrow"] & Z \arrow[dl] \\ & X & \end{tikzcd},$$

and a 2-morphism between two such maps is a 2-morphism $f \rightsquigarrow g$ between the given 1-morphisms $f, g\colon Y \to Z$ commuting with the 2-morphisms.

We will now consider group objects in $\mathtt{gpd}(\mathcal{U})/X$. They together with their products (i.e. fibre products over X) will lie in a subcategory which is equivalent as a 2-category to a 1-category, so it will not be a problem to formulate what we mean by a group object in this case.

Lemma 2.23. *The loop groupoid LX has a natural structure of a group object in $\mathtt{gpd}(\mathcal{U})/X$.*

Proof. We consider the category $\mathcal{E} \in (\mathtt{Sets\text{-}Cat})$ pictured by

$$\begin{tikzcd} \bullet_0 \arrow[loop left, "a"] \arrow[r, "b"] & \bullet_1 \arrow[loop right, "c"] \end{tikzcd},$$

where a, c generate infinite semigroups. By $\tilde{\mathcal{E}}$ we denote the category obtained from $\mathcal{E}$ by adjoining inverses. Then we observe that in the 2-category $\mathtt{gpd}(\mathcal{U})$

$$LX \times_X LX \cong \underline{\mathtt{Hom}}_{\mathtt{Cat}}(\mathcal{E}, X) \cong \underline{\mathtt{Hom}}_{\mathtt{Cat}}(\tilde{\mathcal{E}}, X).$$

We define a functor $j\colon \mathcal{L} \to \tilde{\mathcal{E}}$ which maps $*$ to $\bullet_0$ and σ to $b^{-1} \circ c \circ b \circ a$. The pullback

$$LX \times_X LX \cong \underline{\mathtt{Hom}}_{\mathtt{Cat}}(\tilde{\mathcal{E}}, X) \xrightarrow{j^*} \underline{\mathtt{Hom}}_{\mathtt{Cat}}(\mathcal{L}, X) \cong LX$$

induces the composition law. We leave it to the reader to write out the inverse, the unit and the remaining necessary verifications. □

Definitions, Facts, and Notation 2.24. Let $\mathbf{S}$ be a Grothendieck site. Then we can consider the category of presheaves of sets $\mathtt{PSh}\mathbf{S}$. It is closed under taking arbitrary small limits. The 2-category of strict prestacks $\mathtt{PSt}^{\mathrm{strict}}\mathbf{S}$ on $\mathbf{S}$ is by definition the category $\mathtt{gpd}(\mathtt{PSh}\mathbf{S})$. By Lemma 2.13 in $\mathtt{PSt}^{\mathrm{strict}}\mathbf{S}$ equalizers exist for all pairs of maps.

The category $\mathtt{PSh}\mathbf{S}$ is tensored and cotensored over $\mathtt{Sets}$. Hence we can apply the construction of the loop groupoid in $\mathtt{PSt}^{\mathrm{strict}}\mathbf{S}$. We now consider the full 2-subcategory of strict stacks $\mathtt{St}^{\mathrm{strict}}\mathbf{S} \subset \mathtt{PSt}^{\mathrm{strict}}\mathbf{S}$ of stacks on $\mathbf{S}$. Recall that a stack is a prestack which satisfies descend conditions for objects and morphisms. This subcategory is closed with respect to 2-limits and preserved by the cotensor structure. For all pairs of maps in the category $\mathtt{St}^{\mathrm{strict}}\mathbf{S}$ the equalizer exists by Lemma 2.13. Moreover, the loop object of a stack is again a stack.

While a strict prestack is a strict 2-functor $\mathbf{S}^{\mathrm{op}} \to \mathtt{gpd}(\mathtt{Sets})$, a prestack is a (in general non-strict) 2-functor $\mathbf{S}^{\mathrm{op}} \to \mathtt{gpd}(\mathtt{Sets})$, i.e. it preserves compositions of morphisms in $\mathbf{S}$ up to 2-morphisms which satisfy coherence conditions for triple compositions. The category of stacks is again a full subcategory of the category of prestacks on $\mathbf{S}$ which satisfy certain descend conditions. Note that $\mathtt{PSt}\mathbf{S}$ is cotensored over $(\mathtt{Sets}\text{-}\mathtt{Cat})$; i.e. we have a bifunctor

$$\underline{\mathtt{Hom}}_{\mathtt{Cat}} \colon (\mathtt{Sets}\text{-}\mathtt{Cat}) \times \mathtt{PSt}\mathbf{S} \to \mathtt{PSt}\mathbf{S}.$$

This structure is induced by the corresponding cotensor structure of $(\mathtt{Sets}\text{-}\mathtt{Cat})$, i.e. for a category $\mathcal{D} \in \mathtt{Sets}\text{-}\mathtt{Cat}$ and a prestack X the value of $\underline{\mathtt{Hom}}_{\mathtt{Cat}}(\mathcal{D}, X)$ on $U \in \mathbf{S}$ is given by

$$\underline{\mathtt{Hom}}_{\mathtt{Cat}}(\mathcal{D}, X)(U) := \underline{\mathtt{Hom}}_{\mathtt{Cat}}(\mathcal{D}, X(U)),$$

where $X(U) \in \mathtt{Sets}\text{-}\mathtt{Cat}$. If X is a stack, then $\underline{\mathtt{Hom}}_{\mathtt{Cat}}(\mathcal{D}, X)$ is also a stack.

The 2-categorical fibre product of (pre)stacks is given objectwise in $\mathbf{S}$ by the 2-categorical fibre-product in $\mathtt{gpd}(\mathtt{Sets})$. Therefore, Lemma 2.15 remains true in the categories $\mathtt{PSt}\mathbf{S}$ and $\mathtt{St}\mathbf{S}$. We can furthermore define the loop (pre)stack LX of a (pre)stack as in Definition 2.16 and (2.17) still induces an equivalence of (pre)stacks

$$IX \to LX.$$

Finally, Lemma 2.23 holds in the sense, that for a (pre)stack X the loops LX form a group object in the category of (pre)stacks over X.

Like Lemma 2.14 in the case of inertia stacks we have

Lemma 2.25. *The inertia functor preserves standard* 2*-cartesian diagrams.*

2.3. Loops of topological stacks

We consider the site $\mathtt{Top}$ of topological spaces and open coverings. Let $\mathtt{StTop}$ be the 2-category of stacks in topological spaces. By the observations in 2.24 we can form the loop stack LX of a stack $X \in \mathtt{StTop}$. In the present subsection we show that taking loops preserves topological stacks. Furthermore we show that taking loops commutes with the classifying stack functor from topological groupoids to stacks in topological spaces. We use the latter result in order to verify that LX for an orbispace is what is called the orbispace of twisted sectors in the literature.

We refer to [**19**], [**34**] and also to [**12**] for details about stacks (in topological spaces). Topological spaces are considered as stacks via the Yoneda embedding. A map $a \colon A \to X$ from a topological space to a stack X is called an atlas if it is representable, surjective and admits local sections. A topological stack is a stack which admits an atlas. We shall show that taking loops preserves topological stacks.

Lemma 2.26. *If $X \in \mathtt{StTop}$ is a topological stack, then LX is a topological stack.*

Proof. Let $a\colon A \to X$ be an atlas of X. Then we define a space W by the pull-back diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d, "w"] & A \times_X A \arrow[d, "(\mathtt{pr}_1, \mathtt{pr}_2)"] \\ A \arrow[r, "\mathtt{diag}"] & A \times A. \end{tikzcd}$$

We will construct a canonical map $c\colon W \to LX$ and show that it is an atlas of W.

The map $c\colon W \to LX$ is defined as follows. Let T be a topological space and $(f\colon T \to W) \in W(T)$. By the definition of W this map is given by a pair (g, h) of maps $g\colon T \to A$ and $h\colon T \to A \times_X A$ such that $\mathtt{diag} \circ g = (\mathtt{pr}_1 \circ h, \mathtt{pr}_2 \circ h)$. The map $h\colon T \to A \times_X A$ is given by a pair $h_1, h_2\colon T \to A$ and a 2-isomorphism $\sigma\colon a \circ h_1 \rightsquigarrow a \circ h_2$. Combining these two facts we see that f is given by a pair (g, σ) of a map $g\colon T \to A$ and a 2-automorphism $\sigma\colon a \circ g \rightsquigarrow a \circ g$. Recall that an object of $LX(T)$ is a pair (u, ϕ) of an object $u \in X(T)$ and an automorphism $\phi \in \mathtt{Aut}(u)$. We define $c(f) \in LX(T)$ to be the object $(a \circ g, \sigma) \in LX(T)$.

We now construct a 2-commutative diagram

$$\begin{tikzcd} W \arrow[r, "c"] \arrow[d, "w"] & LX \arrow[d, "i"] \\ A \arrow[r, "a"] \arrow[ur, Rightarrow, dotted, "\phi"] & X \end{tikzcd} \tag{2.27}$$

by defining ϕ is follows. As above let $(f\colon T \to W) \in W(T)$ be given by a pair (g, σ). In $X(T)$ we have the equalities $i \circ c(f) = i(a \circ g, \sigma) = a \circ g$ and $a \circ w(f) = a \circ g$. Therefore we can define $\phi(f) := \sigma$.

We claim that diagram (2.27) is 2-cartesian. In order to see this let T be a space and consider a triple (u, v, θ) consisting of maps $u\colon T \to A$, $v\colon T \to LX$ and a 2-isomorphism $\theta\colon a \circ u \rightsquigarrow i \circ v$. To this data we must associate a unique pair of maps (f, ψ) consisting of a map $f\colon T \to W$ and a 2-isomorphism $\psi\colon c \circ f \rightsquigarrow v$ such that

$$\begin{tikzcd} T \arrow[dr, "f"] \arrow[drr, "v"] \arrow[ddr, "u"'] & & \\ & W \arrow[r, "c"] \arrow[d, "w"] \arrow[ur, phantom, "\psi"] & LX \arrow[d, "i"] \\ & A \arrow[r, "a"] \arrow[ur, Rightarrow, dotted, "\phi", "\theta"'] & X \end{tikzcd}$$

commutes. The map v is given by a pair $(i \circ v, \kappa)$ consisting of an object $i \circ v \in X(T)$ and an automorphism $\kappa \in \mathtt{Aut}(i \circ v)$. Using the description of maps $T \to W$

obtained above we define $f\colon T \to W$ as the map which corresponds to the pair $(u, \theta^{-1} \circ \kappa \circ \theta)$ consisting of an object $u\colon T \to A$ and the automorphism $\theta^{-1} \circ \kappa \circ \theta\colon a \circ u \rightsquigarrow a \circ u$. We furthermore define a 2-isomorphism $\psi\colon c \circ f = (a \circ u, \theta^{-1} \circ \kappa \circ \theta) \overset{\kappa^{-1}\circ\theta}{\rightsquigarrow} (i \circ v, \kappa) = v$. Observe that ψ is uniquely determined by the condition that $i(\psi) \circ \phi(f) = \theta\colon a \circ w \circ f = a \circ u \to i \circ v$. This equality indeed holds for our construction since $\phi(f) = \theta^{-1} \circ \kappa \circ \theta$ and $i(\psi) = \kappa^{-1} \circ \theta$. This finishes the proof of the claim.

Since $A \to X$ is an atlas the map a is representable, surjective and admits local sections. These properties are preserved under pull-back. It follows that $c\colon W \to LX$ is representable, surjective and admits local sections, too. Therefore it is an atlas of LX. □

A topological groupoid $\mathcal{G}$ is a groupoid object in `Top`. It represents the stack of $\mathcal{G}$-principal bundles $\mathcal{B}\mathcal{G}$. If $A \to X$ is an atlas of a topological stack, then we form the topological groupoid $\mathcal{A}\colon A \times_X A \Rightarrow A$. The stack of $\mathcal{A}$-principal bundles is equivalent to X. We can define an equivalence $X \to \mathcal{B}\mathcal{A}$ which maps $(T \to X) \in X(T)$ to $(T \times_X A \to T) \in \mathcal{B}\mathcal{A}$ (we omit to write the action of $\mathcal{A}$ on that space over T).

Observe that finite limits in `Top` exist, and that `Top` is tensored and cotensored over `Sets`. Therefore by 2.13 for any pair of maps in `gpd(Top)` an equalizer exists. Furthermore, we can form the loop groupoid $L\mathcal{A}$ of a topological groupoid $\mathcal{A}$.

Let $A \to X$ be the atlas of a topological stack, and let $\mathcal{A} \in$ `gpd(Top)` denote the associated topological groupoid.

Lemma 2.28. *We have a natural equivalence of stacks $LX \cong \mathcal{B}L\mathcal{A}$.*

Proof. Let $W \to LX$ be as in the proof of Lemma 2.26. Then we can form $\mathcal{W}\colon W \times_{LX} W \Rightarrow W$. If we show that $\mathcal{W} \cong L\mathcal{A}$, then the assertion follows.

From (2.19) we get $W \cong (L\mathcal{A})^0$. Next we calculate using (2.20)

$$\begin{aligned} W \times_{LX} \times W &\cong (A \times_X LX) \times_{LX} (A \times_X LX) \\ &\cong LX \times_X (A \times_X A) \\ &\cong (LX \times_X A) \times_A (A \times_X A) \\ &\cong (L\mathcal{A}^0) \times_A \mathcal{A}^1 \\ &\cong (L\mathcal{A})^1. \end{aligned}$$

These isomorphisms are compatible with the groupoid structures. □

The following result was also shown in [**34**, Cor. 7.6].

Lemma 2.29. *If X is a topological stack, then $LX \to X$ is representable.*

Proof. We must show that for all spaces T and maps $T \to X$ the fibre product $T \times_X LX$ is equivalent to a space. It suffices to verify this in the case that T is an atlas.

