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SPLITTINGS IN THE BURNSIDE RING AND IN SF_G

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Abstract

Let G be a finite p-group, $p \neq 2$. We construct a map from the space J_G, defined as the fiber of $\psi^k - 1\colon B_GO \to B_GSpin$, to the space $(SF_G)_p$, defined as the 1-component of the zeroth space of the equivariant p-complete sphere spectrum. Our map produces the same splitting of the G-connected cover of $(SF_G)_p$ as we have described in previous work, but it also induces a natural splitting of the p-completions of the component groups of fixed point subspaces.

1. Introduction

In his seminal $J(X)$ papers ([**1, 2, 3, 4**]), Adams studied the group of fiber homotopy equivalence classes of virtual real vector bundles over a space X, $J(X)$. Understanding $J(X)$ is important for several reasons. For example, when X is a projective space, $J(X)$ gives information about cross-sections of Stiefel fiberings and the vector fields on spheres question. When X is a sphere, $J(X)$ gives information about the image of the classical J-homomorphism, from the homotopy groups of orthogonal groups to the stable homotopy groups of spheres.

Adams based his study on two auxiliary groups, $J'(X)$ and $J''(X)$, together with surjective maps

$$J''(X) \to J(X) \to J'(X).$$

Thus, $J'(X)$ serves as a *lower bound* for $J(X)$ and $J''(X)$ serves as an *upper bound*. These groups are in principal computable, and with Quillen's proof of the so-called Adams conjecture ([**16**]), Adams' work implies that $J''(X)$ is in fact equal to $J'(X)$. Roughly speaking, Adams' conjecture was that for an integer k, a finite CW-complex X, and a virtual real vector bundle ξ, the underlying virtual spherical fibrations of $\psi^k\xi$ and ξ are equivalent after inverting k. Here, the maps ψ^k are the Adams operations. Having a collection of virtual vector bundles that become trivial in $J(X)$ (after inverting certain integers) is the key ingredient in defining the upper bound $J''(X)$.

A number of the results in Adams' papers were cast in a new geometric light by May in [**12**]. Instead of looking at groups and homomorphisms, like the homomorphism from the group of virtual vector bundles to the group of virtual spherical

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fibrations over a given space, May looked at the representing Hopf spaces for these groups and the Hopf maps between them. May constructed a homotopy commutative diagram, which we call the Adams-May square:

$$\begin{tikzcd} BO \arrow[r, "\psi^k-1"] \arrow[d, "\sigma^k"'] & BSpin \arrow[d, "\rho^k"] \\ BO_\otimes \arrow[r, "\psi^k/1"] & BSpin_\otimes. \end{tikzcd}$$

Here, BO is the classifying space for the group of virtual real vector bundles of virtual dimension 0 under direct sum, while $BO_\otimes$ is the classifying space for the group of virtual real vector bundles of virtual dimension 1 under tensor product. Likewise, $BSpin$ and $BSpin_\otimes$ are the analogous classifying spaces for virtual bundles with Spin structure. The operations $\psi^k - 1$ and $\psi^k/1$ on vector bundles induce the horizontal maps, while the Adams-Bott cannibalistic class induces the map ρ^k on the right. The map σ^k is seemingly something new, defined using geometric constructions like maps of fiber sequences. As it turns out, we can restrict σ^k to BSO and extend ρ^k to BSO, and the resulting maps are homotopic. At an odd prime, of course, the spaces BO, BSO and $BSpin$ all coincide.

If we localize at a prime, then Adams' theorem that $J'(X) = J''(X)$ becomes a corollary of showing that the Adams-May square is a pull-back in the homotopy category, which May does. But May's geometric arguments lead to something more. Let J_p and $J_{\otimes p}$ denote the p-localization of the fibers of the maps $\psi^k - 1$ and $\psi^k/1$. Then the Adams-May square induces a weak equivalence from J_p to $J_{\otimes p}$. Moreover, a closer look at the construction of the Adams-May square shows that the map $J_p \to J_{\otimes p}$ can be made to factor through the p-localization of the space SF of stable degree 1 self-maps of spheres. Thus, one obtains a splitting of the space SF_p. The homotopy groups of the factor J_p contain the image of the classical J-homomorphism.

In the decade that followed the publication of Adams' $J(X)$-papers, questions about equivariant generalizations naturally arose; tom Dieck considered such questions in [**21**]. There he defines a number of variations of the natural equivariant generalization $JO_G(X)$. First, two equivariant real bundles E and F are said to be *stably locally homotopy-equivalent* if for each $H \leqslant G$, there is a G-representation V and fiberwise G-maps $f\colon S(E \oplus V) \to S(F \oplus V)$ and $g\colon S(F \oplus V) \to S(E \oplus V)$, each inducing a fiberwise homotopy equivalence on H-fixed points. This yields a quotient of $JO_G(X)$, denoted $JO_G^{\mathrm{loc}}(X)$, which is more amenable to computation. Also, for a set of primes S, tom Dieck defines a quotient $JO_{G,S}(X)$ of $KO_G(X)$, where two vector bundles E and F are said to be stably S-equivalent if there exist stable maps $f\colon S(E) \to S(F)$ and $g\colon S(F) \to S(E)$ with fiber degrees prime to all elements of S. For a p-group G, he shows [**21**, 11.4.2] that the quotient map from $JO_G^{\mathrm{loc}}(X)_p$ to $JO_{G,\{p\}}(X)_p$ is an isomorphism. In our work below and in [**9**], we consider stable fibrations whose fibers are p-completions of equivariant sphere representations, so when $S = \{p\}$, S-equivalences give rise to equivariant homotopy equivalences.

Most importantly, tom Dieck proved [**21**, 11.4.1] that when G is a p-group, the kernel of the quotient map from $KO_G(X)_p$ to $JO_G^{\mathrm{loc}}(X)_p$ is given by the image of the self-map $\psi^k - 1$ on $KO_G(X)_p$, where k is a positive odd integer generating the p-adic units. Also, tom Dieck proved [**21**, 11.5] that when G is a p-group and q is a prime not equal to p, then the kernel of the J-homomorphism $KO_G(X)_q \to JO_G^{\mathrm{loc}}(X)_q$ is generated by elements of the form $\psi^k\xi - \xi$, where k ranges over a collection of integers. In Adams' language, these statements amount to saying that the computable upper bound of $JO_G^{\mathrm{loc}}(X)_p$ is sharp.

McClure [**15**] considered the case in which G is not a p-group. Here, there is something of a surprise — there exist groups G for which the kernel of the J-homomorphism contains more than just "Adams conjecture elements." That is, there are elements in $KO_G(X)_p$ which cannot be written as linear combinations of elements of the form $\psi^k\xi - \xi$, but which nevertheless vanish in $JO_G(X)_p$. In Adams' language, the upper bound of $JO_G(X)_p$ fails to be sharp (and therefore certainly cannot be equal to the lower bound). McClure's main theorem, however, gives a reduction of the problem of showing two equivariant bundles are stably p-equivalent to the p-group case. For a cyclic subgroup H of G whose order is prime to p, let $P(H)$ denote a p-Sylow subgroup of NH/H. Then two G-vector bundles ξ and η are stably p-equivalent if for each such cyclic subgroup H, the fixed point $P(H)$-bundles ξ^H and η^H are stably p-equivalent.

In [**9**], we considered the equivariant analogue of the Adams-May square, for the sake of obtaining a splitting (at a prime) of an equivariant analogue of SF_G, the space of degree 1 self-maps of equivariant spheres, where G is a finite p-group, $p \neq 2$. Indeed, in light of tom Dieck's work discussed above, we had reason to hope that the equivariant Adams-May square should be a pull-back square in the homotopy category. The key step in proving this was to show that the equivariant analogues of ρ^k and σ^k are again homotopic, the demonstration of which required entirely different techniques from those used by May. We were only able to show that ρ^k and σ^k are homotopic on G-connected covers. (The G-connected cover of a based G-space X is a G-map $i\colon X_0 \to X$ such that i^H induces an equivalence on π_n for all $n \geqslant 1$ and all $H \leqslant G$, and X_0^H is connected for each $H \leqslant G$. Nonequivariantly, of course, all the spaces under consideration are connected.) We used Atiyah and Tall's results on p-adic λ-rings ([**5**]) to show that ρ^k induces a weak equivalence on the equivariant Adams summand of $B_G Spin^{\hat{}}_p$. From there, we completed the argument using the same ideas as May used in the nonequivariant case. As a corollary, we obtained an equivariant splitting of the p-completion of the G-connected cover of SF_G, where G is a p-group, $p \neq 2$. One of the factors in this splitting is given by a map from J_G to SF_G.

Now, in [**19**], Segal used the theory of p-adic λ-rings to study the ring homomorphism $h\colon A(G) \to R_{\mathbb{Q}}(G)$ from the Burnside ring to the ring of rational representations, induced by the permutation representation. He constructed a map from $R_{\mathbb{Q}}(G)$ to $A(G)$ and showed that its composite with h can be described in terms of an Adams-Bott cannibalistic class ρ^k, for a certain k. If $H \leqslant G$, then $\pi_0(SF_G^H)$ and $\pi_0(J_G^H)$ can be described as augmentation ideals in the Burnside ring and the representation ring of H. Thus, one might hope that one could construct the maps

$$(J_G)^{\hat{}}_p \to (SF_G)^{\hat{}}_p \to (J_{\otimes G})^{\hat{}}_p$$

which induce the splitting of the G-connected cover of $(SF_G)\hat{_p}$ so that, on π_0 of fixed point subspaces, these maps yield the splitting studied by Segal. It is to this question we turn in the present paper.

Let us recall in more detail part of the nonequivariant proof in [**12**] of the splitting of SF_p. For a given positive integer k, we have the following diagram of fiber sequences, where all spaces are implicitly localized away from k:

$$
\begin{tikzcd}
J^k \arrow[r] \arrow[d, "\alpha^k"] & BO \arrow[r, "\psi^k - 1"] \arrow[d, "\gamma^k"] & BSpin \arrow[d, equal] & \\
SF \arrow[r] \arrow[d, "\varepsilon^k"] & SF/Spin \arrow[r] \arrow[d, "f"] & BSpin \arrow[r, "Bj"] \arrow[d, "\rho^k"] & BSF \\
J^k_{\otimes} \arrow[r] & BO_{\otimes} \arrow[r, "\psi^k/1"] & BSpin_{\otimes}. &
\end{tikzcd}
$$

The map Bj represents taking the underlying spherical fibration of a bundle. The map γ^k is obtained from the Adams conjecture, which implies that, after inverting k, the underlying spherical fibration of a bundle $\psi^k\xi - \xi$ is trivial. The maps ρ^k and f are produced by means of the Atiyah-Bott-Shapiro orientation of Spin-bundles. We denote $f \circ \gamma^k$ by σ^k. After completing at a prime p not dividing k, we obtain the Adams-May square from the middle two columns of the above diagram. As mentioned above, to prove that this is a pull-back square in the homotopy category, one first shows that σ^k happens to be homotopic to ρ^k. This comes as a surprise, since the map γ^k depends on a choice of null-homotopy for $Bj \circ (\psi^k - 1)$. In this paper we show how to produce an "Adams-conjecture null-homotopy" in the equivariant context (G a p-group, $p \neq 2$) so that the map on components of fixed point subspaces induced by restriction of σ^k to J^k is equal to the map ρ^k described algebraically by Segal in [**19**].

At one point, we had hoped to produce a splitting of $(SF_G)\hat{_p}$ itself. This has proved to be somewhat problematic, since we have been unable to show that the map α^k from J_G to $(SF_G)\hat{_p}$ extends to $(J_G)\hat{_p}$. This is a subtle question — ordinarily, one would assume that one could extend a G map $f\colon X \to Y\hat{_p}$ to $X\hat{_p}$. However, we do not have an equivariant theory of p-completions for spaces which are not even G-connected — the spaces we have referred to as $(J_G)\hat{_p}$ and $(SF_G)\hat{_p}$ are obtained by taking p-completions on the spectrum level, and then passing to zeroth spaces. In order to get an actual splitting of $(SF_G)\hat{_p}$, one would need a stable equivariant null-homotopy of $Bj \circ (\psi^k - 1)$, and we do not know how to extend Quillen's methods in [**16**] to construct this.

In a separate paper [**10**], we have studied the Adams-May square for more general groups G, though in that paper we again restrict to G-connected covers. There we show that if G is any group and p is any prime such that none of the prime divisors of G are congruent to 1 modulo p, then the equivariant Adams-May square becomes a pull-back after passing to p-completions of G-connected covers. As a corollary, we obtain a splitting of the p-completion of the G-connected cover of SF_G.

Our paper is organized as follows. In Section 2, we review some facts about classifying spaces for equivariant bundles and fibrations, for which details can be found in [**9**] and [**22**]. We also describe Brauer lifting maps in our setting. In Section 3, which is the heart of the geometric argument in the paper, we show how to construct a map between the underlying spherical fibrations (with p-local fibers) of a bundle ξ and $\psi^k\xi$. While we rely on the techniques of [**16**] and [**21**], we pay close attention to the degree of our maps over orbits, since this will be essential in understanding the effect of α^k on components of fixed point subspaces. In Section 4, we extend the theory of classifying spaces, showing how to use the geometric maps constructed in the previous section to construct a lift of $\psi^k - 1$ with the right behavior on orbits. We use these constructions to prove our main theorem in Section 5. Our appendix gives some technical information on equivariant completions.

