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MODEL STRUCTURE ON OPERADS IN ORTHOGONAL SPECTRA

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(*communicated by Nicholas J. Kuhn*)

Abstract

We generalize Berger and Moerdijk's results on axiomatic homotopy theory for operads to the setting of enriched symmetric monoidal model categories, and show how this theory applies to orthogonal spectra. In particular, we provide a symmetric fibrant replacement functor for the positive stable model structure.

1. Introduction

Operads (in topological spaces) were introduced in order to describe algebraic structures where the constraints are relaxed up to a system of homotopies. The definition of operads generalizes to any symmetric monoidal category. This raises the question about axiomatic homotopy theory for operads, given that the base category has a monoidal model structure. This question has answers when the base category is simplicial sets by Rezk [**15**], complexes of a module over a ring by Hinich [**5**], a cofibrantly generated symmetric monoidal model category by Spitzweck [**18**], and k-spaces by Vogt [**20**]. Berger and Moerdijk [**1**] construct a Quillen model structure on reduced operads (and their algebras) in a closed symmetric monoidal model category given that the unit is cofibrant and that the base category comes equipped with a symmetric monoidal fibrant replacement functor. This includes the case of operads in spaces.

The aim of this article is to provide Quillen model structures on operads and their algebras when the base category is some symmetric monoidal category of spectra, for instance, orthogonal spectra; see [**11**, Example 4.4]. An argument of Lewis [**9**] shows that no symmetric monoidal model category of spectra can simultaneously have a cofibrant unit and a symmetric monoidal fibrant replacement functor. Thus Berger and Moerdijk's work does not apply directly.

We will weaken the assumptions on the unit in two different ways. It is sufficient to make use of the idea of semicofibrant objects, as defined by Lewis and Mandell [**10**], instead of cofibrant objects. Alternatively, one can assume that the base category is enriched and cotensored over another monoidal model category, in which the unit and Hopf intervals are nicer. The latter approach generalizes the hint given in [**1**, Example 4.6.4]. We follow the strategy and proofs of Berger and Moerdijk's

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paper [**1**] closely and advise the reader to keep a copy of this article at hand while reading Section 2.

The category of orthogonal spectra, with the positive stable model structure (see [**11**, Section 14]), satisfies *a priori* nearly all of the requirements of Section 2. The only missing piece is a symmetric monoidal fibrant replacement functor. We show in Section 3 that the second-most naive guess for a fibrant replacement functor actually is symmetric. The analogous functor in symmetric spectra is not a fibrant replacement.

To summarize, the results in Theorem 2.4, Theorem 3.1, and Theorem 3.5 together imply:

Theorem 1.1. *The category of reduced operads in orthogonal spectra admits a model structure where $\mathcal{P} \to \mathcal{Q}$ is a weak equivalence if and only if it induces an isomorphism of homotopy groups $\pi_q\mathcal{P}(n) \to \pi_q\mathcal{Q}(n)$ for all q and n.*

Furthermore, we also get a model structure on algebras and modules under a fixed reduced operad in orthogonal spectra, and we have a comparison theorem.

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2. Model structures on operads and their left modules

We assume that the reader is familiar with the basic notions of model categories; see for example [**4, 6, 7**].

For the basic definitions of enriched categories, consult [**8**]. We recall a few facts: Throughout, $\mathcal{V}$ denotes a closed symmetric monoidal category with product $\otimes$, unit I, and *internal hom* $[-,-]$. Let $\mathcal{E}$ be a symmetric monoidal category with product $\wedge$, unit S, enriched with hom objects $\mathcal{E}(-,-)$ in $\mathcal{V}$, having tensor $\odot\colon \mathcal{V} \times \mathcal{E} \to \mathcal{E}$, and cotensor written by exponentiation. For A, B in $\mathcal{E}$ and Y in $\mathcal{V}$, there are natural isomorphisms

$$[Y, \mathcal{E}(A,B)] \cong \mathcal{E}(Y \odot A, B) \cong \mathcal{E}(A, B^Y).$$

We will assume that both $\mathcal{V}$ and the underlying category of $\mathcal{E}$ are monoidal model categories (see [**7, 10, 16**]); in particular, the *pushout-product axiom* holds. Relating the model structures on $\mathcal{V}$ and $\mathcal{E}$, we assume the

Pullback-cotensor axiom: If $p\colon A \twoheadrightarrow B$ is an $\mathcal{E}$-fibration and $i\colon X \hookrightarrow Y$ is a $\mathcal{V}$-cofibration, then $p^{\Box i}\colon A^Y \to B^Y \times_{B^X} A^X$ is an $\mathcal{E}$-fibration and, moreover, $p^{\Box i}$ is trivial if either p or i is trivial.

By adjunction this axiom has two equivalent reformulations, [**10**, Proposition 3.4], one of them similar to Quillen's axiom **SM7**. Moreover, our axiom implies that $\mathcal{E}$ is a model enrichment by $\mathcal{V}$, in the sense of [**3**, Section 3.1]. Let I_c be a cofibrant replacement for $I \in \mathcal{V}$. We assume the *unit axiom* for $\mathcal{V}$; i.e. if X is $\mathcal{V}$-cofibrant, then $X \otimes I_c \to X \otimes I$ is a weak equivalence, and the *cotensor unit axiom*, i.e. if A is $\mathcal{E}$-fibrant, then $A^I \to A^{I_c}$ is a weak equivalence.

There is a reduced cotensor $(\mathcal{V}/I)^{\mathrm{op}} \times (\mathcal{E}/S) \to \mathcal{E}/S$ defined by sending $X \to I$ and $A \to S$ to the pullback of $S \to S^X \leftarrow A^X$. We denote the reduced cotensor by $\widetilde{A^X}$. Correspondingly, there is a *reduced cotensor unit axiom* saying that $\widetilde{A^I} \to \widetilde{A^{I_c}}$ is a weak equivalence whenever $A \to S$ is an $\mathcal{E}$-fibration.

Following Lewis and Mandell [**10**], X in $\mathcal{V}$ is called *semicofibrant* if the internal hom from X, $[X, -]$, preserves $\mathcal{V}$-fibrations and acyclic $\mathcal{V}$-fibrations. We call X in $\mathcal{V}$ *cotensor-cofibrant* if X is semicofibrant and cotensoring with X, $(-)^X$, preserves $\mathcal{E}$-fibrations and acyclic $\mathcal{E}$-fibrations. Observe that all cofibrant objects in $\mathcal{V}$ are semicofibrant and cotensor-cofibrant. If the unit in $\mathcal{V}$ is cofibrant, then all semicofibrant objects are cofibrant.

The natural isomorphisms $[X \otimes X', -] \cong [X, [X', -]]$ and $((-)^X)^{X'} \cong (-)^{X \otimes X'}$ ensure that the class of cotensor-cofibrant objects is closed under tensor product. It is a deep result, [**10**, Proposition 6.4(b)], that $X \otimes I_c$ is cofibrant and $X \otimes I_c \to X$ is a weak equivalence whenever X is semicofibrant. We refer to this as the I_c-trick, and prove:

Lemma 2.1.

a) *Suppose that $X \to X'$ is a weak equivalence between semicofibrant objects and Y is semicofibrant; then $X \otimes Y \to X' \otimes Y$ is also a weak equivalence between semicofibrant objects.*

b) *Suppose that $X \to X'$ is a weak equivalence between cotensor-cofibrant objects and A is $\mathcal{E}$-fibrant; then $A^{X'} \to A^X$ is a weak equivalence.*

Proof. To prove b) we inspect the diagram

$$\begin{tikzcd}
A^{X'} \arrow[rrrr] \arrow[d, "\simeq"] & & & & A^X \arrow[d, "\simeq"] \\
(A^{X'})^{I_c} \arrow[r, phantom, "\cong"] & A^{X' \otimes I_c} \arrow[rr, "\simeq"] & & A^{X \otimes I_c} \arrow[r, phantom, "\cong"] & (A^X)^{I_c}.
\end{tikzcd}$$

Since X and X' are cotensor-cofibrant, the vertical arrows are weak equivalences by the cotensor unit axiom. The bottom map is a weak equivalence by the I_c-trick.