We choose an atlas $A \to X$. The assertion then follows from the following two facts:

(1) Diagram (2.27) is cartesian.
(2) W is a space. □

Let us recall some notions related to orbispaces. Orbispaces as particular kinds of topological stacks have previously been introduced in [**12**, Sec. 2.1] and [**34**, Sec. 19.3]. In the present paper we use the set-up of [**12**] but add the additional condition that an orbispace atlas should be separated. This condition is needed in order to show that the loop stack of an orbispace is again an orbispace.

Definitions, Facts, and Notation 2.30.

(1) A topological groupoid $A\colon A^1 \Rightarrow A^0$ is called separated if the identity $\mathbf{1}_A\colon A^0 \to A^1$ of the groupoid is a closed map.

(2) A topological groupoid $A^1 \Rightarrow A^0$ is called proper if $(s,r)\colon A^1 \to A^0 \times A^0$ is a proper map.

(3) A topological groupoid is called étale if the source and range maps $s, r\colon A^1 \to A^0$ are étale.

(4) A proper étale topological groupoid $A^1 \Rightarrow A^0$ is called very proper if there exists a continuous function $\chi\colon A^0 \to [0,1]$ such that
   (a) $r\colon \mathtt{supp}(s^*\chi) \to A^0$ is proper;
   (b) $\sum_{y\in A^x} \chi(s(y)) = 1$ for all $x \in A^0$.

(5) A topological stack is called (very) proper (étale, separated, respectively), if it admits an atlas $A \to X$ such that the topological groupoid $A \times_X A \Rightarrow A$ is (very) proper (étale, separated, respectively).

(6) An orbispace atlas of a topological stack X is an atlas $A \to X$ such that $A \times_X A \Rightarrow A$ is a very proper étale and separated groupoid.

(7) An orbispace X is a topological stack which admits an orbispace atlas.

(8) If X, Y are orbispaces, then a morphism of orbispaces $X \to Y$ is a representable morphism of stacks.

The following lemmas illustrate the meaning of the separatedness and very properness conditions.

Lemma 2.31. *Let $A\colon A^1 \Rightarrow A^0$ be a proper étale groupoid. If A^1, A^0 are locally compact, then A is very proper.*

Proof. The existence of the cut-off function was shown in [**38**, Prop. 6.11].

Lemma 2.32. *Let $A\colon A^1 \Rightarrow A^0$ be a topological groupoid. If A^0 and A^1 are Hausdorff spaces, then A is separated.*

Proof. We define the Hausdorff space Q as the pull-back

$$\begin{tikzcd} Q \arrow[r, "j"] \arrow[d] & A^1 \arrow[d, "{(r,s)}"] \\ A^0 \arrow[r, "\mathtt{diag}"] & A^0 \times A^0. \end{tikzcd}$$

The property of a map between topological spaces being closed is preserved under pull-back. Since A^0 is Hausdorff the diagonal $\mathtt{diag}\colon A^0 \to A^0 \times A^0$ is a closed map.

It follows that $j\colon Q \to A^1$ is a closed map. The composition $\circ$ in A gives the squaring map

$$sq\colon Q \xrightarrow{\mathtt{diag}} Q \times_{A^0} Q \xrightarrow{\circ} Q.$$

Then we have a pull-back

$$\begin{tikzcd} I \arrow[r, "k"] \arrow[d] & Q \arrow[d, "(\mathtt{id}_Q, sq)"] \\ Q \arrow[r, "\mathtt{diag}"] & Q \times Q. \end{tikzcd}$$

Since Q is Hausdorff, it follows that `diag` and hence k are closed maps. The composition $j \circ k\colon I \to A^1$ of closed maps is again closed. In a group the identity is the unique solution of the equation $x^2 = x$. It follows that $j \circ k(I) = \mathbf{1}_A(A^0)$. Therefore $\mathbf{1}_A(A^0) \subseteq A^1$ is closed.

This implies that $\mathbf{1}_A\colon A^0 \to A^1$ is a closed map. If fact, if $K \subseteq A^0$ is a closed subset, then we define the Hausdorff space $A^1_K \subseteq A^1$ as the pull-back

$$\begin{tikzcd} A^1_K \arrow[r, "v"] \arrow[d] & A^1 \arrow[d, "(r,s)"] \\ K \times K \arrow[r, "u"] & A^0 \times A^0. \end{tikzcd}$$

Since u (the obvious embedding) is a closed map, so is v. We apply the discussion above to the restricted groupoid $A^1_K \Rightarrow K$ with identity $\mathbf{1}_{A_K}\colon K \to A^1_K$ in order to show that $\mathbf{1}_{A_K}(K) \subseteq A^1_K$ is closed. Hence $\mathbf{1}_A(K) = v(\mathbf{1}_{A_K}(K)) \subseteq A^1$ is closed. $\square$

Lemma 2.33. *If X is an orbispace, then LX is an orbispace and $LX \to X$ is a morphism of orbispaces.*

Proof. Choose an orbispace atlas $A \to X$. The associated groupoid $\mathcal{A}\colon A \times_X A \to A$ is étale, proper and separated. In order to show that LX is an orbispace it suffices to show by Lemma 2.28 that $L\mathcal{A}$ is étale, proper and separated.

The property of a map between topological spaces being étale is preserved under pull-back. By (2.20) the fact that $s\colon \mathcal{A}^1 \to \mathcal{A}^0$ is étale therefore implies that $s\colon (L\mathcal{A})^1 \to (L\mathcal{A})^0$ is étale. Using the inversion homeomorphism $I\colon (L\mathcal{A})^1 \to (L\mathcal{A})^1$ we can express the range map in terms of the source map: $r = s \circ I$. This implies that $r\colon (L\mathcal{A})^1 \to (L\mathcal{A})^0$ is étale, too. We thus have shown that $L\mathcal{A}$ is étale.

We consider the pull-back

$$\begin{tikzcd} P \arrow[r] \arrow[d, "j"] & \mathcal{A}^1 \arrow[d, "(r,s)"] \\ (L\mathcal{A})^0 \times (L\mathcal{A})^0 \arrow[r] & \mathcal{A}^0 \times \mathcal{A}^0 \end{tikzcd}$$

(compare (2.21)). The property of a map between topological spaces being proper is also preserved by pull-backs. Therefore $j\colon P \to (L\mathcal{A})^0 \times (L\mathcal{A})^0$ is a proper map. The image of $\mathbf{1}_{\mathcal{A}}\colon \mathcal{A}^0 \to \mathcal{A}^1$ is closed. By Lemma 2.22 we can write $(L\mathcal{A})^1$ as a closed subspace $(L\mathcal{A})^1 := m^{-1}(\mathbf{1}_{\mathcal{A}}(\mathcal{A}^0)) \subset P$. In general, the restriction of a proper map to a closed subspace is still proper. Since the restriction of j to the closed subspace $(L\mathcal{A})^1 \subset P$ is exactly $(r,s)\colon (L\mathcal{A})^1 \to (L\mathcal{A})^0 \times (L\mathcal{A})^0$ we see that the groupoid $\mathcal{A}$ is proper.[12]

We now show that $L\mathcal{A}$ is very proper. Since $\mathcal{A}$ is very proper there exists a continuous function $\chi\colon \mathcal{A}^0 \to [0,1]$ such that $r\colon \mathtt{supp}(s^*\chi) \to \mathcal{A}^0$ is proper and

$$\sum_{y\in\mathcal{A}^x} \chi(s(y)) = 1$$

for all $x \in \mathcal{A}^0$. Let $i\colon L\mathcal{A} \to \mathcal{A}$ be the canonical map. Then $i^*\chi\colon L\mathcal{A}^0 \to [0,1]$ has corresponding properties for the groupoid $L\mathcal{A}$.

Finally we show that $\mathbf{1}_{L\mathcal{A}}\colon (L\mathcal{A})^0 \to (L\mathcal{A})^1$ is a closed map. By definition we have the cartesian square

$$\begin{tikzcd}
(L\mathcal{A})^1 \arrow[r] \arrow[d] & \mathcal{A}^1 \arrow[d] \\
(L\mathcal{A})^0 \arrow[r] & \mathcal{A}^0.
\end{tikzcd}$$

Therefore we have an embedding as a subspace $(L\mathcal{A}^1) \subset (L\mathcal{A})^0 \times \mathcal{A}^1$. Let $K \subseteq (L\mathcal{A})^0$ be a closed subset. Then we can write $\mathbf{1}_{L\mathcal{A}}(K) = (L\mathcal{A}^1) \cap (K \times i(\mathcal{A}^0))$. Since $\mathcal{A}$ is separated the subspace $(K \times \mathbf{1}_{\mathcal{A}}(\mathcal{A}^0)) \subseteq (L\mathcal{A})^0 \times \mathcal{A}^1$ is closed. Therefore $\mathbf{1}_{L\mathcal{A}}(K) \subset (L\mathcal{A})^1$ is closed, too.

In order to be a map of orbispaces $LX \to X$ must be representable. This is Lemma 2.29. $\square$

We can replace the site of topological spaces $\mathtt{Top}$ by the site of smooth manifolds $\mathtt{Mf}^\infty$. We will call the corresponding stacks stacks in smooth manifolds. A map $A \to X$ from a manifold to a stack in smooth manifolds is called an atlas if it is representable, surjective and smooth (i.e. a submersion). A stack in smooth manifolds which admits an atlas is called a smooth (or differentiable) stack. An orbifold atlas of a smooth stack is an atlas such that the associated groupoid is étale and proper. An orbifold is a smooth stack which admits an orbifold atlas. Since smooth manifolds are Hausdorff and locally compact a smooth stack is separated. If it is proper, then it is automatically very proper, and the corresponding cut-off function (see 2.30, (4)) can actually be chosen to be smooth.

The obvious problem to extend the proof of Lemma 2.26 from topological spaces to smooth manifolds is that in smooth manifolds fibre products only exist under appropriate transversality conditions. In fact, the map $(\mathtt{pr}_1, \mathtt{pr}_2)\colon A \times_X A \to A \times A$ is in general not transverse to the diagonal $\mathtt{diag}\colon A \to A \times A$.

But it is still true that the loop stack of an orbifold is an orbifold. Proofs of this fact can be found e.g. in [**22**], [**1**], [**14**]. Note that for smooth stacks $LX \to X$ is in general neither smooth nor representable.

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[12] It is because of this argument that in addition to the conditions used in [**12**] we require an orbispace atlas to be separated.

2.4. Loops and principal bundles

Let G be a topological group. The classifying stack $\mathcal{B}G$ of G-principal bundles is given as a quotient stack $\mathcal{B}G := [*/G]$ of the action of G on the one point space $*$ [**19**, Example 2.5]. The map $* \to \mathcal{B}G$ is an atlas and we have a canonical cartesian diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] \arrow[ur, Rightarrow, dotted] & \mathcal{B}G. \end{tikzcd}$$

Hence this atlas gives rise to the groupoid $\mathcal{G}\colon G \Rightarrow *$. We see that $L\mathcal{G}$ is the groupoid $G \times G \Rightarrow G$ of the action of G on itself by conjugations. Therefore by Lemma 2.28 we have $L\mathcal{B}G \cong [G/G]$.

A G-principal bundle over a space Y is by definition an object $p \in \mathcal{B}G(Y)$, or equivalently, by Yoneda's lemma, a map $p\colon Y \to \mathcal{B}G$. The underlying map of spaces $P \to Y$ fits into the cartesian diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & * \arrow[d] \\ Y \arrow[r, "p"] \arrow[ur, Rightarrow, dotted] & \mathcal{B}G. \end{tikzcd}$$

We adopt the same definition for a G-principal bundle over a stack Y. In this case the underlying map $P \to Y$ is a representable map.

Let $a\colon A \to Y$ be an atlas such that the pull-back of the principal bundle $p\colon Y \to \mathcal{B}G$ admits a trivialization. A trivialization is a lift t in the diagram

$$\begin{tikzcd} & & * \arrow[d] \\ A \arrow[r, "a"] \arrow[urr, dotted, "t"] & Y \arrow[r, "p"] \arrow[ur, Rightarrow, dotted] & \mathcal{B}G. \end{tikzcd} \tag{2.34}$$

The cocycle associated to the atlas a and the trivialization t is the induced map

$$\Phi_{a,t}\colon A \times_Y A \to * \times_{\mathcal{B}G} * \cong G.$$

Let $\mathcal{A}\colon A \times_Y A \Rightarrow A$ be the groupoid determined by the atlas and $\mathcal{A}^\bullet$ denote the associated simplicial space. Let

$$C^\bullet(\mathcal{A}; G) := C(\mathcal{A}^\bullet, G), \quad \delta\colon C^\bullet(\mathcal{A}; G) \to C^{\bullet+1}(\mathcal{A}; G)$$

be the associated cochain complex (the part in degree > 2 is only defined if G is abelian). Then $\Phi_{a,t} \in C^1(\mathcal{A}, G)$ is closed, i.e. it satisfies $\delta\Phi_{a,t} = 0$. We refer to [**19**, Sec. 2] for a description of G-principal bundles in terms of cocycles.

Let $p\colon Y \to \mathcal{B}G$ be a G-principal bundle over a stack Y. We apply the loop functor and get the map $Lp\colon LY \to L\mathcal{B}G \cong [G/G]$. It is a homomorphism over the map $Y \to \mathcal{B}G$. If G is abelian, then it induces a homomorphism

$$h\colon LY \to G. \tag{2.35}$$

In the following we give a heuristic description of this homomorphism. Let $f\colon P \to Y$ be the underlying map of stacks of the principal bundle. Furthermore, let $i\colon LY$

$\to Y$ denote the canonical map. For a point $y \in Y$ we get an action of the group $i^{-1}(y)$ on the fibre $f^{-1}(y)$. If $\gamma \in i^{-1}(y)$ and $x \in f^{-1}(y)$, then $\gamma x = xh(\gamma)$. On the left-hand side, $(\gamma, x) \mapsto \gamma x$ denotes the action of $i^{-1}(y)$ on $f^{-1}(y)$. On the right-hand side $(x, g) \to xg$ is the G-action on P given by the principal bundle structure. We see again, that the restriction $h_{|i^{-1}(y)} \colon i^{-1}(y) \to G$ is a homomorphism for all $y \in Y$.