2. Background on classifying spaces

2.1. Bundles

If G is a finite group and A is a compact Lie group, then a principal (G, A)-bundle consists of a G-map $p\colon P \to B$ which is a principal A-bundle, such that the action of each $g \in G$ gives a map of A-bundles. If $\lambda\colon H \to A$ is a homomorphism, we let A_λ denote A with the corresponding left H-action. Then there is a category $\mathcal{C}_G(A)$, whose objects are homomorphisms $\lambda\colon H \to A$, $H \leqslant G$. A morphism in $\mathcal{C}_G(A)$ from $\lambda\colon H \to A$ to $\lambda'\colon H' \to A$ is a commutative square

$$\begin{tikzcd} G \times_H A_\lambda \arrow[r, "\bar{\theta}"] \arrow[d] & G \times_{H'} A_{\lambda'} \arrow[d] \\ G/H \arrow[r, "\tilde{\theta}"] & G/H' \end{tikzcd},$$

where $\tilde{\theta}$ is a G-map and $\bar{\theta}$ is a $G \times A$-map. Let $\mathcal{O}\colon \mathcal{C}_G(A) \to G\mathcal{U}$ be the functor taking λ to G/H and taking $(\bar{\theta}, \tilde{\theta})$ to $\tilde{\theta}$. Then the G-space $B_G A := B(*, \mathcal{C}_G(A), \mathcal{O})$ classifies (G, A)-bundles over finite G-CW-complexes.

Definition 2.1. A G-group is a group A equipped with a distinguished homomorphism $\rho_A\colon G \to A$.

Note that $\rho_A\colon G \to A$ endows $\mathcal{C}_G(A)$ with a distinguished object, which determines a basepoint for $B_G A$.

Remark 2.2. Suppose $H \leqslant G$. For each $\lambda\colon H \to A$, define A^λ to be the set of all elements a in A such that $\lambda(h)a = a\lambda(h)$ for each $h \in H$. Then each $a \in A^\lambda$ determines a self-map of λ in $\mathcal{C}_G(A)$ covering the identity map on G/H. This induces a map $BA^\lambda \to B_G A^H$. Let $R^+(H, A)$ be a set of representatives for the conjugacy classes of homomorphisms $\rho\colon H \to A$. Then we have a homotopy equivalence

$$\coprod_{\rho \in R^+(H,A)} BA^\rho \simeq B_G A^H.$$

Definition 2.3. If $Q \subseteq B$ is a G-invariant subspace, and $p\colon P \to B$ is a bundle, we let $p_Q\colon p^{-1}(Q) \to Q$ denote the restriction of p to Q.

Definition 2.4. If B is a G-space with a G-fixed basepoint b, and A is a G-group, then a based principal (G, A)-bundle over B is a principal (G, A)-bundle $p\colon P \to B$ such that $p_{\{b\}}$ is G-equivalent to A with action ρ_A. If V is a G-representation, then a based G-vector bundle over B with fiber V is a G-vector bundle $p\colon E \to B$ such that $p^{-1}(b)$ is isomorphic to the G-representation V.

If A is a G-group, then a based G-map $B \to B_G A$ represents a based principal (G, A)-bundle. Note that an unbased G-map from a point b to $B_G A$ could represent any homomorphism from G to A.

If V is an orthogonal A-representation with structure map $\alpha\colon A \to O(V)$, then we may also view V as a G-representation. A based principal (G, A)-bundle determines a based G-vector bundle $p^V : P^V = P \times_A V \to B$, with fiber V. This process is classified by the based map $B\alpha_*\colon B_G A \to B_G O(V)$.

Definition 2.5. Suppose ξ^V is a based G-vector bundle over B with fiber V, and Q is a G-orbit in B. Then if $K \leqslant G$ and Q^K is nonempty, we define

$$\dim(\xi^V, Q, K) = \dim(\xi^V_Q)^K.$$

Note that dim is additive on G-vector bundles.

We next construct stabilization maps. On the geometric level, a stable G-vector bundle is an equivalence class of based G-vector bundles. The equivalence relation is generated as follows. If U is any G-representation, we let t^U denote the trivial G-vector bundle $B \times U \to B$. If $\xi^V : P \to B$ is a based G-vector bundle with fiber V, then we identify ξ^V to its fiberwise sum with t^U. We let ξ represent the stable bundle corresponding to ξ^V. Isomorphism classes of stable G-vector bundles form a group under the direct sum operation.

Suppose $\mathcal{U}$ is a complete G-universe; that is, $\mathcal{U}$ is a complex G-representation which contains an infinite number of copies of each irreducible complex G-representation. Since G is finite, we could take $\mathcal{U}$ to be an infinite direct sum of copies of the regular representation. Then, for each finite-dimensional orthogonal G-representation $V \subseteq \mathcal{U}$, we have a classifying space $B_G O(V)$. Note that the G-action on V only determines the basepoint of $B_G O(V)$. In general, we will let $\rho_V : G \to O(V)$ denote the action map of V. If $V \subseteq W$ and $U = W - V$, and $H \leqslant G$, then any homomorphism $\alpha\colon H \to O(V)$ gives rise to a homomorphism

$$H \xrightarrow{\alpha \times \rho_U} O(V) \times O(U) \xrightarrow{\oplus} O(W) .$$

In this way, we get a functor from $\mathcal{C}_G(O(V))$ to $\mathcal{C}_G(O(W))$ taking ρ_V to ρ_W, and thus a based map from $B_G O(V)$ to $B_G O(W)$ classifying fiberwise sum with the trivial bundle $B \times U \to B$. The colimit of these spaces, denoted $B_G O$, classifies stable G-vector bundles.

Definition 2.6. Suppose $\xi^V : E \to B$ is a based G-vector bundle with fiber V, and $Q \subseteq B$ is a G-orbit. Then if $K \leqslant G$, we define

$$\dim(\xi, Q, K) = \dim(\xi^V, Q, K) - \dim(t^V, Q, K).$$

Note that dim is a well-defined homomorphism from the group of isomorphism classes of stable G-vector bundles to the integers.

If A is a G-group, V an A-representation, and $p\colon P \to B$ a based principal (G, A)-bundle, with associated bundle G-vector bundle p^V, then we let $p(V)$ denote the corresponding stable bundle. If $V = V_1 - V_2$ is a virtual A-representation, we define $p(V)$ to be $p^{V_1} - p^{V_2}$.

2.2. Fibrations

An admissible set of fibers is a set $\mathbb{F}$ of based spaces $\{F_\lambda\}$ with left actions of subgroups $H_\lambda \leqslant G$. We assume that each F_λ has the homotopy type of a G-CW-complex, and that the set $\{F_\lambda\}$ is closed under subconjugation. Note that for each space F_λ in $\mathbb{F}$, there is a G-map $p_\lambda\colon G \times_{H_\lambda} F_\lambda \to G/H_\lambda$ with a distinguished section $s_\lambda\colon G/H_\lambda \to G \times_{H_\lambda} F_\lambda$.

Definition 2.7. We let $G\mathcal{F}(\mathbb{F})$ be the category whose objects are all sectioned G-maps $p\colon P \to Q$ such that for some F_λ, there is a diagram of G-spaces

$$\begin{tikzcd} G \times_{H_\lambda} F_\lambda \arrow[r, "\bar\theta"] \arrow[d, "p_\lambda"] & P \arrow[d, "p"] \\ G/H_\lambda \arrow[r, "\tilde\theta"] & Q \end{tikzcd}$$

in which $\tilde\theta$ is a G-homeomorphism and $\bar\theta$ is a section-preserving fiberwise homotopy equivalence. A morphism in $G\mathcal{F}(\mathbb{F})$ is just a section-preserving commutative square.

Then $G\mathcal{F}(\mathbb{F})$ is an equivariant category of fibers with distinguished set of fibers $\mathbb{F}$. Now, a $G\mathcal{F}(\mathbb{F})$-space is a sectioned G-map $\xi\colon P \to B$ which restricts over each G-orbit in B to an object in $G\mathcal{F}(\mathbb{F})$. A $G\mathcal{F}(\mathbb{F})$-fibration is a $G\mathcal{F}(\mathbb{F})$-space satisfying a version of the covering homotopy property. Such fibrations are classified by introducing the following category.

Definition 2.8. Let $\mathcal{A} = \mathcal{A}(\mathbb{F})$ be the full subcategory of $G\mathcal{F}(\mathbb{F})$ consisting only of those objects of the form p_λ. Let $\mathcal{O}\colon \mathcal{A} \to G\mathcal{U}$ be the functor taking p_λ to the orbit G/H_λ. Let $B_G(\mathbb{F})$ be the G-space $B(*, \mathcal{A}, \mathcal{O})$. A distinguished G-space F_ρ endows $B_G(\mathbb{F})$ with a distinguished basepoint.

The based G-space $B_G(\mathbb{F})$ is a classifying space for $G\mathcal{F}(\mathbb{F})$-fibrations over finite G-CW-complexes. The classifying map for a $G\mathcal{F}(\mathbb{F})$-fibration $\xi\colon E \to B$ can be obtained as follows. Let $\mathcal{P}(\xi)\colon \mathcal{A} \to \mathcal{U}$ be the functor given by $\mathcal{P}(\xi)(p_\lambda) = G\mathcal{F}(\mathbb{F})(p_\lambda, \xi)$. Then there is a canonical diagram

$$B \xleftarrow{\simeq} B(\mathcal{P}(\xi), \mathcal{A}, \mathcal{O}) \longrightarrow B(*, \mathcal{A}, \mathcal{O}) = B_G(\mathbb{F})$$

(see [**22**, 2.3.2]). Inverting the equivalence determines a classifying map for ξ.

We can sometimes identify the G-connected cover of $B_G(\mathbb{F})$ as the classifying space of a G-monoid. Let $\tilde{\mathcal{A}}(F_\rho)$ be the G-monoid of based nonequivariant self-maps of F_ρ which are nonequivariant equivalences (G acts through conjugation). If $\pi_0^H(\tilde{\mathcal{A}}(F_\rho))$ is a group for each $H \leqslant G$, then $B(\tilde{\mathcal{A}}(F_\rho))$ is a G-connected cover of $B_G(\mathbb{F})$ (see [**9**, §3]). For example, if V is a G-representation large enough so that $[S^V, S^V]_H$ is isomorphic to the Burnside ring $A(H)$ for each $H \leqslant G$, then as shown in [**9**, §8], $\pi_0^H(\tilde{\mathcal{A}}(S_p^V)) \cong (A(H)_p)^\times$ is a group for each $H \leqslant G$.

If V is an orthogonal G-representation containing a trivial summand, then we define sets of fibers $\mathbb{S}^V_{(p)} = \{S^{V_\lambda}_{(p)}\}$ and $\mathbb{S}^V_p = \{S^{V_\lambda}_p\}$ consisting of the p-localizations and p-completions of the one point compactifications of orthogonal representations with the same underlying real inner product space as V (in both cases, we only consider representations containing a trivial summand). The given G-action on V determines distinguished fibers in both cases, so that $B_G(\mathbb{S}^V_{(p)})$ and $B_G(\mathbb{S}^V_p)$ are based G-spaces.

Fiberwise suspension determines a functor from the category of $G\mathcal{F}(\mathbb{S}^V_{(p)})$-fibrations to that of $G\mathcal{F}(\mathbb{S}^W_{(p)})$-fibrations whenever $V \subseteq W$ are finite-dimensional orthogonal G-representations. Given a complete universe $\mathcal{U}$, we call an object of the colimit category a $G\mathcal{F}(\mathbb{S}_{(p)})$-fibration. On the classifying space level, the stabilization functors are realized by based maps $B_G(\mathbb{S}^V_{(p)}) \to B_G(\mathbb{S}^W_{(p)})$. The colimit, denoted $B_G(\mathbb{S}_{(p)})$, classifies $G\mathcal{F}(\mathbb{S}_{(p)})$-fibrations over finite G-CW-complexes. There is an analagous space $B_G(\mathbb{S}_p)$ classifying stable $G\mathcal{F}(\mathbb{S}_p)$-fibrations over finite G-CW-complexes ([**9**, §7]), though its definition as a colimit is made considerably more complicated by the fact that suspension does not commute with completion.

Remark 2.9. Since the units in $A(G)^\wedge_p$ form a group, we were able to show in [**9**, §8] that $\Omega B_G(\mathbb{S}_p)$ is equivalent to the homotopy units in $\Omega^\infty S_p$, where S_p is the p-complete equivariant sphere spectrum. This argument does not work for the units in the p-localization of $A(G)$, so that we cannot assert (and we do not believe) that $\Omega B_G(\mathbb{S}_{(p)})$ is equivalent to the homotopy units in $\Omega^\infty S_{(p)}$.