The statement in part a) is proved by similar means. □

Remark 2.2.

a) If we take $\mathcal{E} = \mathcal{V}$, we reduce to working in a closed monoidal model category. Here, the classes of cotensor-cofibrant and semicofibrant objects agree. The arguments below show that the unit axiom and the use of semicofibrant objects allow us to construct model categories of reduced (resp. positive) operads in $\mathcal{V}$ and their algebras.

b) In Section 3 we will take $\mathcal{V}$ to be compactly generated spaces and $\mathcal{E}$ to be the category of orthogonal spectra. Observe that the unit I is a cofibrant space. In this case the classes of cofibrant, semicofibrant, and cotensor-cofibrant spaces agree.

Let $\mathrm{Hopf}(\mathcal{V})$ be the category of *commutative Hopf objects* in $\mathcal{V}$; see [**1**, Section 1]. Observe that any Abelian monoid M naturally gives rise to a commutative Hopf object $I[M]$ whose underlying object in $\mathcal{V}$ is $\coprod_M I$. Consider $\mathbb{Z}/2$ multiplicatively. If the folding map $I[\mathbb{Z}/2] \to I$ can be factored in $\mathrm{Hopf}(\mathcal{V})$ as $I[\mathbb{Z}/2] \hookrightarrow H \xrightarrow{\simeq} I$, where the underlying maps in $\mathcal{V}$ are a cofibration and a weak equivalence respectively, then we say that $\mathcal{V}$ *admits a commutative Hopf interval*.

Remark 2.3. Whereas the category of compactly generated spaces obviously admits a commutative Hopf interval, it seems to be quite technical to construct one in the positive stable model structure on orthogonal spectra.

For a finite group G, let $\mathcal{E}^G$ denote the category of objects in $\mathcal{E}$ with a right G-action and G-equivariant maps. A *collection* in $\mathcal{E}$ is a sequence of objects $A(n)$ in $\mathcal{E}$, $n \geqslant 0$, such that $A(n)$ has a right action of the symmetric group Σ_n. This category, $\mathrm{Coll}(\mathcal{E})$, equals the product $\prod_{n=0}^{\infty} \mathcal{E}^{\Sigma_n}$. Assuming that $\mathcal{E}$ is cofibrantly generated, there is a model structure on collections, where $A \to B$ is a weak equivalence (resp. fibration) if each $A(n) \to B(n)$ is a non-equivariant weak equivalence (resp. fibration) in $\mathcal{E}$. The subcategory $\widetilde{\mathrm{Coll}}(\mathcal{E})$ (resp. $\mathrm{Coll}_+(\mathcal{E})$) of *reduced collections* (resp. *positive collections*) consists of those A such that $A(0) = S$ (resp. $A(0) = \emptyset$).

An *operad in* $\mathcal{E}$ is a collection $\mathcal{P}$, together with a unit $S \to \mathcal{P}(1)$, and structure maps

$$\mathcal{P}(k) \wedge \mathcal{P}(n_1) \wedge \cdots \wedge \mathcal{P}(n_k) \to \mathcal{P}(n_1 + \cdots + n_k)$$

satisfying certain conditions; see [**13**]. Alternatively, one can define this category, $\mathrm{Oper}(\mathcal{E})$, as the monoids for Smirnov's non-commutative monoidal product on collections in $\mathcal{E}$; see [**17**] or [**12**, Section I.1.8]. We denote this product by $\circ$, and we will define and study it more thoroughly later in this paper; see Definition 2.10. The unit for $\circ$ is the collection $\mathcal{S}$ with $\mathcal{S}(1) = S$ and $\mathcal{S}(n) = \emptyset$ for $n \neq 1$. An operad $\mathcal{P}$ is called *reduced* (resp. *positive*) if $\mathcal{P}(0) = S$ (resp. $\mathcal{P}(0) = \emptyset$). Denote these categories $\widetilde{\mathrm{Oper}}(\mathcal{E})$ and $\mathrm{Oper}_+(\mathcal{E})$ respectively. Let $\mathbf{Ass}$ and $\mathbf{Com}$ denote the operads for associative and commutative monoids respectively. Their n-ary parts are given by $\mathbf{Ass}(n) = S[\Sigma_n]$ and $\mathbf{Com}(n) = S$. Observe that the category of reduced operads is the subcategory of $\mathrm{Oper}(\mathcal{E})/\mathbf{Com}$ consisting of $\alpha\colon \mathcal{P} \to \mathbf{Com}$ with $\alpha(0)$ being the identity of S. An operad $\mathcal{P}$ is called Σ*-split* if $\mathcal{P}$ is a retract of $\mathcal{P} \wedge \mathbf{Ass}$.

For an arbitrary operad $\mathcal{P}$ we define categories ${}_{\mathcal{P}}\mathrm{Mod}$, ${}_{\mathcal{P}}\mathrm{Alg}$, and ${}_{\mathcal{P}}\mathrm{Form}^d$. A *left* $\mathcal{P}$*-module* is a collection M together with a left action $\mathcal{P} \circ M \to M$. A $\mathcal{P}$*-algebra* A is a left $\mathcal{P}$-module concentrated in arity 0, i.e. $A(n) = *$ for $n > 0$. Explicitly, we have structure maps $\mathcal{P}(n) \wedge A^{\wedge n} \to A$. More generally, we define a *d'th order* $\mathcal{P}$*-form* to be a left $\mathcal{P}$-module truncated above arity d; i.e. $M(n) = *$ for $n > d$. A $\mathcal{P}$*-coalgebra* is an object B of $\mathcal{E}$ together with structure maps $B \wedge \mathcal{P}(n) \to B^{\wedge n}$ satisfying conditions dual to those of a $\mathcal{P}$-algebra.

Let $\mathcal{D}$ be one of the categories $\mathrm{Oper}_+(\mathcal{E})$, $\widetilde{\mathrm{Oper}}(\mathcal{E})$, ${}_{\mathcal{P}}\mathrm{Mod}$, ${}_{\mathcal{P}}\mathrm{Form}^d$, or ${}_{\mathcal{P}}\mathrm{Alg}$. In all cases we have forgetful functors to $\mathrm{Coll}(\mathcal{E})$. We say that $\mathcal{D}$ *admits a transferred model structure* if there is a model structure on $\mathcal{D}$ where $A \to B$ is a weak equivalence (resp. fibration) if and only if the underlying map in $\mathrm{Coll}(\mathcal{E})$ is a weak equivalence (resp. fibration).

