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ON HIGHER NIL GROUPS OF GROUP RINGS

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Abstract

Let G be a finite group and $\mathbb{Z}[G]$ its integral group ring. We prove that the nil groups $N^jK_2(\mathbb{Z}[G])$ do not vanish for all $j \geqslant 1$ and for a large class of finite groups. We obtain from this that the iterated nil groups $N^jK_i(\mathbb{Z}[G])$ are also nonzero for all $i \geqslant 2, j \geqslant i-1$.

1. Introduction

Let Γ be a discrete group and $\mathbb{Z}[\Gamma]$ its integral group ring. The Farrell-Jones Isomorphism Conjecture [**13**] predicts that the algebraic K-theory groups $K_i(\mathbb{Z}[\Gamma])$ may be computed from homological information of Γ and the algebraic K-theory of group rings $R[V]$, where V runs over the virtually cyclic subgroups of Γ. When the Farrell-Jones Isomorphism Conjecture holds there have been explicit examples like [**8, 9, 20**] and it has been the case that these computations may even be reduced further to the case where V runs over the finite subgroups of Γ [**9, 20**]; see Section 4 for a precise formulation. The groups that prevent such reductions are the nil groups of the group rings of finite subgroups of Γ; see H. Bass [**5**] for definitions. In this paper we show that, in principle, such reductions cannot be achieved for $K_i(\mathbb{Z}[\Gamma])$ for $i > 1$. Our main result is the following:

Theorem 1.1. *Let G be a nontrivial finite cyclic group or a split extension of a nontrivial finite cyclic group. Then*

$$N^jK_i(\mathbb{Z}[G]) \neq 0 \quad \textit{for all} \quad i \geqslant 2 \quad \textit{and} \quad j \geqslant i-1.$$

The above is related to an old question by H. Bass [**6**, Prob. VI]. We conjecture that this nonvanishing result must hold for *every* finite group.

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2. Preliminaries

Let R be an associative ring with unity and $K_n(R)$ its nth algebraic K-theory; cf. D. Quillen [**21**]. Let G be a group and $R[G]$ its corresponding group ring. It is natural to compare the K-theory of R to that of its polynomial ring $R[t]$. This leads to the definition of the *nil* groups: Let $\epsilon\colon R[t] \to R$ be the augmentation map, induced by evaluating at $t = 0$. The ith nil group of R is defined as

$$NK_i(R) = \ker(K_i(R[t]) \xrightarrow{\epsilon_*} K_i(R)).$$

By iterating this to polynomial rings in more variables we get the iterated nil groups $N^jK_i(R) = N(N^{j-1}K_i(R))$, $j > 1$. These nil K-groups have geometric significance as they occur as obstructions to geometric problems; see for example [**11**]. We are interested in the study of these nil K-groups in the case when $R = \mathbb{Z}[G]$ and G is a finite group.

Recall that when R is a *regular* ring, it follows that $NK_i(R) = 0$ for all i; cf. [**21**]. The rings $\mathbb{Z}[G]$ are never regular when G is a finite group. However, some vanishing results are available: if G is a finite group of square-free order, then $NK_i(\mathbb{Z}[G]) = 0$ for $i = 0, 1$; see Harmon [**16**]. In fact, $NK_i(\mathbb{Z}[G]) = 0$ for all $i \leqslant -1$ and *any* finite group G; [**5**, XII, 10.2]. It has been conjectured by W.C. Hsiang that this last property should hold for any integral group ring. This has been verified for large classes of groups as a consequence of the Farrell-Jones Conjecture in [**18**].

Another instance in which these nil K-groups appear naturally is in the setup of the Farrell-Jones Conjecture; see Section 4.

3. Nonvanishing results

In [**15**], Guin-Waléry and Loday proved that $NK_2(\mathbb{Z}[C_p]) \cong x\mathbb{Z}/p[x]$, and is generated by Dennis-Stein symbols $\langle (1-\sigma)x^j, (1+\sigma+\cdots\sigma^{p-1})\rangle$, where C_p stands for the finite cyclic group of prime order p and generated by σ.

Theorem 3.1. *Let C_n be a cyclic group of finite order $n \geqslant 2$. Then $NK_2(\mathbb{Z}[C_n]) \neq 0$.*

Proof. As there is a split summand $C_n \twoheadrightarrow C_{p^r}$, we may assume that $n = p^r$ for some prime p and integer $r \geqslant 2$. Let σ be a generator of C_n. Observe that as $(1-\sigma)(1+\sigma+\cdots+\sigma^{p^r-1}) = 1-\sigma^{p^r} = 0$ in $\mathbb{Z}[C_n]$, the symbol

$$\langle (1+\sigma+\cdots+\sigma^{p^r-1})x, (1-\sigma)\rangle$$

is a well-defined Dennis-Stein symbol in $K_2(\mathbb{Z}[C_n][x])$. We will prove that this is not trivial as long as $r \geqslant 1$. Let