The classifying map of a particular $G\mathcal{F}(\mathbb{S}_{(p)})$-fibration can be obtained either by classifying a representative of its equivalence class and then passing to the colimit of the classifying spaces, or inverting the equivalence below:

$$B \xleftarrow{\simeq} \mathrm{colim}_V B(\mathcal{P}(\xi^V), \mathcal{A}(\mathbb{S}^V_{(p)}), \mathcal{O}) \longrightarrow B_G(\mathbb{S}_{(p)}) \ .$$

2.3. Thom classes

Let $E = KO_p$. Then E is commutative unital ring spectrum which comes equipped with a collection of periodicity classes (Bott classes) b^{V_λ} in $\tilde{E}^0_{H_\lambda}(S^{V_\lambda})$ for all Spin H_λ-representations V_λ of dimension divisible by 8. (See [**9**, §5] for details on periodic ring spectra.)

Remark 2.10. If the order of H_λ is odd, then any homomorphism from H_λ to $O(V)$ clearly lifts uniquely to $SO(V)$, and also to $Spin(V)$, since any extension of H_λ by $\mathbb{Z}/2$ splits. Thus, there is no distinction between Spin H_λ-representations and orthogonal H_λ-representations.

In this section, suppose V is a real G-representation of dimension divisible by 8. If ξ^V is a $G\mathcal{F}(\mathbb{S}^V_p)$-fibration over a G-space B, let $T\xi^V$ be the Thom space of ξ^V, obtained by quotienting out the canonical section of ξ^V. Then for each $x \in B^H$, the Thom space $T\xi^V_x$ is equivalent to S^{V_λ}, where V_λ is a real H-representation of dimension divisible by 8. Therefore, $\tilde{E}^0_H(T\xi^V_x)$ is a free E^0_H-module. An E-orientation of a $G\mathcal{F}(\mathbb{S}^V_p)$-fibration ξ^V is an element $\mu \in \tilde{E}^0_G(T\xi^V)$ which restricts to a generator in $\tilde{E}^0_H(T\xi^V_x)$, for each $x \in B^H$.

Now let $F^V E$ be the functor from $\mathcal{A}(\mathbb{S}_p^V)$ to $\mathcal{U}$ which takes the object $G \times_H S_p^{V_\lambda}$ to the subspace of $F_H(S_p^{V_\lambda}, \Omega^\infty E)$ consisting of those components representing generators of $\tilde{E}^0_H(S_p^{V_\lambda})$. Then the space $B(F^V E, \mathcal{A}(\mathbb{S}_p^V), \mathcal{O})$ is a classifying space for E-oriented $G\mathcal{F}(\mathbb{S}_p^V)$-fibrations. A choice of G-map from S_p^V to $\Omega^\infty E$ representing b^V determines a basepoint. The natural map

$$q^V : B(F^V E, \mathcal{A}(\mathbb{S}_p^V), \mathcal{O}) \to B(*, \mathcal{A}(\mathbb{S}_p^V), \mathcal{O})$$

represents forgetting an orientation.

Remark 2.11. If V_λ is a real H-representation of dimension divisible by 8, then $\tilde{E}^0_H(S_p^{V_\lambda})$ and $\tilde{E}^0(S_p^V)$ are isomorphic to $RO(H)_p$ and $\mathbb{Z}_p$ respectively. Since a power of $IO(G)$ is contained in $pIO(G)$ (see [**5**, III.1.1, V.1.3]), every element in $RO(H)_p$ mapping to a unit in $\mathbb{Z}_p$ is itself a unit. By [**9**, §3], this implies that we have a fiber sequence

$$F^V E(p_{\lambda_*}) \xrightarrow{j_*} B(F^V E, \mathcal{A}(\mathbb{S}_p^V), \mathcal{O}) \xrightarrow{q^V} B(*, \mathcal{A}(\mathbb{S}_p^V), \mathcal{O}) \ .$$

Here, $\lambda_* \colon \langle e \rangle \to G$ is the unique homomorphism, and G acts on $F^V E(p_{\lambda_*})$ by conjugation.

Remark 2.12. Suppose V is a real G-representation of dimension divisible by 8. An H-equivariant map $\theta \colon S_p^V \to S_p^V$ determines a morphism in $\mathcal{A}(\mathbb{S}_p^V)$, hence a 1-simplex in $B(*, \mathcal{A}(\mathbb{S}_p^V)_0, \mathcal{O})^H$. This 1-simplex starts and ends at the same point and therefore determines an element $\hat{\theta}$ in $\pi_1(B(*, \mathcal{A}(\mathbb{S}_p^V)_0, \mathcal{O})^H)$. The image of $\hat{\theta}$ in $\pi_0(F^V E(p_{\lambda_*})^H)$ is represented by $b^V \circ \theta$.

By taking a colimit of classifying spaces for E-oriented $G\mathcal{F}(\mathbb{S}_p^V)$-fibrations, we obtain a classifying space $B_G(\mathbb{S}_p; E)$ for stable E-oriented $G\mathcal{F}(\mathbb{S}_p)$-fibrations (see [**9**, §4,7].) There is a natural map $q \colon B_G(\mathbb{S}_p; E) \to B_G(\mathbb{S}_p)$ obtained by taking a colimit of the maps q^V above. The fiber of q is equivalent to the subspace $(\Omega^\infty E)^\times$ of $\Omega^\infty E$ consisting of components representing units in E^0 ([**9**, §4]). We then have a fiber sequence

$$(\Omega^\infty E)^\times \to B_G(\mathbb{S}_p; E) \to B_G(\mathbb{S}_p).$$

2.4. Brauer lifting

Suppose k is a prime different from p, and let $\mathbb{F}_{k^m}$ denote the field of k^m elements. Let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_k$. Suppose $V_\mathbb{F}$ is an $\mathbb{F}$-inner product space. Then if H acts by orthogonal transformations on $V_\mathbb{F}$ through a map $\alpha \colon H \to O(V_\mathbb{F})$, we call $V_\mathbb{F}$ a representation of H *over* $\mathbb{F}$. We let $RO_\mathbb{F}(H)$ denote the ring of virtual orthogonal representations of H over $\mathbb{F}$. By Brauer lifting, there is a natural ring homomorphism

$$\lambda_H \colon RO_\mathbb{F}(H) \to RO(H)$$

(see [**7**, 6.1], [**16**], or [**20**]). If the order of H is prime to k, then λ_H restricts to an isomorphism of semi-rings of actual representations ([**7**, 6.2] or [**20**, 15.5]).

Let $\Pi(m,j) = O(m, \mathbb{F}_{k^j}) \leqslant O(m, \mathbb{F})$, let $W^\mathbb{F}_{m,j}$ be the canonical m-dimensional $\Pi(m,j)$-representation over $\mathbb{F}$, and let $W_{m,j} = \lambda_{\Pi(m,j)}(W^\mathbb{F}_{m,j})$. Any orthogonal G-representation over $\mathbb{F}$ can be realized over $\mathbb{F}_{k^j}$ for j sufficiently large. It follows by the naturality of λ, and surjectivity of λ_G when $|G|$ is prime to k, that for any orthogonal G-representation V of dimension m, there is a map $\rho_V \colon G \to \Pi(m,j)$ such that V is isomorphic to $\rho_V^*(W_{m,j})$. We use this isomorphism to identify the underlying vector spaces of V and $W_{m,j}$. We may then view V as a $\Pi(m,j)$-representation, and we regard $\Pi(m,j)$ as a G-group.

We denote the action map $\Pi(m,j) \to O(V)$ of the $\Pi(m,j)$-representation V as b_V; the map b_V then induces a based map $\beta_V \colon B_G(\Pi(m,j)) \to B_G(O(V))$, representing the operation of replacing a $(G,\Pi(m,j))$-bundle $p\colon P \to B$ by the associated G-vector bundle $p^V \colon P^V \to B$.

Suppose $V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots$ is an increasing sequence of G-representations whose union is a complete G-universe, and let $m_n = \dim V_n$. Let $U_n = V_{n+1} - V_n$, so that $\dim U_n = m_{n+1} - m_n$. Then we may choose an increasing sequence j_n and maps $\rho_{U_n} \colon G \to \Pi(m_{n+1} - m_n, j_{n+1})$ so that $U_n \cong \rho_{U_n}^*(W_{m_{n+1}-m_n}, j_{n+1})$. We may define $\rho_{V_{n+1}}$ inductively as the composite

$$G \xrightarrow{\rho_{V_n} \times \rho_{U_n}} \Pi(m_n, j_n) \times \Pi(m_{n+1} - m_n, j_{n+1}) \xrightarrow{\oplus} \Pi(m_{n+1}, j_{n+1}) \ ,$$

where $\oplus$ denotes block sum of matrices. Then by additivity and naturality of λ, it follows by induction that $\rho_{V_n}^*(W_{m_n,j_n}) \cong V_n$ for each n. We will denote $\Pi(m_n, j_n)$ as Π_n and W_{m_n,j_n} as W_n.

Given a homomorphism $\alpha_n \colon H \to \Pi_n$, we get a map $\alpha_{n+1} \colon H \to \Pi_{n+1}$ defined by

$$H \xrightarrow{\alpha \times \rho_{U_n}} \Pi(m_n, j_n) \times \Pi(m_{n+1} - m_n, j_{n+1}) \xrightarrow{\oplus} \Pi(m_{n+1}, j_{n+1}) = \Pi_{n+1} \ .$$

As in Section 2.1, this gives us based stabilization maps $B_G\Pi_n \to B_G\Pi_{n+1}$. We let $B_GO(\mathbb{F})$ denote the colimit of these maps. Now, for each n, we have the following homotopy commutative diagram:

$$\begin{tikzcd}
B_G(\Pi_n) \arrow[r, "\beta_{V_n}"] \arrow[d] & B_G(O(V_n)) \arrow[d] \\
B_G(\Pi_{n+1}) \arrow[r, "\beta_{V_{n+1}}"] & B_G(O(V_{n+1})).
\end{tikzcd}$$

This yields a map $\beta \colon B_G(O(\mathbb{F})) \to B_G(O)$.

We recall that a $\hat{p}$-cohomology isomorphism is a map inducing an isomorphism on cohomology with coefficients in any p-complete abelian group; see [**13**].

Proposition 2.13. *For each $H \leqslant G$, the map $\beta^H \colon B_G(O(\mathbb{F}))^H \to B_G(O)^H$ induces an isomorphism on π_0 and a $\hat{p}$-cohomology isomorphism on each component.*

Proof. Suppose $\alpha_n \colon H \to O(m_n, \mathbb{F})$ is the action map of an H-representation $V_n^{\alpha_n}$ over $\mathbb{F}$. Then since H is finite, α_n factors through $\Pi(m_n, j)$ for some j. The representation $V_n^{\alpha_n}$ is in the same stable class as $V_{n'}^{\alpha_{n'}}$ for some $n' > n$ where $j_{n'} > j$. Therefore, each stable representation over $\mathbb{F}$ can be represented by a map $\alpha_n \colon H \to \Pi_n$ for some n sufficiently large. It follows by Remark 2.2 that $\pi_0(B_G(O(\mathbb{F}))^H)$ is isomorphic to the set of stable representations of H over $\mathbb{F}$, and by a similar argument $\pi_0(B_G(O)^H)$ is isomorphic to the set of stable representations of H. It follows from the definitions that β^H induces λ_H on π_0. Therefore, β^H induces an isomorphism on π_0 for each $H \leqslant G$.

Furthermore, using Remark 2.2, we see that the component of $B_G(O(\mathbb{F}))^H$ corresponding to the sequence $\alpha_n \colon H \to \Pi_n$ can be represented as a colimit of spaces $B(\Pi(m_n, j_n)^{\alpha_n})$, where $\Pi(m_n, j_n)^{\alpha_n}$ denotes the set of elements in $O(m_n, \mathbb{F})$ which commute with the image of α_n. This colimit is equivalent to the corresponding colimit of the spaces $B(O(m_n, \mathbb{F})^{\alpha_n})$. Elements in $O(m_n, \mathbb{F})^{\alpha_n}$ represent equivariant endomorphisms of the orthogonal H-representation α_n over $\mathbb{F}$. By a similar argument the corresponding component of $B_G(O)^H$ can be represented by the colimit of spaces $B(O(V_n)^{\lambda(\alpha_n)})$, where $\lambda(\alpha_n) = b_{V_n} \circ \alpha_n$.