Generalizing the main results of [**1**] we have:

Theorem 2.4. *Assume that $\mathcal{V}$ admits a commutative Hopf interval, $\mathcal{E}$ is cofibrantly generated, $\mathcal{E}/S$ (resp. $\mathcal{E}$) admits a symmetric monoidal fibrant replacement functor, and the reduced cotensor axiom (resp. the unreduced cotensor axiom) holds. The category of reduced operads in $\mathcal{E}$ (resp. positive operads in $\mathcal{E}$) then admits a transferred model structure.*

Theorem 2.5. *Assume that $\mathcal{E}$ is cofibrantly generated and admits a symmetric fibrant replacement functor. Let $\mathcal{P}$ be an operad in $\mathcal{E}$ and $\mathcal{Q}$ an operad in $\mathcal{V}$. If there exists an operad map $j\colon \mathcal{P} \to \mathcal{Q} \odot \mathcal{P}$ and an interval in $\mathcal{V}$ with a $\mathcal{Q}$-coalgebra structure, then ${}_{\mathcal{P}}\mathrm{Mod}$, ${}_{\mathcal{P}}\mathrm{Form}^d$, and ${}_{\mathcal{P}}\mathrm{Alg}$ admit transferred model structures.*

Corollary 2.6. *Assume that $\mathcal{E}$ is cofibrantly generated and admits a symmetric fibrant replacement functor. If there exists an interval in $\mathcal{V}$ with a coassociative comultiplication, then for all Σ-split operads $\mathcal{P}$ the categories ${}_{\mathcal{P}}\mathrm{Mod}$, ${}_{\mathcal{P}}\mathrm{Form}^d$, and ${}_{\mathcal{P}}\mathrm{Alg}$ admit transferred model structures.*

Corollary 2.7. *Assume that $\mathcal{E}$ is cofibrantly generated and admits a symmetric fibrant replacement functor. If there exists an interval in $\mathcal{V}$ with a coassociative and cocommutative comultiplication, then for all operads $\mathcal{P}$ the categories ${}_{\mathcal{P}}\mathrm{Mod}$, ${}_{\mathcal{P}}\mathrm{Form}^d$, and ${}_{\mathcal{P}}\mathrm{Alg}$ admit transferred model structures.*

Remark 2.8. Berger and Moerdijk construct functorial path-objects by convoluting with an interval. They use the hypothesis of cofibrant unit together with Ken Brown's lemma to show that the first map of the path-object is a weak equivalence. We can bypass this hypothesis in two ways: by placing the interval in another category $\mathcal{V}$, wherein $\mathcal{E}$ is enriched, or by using Lewis and Mandell's semicofibrant objects and replacing Ken Brown's lemma by their [**10**, Theorem 6.2].

Proof. We prove all four results simultaneously and follow Berger and Moerdijk closely in their approach. Hence, we will only outline the arguments to the extent it becomes obvious that everything they do also works in our enriched setting.

Since ${}_{\mathcal{P}}\mathrm{Alg}$ and ${}_{\mathcal{P}}\mathrm{Form}^d$ are truncations of ${}_{\mathcal{P}}\mathrm{Mod}$, we will not mention them again in this proof; i.e. the details are exactly as for left modules. Moreover, the two corollaries follow from Theorem 2.5 by taking $\mathcal{Q} = \mathbf{Ass}$ and $\mathbf{Com}$ respectively.

To put model structures on $\mathrm{Oper}_+(\mathcal{E})$, $\widetilde{\mathrm{Oper}}(\mathcal{E})$, and ${}_{\mathcal{P}}\mathrm{Mod}$, we consider free-forgetful adjunctions

$$\begin{gathered}\mathrm{Coll}_+(\mathcal{E}) \rightleftarrows \mathrm{Oper}_+(\mathcal{E}),\\ \widetilde{\mathrm{Coll}}(\mathcal{E}/S) \rightleftarrows \widetilde{\mathrm{Oper}}(\mathcal{E}), \quad \text{and}\\ \mathrm{Coll}(\mathcal{E}) \rightleftarrows {}_{\mathcal{P}}\mathrm{Mod}\,.\end{gathered}$$

Using the transfer principle and Quillen's path-object argument, as explained in [**1**, Sections 2.5 and 2.6], we have to check that $\mathrm{Oper}_+(\mathcal{E})$, $\widetilde{\mathrm{Oper}}(\mathcal{E})$, and ${}_{\mathcal{P}}\mathrm{Mod}$ have small colimits and finite limits, the free functors preserve small objects, $\mathrm{Oper}_+(\mathcal{E})$, $\widetilde{\mathrm{Oper}}(\mathcal{E})$, and ${}_{\mathcal{P}}\mathrm{Mod}$ have fibrant replacement functors, and $\mathrm{Oper}_+(\mathcal{E})$, $\widetilde{\mathrm{Oper}}(\mathcal{E})$, and ${}_{\mathcal{P}}\mathrm{Mod}$ have functorial path-objects for fibrant objects. See also [**6**, Theorem 11.3.2].

The functorial fibrant replacement functors of $\mathrm{Oper}_+(\mathcal{E})$, $\widetilde{\mathrm{Oper}}(\mathcal{E})$, and ${}_{\mathcal{P}}\mathrm{Mod}$ are defined aritywise using the symmetric fibrant replacement functors of $\mathcal{E}$, $\mathcal{E}/S$, and $\mathcal{E}$ respectively. To get the functorial path-objects we use convolution pairings

$$\begin{gathered}\mathrm{Hopf}(\mathcal{V})^{\mathrm{op}} \times \mathrm{Oper}_+(\mathcal{E}) \to \mathrm{Oper}_+(\mathcal{E}),\\ \mathrm{Hopf}(\mathcal{V})^{\mathrm{op}} \times \widetilde{\mathrm{Oper}}(\mathcal{E}) \to \widetilde{\mathrm{Oper}}(\mathcal{E}), \quad \text{and}\\ \mathrm{Coalg}_{\mathcal{Q}}^{\mathrm{op}} \times {}_{\mathcal{P}}\mathrm{Mod} \to {}_{\mathcal{Q}\odot\mathcal{P}}\mathrm{Mod}\,.\end{gathered}$$

To construct the first pairing, observe that each commutative Hopf object H defines a cooperad TH with $TH(n) = H^{\otimes n}$. The unreduced convolution pairing $\mathcal{P}^{TH}$ is then given by $\mathcal{P}^{TH}(n) = \mathcal{P}(n)^{TH(n)}$. For a commutative Hopf interval H in $\mathcal{V}$ and a fibrant positive operad $\mathcal{P}$, we get a functorial path object

$$\mathcal{P} = \mathcal{P}^{TI} \xrightarrow{\simeq} \mathcal{P}^{TH} \twoheadrightarrow \mathcal{P}^{TI[\mathbb{Z}/2]} \twoheadrightarrow \mathcal{P} \times \mathcal{P}.$$

To see that the first map is a weak equivalence, use Lemma 2.1 and the fact that each $H^{\otimes n} \to I^{\otimes n}$ is a weak equivalence between cotensor-cofibrant objects. By [**10**, Proposition 6.4] each map $(I[\mathbb{Z}/2])^{\otimes n} \to H^{\otimes n}$ is a cofibration. Hence, the middle map is a fibration. The last map, $\mathcal{P}^{TI[\mathbb{Z}/2]} \twoheadrightarrow \mathcal{P} \times \mathcal{P}$, is a projection, whence a fibration since $\mathcal{P}$ is fibrant.

For the second pairing, we are given $H \in \mathrm{Hopf}(\mathcal{V})$ and $\mathcal{P} \to \mathbf{Com}$. Observe that the counit, $\epsilon\colon H \to I$, is a map of commutative Hopf objects. Thus, we can define the reduced convolution pairing $\widetilde{\mathcal{P}^{TH}}$ as the pullback of

$$\mathbf{Com} \xrightarrow{\epsilon^*} \mathbf{Com}^{TH} \leftarrow \mathcal{P}^{TH}.$$

Let $\mathcal{P}$ be a fibrant reduced operad in $\mathcal{E}$, and let $I[\mathbb{Z}/2] \hookrightarrow H \xrightarrow{\simeq} I$ be a commutative Hopf interval in $\mathcal{V}$. By reduced convolution we get

$$\mathcal{P} = \widetilde{\mathcal{P}^{TI}} \xrightarrow{\simeq} \widetilde{\mathcal{P}^{TH}} \twoheadrightarrow \widetilde{\mathcal{P}^{TI[\mathbb{Z}/2]}} \twoheadrightarrow \mathcal{P} \times_{\mathbf{Com}} \mathcal{P}.$$

By the reduced cotensor unit axiom, Lemma 2.1 holds for the reduced cotensor. Hence, the first map is a weak equivalence. The middle map is a fibration by the cotensor-pullback axiom. Finally, the last map is, for $n \geqslant 1$, the projection $\widetilde{\mathcal{P}^{I[\mathbb{Z}/2]}}(n) = \mathcal{P}(n)^{\times_S 2^n} \to \mathcal{P}(n) \times_S \mathcal{P}(n)$ onto the first and last factor, whence a fibration. This yields a functorial path-object for fibrant $\mathcal{P}$.