Assume that we have chosen an atlas $a \colon A \to Y$ and a trivialization t as in (2.34). Let $\mathcal{A} \colon A \times_Y A \Rightarrow A$ be the associated groupoid. Then we get an induced map $h_a \colon L\mathcal{A} \to G$. It is equal to the restriction of the cocycle $\Phi_{a,t}$ to $(L\mathcal{A})^0 \subseteq \mathcal{A}^1$; i.e. we have the equality

$$h_a = (\Phi_{a,t})_{|(L\mathcal{A})^0}. \tag{2.36}$$

The cocycle h_a is closed, i.e. $\delta h_a = 0$, and it represents the function $h \in C(LY; G)$ under the identification $H^0(L\mathcal{A}; G) = C(LY, G)$. Another interpretation of (2.36) is as the equality $h_a = \mathtt{tr}[\Phi_{a,t}]$, where $[\Phi_{a,t}] \in H^1(\mathcal{A}; G)$ is the cohomology class represented by $\Phi_{a,t}$, and $\mathtt{tr} \colon C^{\bullet+1}(\mathcal{A}; G) \to C^{\bullet}(L\mathcal{A}; G)$ is the transgression chain map defined in [**2**], [**27**], [**39**].

Let G be an abelian topological group. In the following lemma we will assume that for all $n \in \mathbb{N}$ the subspace of n-torsion points

$$\mathtt{Tors}_n(G) := \{g \in G | g^n = 1\} \subseteq G$$

is discrete. This is a non-trivial assumption which, for example, is not true for the topological group $\prod_{\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$. Let G^δ denote the group G with the discrete topology. Let $p \colon Y \to \mathcal{B}G$ be a G-principal bundle.

Lemma 2.37. *If Y is an orbispace and the subsets $\mathtt{Tors}_n(G) \subseteq G$ are discrete for all $n \in \mathbb{N}$, then the map $h \colon LY \to G$ (defined in (2.35)) factors over G^δ.*

Proof. We must show that for all spaces T and maps $w \colon T \to LY$ the composition $h \circ w \colon T \to G$ is locally constant. We choose an orbifold atlas $A \to Y$ which gives rise to a very proper separated étale groupoid $\mathcal{A} \colon A \times_Y A \Rightarrow A$.

We consider a point $t \in T$. There exists a neighbourhood $t \in U \subseteq T$ which admits a lift

$$\begin{tikzcd} U \arrow[r, dotted, "\tilde{w}"] \arrow[d] & \mathcal{A}^0 \arrow[d] \\ T \arrow[r, "w \circ i"] \arrow[ur, Rightarrow, dotted, "\sigma"] & Y. \end{tikzcd}$$

By Lemma 2.28 we have the 2-cartesian square in the following diagram:

$$\begin{tikzcd} U \arrow[dr, dotted, "v"] \arrow[drr, bend left, "w"] \arrow[ddr, bend right, "\tilde{w}"'] & & \\ & (L\mathcal{A})^0 \arrow[r] \arrow[d] & LY \arrow[d, "i"] \\ & \mathcal{A}^0 \arrow[r] \arrow[ur, Rightarrow, dotted, "\sigma"] & Y. \end{tikzcd}$$

We get an induced map $v \colon U \to L\mathcal{A}^0 \subseteq \mathcal{A}^1$ such that $\tilde{w} = s \circ v$. Let $a := \tilde{w}(t) \in \mathcal{A}^0$

so that $v(t) \in \mathcal{A}_a^a$. Since the groupoid $\mathcal{A}$ is proper the group $\mathcal{A}_a^a$ is finite. Hence there exists an $n \in \mathbb{N}$ such that $v(t)^n = \mathtt{id}_a$. The map v^n fits into the diagram

$$\begin{tikzcd} \{t\} \arrow[r] \arrow[d] & \mathcal{A}^1 \arrow[d, "s"] \\ U \arrow[r, "\tilde{w}"] \arrow[ur, "v^n"] & \mathcal{A}^0. \end{tikzcd}$$

Note that the map $U \ni u \mapsto \mathtt{id}_{\tilde{w}(u)} \in \mathcal{A}^1$ would fit into the same diagram in the place of v^n. Since $s\colon \mathcal{A}^1 \to \mathcal{A}^0$ is étale we can shrink U further such that $v^n(u) = \mathtt{id}_{\tilde{w}(u)}$ for all $u \in U$. This implies that $h \circ w_{|U} : U \to G$ factors over the discrete subset $\mathtt{Tors}_n(G) \subseteq G$ and is therefore locally constant. $\square$

Let G be a topological abelian group such that $\mathtt{Tors}_n(G) \subseteq G$ is discrete for all $n \in \mathbb{N}$. Furthermore, let $p : Y \to \mathcal{B}G$ be a G-principal bundle over an orbispace Y and $h\colon LY \to G^\delta$ as in Lemma 2.37. Then we have a decomposition

$$LY \cong \bigsqcup_{g \in G} LY_g,$$

where $LY_g := h^{-1}(g)$ is formally defined by the 2-cartesian square

$$\begin{tikzcd} LY_g \arrow[r] \arrow[d] & {[\{g\}/G]} \arrow[d] & \\ LY \arrow[r, "Lp"] \arrow[ur, Rightarrow, dotted] & {[G^\delta/G]} \arrow[r, "\cong"] & \bigsqcup_{l \in G} [\{l\}/G]. \end{tikzcd}$$

Let $f\colon X \to Y$ be the map of stacks underlying the principal bundle p. It fits into the cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & * \arrow[d] \\ Y \arrow[r, "p"] \arrow[ur, Rightarrow, dotted] & \mathcal{B}G. \end{tikzcd} \tag{2.38}$$

Lemma 2.39. *The map $Lf\colon LX \to LY$ factors over the G-principal bundle*

$$LX \to LY_1.$$

Proof. We apply the loop functor to the 2-cartesian diagram (2.38) and get the 2-cartesian diagram (see Lemma 2.25)

$$\begin{tikzcd} LX \arrow[r] \arrow[d, "Lf"] & L\{1\} \arrow[r, equal] \arrow[d] & \{1\} \arrow[d] \\ LY \arrow[r, "p"] \arrow[ur, Rightarrow, dotted] & L\mathcal{B}G \arrow[r, equal] & {[G/G]}. \end{tikzcd} \tag{2.40}$$

It follows from the construction of $h\colon LY \to G$ that $h \circ Lf$ is the constant map with value $1 \in G$. It remains to show that $LX \to LY_1$ is a G-principal bundle. To this end we refine the diagram (2.40) to

$$\begin{tikzcd} LX \arrow[r] \arrow[d, "Lf"] & \{1\} \arrow[d] \\ LY_1 \arrow[r] \arrow[d] & [\{1\}/G] \arrow[d] \\ LY \arrow[r, "p"] \arrow[uur, dotted] \arrow[ur, dotted] & [G/G]. \end{tikzcd}$$

By definition of LY_1 the lower square is 2-cartesian. Since the outer square is the 2-cartesian square (2.40) we conclude that the upper square is 2-cartesian. $\square$

Let Γ be a finite group. The exact segment

$$\begin{tikzcd} \cdots \arrow[r] & H^1(\Gamma;\mathbb{R}^\delta) \arrow[r] \arrow[d, "\cong"] & H^1(\Gamma;U(1)^\delta) \arrow[r, "\cong", "\partial"'] \arrow[d, "\cong"] & H^2(\Gamma;\mathbb{Z}) \arrow[r] & H^2(\Gamma;\mathbb{R}^\delta) \arrow[r] \arrow[d, "\cong"] & \cdots \\ & 0 & \hat{\Gamma} & & 0 & \end{tikzcd}$$

of the Bockstein sequence in group cohomology associated to the sequence of coefficients

$$0 \to \mathbb{Z} \to \mathbb{R}^\delta \to U(1)^\delta \to 0$$

gives rise to a natural identification

$$H^2(\Gamma;\mathbb{Z}) \cong \hat{\Gamma},$$

where $\hat{\Gamma}$ denote the group of $U(1)$-valued characters of Γ.

Let us consider the orbispace $[*/\Gamma]$. Then we have $L[*/\Gamma] \cong [\Gamma/\Gamma]$, where Γ acts on itself by conjugation. A character $\chi \in \hat{\Gamma}$ gives rise to a function

$$\bar{\chi}\colon L[*/\Gamma] \to U(1)^\delta, \quad \gamma \mapsto \chi(\gamma). \tag{2.41}$$

There are various ways to define the integral cohomology of an orbispace B. In order to be able to use results about the classification of $U(1)$-principal bundles over B we use the definition [**12**], where we define

$$H^*(B;\mathbb{Z}) := H^*(|\mathcal{A}|;\mathbb{Z})$$

using the classifying space $|\mathcal{A}|$ of the groupoid $\mathcal{A}$ associated to an orbifold atlas $a\colon A \to B$. Note that by this definition $H^*(\mathcal{B}\Gamma;\mathbb{Z}) \cong H^*(\Gamma;\mathbb{Z})$. In fact, if we choose the atlas $a\colon * \to \mathcal{B}\Gamma$ and let $\mathcal{A}$ be the associated groupoid, then $|\mathcal{A}|$ is the standard model of the classifying space $B\Gamma$ of Γ.

Let $\chi \in H^2(B;\mathbb{Z})$. In this paragraph we generalize the construction (2.41) of the map $\chi \mapsto \bar{\chi}$ to general orbispaces B. We start with describing the values of $\bar{\chi}\colon LB \to U(1)$ at the points of LB. For the moment we do not claim any continuity property, but by Lemma 2.43 we see that it is continuous even if we equip $U(1)$ with the discrete topology.

Consider a point $u\colon * \to LB$. It determines and is determined by a point $p_u\colon * \xrightarrow{u} LB \to B$ in B and an element $\gamma_u \in \mathtt{Aut}(p_u) \cong * \times_B *$. The element γ_u generates a finite cyclic group Γ_u. We obtain an induced map $\tilde{u}\colon [*/\Gamma_u] \to B$. We have $L[*/\Gamma_u] \cong [\Gamma_u/\Gamma_u]$ and consider $\gamma_u \in [\Gamma_u/\Gamma_u]$ (or more formally, as a map $\gamma_u\colon * \to [\Gamma_u/\Gamma_u]$). We have an induced map $L\tilde{u}\colon L[*/\Gamma_u] \to LB$ such that $L\tilde{u}(\gamma_u) = u$. We can now define

$$\bar{\chi}(u) := \overline{\tilde{u}^*\chi}(\gamma_u). \tag{2.42}$$

Let B be an orbispace. By [**12**, Prop. 4.3], the class $\chi \in H^2(B;\mathbb{Z})$ classifies a $U(1)$-principal bundle $P_\chi \to B$. In Lemma 2.43 we will express the corresponding function $h_\chi\colon LB \to U(1)^\delta$ (defined in (2.35) directly in terms of χ.

Lemma 2.43. *We have the equality* $h_\chi = \bar{\chi}$.

Proof. The constructions of h_χ and $\bar{\chi}$ are natural under pull-back. It therefore suffices to show this equality in the case that $B \cong [*/\Gamma]$ for a finite group Γ. In this case we have $P_\chi \cong [U(1)/_\chi\Gamma]$, where Γ acts on $U(1)$ via the character χ. By construction of h_χ we have $h_\chi = \chi\colon [\Gamma/\Gamma] \to U(1)^\delta$. On the other hand, again by construction, we have $\bar{\chi} = \chi\colon [\Gamma/\Gamma] \to U(1)^\delta$. $\square$

Here is another interpretation. Let $a\colon A \to B$ be a good orbifold atlas; i.e. the spaces

$$\underbrace{A \times_B \cdots \times_B A}_{n+1\text{-factors}}$$

have contractible components for all $n \geqslant 0$. We can choose a trivialization t of the pull-back of the $U(1)$-bundle to A and get a cocycle $\Phi_{a,t} \in C^1(\mathcal{A};U(1))$. The definition of an orbifold atlas is in particular made such that $H^i(\mathcal{A};\mathbb{R}_{\mathrm{cont}}) = 0$ for $i \geqslant 1$.[13] Hence the boundary operator in cohomology associated to the sequence $0 \to \mathbb{Z} \to \mathbb{R}_{\mathrm{cont}} \to U(1)_{\mathrm{cont}} \to 0$ induces an isomorphism

$$\partial\colon H^1(\mathcal{A};U(1)_{\mathrm{cont}}) \xrightarrow{\sim} H^2(\mathcal{A};\mathbb{Z}) \cong H^2(B;\mathbb{Z}).$$

Under this identification we have $\chi \cong \partial[\Phi_{a,t}]$. Our construction of $\chi \mapsto \bar{\chi}$ is made such that $\overline{\partial(\phi)} \cong \mathtt{tr}\phi \in H^0(L\mathcal{A};U(1)) \cong C(LB,U(1))$ for every class $\phi \in H^1(\mathcal{A};U(1))$. This assertion is equivalent to Lemma 2.43.