Just as irreducible orthogonal representations over $\mathbb{C}$ come in three types, irreducible orthogonal representations of G over $\mathbb{F}$ come in three analogous types. Following [**7**], we let S_0 consist of those irreducible G-representations over $\mathbb{F}$ which are not isomorphic to their dual, we let S_+ consist of those which admit nondegenerate symmetric bilinear forms, and we let S_- consist of those admitting nondegenerate alternating bilinear forms. By Lemma 3.3 in [**7**], and since $\mathbb{F}$ is algebraically closed, any orthogonal representation (such as α_n) breaks up as an orthogonal direct sum of three types of irreducible orthogonal representations over $\mathbb{F}$. The first type consists of irreducible G-representations in S_+. The second and third types consist of representations of the form $V \oplus V^*$, with $V \in S_0$ or $V \in S_-$, where $V \oplus V^*$ is endowed with the symmetric form

$$\langle (v \oplus f), (w \oplus g) \rangle = g(v) + f(w).$$

It follows that each component of $B_G O(\mathbb{F})^H$ splits into a product of the form

$$\left(\times_{S_+} BO(\mathbb{F})\right) \times \left(\times_{S_0} BGL(\mathbb{F})\right) \times \left(\times_{S_-} BSp(\mathbb{F})\right)$$

with one factor of $BO(\mathbb{F})$, $BGL(\mathbb{F})$, and $BSp(\mathbb{F})$ for each irreducible orthogonal G-representation of each type. (For more details, see, for example, the proof of Proposition 3.4 in [**7**].) The corresponding component of $B_G O^H$ likewise splits into a product

$$\left(\times_{S_+} BO\right) \times \left(\times_{S_0} BGL\right) \times \left(\times_{S_-} BSp\right).$$

Finally, it follows from [**16**] and [**8**, III,§7] (see also [**7**, §6]) that the maps $BO(\mathbb{F}) \to BO$, $BGL(\mathbb{F}) \to BGL$ and $BSp(\mathbb{F}) \to BSp$ induce $\hat{p}$-cohomology isomorphisms. □

3. A map of spherical fibrations

Suppose $p \colon P \to B$ is a (G, A)-bundle, and W is a virtual A-representation. Let $S(p(W))$ be the underlying stable spherical fibration of $p(W)$. Let $S(p(W))_{(p)}$ denote the fiberwise smash product of $S(p(W))$ with a p-local circle. Our goal in this section is to show that for any virtual A-representation V, there is a map from the trivial $G\mathcal{F}(\mathbb{S}_{(p)})$-fibration to $S(p(\psi^k V - V))_{(p)}$ having certain specified degrees on fixed points (see Theorem 3.5).

Let $RO(A)$ and $R(A)$ denote the real and complex representation rings of A respectively, and let $c \colon RO(A) \to R(A)$ and $r \colon R(A) \to RO(A)$ be the complexification and “realification” homomorphisms.

Lemma 3.1. *Suppose A has odd order. Then the map $r\colon R(A) \to RO(A)$ has cokernel $\mathbb{Z}/2$, generated by the one-dimensional trivial representation. In particular, if V is an orthogonal A-representation, then the numbers $\dim_{\mathbb{R}} V^K$ have the same parity for all $K \leqslant A$.*

Proof. By Proposition V.1.3 of [**5**], any nontrivial real irreducible representation of A is in the image of r. Clearly, twice the trivial representation is in the image of r. The last statement follows since either V or $V-1$ is the underlying real representation of a complex representation. □

Given an A-equivariant map of p-local spheres $f\colon X \to Y$, let $d_f(K)$ denote the degree of $f^K\colon X^K \to Y^K$ for each $K \leqslant A$ such that X^K and Y^K have the same dimension.

Lemma 3.2. *Let p and k be distinct odd prime numbers. Suppose that V is an even-dimensional virtual orthogonal representation of a p-group A_p. Then there is a (stable) A_p-equivariant map $f'_V\colon S^0_{(p)} \to S^{\psi^k V - V}_{(p)}$ such that $d_{f'_V}(K) = \sqrt{k}^{\dim V^K}$ for all $K \leqslant A_p$.*

Proof. Of course, the claim means that there is an A_p-representation U such that $U + \psi^k V - V$ is an actual A_p-representation, and there is an A_p-equivariant map $S^U_{(p)} \to S^{U+\psi^k V - V}_{(p)}$ having the claimed degrees. Observe that these degrees are p-local integers by Lemma 3.1. Also, note that $(\psi^k V)^H$ and V^H have the same dimension, so the degrees are defined for all $K \leqslant A_p$.

By Lemma 3.1 and Brauer Induction (cf. [**16**, 2.4]), V is the underlying real representation of an integral linear combination of representations induced from complex one-dimensional representations of subgroups of A_p. So, it suffices to consider such induced representations.

If W is a complex one-dimensional H-representation, $H \leqslant A_p$, then there is a (nonlinear) H-equivariant map of $W \to \psi^k W$ given by $z \to z^k$, inducing an H-equivariant map $S^W_{(p)} \to S^{\psi^k W}_{(p)}$. This map has nonequivariant degree k, and has degree 1 on H'-fixed points for any subgroup $H' \leqslant H$ acting nontrivially on W. Since k and p are relatively prime, this map is an H-equivalence, and also $\psi^k \mathrm{ind}_H^{A_p} W = \mathrm{ind}_H^{A_p} \psi^k W$. So, there is a (nonlinear) map $\mathrm{ind}_H^{A_p} W \to \psi^k \mathrm{ind}_H^{A_p} W$ whose degree on K-fixed points is equal to k^n, where n is the dimension of $(\mathrm{ind}_H^{A_p} W)^K$. □

The following transfer argument will certainly be familiar to the expert.

Lemma 3.3. *Let A_p be a Sylow p-subgroup of a finite group A, and suppose that $f'\colon X \to Y$ is an A_p-equivariant map between A-equivariant spheres of the same dimension. Then there is a stable A-equivariant map $f\colon X \to Y$ such that $d_f(K) = |(A/A_p)^K| d_{f'}(K)$ for all $K \leqslant A_p$.*

Proof. Let $U = \mathbb{C}[A/A_p]$. We will represent f as a map $S^U \wedge X \to S^U \wedge Y$. Let $e = \pi_1, \pi_2, \cdots, \pi_n$ be coset representatives of A/A_p, and let $e_i = \pi_i A_p$ be the associated basis of U. If $u \in U$, let u_i be defined by the formula $u = \sum_{i=1}^n u_i e_i$.

Let ϕ be a homeomorphism from $\mathbb{C}$ to the open disk of radius $1/2$ centered at the origin. Let $i\colon U \to U$ be given by $i(\sum u_i e_i) = e_1 + \sum_{i=1}^n \phi(u_i) e_i$. Then i is an

A_p-equivariant embedding of U in itself, since $ge_1 = e_1$ for any $g \in A_p$. This gives rise to an A-equivariant embedding

$$A \times_{A_p} (U \times X) \to U \times X$$

and by the Pontryagin-Thom construction, we obtain an A-equivariant map

$$S^U \wedge X \to A_+ \wedge_{A_p} (S^U \wedge X).$$

Now any A_p-equivariant map $f'\colon X \to Y$ induces an A-equivariant map

$$A_+ \wedge_{A_p} (S^U \wedge X) \to S^U \wedge Y$$

and hence an A-equivariant map $f\colon S^U \wedge X \to S^U \wedge Y$.

For each $i = 1, 2, \cdots, n$, let $K_i = \pi_i^{-1} K \pi_i$. The K-fixed points of

$$A_+ \wedge_{A_p} (S^U \wedge X)$$

are isomorphic to

$$\bigvee_{\{i | K_i \leqslant A_p\}} (S^U \wedge X)^{K_i}.$$

The number of wedge summands in this wedge product is equal to $|(A/A_p)^K|$. It follows that the degree of f on K fixed points is equal to $|(A/A_p)^K| d_{f'}(K)$. □

Corollary 3.4. *Suppose that A is a finite group, and let k be a prime different from p. Suppose V is a virtual orthogonal A-representation. Then there is a stable A-equivariant map $f_V\colon S^0_{(p)} \to S^{\psi^k V - V}_{(p)}$ such that $d_{f_V}(K) = \sqrt{k}^{\dim V^K}$ for all p-subgroups $K \leqslant A$.*

Proof. As above, the corollary means that there is an A-representation U such that $U + \psi^k V - V$ is an actual A-representation, and that there is an A-equivariant map $f_V\colon S^U_{(p)} \to S^{U+\psi^k V - V}_{(p)}$ with the claimed degrees. By Lemma 3.2, there is an A_p-equivariant map f'_V having the correct degrees.

We view A/A_p as an A_p-set, hence as an element in the Burnside ring of A_p. Let W be an A-representation large enough that $[S^W, S^W]_{A_p}$ is isomorphic to the Burnside ring of A_p. Then there is an A_p-equivariant map $\alpha\colon S^W \to S^W$ with $\deg(\alpha^K) = |(A/A_p)^K|$ for all $K \leqslant A_p$. As a K-set, A/A_p breaks up into orbits, and all nontrivial orbits must have order divisible by p. Since the order A/A_p is not divisible by p, $|(A/A_p)^K|$ is prime to p. Now, $\Sigma^W S^U_{(p)}$ is p-local, so that $\alpha \wedge 1\colon S^W \wedge S^U_{(p)} \to S^W \wedge S^U_{(p)}$ is an equivariant homotopy equivalence. We let $\beta\colon \Sigma^W S^U_{(p)} \to \Sigma^W S^U_{(p)}$ denote the inverse of $\alpha \wedge 1$, so that $d_\beta(K) \cdot |(A/A_p)^K| = 1$ for all $K \leqslant A_p$.

Now, apply Lemma 3.3 to the A_p-equivariant map $\Sigma^W f'_V \circ \beta$ to get an A-equivariant map $f_V\colon \Sigma^W S^U_{(p)} \to \Sigma^W S^{U+\psi^k V - V}_{(p)}$ such that for each $K \leqslant A_p$,

$$d_{f_V}(K) = |(A/A_p)^K| d_\beta(K) d_{f'_V}(K) = d_{f'_V}(K).$$

This suffices since the degree of an A-equivariant map on K fixed points only depends on the conjugacy class of K and all p-groups are subconjugate to A_p. □

Theorem 3.5. *Suppose A is a finite G-group, $p\colon P \to B$ is a based principal (G, A)-bundle, and V is a virtual orthogonal A-representation. Then there is a stable G-map from the trivial $G\mathcal{F}(\mathbb{S}_{(p)})$-fibration to $S(p(\psi^k V - V))$ whose restriction to any orbit $Q \subseteq B$ has degree function*

$$d(K) = \sqrt{k}^{\dim(p(V),Q,K)}.$$

This degree function is defined on all K such that Q^K is nonempty.

Proof. Let U be an A-representation such that $U + \psi^k V - V$ is an actual A-representation. We will construct a (stable) map of $G\mathcal{F}(\mathbb{S}_{(p)})$-fibrations from $S(p(U))$ to $S(p(U + \psi^k V - V))$. First, we represent $p(U)$ and $p(U + \psi^k V - V)$ by actual G-vector bundles, both having the same fiber $2U + \psi^k V - V$, namely

$$p^U + t^{U+\psi^k V-V} \text{ and } p^{U+\psi^k V-V} + t^U.$$

By Corollary 3.4, there is a map

$$S(p^U) \to S(p^{U+\psi^k V-V}).$$

Now, if $Q \cong G/H$ is an orbit in B, then $p^{-1}(Q)$ can be identified with $G \times_H A_\alpha$, where H acts on A_α by a homomorphism $\alpha\colon H \to A$. Then the restriction of $S(p^U)$ to Q can be identified with $G \times_H S_{(p)}^{\alpha^*(U)}$. If Q^K is nonempty for some $K \leqslant G$, then K is conjugate to some $K' \leqslant H$, and by Corollary 3.4, the degree of our map on the K-fixed points of this G-space is $\sqrt{k}^{\dim(\alpha^*(V)^{K'})}$. But $\alpha^*(V)^{K'}$ has the same dimension as $(p_Q^V)^K$.

Also by Corollary 3.4, the map

$$f_V : S_{(p)}^U \to S_{(p)}^{U+\psi^k V-V}$$

is invertible, so that we get a map

$$S(t^{U+\psi^k V-V}) \to S(t^U).$$

If Q is an orbit in B, then the preimages of Q in $S(t^U)$ and $S(t^{U+\psi^k V-V})$ are just $Q \times S_{(p)}^U$ and $Q \times S_{(p)}^{U+\psi^k V-V}$. So, if $K \leqslant G$ is conjugate to some $K' \leqslant H$, then the restriction of our map to the K-fixed points of the preimage of Q has degree $(\sqrt{k})^{-\dim V^K}$. But V^K has the same dimension as $(t_Q^V)^K$. Therefore, adding these two maps (by fiberwise smash product) gives a map with the claimed degrees. $\square$

4. Constructing homotopies from geometric data

4.1. Classifying maps

In what follows, we produce a contractible parameter space of classifying maps for any given $(G, Spin)$-bundle or $G\mathcal{F}(\mathbb{S}_{(p)})$-fibration, and, given a map of $(G, Spin)$-bundles or $G\mathcal{F}(\mathbb{S}_{(p)})$-fibrations, we produce a contractible parameter space of homotopies between the classifying maps of the source and target. We use these constructions to show how to pass from geometric data, concerning maps of $G\mathcal{F}(\mathbb{S}_{(p)}))$-fibrations, to homotopical data, concerning the classifying homotopies of these maps.

In particular, we wish to show that if two maps of $G\mathcal{F}(\mathbb{S}_{(p)}))$ fibrations restrict to equivalent maps on orbits, then their classifying homotopies restrict to equivalent homotopies on orbits.