The last convolution pairing, M^B, between a coalgebra B under an operad $\mathcal{Q}$ in $\mathcal{V}$ and a left $\mathcal{P}$-module M, is defined by the formula $M^B(n) = M(n)^B$. Here, the left $\mathcal{Q} \odot \mathcal{P}$-module structure map

$$(\mathcal{Q} \odot \mathcal{P})(k) \wedge M^B(n_1) \wedge \cdots \wedge M^B(n_k) \to M^B(n)$$

is given as the adjoint of the composition

$$\begin{aligned} &B \odot (\mathcal{Q} \odot \mathcal{P})(k) \wedge M^B(n_1) \wedge \cdots \wedge M^B(n_k)\\ &\quad \cong (B \otimes \mathcal{Q}(k)) \odot \big(\mathcal{P}(k) \wedge M(n_1)^B \wedge \cdots \wedge M(n_k)^B\big)\\ &\quad \to B^{\otimes k} \odot \big(\mathcal{P}(k) \wedge M(n_1)^B \wedge \cdots \wedge M(n_k)^B\big)\\ &\quad \cong \mathcal{P}(k) \wedge (B \odot M(n_1)^B) \wedge \cdots \wedge (B \odot M(n_k)^B)\\ &\quad \to \mathcal{P}(k) \wedge M(n_1) \wedge \cdots \wedge M(n_k) \to M(n). \end{aligned}$$

An interval $I \amalg I \hookrightarrow J \xrightarrow{\simeq} I$ with $\mathcal{Q}$-coalgebra structure gives a path-object

$$M \cong M^I \xrightarrow{\simeq} M^J \twoheadrightarrow M^{I \amalg I} \cong M \times M,$$

for fibrant M, where the first map is a weak equivalence by Lemma 2.1 and the second map is a fibration by the cotensor-pullback axiom. □

The category of collections, $\mathrm{Coll}(\mathcal{E})$, is also known as the category of Σ-modules; see [**12**, Section II.1.2]. An map of operads $\mathcal{P} \to \mathcal{Q}$ is therefore called a Σ-*cofibration* if the underlying map of collections is a cofibration. Furthermore, $\mathcal{P}$ is called Σ-*cofibrant* if the map from the initial operad into $\mathcal{P}$ is a Σ-cofibration. Observe that extra care has to be taken if the unit S is not cofibrant in $\mathcal{E}$: The initial positive operad is $\mathcal{S}$, the unit for the $\circ$-product on collections, while the initial reduced operad $\widetilde{\mathcal{S}}$ is given by $\widetilde{\mathcal{S}}(n) = S$ for $n = 0, 1$ and $\widetilde{\mathcal{S}}(n) = \emptyset$ otherwise. Therefore, a reduced operad $\mathcal{P}$ is Σ-cofibrant if the unique map $\widetilde{\mathcal{S}} \to \mathcal{P}$ is a Σ-cofibration. Similarly, a positive operad $\mathcal{P}$ is Σ-cofibrant if the unique map $\mathcal{S} \to \mathcal{P}$ is a Σ-cofibration. Observe that our notions of Σ-cofibrant differs from the definition found in [**1**, Section 4], but agrees with the definition in [**2**, Section 2.4]. However, all notions coincide if S is cofibrant in $\mathcal{E}$.

Proposition 2.9. *Any cofibrant reduced (resp. positive) operad is Σ-cofibrant.*

Proof. We consider the case of reduced operads first. Since the initial reduced operad, $\widetilde{S}$, is Σ-cofibrant, it is enough to show that Σ-cofibrant reduced operads are closed under cellular extensions. We will now contemplate the difference between reduced and unreduced operads. Let $F\colon \mathrm{Coll}(\mathcal{E}) \to \mathrm{Oper}(\mathcal{E})$ be the free operad functor, and let $\tilde{F}\colon \widetilde{\mathrm{Coll}}(\mathcal{E}/S) \to \widetilde{\mathrm{Oper}}(\mathcal{E})$ be the free reduced operad functor. Given a reduced collection A in $\mathcal{E}/S$, we observe that $\tilde{F}A(n) = FA(n)$ for $n > 0$ while $\tilde{F}A(0) = S$.

[**1**, Corollary 5.2] says that, for any cofibration $A \hookrightarrow B$ of collections and any map of operads $FA \to \mathcal{P}$, the induced map $\mathcal{P} \to \mathcal{P} \cup_{FA} FB$ is a Σ-cofibration. So, if $\mathcal{P}$ is a Σ-cofibrant reduced operad, $A \hookrightarrow B$ a cofibration in $\widetilde{\mathrm{Coll}}(\mathcal{E}/S)$, and $u\colon A \to U(\mathcal{P})$ an arbitrary map, then the only difference between $\mathcal{P} \cup_{\tilde{F}A} \tilde{F}B$ and $\mathcal{P} \cup_{FA} FB$ lies in arity 0. Hence, the map $\mathcal{P} \to \mathcal{P} \cup_{\tilde{F}A} \tilde{F}B$ is a Σ-cofibration.

Next, we consider the case of positive operads. Observe that the free positive operad functor is the restriction of $F\colon \mathrm{Coll}(\mathcal{E}) \to \mathrm{Oper}(\mathcal{E})$ to the subcategory of positive collections. Hence, [**1**, Corollary 5.2] immediately applies and yields the result. □

We now turn towards Smirnov's product on collections; see [**17**]. We begin with the definition of the $\circ$-product and proceed by proving two technical results, namely Propositions 2.11 and 2.12. Berger and Moerdijk prove technicalities of similar flavor in [**2**, Section 2.5].

Let $r_1, \ldots, r_k$ be an increasing sequence of non-negative integers that sum up to n. They determine a partition $R_1 \cup \cdots \cup R_k$ of $\{1, \ldots, n\}$, where R_i contains the elements from $r_1 + r_2 + \cdots + r_{i-1} + 1$ to $r_1 + r_2 + \cdots + r_{i+1} + r_i$ inclusive. Let

$P(r_*)$ be the subgroup of $\Sigma_k \times \Sigma_n^{\mathrm{op}}$ consisting of pairs (τ, σ) with the property that

$$p \in R_{\tau(i)} \quad \text{if and only if} \quad \sigma(p) \in R_i.$$

For each $\tau \in \Sigma_k$ there is a unique τ_{block} in Σ_n^{op} such that $(\tau, \tau_{\mathrm{block}}) \in P(r_*)$ and, for each i, the restriction of τ_{block} to R_i is order-preserving. Given (τ, σ) in $P(r_*)$, let $\sigma_* \in \Sigma_n^{\mathrm{op}}$ denote the unique permutation such that

$$(1, \sigma_*)(\tau, \tau_{\mathrm{block}}) = (\tau, \sigma).$$

Observe that σ_* canonically corresponds to a k-tuple $(\sigma_1, \ldots, \sigma_k)$ in $\Sigma_{r_1}^{\mathrm{op}} \times \cdots \times \Sigma_{r_k}^{\mathrm{op}}$. Let $\Sigma(r_*)$ abbreviate $\Sigma_{r_1} \times \cdots \times \Sigma_{r_k}$. Denote by $\mathrm{Aut}(r_*)$ the permutations in Σ_k that acts trivially on $(r_1, r_2, \ldots, r_k)$. This is exactly the image of $P(r_*)$ projected into Σ_k, and we observe that $P(r_*)$ can be written as a semidirect product $\mathrm{Aut}(r_*) \ltimes \Sigma(r_*)^{\mathrm{op}}$.