2.5. Gerbes and local systems

We consider stacks in topological spaces $\mathtt{StTop}$. Let H be an abelian topological group and $f\colon G \to X$ be a topological gerbe with band H over some topological stack X. We take loops and obtain $Lf\colon LG \to LX$. We further have a canonical map $\tilde{i}\colon LG \to G$, and LG/G is a group in $\mathtt{StTop}/G$ (see Lemma 2.23). Since $i \circ Lf \cong f \circ \tilde{i}$

[13] For a proof see [**15**, Prop. 1] or the corrected version [**12**]. In the original version an orbifold atlas was characterized by the property that it gives rise to a proper étale groupoid. In order to prove this vanishing of real continuous cohomology we added the assumption of being very proper.

we get the dotted arrow

$$\begin{tikzcd}
LG \arrow[dr, dotted, "\pi"] \arrow[drr, "\tilde{i}"] \arrow[ddr, "Lf"'] & & \\
& G_L \arrow[r] \arrow[d] & G \arrow[d, "f"] \\
& LX \arrow[r, "i"] \arrow[ur, Rightarrow, dotted] & X,
\end{tikzcd} \tag{2.44}$$

where the gerbe $G_L \to LX$ is defined by the 2-cartesian square. One way to say that the gerbe $G \to X$ is a topological gerbe with band H is as follows:[14]

(1) The map $\pi\colon LG \to G_L$ is the underlying map of an H-principal bundle classified by $G_L \to BH$.
(2) The sequence of (representable; see 2.29) maps $LG \xrightarrow{\pi} G_L \to G$ is a central extension of groups
$$G \times H/G \to LG/G \to G_L/G \tag{2.45}$$
in $\mathtt{StTop}/G$ (the group stack structures of G_L/G is induced from that of LX/X).

Proposition 2.46. *There exists a canonical central extension*
$$X \times H/X \to \tilde{G}/X \to LX/X$$
of groups in $\mathtt{StTop}/X$ *whose pull-back along* $G \to X$ *is isomorphic to (2.45). It depends functorially on the datum* $G \to X$.

Proof. We first go over to topological groupoids by choosing atlases. Then we construct the required extension in topological groupoids. Finally we pass back to stacks.

We choose an atlas $a\colon A \to X$ which admits a lift

$$\begin{tikzcd}
& G \arrow[d, "f"] \\
A \arrow[ur, dotted, "b"] \arrow[r, "a"'] \arrow[ur, phantom, "\overset{\phi}{\Rightarrow}"] & X
\end{tikzcd} \tag{2.47}$$

to an atlas of G. We get topological groupoids

$$\mathcal{X} : \mathcal{X}^1 := A \times_X A \rightrightarrows \mathcal{X}^0 := A$$
$$\mathcal{G} : \mathcal{G}^1 := A \times_G A \rightrightarrows \mathcal{G}^0 := A,$$

and a central H-extension

[14] The definition given in [**19**, Def. 5.3] expresses these properties using objects.

$$\begin{tikzcd} X^0 \times H \arrow[d] & \\ \mathcal{G}^1 \arrow[r, Rightarrow] \arrow[d] & \mathcal{G}^0 \arrow[d, equal] \\ \mathcal{X}^1 \arrow[r, Rightarrow] & \mathcal{X}^0. \end{tikzcd}$$

Using the description (2.19) of the objects of $L\mathcal{X}$ and $L\mathcal{G}$ we get the pull-back of H-principal bundles

$$\begin{tikzcd} (L\mathcal{G})^0 \arrow[r] \arrow[d] & \mathcal{G}^1 \arrow[d] \\ (L\mathcal{X})^0 \arrow[r] & \mathcal{X}^1. \end{tikzcd}$$

Furthermore, by (2.20) we have the following description of the morphisms $(L\mathcal{G})^1$ as a pull-back

$$\begin{tikzcd} (L\mathcal{G})^1 \arrow[r] \arrow[d, "s"] & \mathcal{G}^1 \arrow[d, "s"] \\ (L\mathcal{G})^0 \arrow[r, "\delta"] & \mathcal{G}^0. \end{tikzcd} \tag{2.48}$$

We see that $(L\mathcal{G})^1$ has two commuting H-actions, the first comes from the action on $(L\mathcal{G})^0$ (the principal bundle structure of the left lower-corner in (2.20)), and the second comes from the action on $\mathcal{G}^1$, the right upper-corner in (2.20).

We now define the groupoid $\mathcal{G}_L$ corresponding to the stack G_L. The obvious definition would be as $L\mathcal{X} \times_{\mathcal{X}} \mathcal{G}$, but we consider the simpler equivalent groupoid $\mathcal{G}_L \colon (\mathcal{G}_L)^1 \Rightarrow (L\mathcal{X})^0$ where the morphisms are given by the cartesian diagram

$$\begin{tikzcd} (\mathcal{G}_L)^1 \arrow[r] \arrow[d] & \mathcal{G}^1 \arrow[d, "{(r,s)}"] \\ (L\mathcal{X})^1 \arrow[r, "{i\circ r, i\circ s}"] & \mathcal{X}^0 \times \mathcal{X}^0. \end{tikzcd} \tag{2.49}$$

We have a natural homomorphism of groupoids $L\mathcal{G} \to \mathcal{G}_L$ which is an H-principal bundle as expected.

Observe that we can define a groupoid $\tilde{\mathcal{G}} \colon \tilde{\mathcal{G}}^1 \to \tilde{\mathcal{G}}^0 = L\mathcal{G}^0$ by taking the quotient of $\tilde{\mathcal{G}}^1 := (L\mathcal{G})^1$ by the second H-action. In other words, we define $\tilde{\mathcal{G}}^1$ by the cartesian diagram

$$\begin{tikzcd} \tilde{\mathcal{G}}^1 \arrow[r] \arrow[d] & (L\mathcal{G})^0 \arrow[d] \\ \mathcal{X}^1 \arrow[r] & \mathcal{X}^0. \end{tikzcd} \tag{2.50}$$

With the natural induced map $\tilde{\mathcal{G}} \to L\mathcal{X}$ is an H-principal bundle over $L\mathcal{X}$. We compose this map with $L\mathcal{X} \to \mathcal{X}$ and observe that the groupoid structure on $\tilde{\mathcal{G}}$ induces on $\tilde{\mathcal{G}} \to \mathcal{X}$ the structure of a group in groupoids over $\mathcal{X}$. It fits into the central extension

$$\mathcal{X} \times H \to \tilde{\mathcal{G}} \to L\mathcal{X}$$

of groups in $\mathtt{gpd}(\mathtt{Top})/\mathcal{X}$.

The bundle $\tilde{\mathcal{G}} \to L\mathcal{X}$ fits into a cartesian diagram

$$\begin{tikzcd} L\mathcal{G} \arrow[r] \arrow[d] & \tilde{\mathcal{G}} \arrow[d] \\ \mathcal{G}_L \arrow[r] \arrow[ur, Rightarrow, dotted] & L\mathcal{X}. \end{tikzcd}$$

We now pass back to stacks. We interpret the H-principal bundle $\tilde{\mathcal{G}}^0 \to (L\mathcal{X})^0$ as an object $(L\mathcal{X})^0 \to \mathcal{B}H$. The action $(L\mathcal{X})^1 \times_{(L\mathcal{X})^0} \tilde{\mathcal{G}}^0 \to \tilde{\mathcal{G}}^0$ gives the descend[15] datum for completing the following diagram by the dotted arrows:

$$\begin{tikzcd} \tilde{G}^0 \arrow[rr] \arrow[d] & & \tilde{G} \arrow[d, dotted] & \\ (L\mathcal{X})^0 \arrow[rr] \arrow[drr] & & {[(L\mathcal{X})^0/(L\mathcal{X})^1]} \arrow[r, "\cong"] \arrow[d, dotted] & LX \\ & & \mathcal{B}H. & \end{tikzcd}$$

The H-principal bundle in groupoids $\tilde{\mathcal{G}} \to L\mathcal{X}$ thus gives rise to an H-principal bundle in topological stacks $LX \to \mathcal{B}H$ with underlying map of stacks $\tilde{G} \to LX$. In fact, it fits into the cartesian diagram

$$\begin{tikzcd} LG \arrow[r] \arrow[d] & \tilde{G} \arrow[d] \\ G_L \arrow[r] \arrow[ur, Rightarrow, dotted] & LX \end{tikzcd}$$

and the central extension

$$X \times H \to \tilde{G} \to LX$$

in $\mathtt{StTop}/X$.

---

[15]Let $\mathcal{B} \colon \mathcal{B}^1 \Rightarrow \mathcal{B}^0$ be a topological groupoid with quotient stack $[\mathcal{B}^1/\mathcal{B}^0]$. Let U be some stack. A descent datum is a diagram

$$\begin{tikzcd} \mathcal{B}^1 \arrow[r, "s"] \arrow[d, "r"] & \mathcal{B}^0 \arrow[d] \\ \mathcal{B}^0 \arrow[r] \arrow[ur, Rightarrow, dotted] & U, \end{tikzcd}$$

which is compatible with the composition in $\mathcal{B}$ in the obvious way. We use the equivalence of the category $\mathtt{Hom}([\mathcal{B}^1/\mathcal{B}^0], U)$ with the category of descent data.

In order to answer the question whether $\tilde{G} \to LX$ is well-defined up to canonical equivalence we must study how it depends on the choice of the atlas $a\colon A \to X$ and its lift (b, ϕ) (see 2.47). We must show that an automorphism of this datum induces the identity on $\tilde{\mathcal{G}} \to L\mathcal{X}$. Now observe that the automorphism group of (a, b, ϕ) is the group of automorphisms of b which induce the identity on a (in order not to change ϕ). By the definition of an H-banded gerbe it is given by $C(A, H)$. It acts trivially on $\tilde{\mathcal{G}} \to L\mathcal{X}$, indeed.

Finally observe that the construction of $\tilde{G} \to LX$ depends functorially on $G \to X$. We leave the details to the reader. $\square$

We now assume that the stack X is an orbispace. We further assume that $\mathtt{Tors}_n(H) \subseteq H$ is discrete. Let H^δ be the group H equipped with the discrete topology.

Lemma 2.51. *The H-bundle $\phi\colon \tilde{G} \to LX$ admits a natural reduction of structure groups $\phi^\delta\colon \tilde{G}^\delta \to LX$ from H to H^δ.*

Proof. Let T be a space and $* \in T$ be a point. We consider the lifting problem

$$\begin{tikzcd} * \arrow[r, "\sigma"] \arrow[d] & \tilde{G} \arrow[d] \\ T \arrow[r] \arrow[ur, dotted] & LX. \end{tikzcd}$$

We must show that this problem has a unique solution after replacing T by some neighbourhood of $*$, if necessary.

Using an orbispace atlas $A \to X$ we translate the problem to an equivalent lifting problem for topological groupoids

$$\begin{tikzcd} * \arrow[r] \arrow[d] & \tilde{\mathcal{G}} \arrow[d] & \\ T \arrow[r, "t"] \arrow[ur, dotted, "\tilde{t}"] & L\mathcal{A} \arrow[r, "i"] & \mathcal{A}. \end{tikzcd}$$

Here we consider T as a groupoid $T \Rightarrow T$ in the canonical way. Let $\gamma := t(*) \in (L\mathcal{A})^0 \cong \mathcal{A}^a_a$, where $a := i(\gamma) \in \mathcal{A}^0 = A$. Since $\mathcal{A}^a_a$ is a finite group there exists $n \in \mathbb{N}$ such that $\gamma^n = \mathtt{id}_{\mathcal{A}^a_a}$. We consider the embedding $\mathcal{A}^0 \subseteq \mathcal{A}^1$ given by the identities. Using the group structure (Lemma 2.23) of $L\mathcal{A} \to \mathcal{A}$ and the fact that the groupoid $\mathcal{A}$ is étale it follows that $1 \equiv t^n\colon T \to L\mathcal{A}$ after replacing T by some neighbourhood of $*$, if necessary (see the proof of Lemma 2.37 for a similar argument). It follows that $t^n\colon T \to L\mathcal{A}$ has a natural lift $\tilde{t}^n$ given by an H-translate of the identity map such that $\sigma^n = \tilde{t}^n(*)$.

It remains to find the n-th root $\tilde{t}$ of $\tilde{t}^n$. We now consider the diagram

$$\begin{tikzcd} \ker(\dots)^n \arrow[r] \arrow[d] & H \arrow[r, "(\dots)^n"] \arrow[d] & H \arrow[d] \\ \ker(\dots)^n \arrow[r] & \tilde{\mathcal{G}} \times_{L\mathcal{A}} T \arrow[r, "(\dots)^n", "c"'] & \tilde{\mathcal{G}} \times_{L\mathcal{A}} T. \end{tikzcd}$$

The map $c\colon \tilde{\mathcal{G}} \times_{L\mathcal{A}} T \to \tilde{\mathcal{G}} \times_{L\mathcal{A}} T$ is étale. Therefore, after replacing T by some neighbourhood of $*$ again, the datum of σ and $\tilde{t}^n$ give the unique lift $\tilde{t}$. □

For smooth gerbes with band $U(1)$ on orbifolds the analog of Lemma 2.51 was shown e.g. in [**40**] or [**28**]. The argument in these papers uses the existence of a geometric structure (connection and curving) on the gerbe G. This geometry naturally induces a connection on the $U(1)$-principal bundle $\tilde{G} \to LX$. By a calculation the curvature of this connection vanishes. This gives the reduction of structure groups.

Let $g\colon Y \to X$ be a map of topological stacks and $f\colon G \to X$ be a topological gerbe with band H over X. We consider a 2-cartesian diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d] & G \arrow[d] \\ Y \arrow[r, "g"] \arrow[ur, phantom, "\Nearrow"] & X. \end{tikzcd}$$

Lemma 2.52. *We have a 2-cartesian diagram*

$$\begin{tikzcd} \widetilde{K} \arrow[r, "\phi"] \arrow[d] & \tilde{G} \arrow[d] \\ LY \arrow[r, "Lg"] \arrow[ur, phantom, "\Nearrow"] & LX. \end{tikzcd} \tag{2.53}$$

Under the assumptions of Lemma 2.51 this diagrams refines to a 2-cartesian diagram

$$\begin{tikzcd} \tilde{K}^\delta \arrow[r] \arrow[d] & \tilde{G}^\delta \arrow[d] \\ LY \arrow[r, "Lg"] \arrow[ur, phantom, "\Nearrow"] & LX. \end{tikzcd}$$

Proof. We get the square (2.53) from the functoriality part of Proposition 2.46. Since the vertical maps are H-principal bundles it is automatically 2-cartesian. The second statement easily follows from Lemma 2.51. □

2.6. The holonomy of $\tilde{G}^\delta$

Let $G \to X$ be a topological gerbe with band $U(1)$ over an orbispace X. In 2.5 we constructed a $U(1)^\delta$-principal bundle $\tilde{G}^\delta \to LX$. It is an instructive exercise to calculate the holonomy of this bundle in terms of the Dixmier-Douady invariant $d \in H^3(X;\mathbb{Z})$ of the gerbe $G \to X$. In the following we consider a special but typical case of this problem.