Remark 4.1. Suppose that $\varepsilon\colon C \to B$ is an acyclic fibration (i.e. a homotopy equivalence and a fibration). Then the map

$$\varepsilon_*\colon \mathrm{Map}_G(B, C) \to \mathrm{Map}_G(B, B)$$

is an acyclic fibration. Therefore, $\varepsilon_*^{-1}(\mathrm{id}_B)$ is contractible. That is, the space of all equivariant sections of ε is contractible.

Remark 4.2. Now suppose given a commutative diagram

$$\begin{tikzcd} C_1 \arrow[r, "\varepsilon_1"] \arrow[d, "f"] & B_1 \arrow[d, "g"] \\ C_2 \arrow[r, "\varepsilon_2"] & B_2 \end{tikzcd},$$

where ε_1 and ε_2 are acyclic fibrations. Then

$$(\varepsilon_2)_*\colon \mathrm{Map}_G(B_1, C_2) \to \mathrm{Map}_G(B_1, B_2)$$

is an acyclic fibration. Therefore $(\varepsilon_2)_*^{-1}(g)$ is contractible. That is, the space of all G-maps h from B_1 to C_2 satisfying $\varepsilon_2 \circ h = g$ is contractible. Note that if η_1 is a section of ε_1 and η_2 is a section of ε_2, then $\varepsilon_2 \circ (f \circ \eta_1) = g$ and $\varepsilon_2 \circ (\eta_2 \circ g) = g$, so there is a homotopy H from $f \circ \eta_1$ to $\eta_2 \circ g$ such that $\varepsilon_2 \circ H$ is constant at g.

Now, suppose given a $(G, Spin)$-bundle ξ over a finite G-CW-complex B. Recall that ξ is represented by a sequence ξ^{V_i} of $(G, Spin(V_i))$-bundles. Let $C(\xi)$ denote the homotopy pullback of the diagram

$$B \longleftarrow \mathrm{colim}_i B(\mathcal{P}(\xi^{V_i}), \mathcal{C}_G(Spin(V_i)), \mathcal{O}) \longrightarrow B_G(Spin)\ .$$

Then we have maps $\varepsilon_\xi\colon C(\xi) \to B$ and $\pi_\xi\colon C(\xi) \to B_G(Spin)$. Note that ε_ξ is an acyclic fibration and $\varepsilon_\xi \times \pi_\xi$ is a fibration.

Similarly, if ξ is a $G\mathcal{F}(\mathbb{S}_{(p)})$-fibration over B, represented by the sequence ξ^{V_i} of $G\mathcal{F}(\mathbb{S}_{(p)}^{V_i})$-fibrations, then let $C'(\xi)$ denote the homotopy pullback of

$$B \longleftarrow \mathrm{colim}_i B(\mathcal{P}(\xi^{V_i}), \mathcal{A}(\mathbb{S}_{(p)}^{V_i}), \mathcal{O}) \longrightarrow B_G(\mathbb{S}_{(p)})\ .$$

Then again we have maps $\varepsilon'_\xi\colon C'(\xi) \to B$ and $\pi'_\xi\colon C'(\xi) \to B_G(\mathbb{S}_{(p)})$, with ε'_ξ an acyclic fibration and $\varepsilon'_\xi \times \pi'_\xi$ is a fibration.

Definition 4.3. If ξ is a $(G, Spin)$-bundle, then a choice of section η_ξ of ε_ξ determines a classifying map $\chi(\xi, \eta_\xi) = \pi_\xi \circ \eta_\xi$ for ξ. Likewise, if ξ is a $G\mathcal{F}(\mathbb{S}_{(p)})$-fibration, then a choice of section η'_ξ of ε'_ξ determines a classifying map $\chi(\xi, \eta'_\xi) = \pi'_\xi \circ \eta'_\xi$. In either case, the space of all such sections is contractible by Remark 4.1, so we have a contractible parameter space of classifying maps for ξ. When η_ξ or η'_ξ is understood, we denote our classifying map by $\chi(\xi)$.

Let $Bj\colon B_G(Spin) \to B_G(\mathbb{S}_{(p)})$ denote the canonical map induced by the inclusion of $\mathcal{C}_G(Spin(V_i))$ in $\mathcal{A}(\mathbb{S}_{(p)}^{V_i})$. Thus, Bj represents replacing a stable $(G, Spin)$-bundle ξ by its underlying stable $G\mathcal{F}(\mathbb{S}_{(p)})$-fibration $S(\xi)$. We have a commutative diagram

$$\begin{tikzcd} B \arrow[d, equal] & C(\xi) \arrow[l, "\varepsilon_\xi"'] \arrow[r, "\pi_\xi"] \arrow[d, "Bj_\xi"] & B_G(Spin) \arrow[d, "Bj"] \\ B & C'(S(\xi)) \arrow[l, "\varepsilon'_{S(\xi)}"'] \arrow[r, "\pi'_{S(\xi)}"] & B_G(\mathbb{S}_{(p)}). \end{tikzcd}$$

Given a section η_ξ of ε_ξ, then $\eta_{S(\xi)} := Bj_\xi \circ \eta_\xi$ is a section for $\varepsilon'_{S(\xi)}$, so that $\chi(S(\xi), \eta_{S(\xi)}) = Bj \circ \chi(\xi, \eta_\xi)$.

Also, let $k_p\colon B_G(\mathbb{S}_{(p)}) \to B_G(\mathbb{S}_p)$ be the fiberwise completion map constructed in [**9**, §7], and let $Bj_p = k_p \circ Bj\colon B_G Spin \to B_G(\mathbb{S}_p)$.

A map $\phi\colon \xi_1 \to \xi_2$ of $(G, Spin)$-bundles, covering $\bar{\phi}\colon B_1 \to B_2$, induces a map $C(\phi)\colon C(\xi_1) \to C(\xi_2)$ in the diagram

$$\begin{tikzcd} B_1 \arrow[d, "\bar{\phi}"'] & C(\xi_1) \arrow[l, "\varepsilon_1"'] \arrow[d, "C(\phi)"] \\ B_2 & C(\xi_2). \arrow[l, "\varepsilon_2"'] \end{tikzcd}$$

If η_1 and η_2 are sections of ε_1 and ε_2, then by Remark 4.2, $C(\phi) \circ \eta_1$ and $\eta_2 \circ \bar{\phi}$ are both points in a contractible parameter space of maps from B_1 to $C(\xi_2)$. We may therefore choose a homotopy between them, with any two choices of homotopy being equivalent. We let H_ϕ denote such a homotopy. Of course, an analogous statement holds for $G\mathcal{F}(\mathbb{S}_{(p)})$-fibrations.

Definition 4.4. Define the homotopy $\chi(\phi)$ from $\chi(\xi_1)$ to $\chi(\xi_2) \circ \bar{\phi}$ to be $\pi_{\xi_2} \circ H_\phi$.

Remark 4.5. If $\phi\colon \xi_1 \to \xi_2$ and $\phi'\colon \xi_1 \to \xi_2$ are two maps of $G\mathcal{F}(\mathbb{S}_{(p)})$-fibrations, and $\eta_1\colon X \to C(\xi_1)$ and $\eta_2\colon X \to C(\xi_2)$ are sections, then it follows from Remark 4.2 that $C(\phi) \circ \eta_1$ and $C(\phi') \circ \eta_1$ are homotopic. Applying π_{ξ_2} to a choice of homotopy would give rise to a path from $\chi(\xi_1)$ to itself, i.e. a loop in $\mathrm{Map}_G(X, B_G(\mathbb{S}_{(p)}))$ based at $\chi(\xi_1)$. This loop may or may not be null-homotopic. If, on the other hand, there is a homotopy H from ϕ to ϕ', then we have a homotopy $C(H)$ from $C(\phi)$ to $C(\phi')$ such that $\pi_{\xi_2} \circ C(\phi)$ is constant. Thus, a homotopy from ϕ to ϕ' determines an equivalence between $\chi(\phi)$ and $\chi(\phi')$.

Remark 4.6. The trivial $G\mathcal{F}(\mathbb{S}_{(p)})$-fibration $S(t)$ over G/H canonically determines a 0-simplex in

$$\mathrm{colim}_i B(\mathcal{P}(t^{V_i}), \mathcal{A}(\mathbb{S}_{(p)}^{V_i}), \mathcal{O})^H,$$

and hence a section $\eta_t\colon G/H \to C(t)$. A self-map $\zeta\colon S(t) \to S(t)$ determines a 1-simplex in the above colimit, which in turn determines a homotopy $\hat{\zeta}\colon G/H \times I \to C(t)$ from η_t to $C(\zeta) \circ \eta_t$ such that $\varepsilon_t \circ \hat{\zeta}$ is constant. Therefore, by Definition 4.4, $\chi(\zeta) = \pi_t \circ \hat{\zeta}$. Note that $\chi(\zeta)$ may be thought of as an element in $\pi_1(B_G(\mathbb{S}_{(p)})^H)$. Since this is abelian, all fundamental groups for all choices of basepoint are canonically isomorphic.

Lemma 4.7. *Given maps $\phi_1\colon \xi_1 \to \xi_2$ and $\phi_2\colon \xi_2 \to \xi_3$, covering $\bar{\phi}_1\colon B_1 \to B_2$ and $\bar{\phi}_2\colon B_2 \to B_3$, the homotopy $\chi(\phi_2 \circ \phi_1)$ is equivalent to $\chi(\phi_1) \cdot (\chi(\phi_2) \circ \bar{\phi}_1)$. This holds for $G\mathcal{F}(\mathbb{S}_{(p)})$-fibrations or $(G, Spin)$-bundles.*

Proof. This follows immediately from the contractibility of our parameter space of maps from B_1 to $C(\xi_3)$ and the fact that $\pi_{\xi_3} \circ C(\phi_2) = \pi_{\xi_2}$. □

Corollary 4.8. *Suppose given maps ϕ, $\phi'\colon \xi_1 \to \xi_2$ of $G\mathcal{F}(\mathbb{S}_{(p)})$-fibrations or $(G, Spin)$-bundles, covering the same inclusion $\bar{\phi}\colon B_1 \to B_2$. Suppose that for each orbit inclusion $\bar{j}_1\colon Q \to B_1$, the restrictions of ϕ and ϕ' to Q are homotopic. Then the homotopies $\chi(\phi)$ and $\chi(\phi')$ are 0-equivalent.*

Proof. Let $\bar{j}_2 = \bar{\phi} \circ \bar{j}_1$. We then have a commutative diagram

$$\begin{tikzcd}
\xi_1|_Q \arrow[r, "j_1"] \arrow[d, "\phi|_Q"'] & \xi_1 \arrow[d, "\phi"] \\
\xi_2|_Q \arrow[r, "j_2"] & \xi_2.
\end{tikzcd}$$

Here j_1 and j_2 are the maps induced by the inclusions $\bar{j}_1$ and $\bar{j}_2$. We have the same commutative diagram with $\phi|_Q$ and ϕ replaced by $\phi'|_Q$ and ϕ'. Now, it follows from Lemma 4.7, that

$$\chi(j_1) \cdot (\chi(\phi) \circ \bar{j}_1) \simeq \chi(\phi|_Q) \cdot (\chi(j_2) \circ \mathrm{id}_Q),$$

and

$$\chi(j_1) \cdot (\chi(\phi') \circ \bar{j}_1) \simeq \chi(\phi'|_Q) \cdot (\chi(j_2) \circ \mathrm{id}_Q).$$

But by Remark 4.5, a homotopy between $\phi|_Q$ and $\phi'|_Q$ induces an equivalence between $\chi(\phi|_Q)$ and $\chi(\phi'|_Q)$, and therefore an equivalence between $\chi(\phi) \circ \bar{j}_1$ and $\chi(\phi') \circ \bar{j}_1$. □

4.2. The main construction

Now suppose that $B = \cup B_n$, where each B_n is a finite G-CW-complex, and the inclusions $\bar{i}_n\colon B_n \to B_{n+1}$ are cofibrations. Suppose given $(G, Spin)$-bundles ξ_n over each B_n such that $\xi_{n+1}|_{B_{n+1}} \cong \xi_n$. Let t_n denote the trivial $(G, Spin)$-bundle on B_n, and let $i_n^t\colon t_n \to t_{n+1}$ denote the map induced by $\bar{i}_n$. We next show how to choose sections yielding compatible classifying maps for the ξ_n, and hence a classifying map $g\colon B \to B_G Spin$. We also show how to choose sections so that $\chi(t_n)$ can be taken to be trivial and $\chi(i_n^t)$ can be taken to be the constant homotopy. We use this to construct and study a null-homotopy of $Bj_p \circ g$.

Remark 4.9. Suppose we choose isomorphisms $i_n^\xi\colon \xi_n \to \xi_{n+1}$. Then, in the diagram below, the left vertical map is a cofibration and the right vertical map is an acyclic fibration:

$$\begin{tikzcd} B_n \arrow[r, "C(i_n^\xi)\circ\eta_{\xi_n}"] \arrow[d, "\bar{i}_n"] & C(\xi_{n+1}) \arrow[d, "\varepsilon_{\xi_{n+1}}"] \\ B_{n+1} \arrow[r, equal] & B_{n+1}. \end{tikzcd}$$

By choosing lifts, we obtain sections $\eta_{\xi_{n+1}}\colon B_{n+1} \to C(\xi_{n+1})$ so that $\eta_{\xi_{n+1}} \circ \bar{i}_n = C(i_n^\xi) \circ \eta_{\xi_n}$. Therefore, $\chi(\xi_{n+1}) \circ \bar{i}_n = \chi(\xi_n)$, and we may take $\chi(i_n^\xi)$ to be constant. Then we get a map $g\colon B \to B_G(Spin)$ so that $g\,|_{B_n} = \chi(\xi_n)$.