Let X be a collection in $\mathcal{E}$. By convention $X(r)$ has a right Σ_r-action. Abbreviate $X(r_1) \wedge \cdots \wedge X(r_k)$ by $X(r_*)$. There is a left $P(r_*)$-action on $X(r_*)$ given by letting $(\tau, \tau_{\mathrm{block}})$ permute the factors $X(r_1), \ldots, X(r_k)$ as τ prescribes, whereas $(1, \sigma_*)$ acts on each individual factor $X(r_i)$ by σ_i. Now induce up to a $\Sigma_k \times \Sigma_n^{\mathrm{op}}$-equivariant object $\mathrm{Ind}_{P(r_*)}^{\Sigma_k \times \Sigma_n^{\mathrm{op}}} X(r_*)$. Define

$$X[k, n] = \coprod_{r_*} \mathrm{Ind}_{P(r_*)}^{\Sigma_k \times \Sigma_n^{\mathrm{op}}} X(r_*),$$

where the coproduct runs over all increasing sequences r_* of length k and with sum n. On $X[k, n]$ we have a left Σ_k-action and a right Σ_n-action.

Definition 2.10. For collections A and X in $\mathcal{E}$ define

$$(A \circ X)(n) = \coprod_{k=0}^{\infty} A(k) \wedge_{\Sigma_k} X[k, n].$$

We will now derive a few properties of this product, but before that let us introduce a piece of terminology coming from Goodwillie's calculus of functors: We call the diagram

$$\begin{array}{ccc} A & \rightarrow & B \\ \downarrow & & \downarrow \\ C & \rightarrow & D \end{array}$$

a cofibration square if the three maps $A \to B$, $A \to C$ and $B \cup_A C \to D$ are cofibrations.

Proposition 2.11. *Let A be cofibrant in the model category of reduced collection under $\widetilde{\mathcal{S}}$ (resp. positive collections under $\mathcal{S}$), and let $X \hookrightarrow Y$ be a cofibration between cofibrant collections. Then $A \circ X \to A \circ Y$ is also a cofibration.*

Proof. Fix n, k, and r_*. We have a chain of groups

$$P(r_*) = \mathrm{Aut}(r_*) \ltimes \Sigma(r_*)^{\mathrm{op}} \subseteq \mathrm{Aut}(r_*) \ltimes \Sigma_n^{\mathrm{op}} \subseteq \Sigma_k \times \Sigma_n^{\mathrm{op}}.$$

This gives a natural isomorphism

$$A(k) \wedge_{\Sigma_k} \left(\mathrm{Ind}_{P(r_*)}^{\Sigma_k \times \Sigma_n^{\mathrm{op}}} X(r_*)\right) \cong A(k) \wedge_{\mathrm{Aut}(r_*)} \left(\mathrm{Ind}_{\Sigma(r_*)^{\mathrm{op}}}^{\Sigma_n^{\mathrm{op}}} X(r_*)\right).$$

In the following discussion we drop the superscript $^{\mathrm{op}}$ and take the view that the action of Σ_n is a right action. By inspection of the definition of the $\circ$-product, it is enough to show that each

$$A(k) \wedge_{\mathrm{Aut}(r_*)} \left(\mathrm{Ind}_{\Sigma(r_*)}^{\Sigma_n} X(r_*)\right) \to A(k) \wedge_{\mathrm{Aut}(r_*)} \left(\mathrm{Ind}_{\Sigma(r_*)}^{\Sigma_n} Y(r_*)\right)$$

is a Σ_n-equivariant cofibration. Clearly, $X(r_*) \to Y(r_*)$ is a $\Sigma(r_*)$-equivariant cofibration and $\mathrm{Ind}_{\Sigma(r_*)}^{\Sigma_n}$ preserves (equivariant) cofibrations. Observe that the map above is tautologically an equivariant cofibration if A is the initial reduced operad $\widetilde{\mathcal{S}}$ (resp. the initial positive operad $\mathcal{S}$). In general, we may assume that $A(k)$ is a cellular $\mathcal{E}^{\Sigma_k}$-object relative to $\widetilde{\mathcal{S}}(k)$ (resp. rel $\mathcal{S}(k)$), i.e. $A(k)$ is a (transfinite) sequential colimit where each step $A(k)_\alpha \hookrightarrow A(k)_{\alpha+1}$ is formed by gluing a generating cofibration $\Sigma_k \times \partial_\alpha \hookrightarrow \Sigma_k \times D_\alpha$. By the pushout product axiom for $\mathcal{E}$, we have a cofibration square

$$\begin{tikzcd}
\coprod_{\Sigma_k/\mathrm{Aut}(r_*)} \partial_\alpha \wedge \left(\mathrm{Ind}_{\Sigma(r_*)}^{\Sigma_n} X(r_*)\right) \arrow[r] \arrow[d] & \coprod_{\Sigma_k/\mathrm{Aut}(r_*)} D_\alpha \wedge \left(\mathrm{Ind}_{\Sigma(r_*)}^{\Sigma_n} X(r_*)\right) \arrow[d] \\
\coprod_{\Sigma_k/\mathrm{Aut}(r_*)} \partial_\alpha \wedge \left(\mathrm{Ind}_{\Sigma(r_*)}^{\Sigma_n} Y(r_*)\right) \arrow[r] & \coprod_{\Sigma_k/\mathrm{Aut}(r_*)} D_\alpha \wedge \left(\mathrm{Ind}_{\Sigma(r_*)}^{\Sigma_n} Y(r_*)\right).
\end{tikzcd}$$

Hence, also

$$\begin{tikzcd}
A(k)_\alpha \wedge_{\mathrm{Aut}(r_*)} \left(\mathrm{Ind}_{\Sigma(r_*)}^{\Sigma_n} X(r_*)\right) \arrow[r] \arrow[d] & A(k)_{\alpha+1} \wedge_{\mathrm{Aut}(r_*)} \left(\mathrm{Ind}_{\Sigma(r_*)}^{\Sigma_n} X(r_*)\right) \arrow[d] \\
A(k)_\alpha \wedge_{\mathrm{Aut}(r_*)} \left(\mathrm{Ind}_{\Sigma(r_*)}^{\Sigma_n} Y(r_*)\right) \arrow[r] & A(k)_{\alpha+1} \wedge_{\mathrm{Aut}(r_*)} \left(\mathrm{Ind}_{\Sigma(r_*)}^{\Sigma_n} Y(r_*)\right)
\end{tikzcd}$$

is a cofibration square. The conclusion follows. □

Proposition 2.12. *Let $A \xrightarrow{\simeq} B$ be a weak equivalence between cofibrant reduced collections under $\widetilde{\mathcal{S}}$ (resp. cofibrant positive collections under $\mathcal{S}$), and let X be a cofibrant collection. Then $A \circ X \to B \circ X$ is also a weak equivalence.*