$$\varphi\colon K_2(\mathbb{Z}[C_n][x]) \to K_2(\mathbb{F}_p[C_n][x])$$

be induced from mod p reduction. We see that

$$\mathbb{F}_p[C_n][x] \cong \mathbb{F}_p\left[\varepsilon\right]/\varepsilon^{p^r}[x] \cong \mathbb{F}_p[\varepsilon, x]/(\varepsilon^{p^r}),$$

and, under the above identifications, σ is taken to $1-\varepsilon$. On the other hand, let I be the ideal generated by ε in $\mathbb{F}_p[\varepsilon, x]/(\varepsilon^{p^r})$; thus $\left(\mathbb{F}_p[\varepsilon, x]/(\varepsilon^{p^r})\right)/I \cong \mathbb{F}_p[x]$. From the long exact sequence associated to the pair $(\mathbb{F}_p[\varepsilon, x]/(\varepsilon^{p^r}), I)$, the fact that $K_3(\mathbb{F}_p[x])$ is isomorphic to $K_3(\mathbb{F}_p)$, which is a group of order prime to p, and by [**19**, Corollary 2.7], the first group below is a p-group, so we have a *monomorphism*

$$K_2(\mathbb{F}_p[\varepsilon, x]/(\varepsilon^{p^r}), I) \hookrightarrow K_2(\mathbb{F}_p[\varepsilon, x]/(\varepsilon^{p^r})).$$

Now observe that

$$\begin{aligned}
&\varphi(\langle(1 + \sigma + \cdots + \sigma^{p^r-1})x, (1 - \sigma)\rangle) \\
&= \langle(1 + (1 - \varepsilon) + (1 - \varepsilon)^2 + \cdots + (1 - \varepsilon)^{p^r-1})x, \varepsilon\rangle \\
&= \langle\varepsilon^{p^r-1}x, \varepsilon\rangle.
\end{aligned}$$

This element $\langle\varepsilon^{p^r-1}x, \varepsilon\rangle$ as an element of $K_2(\mathbb{F}_p[\varepsilon, x]/(\varepsilon^{p^r}))$ comes from the relative group $K_2(\mathbb{F}_p[\varepsilon, x]/(\varepsilon^{p^r}), I)$ and is a generator of order p in the relative group by the computations of W. van der Kallen and J. Stienstra in [**19**, Corollary 2.7]. Finally, observe that this element really is in $NK_2(\mathbb{F}_p[\varepsilon]/(\varepsilon^{p^r}), (\varepsilon)) \cong NK_2(\mathbb{F}_p[\varepsilon]/(\varepsilon^{p^r}))$. □

C. Weibel, proved (see [**23**, Application III.3.4.2]) that for any ring if $N^sK_i(R) = 0$, it follows that $N^jK_i(R) = 0$ for $j = 1, 2, \ldots, s - 1$. As a corollary we have

Corollary 3.2. *Let C_n be a nontrivial finite cyclic group, then*

$$N^jK_2(\mathbb{Z}[C_n]) \neq 0, \text{ for all } j \geqslant 1.$$

This contrasts with $NK_1(\mathbb{Z}[C_n])$ where it is known [**7**] that

$$NK_1(\mathbb{Z}[C_n]) = 0 \text{ if and only if } n \text{ is square-free.}$$

By the fundamental theorem in algebraic K-theory, the NK_2 terms are direct summands in the group $K_2(\mathbb{Z}[C_n \times T^s])$, where T^s is the free abelian group of rank s, and $s \geqslant 1$. Moreover, by [**12**] we see that if NK_2 is nontrivial then it is not finitely generated. Thus we have the following

Corollary 3.3. *Let C_n be a nontrivial finite cyclic group and T^s a free abelian group of rank s, $s \geqslant 1$. Then $K_2(\mathbb{Z}[C_n \times T^s])$ is not finitely generated.*

A ring R is called K_n-regular if $N^jK_n(R) = 0$ for all $j \geqslant 1$. T. Vorst proved in [**22**, Proposition 2.1] that if $N^2K_n(R) = 0$ then $NK_{n-1}(R) = 0$. Hence, if R is such that $NK_2(R) \neq 0$ it follows that $N^jK_i(R) \neq 0$ for $j \geqslant i - 1$ and for all $i > 2$. From this we have the following corollary:

Corollary 3.4. *Let C_n and T^s be a finite cyclic group of order n, $n \geqslant 2$, and the free abelian group of rank $s \geqslant 1$, respectively. Then*

1. *$N^jK_i(\mathbb{Z}[C_n]) \neq 0$ for all $i \geqslant 2$ and all $j \geqslant i - 1$,*
2. *$K_i(\mathbb{Z}[C_n \times T^s])$ is not finitely generated for all $i \geqslant 2$, and $s \geqslant i - 1$.*

The results in the previous section immediately give the proof of Theorem 1.1.