Lemma 4.10. *There exist sections $\eta_{t_n}\colon B_n \to C(t_n)$ for each n such that*

$$\eta_{t_{n+1}} \circ \bar{i}_n = C(i_n^t) \circ \eta_{t_n}$$

and $\chi(t_n, \eta_{t_n})$ is the trivial map for each n. In particular, the homotopy $\chi(i_n^t)$ may be taken to be constant.

Proof. Let $\langle e\rangle$ denote the trivial group. Note that the category $\mathcal{C}_G(\langle e\rangle)$ is isomorphic to the orbit category $\mathcal{O}_G$. Moreover, the composite

$$\mathcal{C}_G(\langle e\rangle) \hookrightarrow \mathcal{C}_G(Spin(V_i)) \xrightarrow{\mathcal{P}(t^{V_i})} \mathcal{U}$$

may be identified with the functor $\underline{B_n}\colon \mathcal{O}_G \to \mathcal{U}$ taking G/H to B_n^H. Now let $C^t(B_n)$ be the homotopy limit of the diagram

$$B_n \longleftarrow B(\underline{B_n}, \mathcal{O}_G, \mathcal{O}) \longrightarrow B(*, \mathcal{O}_G, \mathcal{O})\ .$$

Let $\pi_n\colon C^t(B_n) \to B(*, \mathcal{O}_G, \mathcal{O})$ and $\varepsilon_n\colon C^t(B_n) \to B_n$ be the induced maps. Since $B(*, \mathcal{O}_G, \mathcal{O})$ is contractible, $\pi_n \times \varepsilon_n\colon C^t(B_n) \to B(*, \mathcal{O}_G, \mathcal{O}) \times B_n$ is an acyclic fibration for each n. Now, by choosing lifts in the diagram below, we can inductively define sections $\eta_n^t\colon B_n \to C^t(B_n)$ such that $\pi_n \circ \eta_n^t$ is trivial and $C^t(i_n) \circ \eta_n^t = \eta_{n+1}^t \circ \bar{i}_n$:

$$\begin{tikzcd} B_n \arrow[r, "C^t(\bar{i}_n)\circ\eta_n^t"] \arrow[d, "\bar{i}_n"] & C^t(B_{n+1}) \arrow[d, "\pi_n\times\varepsilon_n"] \\ B_{n+1} \arrow[r] \arrow[ur, dotted, "\eta_{n+1}^t"] & B(*, \mathcal{O}_G, \mathcal{O}) \times B_n. \end{tikzcd}$$

Finally, let η_{t_n} be the composite of η_n^t with the natural inclusion $C^t(B_n) \to C(t_n)$. $\square$

Definition 4.11. If X and Y are G-spaces, then two homotopies F_1, F_2 are equivalent $(F_1 \sim F_2)$ if the corresponding paths in $\mathrm{Map}_G(X, Y)$ are path homotopic. Two homotopies F_1, F_2 are 0-equivalent $(F_1 \sim_0 F_2)$ if for each orbit $Q \subseteq X$, the restrictions of F_1 and F_2 to Q are equivalent.

Theorem 4.12. *Suppose given maps $h_n\colon S(t_n) \to S(\xi_n)$ for each n such that $h_{n+1} \circ S(i_n^t)$ and $S(i_n^\xi) \circ h_n$ restrict to homotopic maps on all orbits $Q \subseteq B_n$. Then there is a null-homotopy $\chi(h)$ of $Bj_p \circ g$ whose restriction to B_n is 0-equivalent to $k_p \circ \chi(h_n)$ for each n. In particular, there is a lift $\tilde{g}$ of g to $Fib(Bj_p)$.*

Proof. Since $\chi(\xi_n) = g\,|_{B_n}$ and $\chi(t_n)$ is trivial, it follows that $\chi(S(\xi_n)) = Bj \circ g\,|_{B_n}$ and $\chi(S(t_n))$ is trivial. By Lemma 4.7 and Corollary 4.8,

$$\chi(h_n) \cdot \chi(S(i_n^\xi)) \sim \chi(S(i_n^\xi) \circ h_n) \sim_0 \chi(h_{n+1} \circ S(i_n^t)) \sim \chi(S(i_n^t)) \cdot (\chi(h_{n+1}) \circ \bar{i}_n).$$

By Remark 4.9 and Lemma 4.10, $\chi(S(i_n^t)) = Bj(\chi(i_n^t))$ and $\chi(S(i_n^\xi)) = Bj(\chi(i_n^\xi))$ are constant, so

$$\chi(h_{n+1}) \circ \bar{i}_n \sim_0 \chi(h_n)$$

for each n. The result will follow from Theorem 4.15, which can be found in the next subsection. $\square$

Given the hypotheses of Theorem 4.12, we have a diagram

$$\begin{tikzcd}
Fib(g) \arrow[r] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\tilde{g}"] & B_G Spin \arrow[d, equal] & \\
\Omega B_G(\mathbb{S}_p) \arrow[r] & Fib(Bj_p) \arrow[r] & B_G Spin \arrow[r, "Bj_p"] & B_G(\mathbb{S}_p).
\end{tikzcd}$$

We need to understand the effect of α on components of $Fib(g)^H$ for $H \leqslant G$. Since $\pi_1(B_G Spin^H)$ is trivial for all $H \leqslant G$, $\pi_0(Fib(g)^H)$ is equal to the kernel of $\pi_0(g^H)$. Therefore, an element $x \in \pi_0(Fib(g)^H)$ is determined by an orbit $\hat{x}\colon G/H \to B_n$ for some n such that $\xi_n\,|_{G/H} \cong t_n\,|_{G/H}$. If $\delta\colon \xi_n\,|_{G/H} \to t_n\,|_{G/H}$ is any such isomorphism, then $S(\delta) \circ h_n\,|_{G/H}$ is a self-map of the trivial $G\mathcal{F}(\mathbb{S}_{(p)})$-fibration over G/H, which is equivalent to an H-equivariant stable self-equivalence of $S^0_{(p)}$. By Remark 4.6, such a map determines an element $\beta(x)$ in $\pi_0(\Omega B_G(\mathbb{S}_{(p)})^H)$.

Lemma 4.13. *The completion map $k_p\colon \Omega B_G(\mathbb{S}_{(p)}) \to \Omega B_G(\mathbb{S}_p)$ takes $\beta(x)$ to $\alpha(x)$.*

Proof. First, $k_p \circ \chi(h_n)\,|_{G/H}$ is a null-homotopy of $Bj_p \circ g \circ \hat{x}$ which by Theorem 4.12 is equivalent to $\chi(h) \circ \hat{x}$. Now, let $j_n^\xi\colon \xi_n\,|_{G/H} \to \xi_n$ and $j_n^t\colon t_n\,|_{G/H} \to t_n$ be the maps induced by the inclusion $\hat{x}\colon G/H \to B_n$. Then, $\chi(j_n^\xi)^{-1} \cdot \chi(\delta) \cdot \chi(j_n^t)$ is a null-homotopy of $\chi(\xi_n) \circ \hat{x} = g \circ \hat{x}$. Since any two null-homotopies of $g \circ \hat{x}$ are equivalent, it follows that $\alpha(x)$ is given by

$$\begin{aligned}
&(k_p \circ \chi(h_n)\,|_{G/H}) \cdot (Bj_p \circ (\chi(j_n^\xi)^{-1} \cdot \chi(\delta) \cdot \chi(j_n^t))) \\
&\qquad = k_p \circ (\chi(h_n)\,|_{G/H} \cdot \chi(S(j_n^\xi))^{-1} \cdot \chi(S(\delta)) \cdot \chi(S(j_n^t))).
\end{aligned}$$

Now,

$$\begin{aligned}
&\chi(h_n)\,|_{G/H} \cdot \chi(S(j_n^\xi))^{-1} \cdot \chi(S(\delta)) \cdot \chi(S(j_n^t)) \\
&\qquad \sim \chi(S(j_n^t))^{-1} \cdot \chi(S(j_n^t)) \cdot \chi(h_n)\,|_{G/H} \cdot \chi(S(j_n^\xi))^{-1} \cdot \chi(S(\delta)) \cdot \chi(S(j_n^t)).
\end{aligned}$$

By Lemma 4.7, this is equivalent to

$$\begin{aligned}
&\chi(S(j_n^t))^{-1} \cdot \chi(h_n\,|_{G/H}) \cdot \chi(S(\delta)) \cdot \chi(S(j_n^t)) \\
&\qquad \simeq \chi(S(j_n^t))^{-1} \cdot \chi(S(\delta) \circ h_n\,|_{G/H}) \cdot \chi(S(j_n^t)).
\end{aligned}$$

By definition, this is $\beta(x)$. $\square$

4.3. Replacing homotopies

To complete the proof of Theorem 4.12, we need to show that given a G-map $f\colon B \to B_G(\mathbb{S}_p)$, together with null-homotopies H_n of $f_n = f\,|_{B_n}$ such that for each n, $H_{n+1} \circ \bar{i}_n \sim_0 H_n$, we can construct a null-homotopy H of f such that $H\,|_{B_n} \sim_0 H_n$ for each n. As shorthand, we let $Z = B_G(\mathbb{S}_p)$. Then Z is an equivariant commutative Hopf space, or Hopf G-space, and $\pi_0(Z^H)$ is a group for each $H \leqslant G$. This implies, in particular, that Z has a homotopy inverse map, so that $[A, Z]_G$ is a group for each G-space A. Also, $\pi_n(Z^H)$ is finite for all $n \geqslant 2$ and all $H \leqslant G$.

If $H\colon B \to Z^I$ is a homotopy from f to g, let $(-H)\colon B \to Z^I$ be the homotopy from g to f given by $(-H)(x)(t) = H(x)(1-t)$. If $H_1\colon B \to Z^I$ and $H_2\colon B \to Z^I$ are homotopies from f to g and g to h respectively, then we let $(H_1 \cdot H_2)$ denote the usual homotopy from f to h. Given a homotopy $H\colon B \to Z^I$ from $f\colon B \to Z$ to $g\colon B \to Z$ and an arbitrary map $E\colon B \to \Omega Z$, let $(H + E)\colon B \to Z^I$ be given by

$$(H + E)(x)(t) = \mu_Z(H(x)(t), E(x)(t)).$$

Note that $(H + E)$ is a homotopy from f to g since the unit in Z is strict. Let $(H - E) = (H + (-E))$.

Lemma 4.14. *The homotopies below are all equivalent:*

$$((H_1 \cdot H_2) + E), \quad ((H_1 + E) \cdot H_2) \quad \textit{and} \quad (H_1 \cdot (H_2 + E)).$$

Also, $-(H_1 + E)$ is equivalent to $(-H_1) + (-E)$.

Proof. An explicit path homotopy in the first case is given by

$$(s,t) \to \begin{cases} \mu_Z(H_1(x)(2t), E(x)(\frac{2t}{2-s})) & 0 \leqslant t \leqslant \frac{1}{2} \\ \mu_Z(H_2(x)(2t-1), E(x)(\frac{2t}{2-s})) & \frac{1}{2} \leqslant t \leqslant \frac{2-s}{2} \\ H_2(x)(2t-1) & \frac{2-s}{2} \leqslant t \leqslant 1. \end{cases}$$

A similar homotopy can be constructed in the second case. The last statement is obvious. □

Theorem 4.15. *Suppose given null-homotopies $H_n\colon f_n \simeq *$ such that for each n, $H_{n+1} \circ \bar{i}_n \sim_0 H_n$. Then there exists a null-homotopy H of f such that $H\,|_{B_n} \sim_0 H_n$ for each n.*

Proof. Let $i_Z\colon \tilde{Z} \to Z$ be the G-universal cover of Z. That is, for each $H \leqslant G$, $\pi_n(\tilde{Z}^H) = 0$ for $n = 0$ and $n = 1$, and $\pi_n(i_Z^H)$ is an isomorphism for $n \geqslant 2$. Note that $\Omega\tilde{Z}$ is then the G-connected cover of ΩZ.