We consider a $U(1)$-principal bundle $\pi\colon E \to B$ in orbispaces and a topological gerbe $f\colon G \to E$ with band $U(1)$. Let $h\colon LB \to U(1)^\delta$ be the function associated to the principal bundle $E \to B$ as in Lemma 2.37 and define $LB_1 := h^{-1}(1)$. Then by Lemma 2.39 we have an induced $U(1)$-principal bundle $L\pi\colon LE \to LB_1$. The holonomy of the bundle $\tilde{G}^\delta \to LE$ along the fibres of $L\pi$ gives rise to a function

$$g\colon LB_1 \to U(1)^\delta$$

(see 2.55 for a precise construction).

The gerbe $f\colon G \to B$ is classified by a Dixmier-Douady class $d \in H^3(E;\mathbb{Z})$. Let $\pi_!\colon H^3(E;\mathbb{Z}) \to H^2(B;\mathbb{Z})$ be the integration map. According to (2.42) the class $\pi_!(d) \in H^2(B;\mathbb{Z})$ gives rise to a function

$$\overline{\pi_!(d)}\colon LB \to U(1)^\delta .$$

The main result of the present subsection is the following proposition.

Proposition 2.54. *We have the equality*

$$g = \overline{\pi_!(d)}_{|LB_1} .$$

Construction 2.55. Here is the precise construction of the function $g\colon LB_1 \to U(1)^\delta$. Let T be a space and $T \to LB_1$ be a map. The pull-back

$$\begin{tikzcd} W \arrow[r] \arrow[d] & \tilde{G}^\delta \arrow[d] \\ S \arrow[r] \arrow[d] \arrow[ur, dotted] & LE \arrow[d] \\ T \arrow[r] \arrow[ur, dotted] & LB_1 \end{tikzcd}$$

defines a $U(1)$-principal bundle $S \to T$ and a $U(1)^\delta$-principal bundle $W \to S$. We choose an open covering $(T_\alpha \to T)_{\alpha \in I}$ such that for all $\alpha \in I$ there exists a section

$$\begin{tikzcd} & S \arrow[d] \\ T_\alpha \arrow[r] \arrow[ur, dotted, "s_\alpha"] & T. \end{tikzcd}$$

The section s_α gives rise to a map $T_\alpha \times \mathbb{R} \to S$ by $(t,x) \mapsto s_\alpha(t)x$, where $\mathbb{R}$ acts on S via the covering $\mathbb{R} \to U(1)$. We can now (after refining the covering $(T_\alpha \to T)$ if necessary) choose a lift

$$\begin{tikzcd} & W \arrow[d] \\ T_\alpha \times \mathbb{R} \arrow[r] \arrow[ur, dotted, "w_\alpha"] & S. \end{tikzcd}$$

Then we define a map $g_{T_\alpha}\colon T_\alpha \to U(1)^\delta$ such that $w_\alpha(t,0) = w_\alpha(t,1)g_{T_\alpha}(t)$. Observe that g_{T_α} does not depend on the choices of s_α and w_α. One easily checks that the family of maps $(g_{T_\alpha})_{\alpha \in I}$ determines a map $g_T\colon T \to U(1)^\delta$ which depends functorially on $T \to LB_1$. It therefore defines a map $g\colon LB_1 \to U(1)^\delta$.

We now turn to the actual proof of Proposition 2.54. We first consider a special case. Let Γ be a finite cyclic group which we write additively. We let Γ act trivially on $U(1)$ and consider the orbispace $E := [U(1)/\Gamma]$. The projection $U(1) \to *$ induces a $U(1)$-principal bundle $\pi\colon E \to B := [*/\Gamma]$. We calculate $H^3(E;\mathbb{Z})$ using the Künneth formula and the product decomposition $E = U(1) \times B$. Note that $H^*(B;\mathbb{Z}) \cong$

$H^*(\Gamma;\mathbb{Z})$. In particular we have $H^3(B;\mathbb{Z}) \cong 0$ and a canonical isomorphism $H^2(B;\mathbb{Z}) \cong \hat{\Gamma}$. It follows that

$$H^3(E;\mathbb{Z}) \cong H^1(U(1);\mathbb{Z}) \otimes H^2(B;\mathbb{Z}) \cong \hat{\Gamma}, \tag{2.56}$$

using the canonical orientation $H^1(U(1);\mathbb{Z}) \cong \mathbb{Z}$ of $U(1)$.

The group $H^3(E;\mathbb{Z})$ classifies topological $U(1)$-gerbes over E. In the following we present a construction which associates to every character $\phi \in \hat{\Gamma}$ a $U(1)$-gerbe $G_\phi \to E$. We construct these gerbes in terms of representing groupoids.

The canonical covering $\mathbb{R} \to U(1)$ induces an atlas $\mathbb{R} \to E$. The corresponding topological groupoid is the action groupoid for the action of $\mathbb{Z} \times \Gamma$ on $\mathbb{R}$ by $(n,\gamma)t := t+n$. It is given by

$$\mathbb{R} \times \mathbb{Z} \times \Gamma \rightrightarrows \mathbb{R} \tag{2.57}$$

with range $r(t,n,\gamma) := t+n$, source $s(t,n,\gamma) := t$, and the composition $(t+m,n,\gamma) \bullet (t,m,\gamma') := (t,n+m,\gamma+\gamma')$.

The character $\phi \in \hat{\Gamma}$ determines a $U(1)$-central extension

$$0 \to U(1) \to \widehat{\mathbb{Z} \times \Gamma}_\phi \to \mathbb{Z} \times \Gamma \to 0. \tag{2.58}$$

If we identify $\widehat{\mathbb{Z} \times \Gamma}_\phi \cong \mathbb{Z} \times \Gamma \times U(1)$ as sets, then the multiplication is given by $(n,\gamma,z)(n',\gamma',z') = (n+n',\gamma+\gamma',\phi(\gamma)^{n'}zz')$. This central extension acts on $\mathbb{R}$ via its projection $\widehat{\mathbb{Z} \times \Gamma}_\phi \to \mathbb{Z} \times \Gamma$, $(n,\gamma,z) \mapsto (n,\gamma)$. The gerbe $G_\phi \to E$ is then given by

$$[\mathbb{R}/\widehat{\mathbb{Z} \times \Gamma}_\phi] \to [\mathbb{R}/\mathbb{Z} \times \Gamma].$$

In terms of groupoids, G_ϕ is given as the $U(1)$-central extension of the groupoid (2.57) which on the level of morphisms is the trivial $U(1)$-bundle

$$\mathbb{R} \times \mathbb{Z} \times \Gamma \times U(1) \to \mathbb{R} \times \mathbb{Z} \times \Gamma,$$

whose source and range maps are

$$s(t,n,\gamma,z) := t, \quad r(t,n,\gamma,z) := t+n,$$

and whose composition is given by

$$(t+m,n,\gamma,z')(t,m,\gamma',z) := (t,n+m,\gamma+\gamma',\phi(\gamma)^m z'z).$$

We now calculate the bundle $\tilde{G}^\delta_\phi \to LE$. First of all note that

$$LE \cong [\Gamma \times U(1)/\Gamma],$$

where Γ acts trivially on $\Gamma \times U(1)$. The map $\Gamma \times \mathbb{R} \to \Gamma \times U(1)$ gives an atlas of LE. The associated groupoid is the action groupoid of the action of $\mathbb{Z} \times \Gamma$ on $\Gamma \times \mathbb{R}$ by $(n,\gamma)(\sigma,t) = (\sigma,t+n)$. It is given by

$$\Gamma \times \mathbb{R} \times \mathbb{Z} \times \Gamma \rightrightarrows \Gamma \times \mathbb{R}$$

with range and source given by

$$r(\sigma,t,n,\gamma) := (\sigma,t+n), \quad s(\sigma,t,n,\gamma) := (\sigma,t),$$

and with the composition

$$(\sigma,t+m,n,\gamma) \circ (\sigma,t,m,\gamma') := (\sigma,t,n+m,\gamma+\gamma').$$

We can now read off a groupoid presentation of the $U(1)^\delta$-principal bundle $\tilde{G}^\delta_\phi \to LE$. It is presented by the $U(1)^\delta$-principal bundle in groupoids

$$\begin{array}{ccc} \Gamma \times \mathbb{R} \times U(1)^\delta \times \mathbb{Z} \times \Gamma & \Longrightarrow & \Gamma \times \mathbb{R} \times U(1)^\delta \\ \downarrow & & \downarrow \\ \Gamma \times \mathbb{R} \times \mathbb{Z} \times \Gamma & \Longrightarrow & \Gamma \times \mathbb{R}. \end{array}$$

The range and source maps in the upper horizontal line are given by

$$r(\sigma, t, z, n, \gamma) := (\sigma, t + n, \phi(\sigma)^n z), \quad s(\sigma, t, z, n, \gamma) := (\sigma, t, z),$$

and with the composition

$$(\sigma, t + m, \phi(\sigma)^m z, n, \gamma) \circ (\sigma, t, z, n, \gamma') := (\sigma, t, z, n + m, \gamma + \gamma').$$

In particular, the holonomy of $\tilde{G}^\delta_\phi$ along the fibre of LE over $[\{\sigma\}/\Gamma]$ is given by $\phi(\sigma)$.

In our example we have $LB_1 = [\Gamma/\Gamma] = LB$, where Γ acts trivially on itself. The function $g_\phi \colon LB_1 \to U(1)^\delta$, which measures the holonomy of $\tilde{G}_\phi \to LE$ along the fibres of $LE \to LB_1$, is given by the calculation above by

$$g_\phi = \phi \colon \Gamma \to U(1)^\delta. \tag{2.59}$$

By (2.56) the character ϕ gives rise to a class $d_\phi \in H^3(E; \mathbb{Z})$ such that

$$\pi_!(d_\phi) = \phi$$

(using the isomorphism $\hat{\Gamma} \cong H^2(B; \mathbb{Z})$). Furthermore we have the function

$$\overline{\pi_!(d_\phi)} = \phi \colon \Gamma \to U(1)^\delta$$

defined in (2.42).

In order to finish the proof of Proposition 2.54 in the special case we must show that d_ϕ is the Dixmier-Douady class $d(G_\phi)$ of G_ϕ. We will use the following two general facts:

(1) Let
$$1 \to U(1) \to \hat{G} \to G \to 1$$
be a $U(1)$-central extension of a discrete group G classified by
$$e \in \mathtt{Ext}(G; U(1)) := H^2(G; U(1)).$$
Furthermore, let $\delta \colon H^2(G; U(1)) \to H^3(G; \mathbb{Z})$ be the boundary operator in the Bockstein sequence in group cohomology associated to the exact sequence of coefficients $0 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 0$. Then the Dixmier-Douady class of the gerbe $[*/\hat{G}] \to [*/G]$ is given by the image of $\delta(e) \in H^3(G; \mathbb{Z})$ under the isomorphism $H^3(G; \mathbb{Z}) \cong H^3([*/G]; \mathbb{Z})$.

(2) Let $\phi \colon G \to U(1)$ be a character of a finite group G. It gives rise to a class $\phi \in H^1(G; U(1))$ and an extension $1 \to U(1) \to \widehat{\mathbb{Z} \times G} \to \mathbb{Z} \times G \to 1$. We can identify $\widehat{\mathbb{Z} \times G} \cong \mathbb{Z} \times G \times U(1)$ as sets. Its multiplication is then given by
$$(n, g, z)(n', g', z') = (n + n', gg', \phi(g)^{n'} zz').$$
The class $e \in \mathtt{Ext}(\mathbb{Z} \times G; U(1))$ of the extension is then given by image of $\mathtt{id}_{\mathbb{Z}} \times$

$\phi \in H^1(\mathbb{Z};\mathbb{Z}) \times H^1(G;U(1))$ under the product

$$\times \colon H^1(\mathbb{Z};\mathbb{Z}) \times H^1(G;U(1)) \to H^2(\mathbb{Z}\times G;U(1)),$$

where $\mathtt{id}_{\mathbb{Z}} \in H^1(\mathbb{Z};\mathbb{Z})$ is the identity homomorphism.

We now specialize these facts to the present situation. The Künneth formula gives an isomorphism

$$\mathtt{Ext}(\mathbb{Z}\times\Gamma;U(1)) := H^2(\mathbb{Z}\times\Gamma;U(1)) \cong H^1(\mathbb{Z};\mathbb{Z}) \otimes H^1(\Gamma;U(1)) \cong \mathbb{Z}\otimes\hat{\Gamma} \cong \hat{\Gamma}, \tag{2.60}$$

where we use the generator $\mathtt{id}_{\mathbb{Z}} \in H^1(\mathbb{Z};\mathbb{Z})$ in order to identify $H^1(\mathbb{Z};\mathbb{Z}) \cong \mathbb{Z}$. The class $e_\phi \in \mathtt{Ext}(\mathbb{Z}\times\Gamma;U(1))$ of the extension (2.58) corresponds under this isomorphism to $\phi \in \hat{\Gamma}$ (by (2)).