Proof. The cyclic case is Theorem 3.1. Let $r\colon G \to C$ be a split surjection onto a cyclic group C. Then the splitting $s\colon C \to G$ induces an injection $N^jK_i(\mathbb{Z}[C]) \hookrightarrow N^jK_i(\mathbb{Z}[G])$; thus our result follows from Theorem 3.1 and the above. □

4. Examples

Our results contrast with those for *lower* K-theory where it is known that $NK_{-i}(\mathbb{Z}[G]) = 0$ for all $i \geqslant 1$ and all finite groups G; see H. Bass [**5**, XII, 10.2]. On the other hand, if G is a finite group of square-free order it is known that $NK_i(\mathbb{Z}[G]) = 0$ for $i = 0, 1$; see [**16**]. The following examples show consequences of our results for infinite groups of geometric relevance.

We begin by recalling some terminology: a group V is called *virtually cyclic* if it contains a cyclic group of finite index. It follows that either V is finite or it contains a unique maximal normal *finite* subgroup F such that either:

1. V/F is infinite cyclic or
2. V/F is infinite dihedral.

We call V *orientable* if the first case above holds and is *nonorientable* in the second case; see [**17**].

Example 4.1. Let G be a word hyperbolic group in the sense of Gromov; see [**14**]. Assume that all finite subgroups of G satisfy the hypotheses of Theorem 1.1 and that the only infinite virtually cyclic subgroups of G are of the form $F \times \mathbb{Z}$, where F is a finite subgroup of G. Let R be an associative ring, and $\mathbb{K}_R$ be the algebraic K-theory spectrum defined in [**10**]. Given X a G-CW-complex, write $H_*^G(X; \mathbb{K}_R)$ for the associated equivariant homology theory applied to X. This theory is such that for any $H \leqslant G$, $H_*^G(G/H; \mathbb{K}_R)$ is naturally isomorphic to the algebraic K-groups $K_*(RH)$. The following description of this equivariant homology for hyperbolic groups is found in [**17**, Corollary 19 and Remark 7]: For any word hyperbolic group G as above, there is an isomorphism

$$H_n^G(\underline{\underline{\mathcal{E}}}G; \mathbb{K}_\mathbb{Z}) \cong H_n^G(\underline{\mathcal{E}}G; \mathbb{K}_\mathbb{Z}) \oplus \bigoplus_{\mathrm{Conj}(V)} NK_n(\mathbb{Z}[\mathrm{fin}(V)]) \oplus NK_n(\mathbb{Z}[\mathrm{fin}(V)]),$$

where

- $\mathrm{Conj}(V)$ denotes representatives of conjugacy classes of maximal infinite virtually cyclic subgroups of G,
- $\mathrm{fin}(V)$ is the finite maximal subgroup of V,
- the spaces $\underline{\underline{\mathcal{E}}}G$ and $\underline{\mathcal{E}}G$ are universal spaces for actions with virtually cyclic and finite isotropy respectively.

As a corollary of the above, we have:

Corollary 4.2. *Let G be as in Example 4.1; then $H_2^G(\underline{\underline{\mathcal{E}}}G; \mathbb{K}_\mathbb{Z})$ is not finitely generated.*

Remark 4.3. It has been announced by A. Bartels, H. Reich and W. Lück that the Farrell-Jones Isomorphism Conjecture in K-theory holds for hyperbolic groups [**3**]; hence $H_n^G(\underline{\underline{\mathcal{E}}}G;\mathbb{K}_{\mathbb{Z}})$ is really $K_n(\mathbb{Z}[G])$.

Example 4.4. Let Γ be a discrete group. Assume that Γ has nontrivial torsion, that the finite subgroups of Γ satisfy the hypotheses of Theorem 1.1, and that the Farrell-Jones Isomorphism Conjecture holds for $\mathbb{Z}[\Gamma]$ [**13**]. Then, the algebraic K-theory of $\mathbb{Z}[\Gamma]$ is isomorphic to the generalized equivariant homology theory (Example 4.1):

$$H_n^\Gamma(\underline{\underline{\mathcal{E}}}\Gamma;\mathbb{K}_{\mathbb{Z}}),$$

where $\underline{\underline{\mathcal{E}}}\Gamma$ denotes the *universal space for actions with virtually cyclic* isotropy.

On the other hand, we may take $H_n^\Gamma(\underline{\mathcal{E}}\Gamma;\mathbb{K}_{\mathbb{Z}})$, where $\underline{\mathcal{E}}\Gamma$ denotes the *universal space for actions with finite* isotropy. There is a natural map induced by inclusions

$$\mathcal{A}\colon H_n^\Gamma(\underline{\mathcal{E}}\Gamma;\mathbb{K}_{\mathbb{Z}})\to H_n^\Gamma(\underline{\underline{\mathcal{E}}}\Gamma;\mathbb{K}_{\mathbb{Z}}).$$

We say that the K-theory of $\mathbb{Z}[\Gamma]$ *reduces* to finite groups of Γ if $\mathcal{A}$ is an isomorphism.

A. Bartels shows in [**1**] that $\mathcal{A}$ is a split injection and its cokernel is built from the nil K-groups of the rings $\mathbb{Z}[G]$, where G runs over the finite subgroups of Γ and other types of nil K-groups. By our results, these nil K-groups rarely vanish in higher K-theory, hence, in principle, higher K-theory does not reduce to finite groups.

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