Proof. By Ken Brown's lemma, it is enough to consider the case where $A \to B$ is an acyclic cofibration between reduced (resp. positive) collections. Fix n, k and r_* as above. We may assume that $B(k)$ is cellular relative to $A(k)$, i.e. we write $B(k)$ as a (transfinite) sequential colimit starting with $B(k)_0 = A(k)$ and such that each step $B(k)_\alpha \to B(k)_{\alpha+1}$ is the pushout along a generating acyclic cofibration, $\Sigma_k \times \partial_\alpha \hookrightarrow \Sigma_k \times D_\alpha$. We now get a pushout diagram

$$\begin{tikzcd}
\coprod_{\Sigma_k/\,\mathrm{Aut}(r_*)} \partial_\alpha \wedge \left(\mathrm{Ind}_{\Sigma(r_*)}^{\Sigma_n} X(r_*)\right) \arrow[r] \arrow[d] & \coprod_{\Sigma_k/\,\mathrm{Aut}(r_*)} D_\alpha \wedge \left(\mathrm{Ind}_{\Sigma(r_*)}^{\Sigma_n} X(r_*)\right) \arrow[d] \\
B(k)_\alpha \wedge_{\mathrm{Aut}(r_*)} \left(\mathrm{Ind}_{\Sigma(r_*)}^{\Sigma_n} X(r_*)\right) \arrow[r] & B(k)_{\alpha+1} \wedge_{\mathrm{Aut}(r_*)} \left(\mathrm{Ind}_{\Sigma(r_*)}^{\Sigma_n} X(r_*)\right),
\end{tikzcd}$$

where the top map is an acyclic cofibration by the pushout-product axiom. Hence the bottom map also is an acyclic cofibration. □

The free functor $F_{\mathcal{P}}\colon \mathrm{Coll}(\mathcal{E}) \to {}_{\mathcal{P}}\mathrm{Mod}$ is given by the $\circ$-product, i.e. $F_{\mathcal{P}}(X) = \mathcal{P} \circ X$. This extends the Schur functor defining the free $\mathcal{P}$-algebra.

Theorem 2.13. *Under the assumptions of Theorem 2.5, assume additionally that $\mathcal{E}$ is left proper and that the domains of the generating cofibrations are cofibrant. If $\phi\colon \mathcal{P} \to \mathcal{Q}$ is a weak equivalence between Σ-cofibrant reduced (resp. positive) operads, then the base-change adjunctions ${}_{\mathcal{P}}\mathrm{Mod} \rightleftarrows {}_{\mathcal{Q}}\mathrm{Mod}$, ${}_{\mathcal{P}}\mathrm{Alg} \rightleftarrows {}_{\mathcal{Q}}\mathrm{Alg}$, and ${}_{\mathcal{P}}\mathrm{Form}^d \rightleftarrows {}_{\mathcal{Q}}\mathrm{Form}^d$ are Quillen equivalences.*

Proof. We prove this for left modules; the other cases are similar. The categories ${}_{\mathcal{P}}\mathrm{Mod}$ and ${}_{\mathcal{Q}}\mathrm{Mod}$ both carry transferred model structures. Hence, the base-change adjunction is a Quillen pair by inspection. Since the forgetful functor ϕ^* reflects weak equivalences, it is enough to show that the unit of the adjunction, $M \to \phi^*\phi_!M$, is a weak equivalence for each cellular left $\mathcal{P}$-module M. Cellular means that M is a (transfinite) sequential colimit starting from the initial left $\mathcal{P}$-module and where each $M_{\alpha+1}$ is the pushout of $M_\alpha \leftarrow \mathcal{P} \circ \partial_\alpha \hookrightarrow \mathcal{P} \circ D_\alpha$ for some generating cofibration $\partial_\alpha \to D_\alpha$ in $\mathrm{Coll}(\mathcal{E})$. Observe that $\phi^*\phi_!M$ inherits a similar description; i.e. $\phi^*\phi_!M_{\alpha+1}$ is the pushout of $\phi^*\phi_!M_\alpha \leftarrow \phi^*(\mathcal{Q} \circ \partial_\alpha) \to \phi^*(\mathcal{Q} \circ D_\alpha)$. Recall from Proposition 2.12 that $\mathcal{P} \circ \partial_\alpha \to \mathcal{Q} \circ \partial_\alpha$ and $\mathcal{P} \circ D_\alpha \to \mathcal{Q} \circ D_\alpha$ are weak equivalences, while $\mathcal{P} \circ \partial_\alpha \to \mathcal{P} \circ D_\alpha$ and $\mathcal{Q} \circ \partial_\alpha \to \mathcal{Q} \circ D_\alpha$ are Σ-cofibrations by Proposition 2.11. Thus, inductively, all $M_\alpha \to \phi^*\phi_!M_\alpha$ are weak equivalences, and the conclusion follows. □

3. The case of orthogonal spectra

It is convenient to replace the category of all topological spaces by compactly generated spaces (= weak Hausdorff k-spaces; see [**14**]). We define the reduced homotopy colimit of a sequence $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to \cdots$ of based spaces as the reduced mapping telescope. $\mathrm{hocolim}_n X_n$ has the topology of the union $\bigcup_{n=0}^\infty F_n$, where the n'th space of the filtration is

$$F_n = (X_1 \wedge I_+) \cup_{f_1} (X_2 \wedge I_+) \cup_{f_2} (X_3 \wedge I_+) \cup \cdots \cup (X_{n-1} \wedge I_+) \cup_{f_{n-1}} X_n.$$

Since each X_n is compactly generated, all base points $* \in X_n$ are closed, so any compact subset of $\mathrm{hocolim}_n X_n$ is contained in some F_k; see [**19**, Lemma 9.3]. Using the projections $F_n \to X_n$, it is easy to prove that we have natural group isomorphisms

$$\operatorname*{colim}_n \pi_q X_n \xrightarrow{\cong} \pi_q \operatorname*{hocolim}_n X_n.$$

An *orthogonal spectrum* X consists of a based $O(V)$-equivariant space $X(V)$, for every finite-dimensional real inner product space V, together with $O(V) \times O(W)$-equivariant suspension maps $\sigma\colon X(V) \wedge S^W \to X(V \oplus W)$ satisfying the obvious coherence condition; see [**11**, Example 4.4]. Fix orthogonal spectra X and Y. Now consider pairs (Z, μ) where Z is an orthogonal spectrum and μ is a family of maps $\mu(V,W)\colon X(V)\wedge Y(W)\to Z(V\oplus W)$ such that $\mu(V,W)$ is $O(V)\times O(W)$-equivariant and the following diagram commutes for all U, V and W:

$$\begin{tikzcd}
 & & X(U)\wedge Y(V\oplus W) \arrow[drr, "\mu(U,V\oplus W)"] & & \\
X(U)\wedge Y(V)\wedge S^W \arrow[urr, "1\wedge\sigma_Y"] \arrow[r, "\mu(U,V)\wedge 1"'] \arrow[d, "\text{twist}"'] & & Z(U\oplus V)\wedge S^W \arrow[rr, "\sigma_Z"'] & & Z(U\oplus V\oplus W) \\
X(U)\wedge S^W\wedge Y(V) \arrow[rr, "\sigma_X\wedge 1"] & & X(U\oplus W)\wedge Y(V) \arrow[rr, "\mu(U\oplus W,V)"] & & Z(U\oplus W\oplus V). \arrow[u, "\cong"']
\end{tikzcd} \qquad (*)$$

The *smash product*, $X\wedge Y$, is initial among such Z. Thus, any such pair (Z,μ) determines a unique map $X\wedge Y\xrightarrow{\mu} Z$. We denote the category of orthogonal spectra by $\mathcal{S}p^O$. The smash product, $\wedge$, is symmetric monoidal with unit the sphere spectrum S. Moreover, $\mathcal{S}p^O$ is enriched, tensored and cotensored over compactly generated spaces. The *stable homotopy groups* of an orthogonal spectrum X are defined, for all integers q, in terms of homotopy groups of spaces by the formula $\pi_q X=\operatorname{colim}_n \pi_{q+n}X(\mathbb{R}^n)$. A map $X\to Y$ of orthogonal spectra is a *weak equivalence* if it induces isomorphisms $\pi_q X\cong\pi_q Y$ of all q. This definition of weak equivalence is part of a model structure on $\mathcal{S}p^O$:

Theorem 3.1 ([**11**, Section 14])**.** *There is a model structure on $\mathcal{S}p^O$, called the* positive stable model structure*, where the weak equivalences are as above. The model structure is cofibrantly generated, left and right proper, topological, and satisfies the pullback-cotensor and pushout-product axioms. Furthermore, the domains of the generating cofibrations are cofibrant. The fibrations are characterized as the maps $E\to B$ such that for all V of positive dimension $E(V)\to B(V)$ is a Serre fibration and the diagram*

$$\begin{tikzcd}
E(V) \arrow[r] \arrow[d] & \Omega E(V\oplus\mathbb{R}) \arrow[d] \\
B(V) \arrow[r] & \Omega B(V\oplus\mathbb{R})
\end{tikzcd}$$

is homotopy pullback.