By (1) the Dixmier-Douady class $d(\phi) \in H^3([*/(\mathbb{Z}\times\Gamma)];\mathbb{Z})$ of the gerbe

$$[*/\widehat{\mathbb{Z}\times\Gamma}_\phi] \to [*/(\mathbb{Z}\times\Gamma)]$$

corresponds to

$$\delta(e_\phi) \in H^3(\mathbb{Z}\times\Gamma;\mathbb{Z})$$

under the identification

$$H^3([*/(\mathbb{Z}\times\Gamma)];\mathbb{Z}) \cong H^3(\mathbb{Z}\times\Gamma;\mathbb{Z}).$$

Let $p\colon B = [\mathbb{R}/(\mathbb{Z}\times\Gamma)] \to [*/(\mathbb{Z}\times\Gamma)]$ be the canonical projection. Then we have $d(G_\phi) = p^* d(\phi)$. We now observe that the following diagram commutes,

$$\begin{tikzcd}
d(\phi) \arrow[d, mapsto, bend left] & & & e_\phi \arrow[lll, mapsto, bend left] \arrow[dddd, mapsto, bend right=60, "(2.60)"'] \\
& H^3([*/(\mathbb{Z}\times\Gamma)];\mathbb{Z}) \arrow[d, "\cong"', "p^*"] & H^3(\mathbb{Z}\times\Gamma;\mathbb{Z}) \arrow[l, "\cong"] & H^2(\mathbb{Z}\times\Gamma;U(1)) \arrow[l, "\cong", "\delta"'] \arrow[d, "\cong"] \\
d(G_\phi) \arrow[dd, mapsto, bend left] & H^3([\mathbb{R}/(\mathbb{Z}\times\Gamma)];\mathbb{Z}) \arrow[d, "\pi_!"] & & H^1(\mathbb{Z};\mathbb{Z})\otimes H^1(\Gamma;U(1)) \arrow[d, "\cong"] \\
& H^2([*/\Gamma];\mathbb{Z}) & H^2(\Gamma;\mathbb{Z}) \arrow[l, "\cong"] & H^1(\Gamma;U(1)) \arrow[l, "\delta"] \\
\pi_!(d_\phi) & & & \phi \arrow[lll, mapsto, bend right, "\text{definition of } d_\phi"']
\end{tikzcd}$$

and that the elements are mapped as indicated.

We show how the general case of Proposition 2.54 can be reduced to the special case discussed above. The constructions of g and $\overline{\pi_!(d)}_{|LB_1}$ are natural with respect to pull-back. Therefore in order to verify Proposition 2.54 it suffices to show the desired equality over each point in LB separately. A point $u \in LB$ is given by a point $p \in B$ and an element $\gamma \in \mathtt{Aut}(p)$ (in the present subsection we omit the subscript u in order to simplify the notation). Let $\Gamma \subset \mathtt{Aut}(p)$ be the cyclic group generated by

γ and $\chi \in \hat{\Gamma}$ be the character by which Γ acts on the fibre $\pi^{-1}(p)$. Note that

$$\chi(\gamma) = h(u). \tag{2.61}$$

We get a cartesian diagram

$$\begin{tikzcd} v^*G \arrow[r] \arrow[d] & G \arrow[d, "f"] \\ {[U(1)/_\chi\Gamma]} \arrow[r, "v"] \arrow[d, "q"] & E \arrow[d, "\pi"] \\ {[*/\Gamma]} \arrow[r, "\tilde{u}"] & B \end{tikzcd} \tag{2.62}$$

such that $L\tilde{u}(\gamma) = u$, where we consider $\gamma \in [\Gamma/\Gamma] \cong L[*/\Gamma]$. In particular, v^*d is the Dixmier-Douady class of the gerbe $v^*G \to [U(1)/_\chi\Gamma]$ and we have

$$\overline{\pi_!(d)}(u) = \overline{q_! v^*(d)}(\gamma).$$

Observe that $L[*/\Gamma]_1 = [\ker(\chi)/\Gamma]$. Let $g_{v^*G}\colon L[*/\Gamma]_1 \to U(1)^\delta$ denote the function (2.55) which measures the holonomy of $\widetilde{v^*G}^\delta \to L[U(1)/_\chi\Gamma]$ along the fibres of q. If $u \in LB_1$, then by 2.61 we have $\chi(\gamma) = 1$ and

$$g(u) = g_{v^*G}(\gamma).$$

The equation

$$\overline{\pi_!(d)}(u) = g(u)$$

now follows from the equation

$$g_{v^*G}(\gamma) = \overline{q_! v^*(d)}(\gamma),$$

which was already shown above. □

In the smooth case (i.e. for orbifolds) holonomy questions can be addressed using Deligne cohomology. In fact, Deligne cohomology $H^*_{\mathrm{Del}}(X)$ for orbifolds has been introduced in [**27**]. The choice of a connection on the gerbe G leads to a lift of the Dixmier-Douady class $d \in H^3(X;\mathbb{Z})$ of $G \to X$ to a Deligne cohomology class $d_{\mathrm{Del}} \in H^3_{\mathrm{Del}}(X)$ under the natural forgetful map $H^3_{\mathrm{Del}}(X) \to H^3(X;\mathbb{Z})$. The transgression of d_{Del} according to [**27**, Thm. 6.0.2] is a class $\mathtt{Tr}(d_{\mathrm{Del}}) \in H^2_{\mathrm{Del}}(LX)$. Its integral $(L\pi)_!(\mathtt{Tr}(d_{\mathrm{Del}})) \in H^1_{\mathrm{Del}}(LX_1)$ should[16] give the function $g\colon LX_1 \to U(1)$.

3. Delocalized cohomology of orbispaces and orbifolds

3.1. Definition of delocalized twisted cohomology

A topological stack X gives rise to a site $\mathtt{Site}(X) = \mathbf{X}$. The underlying category of $\mathbf{X}$ is the subcategory of $\mathtt{Top}/X$ of maps $(U \to X)$ which are representable and have local sections. The covering families $(U_i \to U)$ are families of maps $U_i \to U$ in

[16] We have not checked the details here. In this picture it is also not obvious that $(L\pi)_!(\mathtt{Tr}(d_{\mathrm{Del}}))$ only depends on $d \in H^3(X;\mathbb{Z})$, and not on the choice of its lift $d_{\mathrm{Del}} \in H^3_{\mathrm{Del}}(X)$.

$\mathbf{X}$ which have local sections[17] and are such that $\sqcup_i U_i \to U$ is surjective. One can actually restrict to covering families by open subsets without changing the induced topology (the argument is similar as for [**13**, Lemma 2.47]). If X is a space, then the small site (X) of X is the category of open subsets of X with the usual notion of covering families.

To the site $\mathbf{X}$ we associate categories of presheaves and sheaves $\mathtt{PSh}\mathbf{X}$ and $\mathtt{Sh}\mathbf{X}$ in the usual way. A map $p\colon X \to Y$ of topological stacks induces a pair of adjoint functors

$$p^*\colon \mathtt{Sh}\mathbf{Y} \Leftrightarrow \mathtt{Sh}\mathbf{X}\colon p_*.$$

We use this framework of sheaf theory on topological stacks in order to define the delocalized cohomology of an orbispace twisted by a gerbe.

For details of the sheaf theory we refer to [**13**] and [**11**].

For a site $\mathbf{X}$ let $i\colon \mathtt{Sh}\mathbf{X} \to \mathtt{PSh}\mathbf{X}$ denote the canonical embedding of the category of presheaves into the category of sheaves, and let $i^\sharp\colon \mathtt{PSh}\mathbf{X} \to \mathtt{Sh}\mathbf{X}$ denote its left-adjoint, the sheafification functor. We use the same symbols in order to denote the restriction of these functors to the categories $\mathtt{PSh}_{\mathtt{Ab}}\mathbf{X}$ and $\mathtt{Sh}_{\mathtt{Ab}}\mathbf{X}$ of presheaves and sheaves of abelian groups.

Construction 3.1. Let H be a topological abelian group. We assume that $\mathtt{Tors}_n(H) \subseteq H$ is discrete for all $n \in \mathbb{N}$. Let H^δ denote the group H with the discrete topology. Furthermore, let Z be a discrete abelian group and $\lambda\colon H^\delta \to \mathtt{Aut}(Z)$ be a homomorphism.

Let $P \to X$ be the underlying map of stacks of an H^δ-principal bundle over a topological stack X. If $(U \to X) \in \mathbf{X}$, then $U \times_X P \to U$ is an ordinary H^δ-principal bundle. We define the abelian group $\mathcal{Z}_{P,\lambda}(U)$ to be the group of continuous sections of the associated bundle $(U \times_X P) \times_{H^\delta,\lambda} Z \to U$ under pointwise multiplication. If $(U' \to X) \to (U \to X)$ is a morphism in $\mathbf{X}$, then we have an induced morphism $U' \times_X P \to U \times_X P$ of H^δ-principal bundles over $U' \to U$. It induces a homomorphism $\mathcal{Z}_{P,\lambda}(U) \to \mathcal{Z}_{P,\lambda}(U')$. In this way we obtain a presheaf of abelian groups $\mathcal{Z}_{P,\lambda} \in \mathtt{PSh}_{\mathtt{Ab}}\mathbf{X}$, $U \mapsto \mathcal{Z}_{P,\lambda}(U)$. Note that $\mathcal{Z}_{P,\lambda}$ is actually a sheaf; i.e. we have $\mathcal{Z}_{P,\lambda} \in \mathtt{Sh}_{\mathtt{Ab}}\mathbf{X}$.

Let $f\colon G \to X$ be a gerbe with band H over an orbispace X. Then by Lemma 2.51 we have the H^δ-principal bundle $\tilde{G}^\delta \to LX$. By the construction 3.1 it gives rise to the presheaf $\mathcal{Z}_{\tilde{G}^\delta,\lambda} \in \mathtt{PSh}_{\mathtt{Ab}}\mathbf{LX}$.

We define a gerbe $f_L\colon G_L \to LX$ with band H as the pull-back of the gerbe $f\colon G \to X$ along the canonical map $i\colon LX \to X$ (see 2.44). We have a diagram

$$\begin{tikzcd} * & G_L \arrow[l, "p"'] \arrow[r] \arrow[d, "f_L"] & G \arrow[d, "f"] \\ & LX \arrow[r, "i"] \arrow[ur, dotted] & X. \end{tikzcd}$$

We consider $f_L^*\mathcal{Z}_{\tilde{G}^\delta,\lambda} \in \mathtt{Sh}_{\mathtt{Ab}}\mathbf{G_L}$.

[17]A map of topological spaces $f\colon V \to W$ has local sections if for every $w \in f(V)$ there exists an open neighbourhood $W_w \subseteq W$ and a map $\sigma\colon W_w \to V$ such that $\mathtt{id}_{W_w} = f \circ \sigma$.

Let $\mathtt{ev} := \mathtt{ev}_{*\to *}\colon \mathtt{Sh}_{\mathtt{Ab}}\mathtt{Site}(*) \to \mathtt{Ab}$ be the functor, which evaluates a sheaf of abelian groups on $\mathtt{Site}(*)$ at the object $(* \to *) \in \mathtt{Site}(*)$.

Lemma 3.2. *The functor* $\mathtt{ev}\colon \mathtt{Sh}_{\mathtt{Ab}}\mathtt{Site}(*) \to \mathtt{Ab}$ *is exact.*

Proof. A basic observation lying at the heart of sheaf theory is that evaluation functors are not exact in general. Therefore, a proof of exactness of the evaluation $\mathtt{ev}$ is required. First note that $\mathtt{Site}(*)$ is the big site of $*$ which can be identified with the category of all topological spaces. Every non-empty collection of non-empty spaces is a covering family of $*$.

The small site $(*)$ of $*$ has one object $* \to *$. In [**11**] (see also [**13**, Prop. 2.46], the arguments works equally well in the smooth and topological contexts) we have seen that the restriction functor $\nu_*\colon \mathtt{ShSite}(*) \to \mathtt{Sh}(*)$ is exact. Let $\tilde{\mathtt{ev}}\colon \mathtt{Sh}_{\mathtt{Ab}}(*) \to \mathtt{Ab}$ denote the corresponding evaluation functor. It is actually an isomorphism of categories, and in particular exact. We have $\tilde{\mathtt{ev}} \circ \nu_* \cong \mathtt{ev}$. We see that $\mathtt{ev}$ is exact, since it is a composition of exact functors. □

The functor $p_*\colon \mathtt{Sh}_{\mathtt{Ab}}(\mathbf{G_L}) \to \mathtt{Sh}_{\mathtt{Ab}}\mathtt{Site}(*)$ is left-exact and thus admits right-derived functors

$$Rp_*\colon D^+(\mathtt{Sh}_{\mathtt{Ab}}\mathbf{G_L}) \to D^+(\mathtt{Sh}_{\mathtt{Ab}}\mathtt{Site}(*))$$

between the lower-bounded derived categories. The functor $ev\colon \mathtt{Sh}_{\mathtt{Ab}}\mathtt{Site}(*) \to \mathtt{Ab}$ is exact and thus descends to the lower-bounded derived categories.

Definition 3.3. We define the delocalized G-twisted cohomology of X with coefficients in (Z, λ) by

$$H^*_{\mathrm{deloc}}(X; G, (Z, \lambda)) := H^*(\mathtt{ev} \circ Rp_*(f_L^* \mathcal{Z}_{\tilde{G}^\delta,\lambda})).$$

The most important example for us is the case where $Z = \mathbb{C}^\delta$ and $H = U(1)$ with $\lambda\colon H^\delta \to Z \to \mathtt{End}(Z)$ being the obvious embedding $U(1)^\delta \to \mathtt{End}(\mathbb{C}^\delta)$. Recall the construction 3.1 of $\mathcal{Z}_{\tilde{G}^\delta,\lambda}$.

Definition 3.4. We define the sheaf $\mathcal{L} := \mathcal{Z}_{\tilde{G}^\delta,\lambda}$.

We will also write $\mathcal{L}_G$ for $\mathcal{L}$, if a reference to G is necessary.