Let $K_n\colon B_n \to \Omega Z$ be given by $(-H_n) \cdot (H_{n+1}\,|_{B_n})$. Our hypothesis implies that for each $H \leqslant G$, the map $\pi_0(K_n^H)\colon \pi_0(X_n^H) \to \pi_0(\Omega Z^H)$ is trivial, so K_n factors up to homotopy through $\Omega\tilde{Z}$. For each n, we choose a factorization $\tilde{K}_n$ of K_n, yielding an element

$$\{\tilde{K}_n\} \in \prod_{n=1}^{\infty} [X_n, \Omega\tilde{Z}]_G.$$

Since $\Omega\tilde{Z}$ has finite homotopy groups, each group $[B_n, \Omega\tilde{Z}]_G$ is finite, so that

$\lim^1[B_n, \Omega\tilde{Z}]_G = 0$. Therefore, the map

$$\prod_{n=1}^{\infty}[B_n, \Omega\tilde{Z}]_G \to \prod_{n=1}^{\infty}[B_n, \Omega\tilde{Z}]_G$$

taking $\{\tilde{E}_n\}$ to $\{\tilde{E}_{n+1}|_{B_n} - \tilde{E}_n\}$ is surjective. Thus, there is a sequence of maps $\tilde{E}_n \colon B_n \to \Omega\tilde{Z}$ such that $\tilde{K}_n$ is homotopic to $\tilde{E}_{n+1}|_{B_n} - \tilde{E}_n$ for each n. We let $E_n = \Omega i_Z \circ \tilde{E}_n \colon B_n \to \Omega Z$, so that

$$K_n = E_{n+1}|_{B_n} - E_n. \tag{1}$$

Now let $H'_n = (H_n + E_n)$, and let $K'_n = (-H'_n) \cdot (H'_{n+1}|_{B_n})$. Since E_n factors through $\Omega\tilde{Z}$, H'_n is 0-equivalent to H_n. Also, by Lemma 4.14, we have

$$\begin{aligned} K'_n &= (-H_n - E_n) \cdot ((H_{n+1} + E_{n+1})|_{B_n}) \\ &\simeq \big(-H_n \cdot (H_{n+1}|_{B_n})\big) - (E_n - E_{n+1}|_{B_n}) \\ &= K_n - (E_n - E_{n+1} \circ j_n). \end{aligned}$$

So, K'_n is null-homotopic by equation 1. It follows from this that $H'_{n+1}|_{B_n}$ is equivalent to H'_n for each n.

Now since the inclusion of B_n in B_{n+1} is a cofibration, it follows that H'_{n+1} is equivalent to a homotopy which restricts under the inclusion $B_n \to B_{n+1}$ to H'_n. We can then inductively define the required null-homotopy H on each B_n, and hence on B. $\square$

5. Proof of the main theorem

Now let Π_n and W_n be the G-group and the Π_n-representation constructed in Section 2.4. Choose finite subcomplexes B_n of $B_G(\Pi_n)$ so that colim $B_n = B_G O(\mathbb{F})$ and each map $\bar{i}_n \colon B_n \to B_{n+1}$ is a cofibration. Let $p_n \colon P_n \to B_n$ be the based principal (G, Π_n)-bundle associated to the inclusion $B_n \subseteq B_G\Pi_n$. Note that $\psi^k W_n - W_n$ is a virtual Π_n-representation of dimension zero. Let $\xi_n = p_n(\psi^k W_n - W_n) = (\psi^k - 1)p_n^{W_n}$ be the associated stable G-vector bundle over B_n. We use Remark 4.9 to choose classifying maps $\chi(\xi_n)$ yielding a map $g \colon B_G O(\mathbb{F}) \to B_G Spin$ with $g|_{B_n} = \chi(\xi_n)$ for each n.

By Theorem 3.5, there is a (stable) map h_n for each n from the trivial $G\mathcal{F}(\mathbb{S}_{(p)})$-fibration $S(t_n)$ over B_n to $S(\xi_n)$, and since the restrictions of these trivializations to orbits of B_n are determined up to homotopy by their degrees, they are compatible with n. Therefore, by Theorem 4.12, there is a null-homotopy of $Bj_p \circ g$ whose restriction to B_n is 0-equivalent to $k_p \circ \chi(h_n)$ for each n, and in particular a lift $\tilde{g}$ of g to $Fib(Bj_p)$.

Remark 5.1. Let $\psi^k - 1 \colon B_G O \to B_G Spin$ be the map classifying the operation of replacing ξ by $(\psi^k - 1)\xi$. (See [**9**, §9] for the proof that $\psi^k - 1$ lifts to $B_G Spin$.) We wish to produce a lift γ^k of $\psi^k - 1$ to $Fib(Bj_p)$ so that $\gamma^k \circ \beta$ and $\tilde{g}$ induce the same map on components of fixed point subspaces. First, note that both composites in the diagram below, restricted to B_n, classify the $(G, Spin)$-bundle $(\psi^k - 1)p_n^{W_n}$, so that the following diagram commutes up to homotopy when restricted to B_n:

$$\begin{tikzcd} B_G O(\mathbb{F}) \arrow[d, "\beta"] \arrow[dr, "g"] & \\ B_G O \arrow[r, "\psi^k - 1"] & B_G Spin. \end{tikzcd}$$

Since $\pi_1(B_G Spin^H) = 0$ for all $H \leqslant G$, any two homotopies between $(\psi^k - 1) \circ \beta|_{B_n}$ and $g|_{B_n}$ must be 0-equivalent. Therefore, there are homotopies between $Bj_p \circ (\psi^k - 1) \circ \beta|_{B_n}$ and $Bj_p \circ g|_{B_n}$, which are compatible with n. Since we have already constructed a null-homotopy of $Bj_p \circ g$, we can use Theorem 4.15 and Lemma A.1 to get a null-homotopy of $Bj_p \circ (\psi^k - 1)$ whose restriction to $B_n \subseteq B_G(O(\mathbb{F}))$ is 0-equivalent to the composite null-homotopy

$$Bj_p \circ (\psi^k - 1) \circ \beta|_{B_n} \simeq Bj_p \circ g|_{B_n} \simeq *.$$

Therefore, as needed, we get a lift $\gamma^k \colon B_G O \to Fib(Bj_p)$ of $\psi^k - 1$, and the maps $\gamma^k \circ \beta$ and $\tilde{g}$ induce the same map from $\pi_0(B_G(O(\mathbb{F}))^H)$ to $\pi_0(Fib(Bj_p)^H)$.

In [**9**], we showed that the Atiyah-Bott-Shapiro orientation on $(G, Spin)$-bundles induces a map

$$g_p \colon B_G Spin \to B_G(\mathbb{S}_p; KO_p).$$

Since $B_G Spin$ is nonequivariantly connected, g_p factors through the component $B_G(\mathbb{S}_p; KO_p)_0$ of the basepoint. By Remark 2.9, the homotopy groups $\pi_n(B_G(\mathbb{S}_p)^H)$ are p-complete for all $n \geqslant 1$ (see also [**9**, 8.4]). There are therefore no phantom maps from $B_G Spin$ to $B_G(\mathbb{S}_p)$. Therefore, the right square of the diagram below, which clearly commutes up to homotopy on finite subcomplexes, commutes up to homotopy. From this, we get the induced map f on the left:

$$\begin{tikzcd} Fib(Bj_p) \arrow[r, "q"] \arrow[d, dotted, "f"] & B_G Spin \arrow[r, "Bj_p"] \arrow[d, "g_p"] & B_G(\mathbb{S}_p) \arrow[d, equal] \\ Fib(q) \arrow[r] & B_G(\mathbb{S}_p; KO_p) \arrow[r, "q"] & B_G(\mathbb{S}_p). \end{tikzcd}$$

Recall from Section 2.3 that $Fib(q) \simeq (\Omega^\infty KO_p)^\times$.

Letting J denote the fiber of $\psi^k - 1$ as usual, we now have the following homotopy commutative diagram:

$$\begin{tikzcd} J \arrow[r, "\pi"] \arrow[d, "\alpha^k"] & B_G O \arrow[d, "\gamma^k"'] \arrow[dr, "\psi^k - 1"] & & \\ \Omega B_G(\mathbb{S}_p) \arrow[r, "\tau"] & Fib(Bj_p) \arrow[r, "q"] \arrow[d, "f"'] & B_G(Spin) \arrow[r, "Bj"] \arrow[d, "g_p"] & B_G(\mathbb{S}_p) \arrow[d] \\ & (\Omega^\infty KO_p)^\times \arrow[r] & B_G(\mathbb{S}_p; KO_p) \arrow[r, "q"] & B_G(\mathbb{S}_p). \end{tikzcd}$$

Since J and $B_G O$ are nonequivariantly connected, α^k and $f \circ \gamma^k$ each factor through a component in $\Omega B_G(\mathbb{S}_p) \simeq \Omega^\infty S_p^\times$ and $(\Omega^\infty KO_p)^\times$ respectively. It will follow from

Theorems 5.2 and 5.3 below that these are both the component of 1. We denote these components SF_p and $(B_GO_\otimes)_p$. Let $\sigma^k\colon B_GO \to (B_GO_\otimes)_p$ denote the factorization of $f \circ \gamma^k$.

In [**9**, §9,10], we constructed a map $c(\psi^k)$ such that the composite below is the map induced by $\psi^k/1$:

$$(\Omega^\infty KO_p)^\times \longrightarrow B_G(\mathbb{S}_p; KO_p) \xrightarrow{c(\psi^k)} (\Omega^\infty KO_p)^\times \ .$$

Restricting to the component of 1, we get a self-map $\psi^k/1$ of $(B_GO_\otimes)_p$. Let $\pi\colon (J_\otimes)_p \to (B_GO_\otimes)_p$ denote the homotopy fiber of this self-map. We now have the following homotopy commutative diagram, where the rows are fiber sequences:

$$\begin{tikzcd}
J \arrow[r, "\pi"] \arrow[d, "\alpha^k"] & B_GO \arrow[r, "\psi^k - 1"] \arrow[d, "\sigma^k"'] & B_GSpin \arrow[d, "g_p"] \\
SF_p \arrow[r] \arrow[d, "\varepsilon^k"] & (B_GO_\otimes)_p \arrow[r] \arrow[d, equal] & B_G(\mathbb{S}_p; KO_p)_0 \arrow[d, "c(\psi^k)"] \\
(J_\otimes)_p \arrow[r, "\pi"] & (B_GO_\otimes)_p \arrow[r, "\psi^k/1"] & (B_GO_\otimes)_p.
\end{tikzcd}$$

Our next goal is to analyze the effects of α^k and ε^k on sets of components of fixed point subspaces. We will show in particular that $\varepsilon^k \circ \alpha^k$ induces the same map from $\pi_0(J^H)$ to $\pi_0((J_\otimes)_p^H)$ as the classical map ρ^k, and from there we can easily show that $\varepsilon^k \circ \alpha^k$ induces a p-completion on $\pi_0(J^H)$.

Since $\pi_1(B_GSpin^H) = 0$, an element in $\pi_0(J^H)$ may be identified with a 0-dimensional virtual orthogonal representation of H which is fixed by ψ^k. Moreover, a stable H-equivariant self-map of the p-local sphere spectrum determines an element in $\pi_0(\Omega B_G(\mathbb{S}_{(p)})^H)$ by Remark 4.6, and hence an element in $\pi_0(\Omega B_G(\mathbb{S}_p)^H)$ using k_p. Our next theorem then follows from Lemma 4.13, Theorem 3.5, and Remark 5.1.

Theorem 5.2. *Suppose that V is a 0-dimensional virtual H-representation which is fixed by ψ^k. Then $\alpha^k(V)$ can be represented by the stable self-map of the p-local H-equivariant sphere spectrum $S^0_{(p)}$ whose degree on K-fixed points is given by $\sqrt{k}^{\dim V^K}$.*

Any stable H-equivariant self-map of the p-complete sphere spectrum can be represented by an H-equivariant self-map $\tilde{x}$ of S^V_p. where V is the underlying real orthogonal G-representation of a complex representation of dimension divisible by 4, whose action map factors through $SU(V)$. If $\tilde{x}$ has nonequivariant degree one, it determines an element in $\pi_0(SF_p^H)$.

Theorem 5.3. *Suppose an element $x \in \pi_0(SF_p^H)$ is determined by a map $\tilde{x}\colon S^V_p \to S^V_p$ where V is as above. Then the element $\varepsilon^k(x)$ in $\pi_0((J_\otimes)_p^H) \simeq RO(H)_p^{\psi^k}$ represents the H-representation whose $\mathbb{Z}_p \otimes \mathbb{C}$-valued character is given by*

$$\chi_h(\varepsilon^k(x)) = \deg(\tilde{x}\,|_{(S^V_p)^h}).$$

Proof. The map $\tilde{x}$ induces a self-map $\tilde{x}^*$ of $\widetilde{KO}_p^H(S_p^V)$. By Remark 2.12, $\varepsilon^k(x) \in RO(H)_p^{\psi^k}$ is the representation $\tilde{x}^*(b^V)/b^V$, where b^V is the Bott class of V. By Proposition 5.7 in [**9**], if V is a complex representation of dimension divisible by 4 whose action map factors through $SU(V)$, and $b^V \in \widetilde{KO}_p^G(S_p^V)$ denotes the Bott class of the underlying *Spin* representation, then the complexification of b^V is equal to b_c^V, the Bott class of V in $\tilde{K}_p^G(S_p^V)$. By naturality of complexification, $c(\tilde{x}^*(b^V)/b^V) = \tilde{x}^*(b_c^V)/b_c^V$. We may now restrict to the cyclic subgroup H' generated by h.