Definition 3.2. Abbreviate $\mathbb{R}^n\otimes V\cong V\oplus V\oplus\cdots\oplus V$ by ${}_nV$. Define the functor T by the formula $TX(V)=\operatorname{hocolim}_n \Omega^{nV}X({}_{(n+1)}V)$.

In order to see that T is a functor from spectra to spectra, we need a precise understanding of both suspension maps and the structure maps of the homotopy colimit. The key point is that the latter maps comes from adding copies of V to the left side of the spectrum index, whereas suspension occurs on the right.

Given an orthogonal spectrum X, we are interested in the spaces $\Omega^V X(W)$ as V and W varies. Moreover, we will describe the functorality induced by the adjoint of the suspension $\tilde{\sigma}\colon X(W)\to\Omega^U X(W\oplus U)$.

Let $\mathscr{K}$ be the topological category with objects pairs (V,W) and morphisms $(U,\alpha,\beta)\colon(V,W)\to(V',W')$ consisting of a finite-dimensional real inner product space U together with linear isometric isomorphisms

$$\alpha\colon V \oplus U \cong V' \quad \text{and} \quad \beta\colon W \oplus U \cong W'.$$

In $\mathscr{K}$ we identify two morphisms if they only differ by the choice of U up to isomorphism. If V, W, V', W', and U are oriented, then we call a morphism in $\mathscr{K}$ *positively oriented* if α and β are of the same orientation. This gives a topological category $\mathscr{K}_+$ of pairs of oriented inner product spaces and positively oriented morphisms. It is easily seen that the space $\mathscr{K}_+\big((V,W),(V',W')\big)$ is connected whenever $\dim V < \dim V'$.

Now observe that each orthogonal spectrum X gives rise to a continuous functor from $\mathscr{K}$ to spaces by sending (V,W) to $\Omega^V X(W)$. The induced map of (U,α,β) is given as

$$\Omega^V X(W) \xrightarrow{\Omega^V \tilde{\sigma}} \Omega^{V\oplus U} X(W \oplus U) \cong \Omega^{V'} X(W')$$

where the homeomorphism is induced by α and β.

To get the structure maps of the homotopy colimit defining T, we consider the pair $({}_nV, {}_{(n+1)}V)$ and add a copy of V to the left. This specifies a morphism

$$({}_nV, {}_{(n+1)}V) \to (V \oplus {}_nV, V \oplus {}_{(n+1)}V) \cong \big({}_{(n+1)}V, {}_{(n+2)}V\big),$$

which induces the homotopy colimit structure map

$$\Omega^{nV} X\big({}_{(n+1)}V\big) \to \Omega^{(n+1)V} X\big({}_{(n+2)}V\big).$$

To get the spectrum suspension maps, we add $n+1$ copies of W to the right side of $({}_nV, {}_{(n+1)}V)$ and shuffle:

$$\begin{aligned}({}_nV, {}_{(n+1)}V) \to ({}_nV \oplus {}_{(n+1)}W, {}_{(n+1)}V \oplus {}_{(n+1)}W)& \\ \cong \big({}_n(V \oplus W) \oplus W, {}_{(n+1)}(V \oplus W)\big).&\end{aligned}$$

When shuffling the W's into the V's we match the copies of V with copies of W starting from the left. Thus the rightmost copy of W is unmatched in the left component of the pair. We get an induced map

$$\Omega^{nV} X\big({}_{(n+1)}V\big) \to \Omega^{n(V\oplus W)} \Omega^W X\big({}_{(n+1)}(V \oplus W)\big).$$

The suspension-loop adjunction can be performed partially, and with our conventions this gives a bijection between maps $A \to \Omega^U \Omega^W Y$ and $A \wedge S^W \to \Omega^U Y$. Thus the rightmost and unmatched copy of W gives us the suspension

$$\big(\Omega^{nV} X\big({}_{(n+1)}V\big)\big) \wedge S^W \to \Omega^{n(V\oplus W)} X\big({}_{(n+1)}(V \oplus W)\big).$$

This entitles us to call the assignment $V \mapsto \Omega^{nV} X\big({}_{(n+1)}V\big)$ a spectrum, and we denote it by $\Omega^{\otimes n} X$.

Now observe that these suspension maps commutes with the homotopy colimit structure maps. Hence, TX is a spectrum as claimed.

Next, we want to specify the symmetric monoidal structure of T. Fix orthogonal spectra X and Y. Given that $k \geqslant \max(m,n)$, we will define spectrum maps

$$\mu^k_{m,n}\colon \Omega^{\otimes m} X \wedge \Omega^{\otimes n} Y \to \Omega^{\otimes k}(X \wedge Y).$$

To do this we will specify a family $\mu^k_{m,n}(V,W)$ satisfying (*). The idea is first to go from $\Omega^{mV}X\big((m{+}1)V\big)$ and $\Omega^{nW}Y\big((n{+}1)W\big)$ to $\Omega^{kV}X\big((k{+}1)V\big)$ and $\Omega^{kW}Y\big((k{+}1)W\big)$ respectively, using the homotopy colimit structure maps and then to shuffle V's and W's. To be precise we define $\mu^k_{m,n}(V,W)$ as the composition

$$\begin{aligned}
&\Omega^{mV}X\big((m{+}1)V\big)\wedge\Omega^{nW}Y\big((n{+}1)W\big)\\
&\qquad\to\Omega^{kV}X\big((k{+}1)V\big)\wedge\Omega^{kW}Y\big((k{+}1)W\big)\\
&\qquad\to\Omega^{kV\oplus kW}\Big(X\big((k{+}1)V\big)\wedge Y\big((k{+}1)W\big)\Big)\\
&\qquad\to\Omega^{k(V\oplus W)}(X\wedge Y)\big((k{+}1)(V\oplus W)\big).
\end{aligned}$$

Observe that the $\mu^k_{m,n}$'s commute with the homotopy colimit structure maps as follows:

$$\begin{tikzcd}
\Omega^{\otimes m}X\wedge\Omega^{\otimes n}Y \arrow[r, "\mu^k_{m,n}"] \arrow[d] & \Omega^{\otimes k}(X\wedge Y)\\
\Omega^{\otimes m+1}X\wedge\Omega^{\otimes n}Y \arrow[ur, "\mu^k_{m+1,n}"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Omega^{\otimes m}X\wedge\Omega^{\otimes n}Y \arrow[r, "\mu^k_{m,n}"] \arrow[d] & \Omega^{\otimes k}(X\wedge Y) \arrow[d]\\
\Omega^{\otimes m+1}X\wedge\Omega^{\otimes n}Y \arrow[r, "\mu^{k+1}_{m+1,n}"] & \Omega^{\otimes k+1}(X\wedge Y).
\end{tikzcd}$$