Definition 3.5. The G-twisted complex delocalized cohomology of X is defined by

$$H^*_{\mathrm{deloc}}(X; G) := H^*_{\mathrm{deloc}}(X; G, \mathcal{L}).$$

Another example related to Spin-structures is the case where $Z = \mathbb{Z}$, $H = \mathbb{Z}^* = \{1, -1\}$, and $\lambda\colon \mathbb{Z}^* \to \mathtt{End}(\mathbb{Z})$ is again the canonical embedding.

We now discuss the functorial behaviour of the delocalized twisted cohomology. We defined the sheaf $\mathcal{Z}_{\tilde{G}^\delta,\lambda}$ on LX in order to connect with usual conventions in the literature on inner local systems and twisted torsion, and in order to have the formula (3.12) below. This construction depends on descending the H^δ-bundle $LG \to G_L$ to the bundle $\tilde{G} \to LX$. The quite complicated construction was carried out in Proposition 2.46. In the definition of twisted cohomology we then use the pull-back $f_L^* \mathcal{Z}_{\tilde{G}^\delta,\lambda}$.

It would be more natural to construct the sheaf

$$\tilde{\mathcal{Z}}_{LG^\delta,\lambda} \cong f_L^* \mathcal{Z}_{\tilde{G}^\delta,\lambda} \tag{3.6}$$

directly starting from the H^δ-principal bundle $LG^\delta \to G_L$. We can proceed as in the definition of $\mathcal{Z}_{\tilde{G}^\delta,\lambda}$. For an object $(U \to G_L) \in \mathbf{G_L}$ we define $\tilde{\mathcal{Z}}_{LG^\delta;\lambda}(U) \in \mathtt{Ab}$ as the group of continuous sections of $(U \times_{G_L} LG^\delta) \times_{H^\delta,\lambda} Z$ under pointwise multiplication. For a morphism $U' \to U$ we then have a natural homomorphism $\tilde{\mathcal{Z}}_{LG^\delta;\lambda}(U) \to \tilde{\mathcal{Z}}_{LG^\delta;\lambda}(U')$ induced by a corresponding morphism of principal bundles over $U' \to U$.

We have a canonical isomorphism

$$H_{\mathrm{deloc}}(X; G, (Z,\lambda)) \cong H^*(\mathtt{ev} \circ Rp_*(\tilde{\mathcal{Z}}_{LG^\delta,\lambda})).$$

In the case $H = U(1)$ and $Z = \mathbb{C}^\delta$ we set $\tilde{\mathcal{Z}}_{LG^\delta,\lambda} := \tilde{\mathcal{L}}$.

We consider a 2-cartesian diagram

$$\begin{tikzcd} G' \arrow[r, "h"] \arrow[d, "f'"] & G \arrow[d, "f"] \\ X' \arrow[r, "g"] \arrow[ur, Rightarrow, dotted] & X, \end{tikzcd}$$

where g is a map of orbispaces, i.e. a representable map.

Lemma 3.7. *We have a canonical functorial map*

$$(g,h)^*\colon H^*_{\mathrm{deloc}}(X;G) \to H^*_{\mathrm{deloc}}(X';G').$$

Proof. Since the loop functor preserves two-cartesian diagrams we get an induced 2-cartesian diagram

$$\begin{tikzcd} LG'^\delta \arrow[r, "Lh"] \arrow[d] & LG \arrow[d] \\ G'_L \arrow[r, "h_L"] \arrow[d, "f'_L"] \arrow[ur, Rightarrow, dotted] & G_L \arrow[d, "f_L"] \\ LX' \arrow[r, "Lg"] \arrow[ur, Rightarrow, dotted] & LX. \end{tikzcd} \tag{3.8}$$

Let $\tilde{\mathcal{L}} = \tilde{\mathcal{Z}}_{LG^\delta,\lambda} \in \mathtt{Sh}_{\mathtt{Ab}}\mathbf{G_L}$ and $\tilde{\mathcal{L}}' := \tilde{\mathcal{Z}}_{LG'^\delta,\lambda}\mathtt{Sh}_{\mathtt{Ab}}\mathbf{G_L}'$ denote the sheaves of abelian groups associated to G and G' and (Z,λ) as in (3.6). Diagram (3.8) induces an isomorphism

$$h_L^*\tilde{\mathcal{L}} \xrightarrow{\sim} \tilde{\mathcal{L}}' \tag{3.9}$$

of sheaves on G'_L. We now consider the diagram

$$\begin{tikzcd} G'_L \arrow[rr, "h_L"] \arrow[dr, "p'"'] & & G_L \arrow[dl, "p"] \\ & * & \end{tikzcd} .$$

The unit $\mathtt{id} \to R(h_L)_* \circ h_L^*$ of the adjoint pair

$$h_L^* \colon D^+(\mathtt{Sh}_{\mathtt{Ab}}\mathbf{G_L}) \Leftrightarrow D^+(\mathtt{Sh}_{\mathtt{Ab}}\mathbf{G_L}') \colon R(h_L)_*$$

induces a natural transformation

$$Rp_* \to Rp_* \circ R(h_L)_* \circ h_L^* \colon D^+(\mathtt{Sh}_{\mathtt{Ab}}\mathbf{G_L}) \to D^+(\mathtt{Sh}_{\mathtt{Ab}}\mathtt{Site}(*)). \tag{3.10}$$

Since $p \circ h_L = p'$ and $Rp_* \circ R(h_L)_* \cong R(p \circ h_L)_*$ (see [**11**] and also [**13**, Lemma 2.26] for an argument in the smooth case) we have an isomorphism

$$Rp_* \circ R(h_L)_* \cong R(p \circ h_L)_* \cong Rp'_*.$$

We insert this into (3.10) and get the natural transformation

$$Rp_* \to Rp'_* \circ h_L^* \colon D^+(\mathtt{Sh}_{\mathtt{Ab}}\mathbf{G_L}) \to D^+(\mathtt{Sh}_{\mathtt{Ab}}\mathtt{Site}(*)). \tag{3.11}$$

We define

$$(g,h)^* \colon H^* \circ \mathtt{ev} \circ Rp_*(\tilde{\mathcal{L}}) \xrightarrow{(3.11)} H^* \circ \mathtt{ev} \circ Rp'_* \circ h_L^*(\tilde{\mathcal{L}}) \overset{(3.9)}{\cong} H^* \circ \mathtt{ev} \circ Rp'_*(\tilde{\mathcal{L}}').$$

We leave it to the reader to write out the argument for functoriality. The basic input is the functoriality of the units for a composition $f \circ g$ which can be expressed as the commutativity of

$$\begin{tikzcd} \mathtt{id} \arrow[r] \arrow[rrr, bend left] & Rf_* \circ f^* \arrow[r] & Rf_* \circ Rg_* \circ g^* \circ f^* \arrow[r, "\cong"] & R(f \circ g)_* \circ (f \circ g)^* \end{tikzcd}$$

(see [**11**] for a proof). □

From now on we consider the case $H := U(1)$ and $Z := \mathbb{C}^\delta$. We can decompose $p = q \circ f_L$, where $q \colon LX \to *$. Since f_L has local sections we have an isomorphism

$$Rp_* \cong Rq_* \circ R(f_L)_*$$

by [**13**, 2.26]. We have

$$Rp_* \circ f_L^*(\mathcal{L}) \cong Rq_* \circ R(f_L)_* \circ f_L^*(\mathcal{L})$$

and the projection formula (see [**11**])

$$R(f_L)_*(\tilde{\mathcal{L}}) \cong R(f_L)_* \circ f_L^*(\mathcal{L}) \cong R(f_L)_*(\mathbb{C}_{\mathbf{G}_L}) \otimes_{\mathbb{C}} \mathcal{L}. \tag{3.12}$$

Therefore we can write

$$H^*_{\mathrm{deloc}}(X;G) \cong H^*(\mathtt{ev} \circ Rq_* \circ (R(f_L)_*(\mathbb{C}_{\mathbf{G}_I}) \otimes_{\mathbb{C}} \mathcal{L})). \tag{3.13}$$

3.2. Twisted de Rham cohomology

The theory developed in the Sections 2.2, 2.3, 2.5 and 3.1 has a counterpart in the world of stacks in smooth manifolds though there is one essential difference. The map $LX \to X$ is not representable as a map of stacks in smooth manifolds. Therefore the proof of the fact that LX is a smooth stack is quite different from the topological case[18]. But note that we have not used representability of $LX \to X$ otherwise.

[18] One could save our argument by introducing the notion of a smoothly representable map between stacks in smooth manifolds and showing that $LX \to X$ is smoothly representable. A map $X \to Y$ between stacks in smooth manifolds is called smoothly representable, if the fibre product $A \times_Y X$ is a manifold for every *submersion* $A \to Y$.

In the following we explain the replacements which lead to a precisely analogous theory.

(1) The category of topological spaces `Top` is replaced by the category of $\mathtt{Mf}^\infty$ of smooth manifolds.
(2) Stacks in topological spaces are replaced by stacks in manifolds.
(3) The condition on a map of having local sections is replaced by the condition of being a submersion (following the conventions from algebraic geometry we will use the term "smooth" synonymously with "submersion").
(4) Topological stacks are replaced by smooth stacks. A stack in smooth manifolds X is called smooth if it admits an atlas $a\colon A \to X$, i.e. a representable, surjective, and submersive (which replaces the local section condition by the preceding point) map from a manifold A.
(5) The notion of a topological groupoid is replaced by the notion of a Lie groupoid. In particular, we require that range and source maps are submersions.
(6) Orbispaces are replaced by orbifolds. A smooth stack is an orbifold if it admits an orbifold atlas. An orbifold atlas is an atlas which gives rise to a proper and étale groupoid in smooth manifolds. Since manifolds are locally compact and Hausdorff the conditions "separated" and "very proper"[19] hold automatically (see Lemma 2.32).
(7) The group H in 3.1 must be a Lie group.
(8) For a smooth stack X the site $\mathbf{X}$ is the subcategory of $\mathtt{Mf}^\infty/X$ of maps $(U \to X)$, which are representable submersions. The covering families are families $(U_i \to U)$ of submersions such that $\sqcup_i U_i \to U$ is surjective.

One problem with the category $\mathtt{Mf}^\infty$ is that fibre products only exist under additional conditions (e.g. if one map is a submersion). We leave it to the interested reader to check that all fibre products used in Sections 2.3, 2.5 and 3.1 exists in manifolds.[20]

Let X be an orbifold and $G \to X$ be a smooth gerbe with band $U(1)$. Then by 3.5 we have a well-defined twisted delocalized cohomology

$$H^*_{\mathrm{deloc}}(X; G).$$

The main goal of the present section is to calculate this cohomology in terms of a twisted de Rham complex. This generalizes the main result of [**13**] from smooth manifold X to orbifolds X.

The first goal of the present subsection is to define the de Rham complex associated to a locally constant sheaf of complex vector spaces on an orbifold in two equivalent (according to Lemma 3.17) ways. In the first picture we define a sheaf of de Rham complexes on the site of the orbifold and then take its global sections. The second picture uses the calculus of differential forms on the orbifold itself. While the first picture belongs to the philosophy of the present paper this second definition is mainly used to compare with other constructions in the literature.

[19]The condition "very proper" is as in 2.30 with the difference that the cut-off function must be smooth.
[20]with the exception that the construction of the simpler model of $L\mathcal{X} \times_{\mathcal{X}} \mathcal{G}$ in 2.5 needs different arguments since (2.49) may not be a transversal pull-back.

In the second part we apply this construction to the local system $\mathcal{L} \in \mathtt{Sh_{Ab}}\mathbf{LX}$ associated to a $U(1)$-gerbe $G \to X$ on an orbifold.

Consider a smooth stack X in smooth manifolds. Let $\mathcal{E}$ be a locally constant sheaf on $\mathbf{X}$ of finite-dimensional complex vector spaces. If $(U \to X) \in \mathbf{X}$, then $\mathcal{E}_{|U}$ is the sheaf of parallel sections of a canonically determined complex vector bundle with flat connection (E_U, ∇^{E_U}). Let $\Omega^k(U, E_U)$ denote the space of global sections of $\Lambda^k_{\mathbb{C}} T^*U \otimes E_U$. The de Rham differential d_{dR} and the connection ∇^{E_U} together induce a differential $d^{E_U} : \Omega^k(U, E_U) \to \Omega^{k+1}(U, E_U)$. Observe that $(\Omega^{\cdot}(U, E_U), d^{E_U})$ is a $(\Omega^{\cdot}(U), d_{dR})$-DG-module.

If $f : (U' \to X) \to (U \to X)$ is a morphism in $\mathbf{X}$, then we have a morphism of sheaves $f^*\mathcal{E}_{|U} \to \mathcal{E}_{|U'}$. This induces a morphism of flat vector bundles $f^*E_U \to E_{U'}$ and finally a morphism of complexes $(\Omega^{\cdot}(U, E_U), d^{E_U}) \to (\Omega^{\cdot}(U', E_{U'}), d^{E_{U'}})$.

We define the sheaf $\Omega^{\cdot}_X(\mathcal{E})$ of $(\Omega^{\cdot}_X, d_{dR})$-DG-modules which associates to $(U \to X)$ in $\mathbf{X}$ the complex $(\Omega^{\cdot}(U, E_U), d^{E_U})$.

Lemma 3.14. *$\mathcal{E} \to \Omega^{\cdot}_X(\mathcal{E})$ is a flabby resolution.*

Proof. This is shown by adapting the arguments of [**13**, Sec. 3.1] to differential forms twisted by a flat vector bundle. □

Let $p : X \to *$ be the projection. If $F \in \mathtt{Sh}\mathbf{X}$, then we define its global sections by

$$\Gamma_X F := \mathtt{ev} \circ p_*(F). \tag{3.15}$$

Construction 3.16. Assume now that X is an orbifold. The sheaf $\mathcal{E}$ gives rise to a flat vector bundle $E \to X$ in the orbifold sense. We can consider the de Rham complex $\Omega(X, E)$ of E-valued forms on X which are smooth in the orbifold sense.