Let $\mathcal{P}$ be the kernel of $\chi_h\colon R(H')_p \to \mathbb{Z}_p \otimes \mathbb{C}$, and consider the sequence of $R(H')_p$-module maps

$$\tilde{K}_p^{H'}(S_p^V) \longrightarrow \tilde{K}_p^{H'}(S_p^V)_{\mathcal{P}} \xrightarrow{\cong} \tilde{K}_p^{H'}((S_p^V)^{H'})_{\mathcal{P}} \xrightarrow{\cong} \Big(R(H')_p \otimes_{\mathbb{Z}_p} \tilde{K}_p((S_p^V)^{H'})\Big)_{\mathcal{P}}.$$

The second map is an isomorphism by the localization theorem (see the proof of Proposition 4.1 in [**17**], which also applies to p-complete K-theory). The last map is the isomorphism of Proposition 2.2 in [**17**]. Thus, the sequence takes b_c^V to some generator b of the free $(R(H')_p)_{\mathcal{P}}$-module $\Big(R(H')_p \otimes_{\mathbb{Z}_p} \tilde{K}_p((S_p^V)^{H'})\Big)_{\mathcal{P}}$. Now, an H'-map $\tilde{x}\colon S_p^V \to S_p^V$ induces endomorphisms of each term in the sequence, commuting with the maps of the sequence. Moreover,

$$\tilde{x}^*\colon \tilde{K}_p((S_p^V)^{H'}) \to \tilde{K}_p((S_p^V)^{H'})$$

is multiplication by $d = \deg\, \tilde{x}\Big|_{(S_p^V)^{H'}}$. Therefore, $\tilde{x}^*(b_c^V)$ maps under the sequence to $d \cdot b$, so the image of $(\tilde{x}^*(b_c^V))/b_c^V$ in $(R(H')_p)_{\mathcal{P}}$ is d. Since χ_h factors through $(R(H')_p)_{\mathcal{P}}$, it follows that

$$\chi_h(\tilde{x}^*(b_c^V)/b_c^V) = \deg\, \tilde{x}\Big|_{(S_p^V)^{H'}}. \qquad \square$$

Theorem 5.4. *Suppose V is a virtual H-representation of virtual dimension 0, which is fixed by ψ^k, representing an element V in $\pi_0(J^H)$. Then, identifying $\pi_0((J_\otimes)_p^H)$ with the set of virtual H-representations of virtual dimension 1 fixed by ψ^k, we have*

$$\varepsilon^k \circ \alpha^k(V) = \rho^k(V).$$

Proof. We will show that for a complex representation V, $\chi_h(\rho_c^k(V)) = k^{\dim V^h}$, and using this, we will show that for any real virtual H-representation V, we have $\chi_h(\rho^k(V))^2 = k^{\dim V^h}$. Assuming this for the moment, the theorem is proved as follows. We can represent a virtual representation V as a difference $V_1 - V_2$, where V_1 and V_2 are Spin representations of dimension divisible by 4 (see 2.10). By Theorems 5.2 and 5.3, together with the above formula,

$$\chi_h(\varepsilon^k \circ \alpha^k(V))^2 = k^{\dim V^h} = \chi_h(\rho^k(V))^2.$$

Thus, $\varepsilon^k \circ \alpha^k(V) = \pm\rho^k(V)$. But if $\dim V = 0$, then $\varepsilon^k \circ \alpha^k(V)$ and $\rho^k(V)$ both map to 1 under the augmentation homomorphism, so $\varepsilon^k \circ \alpha^k(V) = \rho^k(V)$.

Now, for our claim, we may restrict to the cyclic subgroup $\langle h \rangle \cong \mathbb{Z}/p^n$ generated by h. The irreducible representations of $\langle h \rangle$ are all powers of a one-dimensional representation L, and $\psi^k(L^r) = L^{rk}$ for all r. By considering the invariants of ψ^k, we reduce to the case where either V is trivial (where the claim is obvious), or $V = \oplus_{i=1}^{m/p^s} L^{p^s k^i}$, where $m = p^n - p^{n-1}$ and $0 \leqslant s \leqslant n-1$. Let $\zeta = \chi_h L^{p^s}$, and note that $\zeta^{p^{n-s}} = 1$. Then

$$\chi_h \rho_c^k V = \prod_{i=1}^{m/p^s} \frac{\zeta^{k^i \cdot k} - 1}{\zeta^{k^i} - 1}.$$

Since $k^{(m/p^s)+1} \equiv k \mod p^{n-s}$, all numerators and denominators cancel, and we get $\chi_h \rho_c^k V = 1 = k^0$. This implies our first claim, since $\dim V^h = 0$.

By Lemma 9.1 in [**9**], $c\rho^k r(\zeta) = \rho_c^k(\zeta)$ if ζ is a $(G, SU(W))$-bundle of complex dimension divisible by 4, and c and r are the complexification and realification homomorphisms. In particular, this holds if ζ is a bundle over a point, or a complex G-representation V whose action map factors through $SU(V)$. But if $\psi^k V = V$, then the determinant representation of V coincides with its kth tensor power, and since $k-1$ is relatively prime to p, it follows that this determinant representation must be trivial, whence the action map of V factors through $SU(V)$. Thus,

$$\chi_h(\rho^k(2V)) = \chi_h(c\rho^k(2V)) = \chi_h(c\rho^k(rcV)) = \chi_h(\rho_c^k(cV)) = k^{\dim V^h},$$

where the last equality follows from our initial claim and the fact that $\dim_{\mathbb{C}} cV = \dim_{\mathbb{R}} V$. Since ρ^k is exponential, we have

$$\chi_h(\rho^k(V))^2 = k^{\dim V^h}$$

for all virtual representations V. $\square$

We showed in [**9**, §10] that if X is a compact G-space, then the action of the Adams operations on the augmentation ideal of $KO_G(X)\left[\frac{1}{2}\right]$ extend to the p-adic numbers, so that we get an action of the multiplicative group of units in $\mathbb{Z}_p$ on $IKO_G(X)_p$. Letting α be a primitive $p-1$st root of unity, ψ^α generates an action of $\mathbb{Z}/p-1$ on $IKO_G(X)_p$. Let $IKO_G(X)_p^{\psi^\alpha}$ denote the fixed points under this action. We showed in [**9**, 10.15] that

$$\rho^k \colon IKO_G(X)_p^{\psi^\alpha} \to (1 + IKO_G(X)_p)^{\psi^\alpha}$$

is an isomorphism. This implies that ρ^k induces an isomorphism on the sets of elements fixed by ψ^k. In particular, this holds if $X \cong G/H$. Our main theorem is now a corollary of Theorem 5.4.

Corollary 5.5. *For each $H \leqslant G$, the map $\varepsilon^k \circ \alpha^k$ induces a p-completion from $\pi_0(J^H)$ to $\pi_0(J^H_{\otimes p})$.*

Appendix A. Completions

By Propositions 10 and 11 in [**13**], if $f\colon X \to Y$ is a G-map between spaces having the homotopy type of G-CW complexes, and f^H induces a $\hat{p}$-cohomology isomorphism for each $H \leqslant G$, and Z is a p-complete G-nilpotent G-CW complex, then any G-map from X to Z extends uniquely to a map from Y to Z. In particular, if a map from Y to Z becomes null-homotopic after precomposing with f, then the map must have been null-homotopic to begin with. The following lemma gives us added control over our choice of null-homotopy. Observe that we do not assume that Z is G-connected.

Lemma A.1. *Suppose given a G-map $f\colon X \to Y$ such that*

1. *For each $H \leqslant G$, $f_*^H\colon \pi_0(X^H) \to \pi_0(Y^H)$ is an isomorphism, and*
2. *The restriction of f^H to each component of X^H is a $\hat{p}$-cohomology equivalence to the corresponding component in Y^H.*

Suppose also that the G-connected cover of Z has the homotopy type of a p-complete nilpotent G-space. Finally suppose that $g\colon Y \to Z$ is a G-map and $K\colon X \times I \to Z$ is an equivariant null-homotopy of $g \circ f$. Then there is a null-homotopy K' of g whose restriction to X is equivalent to K.

Proof. We first show that g is null-homotopic. Let $i\colon \tilde{Z} \to Z$ be the G-connected cover of Z. Our hypothesis implies that $f^H\colon X^H \to Y^H$ induces a $\hat{p}$-cohomology isomorphism since cohomology takes disjoint unions to direct products, so f is a $\hat{p}$-cohomology isomorphism. Moreover, $\pi_n(\tilde{Z})$ is p-complete for all n. Therefore, by the above remarks, $f^*\colon [Y, \tilde{Z}]_G \to [X, \tilde{Z}]_G$ is an isomorphism.

Now, since $f^H\colon X^H \to Y^H$ induces an equivalence on π_0, the map $g^H\colon Y^H \to Z^H$ induces the trivial map on π_0 for each $H \leqslant G$. Therefore, there is a lift $\tilde{g}\colon Y \to \tilde{Z}$ of g. Now, since $i \circ \tilde{g} \circ f \simeq g \circ f \simeq *$, and since the fiber of i is null-homotopic, it follows that $\tilde{g} \circ f$ is null-homotopic. But since $f^*\colon [Y, \tilde{Z}]_G \to [X, \tilde{Z}]_G$ is an isomorphism, it follows that $\tilde{g}$ (and therefore $g = i \circ \tilde{g}$) is null-homotopic.

Now, suppose that $K''\colon Y \times I \to Z$ is a null-homotopy of g. Then $K'' \circ (f \times I)$ and K are two null-homotopies of $g \circ f$. The difference of these null-homotopies determines a map $X \to \Omega Z \simeq \Omega \tilde{Z}$. Since $\Sigma f\colon \Sigma X \to \Sigma Y$ induces a $\hat{p}$-cohomology isomorphism, it follows that any map $\Sigma X \to \tilde{Z}$ lifts to ΣY, so any map $X \to \Omega Z$ lifts to Y. Adding this to K'' gives a null-homotopy K' of g whose restriction to X is equivalent to K. □

References

[**1**] J. F. Adams, On the groups $J(X)$–I, *Topology* **2** (1963), 181–195.

[**2**] J. F. Adams, On the groups $J(X)$–II, *Topology* **3** (1965), 137–171.

[**3**] J. F. Adams, On the groups $J(X)$–III, *Topology* **3** (1965), 193–222.

[**4**] J. F. Adams, On the groups $J(X)$–IV, *Topology* **3** (1966), 21–71.

[**5**] M. F. Atiyah and D. O. Tall, Group representations, λ-rings and the J-homomorphism, *Topology* **8** (1969) 253–297.

[**6**] J. Caruso, F. R. Cohen, J. P. May, and L. R. Taylor, James maps, Segal maps, and the Kahn-Priddy theorem, *Trans. Amer. Math. Soc.* **281** (1984), 243–283.

[**7**] Z. Fiedorowicz, H. Hauschild, and J. P. May, Equivariant algebraic K-theory, in *Algebraic K-theory, Part II* (Oberwolfach, 1980), Lecture Notes in Math. **967**, 23–80, Springer-Verlag, New York, 1982.

[8] Z. Fiedorowicz and S. Priddy, *Homology of Classical Groups over Finite Fields and their Associated Infinite Loop Spaces*, Lecture Notes in Math. **674**, Springer-Verlag, New York, 1978.

[9] C. P. French, The equivariant J-homomorphism, *Homology Homotopy Appl*. **5** (2003), 161–212.

[10] C. P. French, The equivariant J-homomorphism for finite groups at certain primes, preprint.

[11] J. P. May, *Classifying Spaces and Fibrations*, Memoirs Amer. Math. Soc. **1**, no. 155, A. M. S., Providence, RI (1975).

[12] J. P. May, *E_∞-Ring Spaces and E_∞ Ring-Spectra*, Lecture Notes in Math. **577**, Springer-Verlag, New York, 1977.

[13] J. P. May, Equivariant completion, *Bull. London Math. Soc*. **14** (1982), 231–237.

[14] J. P. May., J. McClure, and G. Triantafillou, Equivariant localization, *Bull. London Math. Soc*. **14** (1982), 223–230.

[15] J. McClure, On the groups JO_GX. I, *Math. Z*. **183** (1983), 229–253.

[16] D. Quillen, The Adams conjecture, *Topology* **10** (1971), 67–80.

[17] G. Segal, Equivariant K-theory, *Publications Mathématiques de Inst. Hautes Études Sci*. **34** (1968), 129–151.

[18] G. Segal, Equivariant stable homotopy theory, *Actes du Congrès International des Mathématiciens* **2**, 59–63 (Nice, 1970), Gauthier-Villars, Paris, 1971.

[19] G. Segal, Permutation Representations of finite p-groups, *Quart. J. Math. Oxford Ser*. **23** (1972), 375–381.

[20] J-P. Serre, *Linear Representations of Finite Groups*, Graduate Texts in Math. **42**, Springer-Verlag, New York, 1977.

[21] T. tom Dieck, *Transformation Groups and Representation Theory*, Lecture Notes in Math. **766**, Springer-Verlag, New York, 1979.

[22] S. Waner, Equivariant classifying spaces and fibrations, *Trans. Amer. Math. Soc.* **258** (1980), 385–405.

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