Clearly, we have a natural map $TX\wedge TY\to\operatorname{hocolim}_{m,n}\left(\Omega^{\otimes m}X\wedge\Omega^{\otimes n}Y\right)$. By sending the pair (m,n) to $k=\max(m,n)$ and applying $\mu^k_{m,n}$, the diagrams above ensure that we have a well-defined map

$$\operatorname*{hocolim}_{m,n}\left(\Omega^{\otimes m}X\wedge\Omega^{\otimes n}Y\right)\to\operatorname*{hocolim}_{k}\Omega^{\otimes k}(X\wedge Y)=T(X\wedge Y).$$

Hence, we have defined the monoidal structure $\mu\colon TX\wedge TY\to T(X\wedge Y)$. To verify associativity we observe that the following diagram commutes:

$$\begin{tikzcd}
\Omega^{\otimes n_1}X\wedge\Omega^{\otimes n_2}Y\wedge\Omega^{\otimes n_3}Z \arrow[r, "1\wedge\mu^{k_2}_{n_2,n_3}"] \arrow[d, "\mu^{k_1}_{n_1,n_2}\wedge 1"] & \Omega^{\otimes n_1}X\wedge\Omega^{\otimes k_2}(Y\wedge Z) \arrow[d, "\mu^k_{n_1,k_2}"]\\
\Omega^{\otimes k_1}(X\wedge Y)\wedge\Omega^{\otimes n_3}Z \arrow[r, "\mu^k_{k_1,n_3}"] & \Omega^{\otimes k}(X\wedge Y\wedge Z).
\end{tikzcd}$$

Furthermore, the symmetry of T comes from commutativity of the diagram

$$\begin{tikzcd}
\Omega^{\otimes m}X\wedge\Omega^{\otimes n}Y \arrow[r, "\mu^k_{m,n}"] \arrow[d, "\mathrm{twist}"] & \Omega^{\otimes k}(X\wedge Y) \arrow[d, "\mathrm{twist}"]\\
\Omega^{\otimes n}Y\wedge\Omega^{\otimes m}X \arrow[r, "\mu^k_{n,m}"] & \Omega^{\otimes k}(Y\wedge X).
\end{tikzcd}$$

Theorem 3.3. *T is a symmetric fibrant replacement functor for the positive stable model structure on orthogonal spectra.*

Proof. We have already argued that T is a symmetric monoidal functor. It remains to show that TX is fibrant, for all X, and that the coaugmentation $X \to TX$ is a weak equivalence.

Because of Theorem 3.1 we only consider V of positive dimension. The q'th homotopy group of $TX(V)$ is calculated as $\mathrm{colim}_n\, \pi_q \Omega^{nV} X({}_{(n+1)}V)$. Now consider the following solid diagram in $\mathscr{K}_+$:

$$\begin{tikzcd}
\big({}_nV, {}_{(n+1)}V\big) \arrow[rr] \arrow[d] \arrow[r, phantom, "\circ"] & & \big({}_{(2n+1)}V, {}_{(2n+2)}V\big) \arrow[d] \\
\big({}_n(V \oplus \mathbb{R}) \oplus \mathbb{R}, {}_{(n+1)}(V \oplus \mathbb{R})\big) \arrow[rr] \arrow[urr, dotted] \arrow[r, phantom, "\simeq"] & & \big({}_{(2n+1)}(V \oplus \mathbb{R}) \oplus \mathbb{R}, {}_{(2n+2)}(V \oplus \mathbb{R})\big).
\end{tikzcd}$$

Here, the horizontal maps induces homotopy colimit structure maps, whereas the vertical maps correspond to suspension by $\mathbb{R}$. For dimensional reasons, the dotted map exists in $\mathscr{K}_+$ making the upper-left triangle commute. Since the morphism spaces of $\mathscr{K}_+$ are connected, the lower-right triangle will commute up to homotopy. Thus, the induced diagram of homotopy groups

$$\begin{tikzcd}
\pi_q \Omega^{nV} X\big({}_{(n+1)}V\big) \arrow[r] \arrow[d] & \pi_q \Omega^{(2n+1)V} X\big({}_{(2n+2)}V\big) \arrow[d] \\
\pi_q \Omega^{n(V\oplus\mathbb{R})} \Omega^{\mathbb{R}} X\big({}_{(n+1)}(V \oplus \mathbb{R})\big) \arrow[r] \arrow[ur] & \pi_q \Omega^{(2n+1)(V\oplus\mathbb{R})} \Omega^{\mathbb{R}} X\big({}_{(2n+2)}(V \oplus \mathbb{R})\big)
\end{tikzcd}$$

commutes. This shows that $TX(V) \to \Omega TX(V \oplus \mathbb{R})$ is a weak equivalence. Hence, $TX(V)$ is fibrant in the positive stable model structure.

A similar argument shows that the map $X \to TX$ is a weak equivalence. □

Remark 3.4. The construction of the functor T can also be a carried out in the category of symmetric spectra. However, in this case TX will in general not be a positive Ω-spectrum. For a counterexample consider $X = F_1 S^1$, the free symmetric spectrum generated by a circle S^1 in level 1. The reason that the proof fails for symmetric spectra is that the category corresponding to $\mathscr{K}_+$ is discrete and, hence, allows no non-trivial homotopies.

In order to construct a symmetric fibrant replacement functor for symmetric spectra, other techniques are required. Perhaps a modified version of the small object argument would do this.

Theorem 3.5. *For an orthogonal spectrum X over S, let $\tilde{T}X$ be the levelwise homotopy pullback of $S \to TS \leftarrow TX$. This defines a symmetric fibrant replacement functor, $\tilde{T}$, for the positive stable model structure on orthogonal spectra over S.*

Proof. The natural transformation $\tilde{\mu} \colon \tilde{T}X \wedge \tilde{T}Y \to \tilde{T}(X \wedge Y)$ is the canonical map into the pullback for the diagram

$$\begin{tikzcd}
\tilde{T}X \wedge \tilde{T}Y \arrow[rr] \arrow[d] & & TX \wedge TY \arrow[rr, "\mu"] & & T(X \wedge Y) \arrow[d] \\
S \wedge S \arrow[r, phantom, "\cong"] & S \arrow[rr] & & T(S) \arrow[r, phantom, "\cong"] & T(S \wedge S).
\end{tikzcd}$$

It is straightforward to verify that this gives a symmetric monoidal structure on $\tilde{T}$.

By construction $\tilde{T}X \to S$ is a level fibration. Fix some positive dimensional V. Consider the diagram

$$\begin{tikzcd}
\tilde{T}X(V) \arrow[r] \arrow[d] & TX(V) \arrow[r, "\simeq"] \arrow[d] & \Omega TX(V \oplus \mathbb{R}) \arrow[d] \\
S(V) \arrow[r] & TS(V) \arrow[r, "\simeq"] & \Omega TS(V \oplus \mathbb{R}).
\end{tikzcd}$$

The left square is homotopy pullback by definition of $\tilde{T}X$, while the right square is homotopy pullback since the top and bottom maps are weak equivalences. Consequently, the outer square is homotopy pullback. Now look at the diagram

$$\begin{tikzcd}
\tilde{T}X(V) \arrow[r] \arrow[d] & \Omega\tilde{T}X(V \oplus \mathbb{R}) \arrow[r] \arrow[d] & \Omega TX(V \oplus \mathbb{R}) \arrow[d] \\
S(V) \arrow[r] & \Omega S(V \oplus \mathbb{R}) \arrow[r] & \Omega TS(V \oplus \mathbb{R}).
\end{tikzcd}$$

The outer squares of this and the previous diagram are the same. Since $\Omega(-)$ commutes with homotopy pullback, and by the definition of $\tilde{T}X$, the right square is homotopy pullback. It follows that the left square is homotopy pullback; hence we are done. □

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