Lemma 3.17. *We have a natural isomorphism $\Gamma_X \Omega^{\cdot}_X(\mathcal{E}) \cong \Omega(X, E)$.*

Proof. We choose an orbifold atlas $A \to X$, i.e. A is a smooth manifold, $A \to X$ is an atlas, and the smooth groupoid $A \times_X A \Rightarrow A$ is very proper, separated and étale. By the definition of smooth forms in the orbifold sense we have the exact sequence

$$0 \longrightarrow \Omega(X, E) \longrightarrow \Omega(A, E_A) \xrightarrow{r^* - s^*} \Omega(A \times_X A, E_{A \times_X A}) .$$

The composition $A \to X \to *$ is clearly representable. By [**13**, Lemma 2.36] we have an exact sequence

$$0 \longrightarrow \Gamma_X \Omega^{\cdot}_X(\mathcal{E}) \longrightarrow \Omega(A, E_A) \xrightarrow{r^* - s^*} \Omega(A \times_X A, E_{A \times_X A}) .$$ □

In order to indicate that the local system $\mathcal{E}$ is the initial datum and the vector bundle $E \to X$ is secondary we use the following notation.

Definition 3.18. We define

$$\Omega^{\cdot}(X, \mathcal{E}) := \Gamma_X \Omega^{\cdot}_X(\mathcal{E}).$$

It is an $\Omega^{\cdot}(X)$-DG-module. Its differential will be denoted by $d^{\mathcal{E}}$.

The twisted de Rham cohomology of X with coefficients in $\mathcal{E}$ depends on the choice of a closed form $\lambda \in \Omega^3(X)$. Let z be a formal variable of degree 2. Then we form the complex $\Omega^{\cdot}(X, \mathcal{E})[[z]]_\lambda$ given by

$$\Omega^{\cdot}(X, \mathcal{E})[[z]], \quad d_\lambda := d^{\mathcal{E}} + \lambda T,$$

where $T := \frac{d}{dz}$.

Definition 3.19. The λ-twisted cohomology $H^*(X; \mathcal{E}, \lambda)$ of X with coefficients in $\mathcal{E}$ is defined as the cohomology of the complex $\Omega^{\cdot}(X, \mathcal{E})[[z]]_\lambda$.

Construction 3.20. We can also define a sheaf $\Omega^{\cdot}_X(\mathcal{E})[[z]]_\lambda$ of $(\mathcal{E}, \lambda)$-twisted de Rham complexes on $\mathbf{X}$ such that for $(U \xrightarrow{\phi} X) \in \mathbf{X}$ we have $\Omega^{\cdot}_X(\mathcal{E})[[z]]_\lambda(U) := \Omega^{\cdot}(U, E_U)[[z]]$ with the differential $d_{\phi^*\lambda}$. By Definition 3.18 have an isomorphism of complexes

$$\Omega^{\cdot}(X, \mathcal{E})[[z]]_\lambda \cong \Gamma_X \Omega^{\cdot}_X(\mathcal{E})[[z]]_\lambda.$$

We now take twists into account. Let X be an orbifold and $f\colon G \to X$ be a smooth gerbe with band $U(1)$. Then we can form the orbifold of loops $LX \to X$ and the pull-back $f_L\colon G_L \to LX$ of the gerbe $f\colon G \to X$. We choose an atlas $(A \to G_L) \in \mathbf{G_L}$. It gives rise to a simplicial object $\mathbf{A}^{\cdot}_{G_L} \in \mathbf{G_L}^{\Delta^{\mathrm{op}}}$ such that

$$\mathbf{A}^n_{G_L} := \underbrace{A \times_{G_L} \cdots \times_{G_L} A}_{n+1\text{-factors}}.$$

Let $\Omega^{\cdot}_{G_L}$ denote the de Rham complex (see [**13**, 3.1.2]) of the smooth stack G_L. The associated chain complex of $\Omega^{\cdot}_{G_L}(\mathbf{A}^{\cdot}_{G_L})$ is a double complex with the de Rham differential d_{dR} and the Čech differential δ.

Note that $A \to G_L \to LX$ is an atlas. We form the simplicial object $\mathbf{A}^{\cdot}_{LX} \in \mathbf{LX}^{\Delta^{\mathrm{op}}}$ such that

$$\mathbf{A}^n_{LX} := \underbrace{A \times_{LX} \cdots \times_{LX} A}_{n+1\text{-factors}}.$$

We consider the double complex $\Omega^{\cdot}_{LX}(\mathbf{A}^{\cdot}_{LX})$ Note that by [**13**, Lemma 2.36] we have

$$\Omega^{\cdot}(LX) \overset{\text{Lemma 3.17}}{\cong} \Gamma_{LX}\Omega^{\cdot}_{LX} \cong \ker(\delta\colon \Omega^{\cdot}_{LX}(A^0_{LX}) \to \Omega^{\cdot}_{LX}(A^1_{LX})).$$

The property that $G_L \to LX$ is a smooth gerbe with band $U(1)$ can be expressed as the fact that the diagram

$$\begin{tikzcd} A \times_{G_L} A \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & A \arrow[d, equal] \\ A \times_{LX} A \arrow[r, shift left] \arrow[r, shift right] & A \end{tikzcd}$$

is a central $U(1)$-extension of smooth groupoids. In particular, we see that the canonical map

$$\ker(\delta\colon \Omega^{\cdot}_{LX}(A^0_{LX}) \to \Omega^{\cdot}_{LX}(A^1_{LX})) \to \ker(\delta\colon \Omega^{\cdot}_{G_L}(A^0_{G_L}) \to \Omega^{\cdot}_{G_L}(A^1_{G_L}))$$

is an isomorphism; i.e. we see that

$$\Gamma_{G_L}(\Omega^{\cdot}_{G_L}) \cong \Gamma_{LX}(\Omega^{\cdot}_{LX}) \cong \Omega^{\cdot}(LX). \tag{3.21}$$

Definitions, Facts, and Notation 3.22. A connection on the gerbe $f_L\colon G_L \to LX$ consists of a pair (α, β), where $\alpha \in \Omega^1(A \times_{G_L} A)$ is a connection one-form on the $U(1)$-bundle $A \times_{G_L} A \to A \times_{LX} A$, and $\beta \in \Omega^2(A)$. We consider $\alpha \in \Omega^1_{G_L}(\mathbf{A}^1_{G_L})$ and $\beta \in \Omega^2_{G_L}(\mathbf{A}^0_{G_L})$. The pair is a connection (α, β) if it satisfies:

(1) $\delta\beta = d_{dR}\alpha$,

(2) $\delta\alpha = 0$.

Note that $\delta d_{dR}\beta = 0$ so that there is a unique $\lambda \in \Gamma_{G_L}\Omega^3_{G_L} \overset{(3.21)}{\cong} \Omega^3(LX)$ which restricts to $d_{dR}\beta$. We have $d_{dR}\lambda = 0$.

Let us choose a connection (α, β), and let $\lambda \in \Omega^3(LX)$ be the associated closed three-form. In 3.4 we have introduced the locally constant sheaf $\mathcal{L}$ on LX. The construction 3.20 gives the complex of sheaves

$$(\Omega^{\cdot}_{LX}(\mathcal{L})[[z]]_\lambda, d_\lambda).$$

Furthermore we set

$$\Omega^{\cdot}(LX, \mathcal{L})[[z]]_\lambda := \Gamma_{LX}\Omega^{\cdot}_{LX}(\mathcal{L})[[z]]_\lambda.$$

Definition 3.23. The delocalized (G, λ)-twisted de Rham cohomology of X is defined by

$$H^*_{dR,\mathrm{deloc}}(X, (G, \lambda)) := H^*(\Omega^{\cdot}(LX, \mathcal{L})[[z]]_\lambda, d_\lambda).$$

In view of Lemma 3.17 this is the definition given in [**40**, 3.10]. Note that the delocalized twisted de Rham cohomology $H^*_{dR,\mathrm{deloc}}(X, (G, \lambda))$ depends on the choice of the connection, though these groups are isomorphic for different choices (see [**40**, 3.11]).

3.3. Comparison

In this subsection we prove

Theorem 3.24. *There is an isomorphism*

$$H^*_{\mathrm{deloc}}(X; G) \cong H^*_{dR,\mathrm{deloc}}(X, (G, \lambda)).$$

Actually, this theorem follows from the following stronger statement. Recall that $f_L\colon G_L \to LX$ is the pull-back of $f\colon G \to X$ via the canonical map $LX \to X$. Let $\mathbb{R}_{\mathbf{G_L}} \in \mathtt{Sh}_{\mathtt{Ab}}\mathbf{G_L}$ denote the constant sheaf with value $\mathbb{R}$.

Theorem 3.25. *There is an isomorphism in* $D^+(\mathtt{Sh}_{\mathtt{Ab}}\mathbf{LX})$

$$R(f_L)_*(\mathbb{R}_{\mathbf{G}_L}) \otimes_{\mathbb{R}} \mathcal{L} \cong \Omega^{\cdot}_{LX}(\mathcal{L})[[z]]_\lambda.$$

Note that this isomorphism depends on a choice of a connection on the gerbe $G_L \to LX$. The remainder of the present subsection is devoted to the proofs of Theorems 3.24 and 3.25.

Proof of Theorem 3.25. First observe that $\Omega^{\cdot}_{LX}(\mathcal{L})[[z]]_\lambda \cong \Omega^{\cdot}_{LX}[[z]]_\lambda \otimes_{\mathbb{R}} \mathcal{L}$. Therefore it suffices to show that

$$R(f_L)_*(\mathbb{R}_{\mathbf{G_L}}) \cong \Omega^{\cdot}_{LX}[[z]]_\lambda.$$

This is exactly the assertion of [**13**, Theorem 1.1], with the difference, that now LX is an orbifold instead of a smooth manifold. We repeat the proof of [**13**, Theorem 1.1] given by [**13**, subsection 3.2] with the following modifications (the numbers refer to the paragraphs in [**13**, subsection 3.2]:

(1) 3.2.1: The manifold X is replaced by the orbifold LX. The gerbe $G \to X$ is replaced by the gerbe $G_L \to LX$. Furthermore, $A \to G_L$ is some atlas. It induces an atlas $A \to G_L \to LX$. The $U(1)$-central extension of groupoids $(A \times_{G_L} A \Rightarrow A) \to (A \times_{LX} A \to A)$ represents a gerbe in the language of groupoids, but we can no longer refer to the paper [**20**]. For existence of a connection we now refer to [**40**, Prop. 3.6].
(2) 3.2.2: We use the notation Ω_{G_L} instead of $\Omega(G_L)$ for the de Rham complex of the smooth stack G_L. $\Omega^{\cdot}(LX)$ must be interpreted as in 3.16. For the existence of connections we refer to [**40**]. The construction of the three-form associated to a connection (α, β) was explained in 3.22.
(3) 3.2.6: We must show that the map $\phi\colon \Omega^{\cdot}[[z]]_\lambda \to i^\sharp C_A(\Omega^{\cdot}(G_L))$ is a quasi-isomoriphism. This can be shown locally. Since we can cover LX by smooth manifolds the local isomorphism immediately follows from the result proved in [**13**]. This argument avoids repeating the proof of [**13**, Prop. 3.4]. □

We now show Theorem 3.24. We need the following fact. Let X be an orbispace or orbifold and $p\colon X \to *$ be the projection. Recall that $\mathtt{ev} \circ p_* = \Gamma_X\colon \mathtt{Sh}_{\mathtt{Ab}}\mathbf{X} \to \mathtt{Ab}$. This functor is left exact and can thus be derived. Let $\mathcal{O}_X$ be the sheaf of continuous or smooth real functions on X, i.e. $\mathcal{O}_X = \Omega^0_X$ in the smooth case.

Lemma 3.26. *If $F \in \mathtt{Sh}_{\mathtt{Ab}}\mathbf{X}$ is a flabby sheaf and a sheaf of $\mathcal{O}_X$-modules, then*

$$R^i\Gamma_X(F) = 0$$

for all $i \geqslant 1$.

Proof. Let $A \to X$ be an orbispace (orbifold) atlas. Then $A \times_X A \Rightarrow A$ is a very proper, separated, and étale groupoid. Let $A^{\cdot}$ be the associated simplicial space (manifold). The complex $F(A^{\cdot})$ represents $R\Gamma_X(F)$ by [**13**, Lemma 2.41]. We now employ the method of [**12**, Sec. 4.1] in order to show that $H^i(F(A^{\cdot})) = 0$ for $i \geqslant 1$. We use the $\mathcal{O}_X$-module structure in order to multiply by cut-off function. □

Proof of Theorem 3.24. We first observe that $\Omega^{\cdot}_{LX}(\mathcal{L})[[z]]_\lambda$ is a complex of flabby sheaves and of $\mathcal{O}_{LX} = \Omega^0_{LX}$-modules. Therefore by Lemmas 3.26 and 3.17 we have (see [**8**, Cor. 25] for a related result)

$$R\Gamma_{LX}(\Omega^{\cdot}_{LX}(\mathcal{L})[[z]]_\lambda) \cong \Omega^{\cdot}(LX, \mathcal{L})[[z]]_\lambda .$$

By Definition 3.23 the cohomology of the right-hand side is $H^*_{\mathrm{deloc},dR}(X,(G,\lambda))$. On the other hand by Theorem 3.24

$$R\Gamma_{LX}(\Omega^{\cdot}_{LX}(\mathcal{L})[[z]]_\lambda) \cong \mathtt{ev} \circ Rp_* \left(R(f_I)_*(\mathbb{R}_{\mathbf{G_L}}) \otimes_{\mathbb{R}} \mathcal{L}\right) .$$

Its cohomology is by (3.13) isomorphic to $H^*_{\mathrm{deloc}}(X;G)$. □

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