

CLASSIFICATION OF DI-EMBEDDINGS OF
THE n -CUBE INTO \mathbb{R}^n

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Abstract

A di-embedding of the n -cube I^n into \mathbb{R}^n is a map $I^n \rightarrow \mathbb{R}^n$ which is a dihomeomorphism onto its image. We show that such a map is, up to a permutation of coordinates, an n -fold product of 1-dimensional orientation preserving embeddings $I^1 \rightarrow \mathbb{R}$.

Introduction

Geometric models for concurrent computing called *progress graphs* first appeared in [2] where they were attributed to Dijkstra. Progress graphs were used in [1] to prove the existence of deadlocks in certain types of concurrent systems. More recently, progress graphs appear in the work of Fajstrup, Goubault, Raussen and others [3, 5, 6] where various tools from algebraic topology are adapted to their study. One works in the category of *directed spaces*, also known as *locally partially ordered spaces* (abbreviated here as “LPO-spaces”), where execution paths in a concurrent system correspond to dipaths (“directed paths”) in an LPO-space. In this language, progress graphs are partially ordered spaces whose partial order is induced by the Euclidean product partial order. An important problem is to determine when two execution paths give the same result; in [3], it is argued that this is the case when the corresponding dipaths are dihomotopic, i.e., homotopic through a family of dipaths. The *fundamental category* of an LPO-space, analogous to the classical fundamental groupoid, is constructed in [3]. Dihomeomorphism is the categorical notion of isomorphism in the category of LPO-spaces, and the fundamental category is a dihomeomorphism invariant. Various categories of fractions of the fundamental category, as well as their corresponding component categories are studied in [7]. The reader is referred to [3, 5] for a treatment of ditopology (“directed topology”).

In this paper we show that the notion of dihomeomorphism is rather restrictive. A *di-embedding* of the n -cube $I^n = \prod_{i=1}^n [0, 1]$ into \mathbb{R}^n is a map $I^n \rightarrow \mathbb{R}^n$ which is a dihomeomorphism onto its image. Given a permutation σ of $\{1, \dots, n\}$, the induced *coordinate permutation* $\sigma_*: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the dihomeomorphism given by $\sigma_*(x_1, \dots, x_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$. We prove (see Theorem 2.5):

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Theorem. Let $f: I^n \rightarrow \mathbb{R}^n$ be a di-embedding. Then there exists a coordinate permutation $\sigma_*: \mathbb{R}^n \rightarrow \mathbb{R}^n$ and orientation preserving embeddings $q_i: I^1 \rightarrow \mathbb{R}$ such that $f = \sigma_*\left(\prod_{i=1}^n q_i\right)$.

This theorem reinforces the general belief that a workable notion of equivalence between LPO-spaces which is weaker than dihomoemorphism would be useful.

As a corollary, we determine the structure of the group $\text{DiHomeo}(I^n)$ of dihomoemorphisms of I^n (see Corollary 2.10):

Corollary. There is an isomorphism of topological groups:

$$\text{DiHomeo}(I^n) \cong (\text{Homeo}^+(I^1))^n \rtimes \Sigma_n \quad (\text{semidirect product})$$

where the symmetric group Σ_n of permutations of $\{1, 2, \dots, n\}$ acts on $(\text{Homeo}^+(I^1))^n$, the n -fold product of the group of orientation preserving homeomorphisms of the unit interval, by permuting factors.

1. Ditopology

For the convenience of the reader and to establish some basic terminology, we recall some basic definitions concerning topological spaces with a compatible partial order, [3, 5].

Given a set X with a partial order denoted by “ \leq ”, the associated *partial order relation* is the set $R = \{(x, y) \in X \times X \mid x \leq y\}$.

Definition 1.1. A *PO-space* (“partially ordered space”) is a pair (X, \leq) consisting of a topological space X and a partial order \leq whose associated partial order relation is closed as a subset of $X \times X$ with the product topology.

A subset $Z \subset X$ of a PO-space (X, \leq) inherits a PO-space structure by giving Z the subspace topology and by restricting the partial order to Z .

Example 1.2. The real line \mathbb{R}^1 is a PO-space with the usual topology and the usual partial order (given by $x \leq y$ if and only if $y - x$ is non-negative). For a positive integer n , let $\pi_i: \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1, \dots, n$, denote projection to the i -th factor. The (*Euclidean*) *product partial order* on \mathbb{R}^n is defined coordinatewise, i.e., for $x, y \in \mathbb{R}^n$, $x \leq y$ if and only if $\pi_j(x) \leq \pi_j(y)$ for $j = 1, \dots, n$. The partial order relation, $R \subset \mathbb{R}^n \times \mathbb{R}^n$, for this partial order is given by $R = \bigcap_{j=1}^n \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid \pi_j(x) \leq \pi_j(y)\}$ which is a closed subset of $\mathbb{R}^n \times \mathbb{R}^n$ since each $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid \pi_j(x) \leq \pi_j(y)\}$ is clearly closed. Hence \mathbb{R}^n with the usual topology and the product partial order is a PO-space.

Definition 1.3. A *dimap* $f: (X, \leq) \rightarrow (Y, \leq)$ between two PO-spaces is a continuous map $f: X \rightarrow Y$ which preserves partial order, i.e. for all $u, v \in X$, $u \leq v$ implies $f(u) \leq f(v)$.

The category of PO-spaces is the category whose objects are PO-spaces and whose morphisms are dimaps (categorical composition is composition of functions). Isomorphisms in this category are called *dihomoemorphisms*, i.e.,

Definition 1.4. A dimap $f : X \rightarrow Y$ is a *dihomeomorphism* if there is a dimap $g : Y \rightarrow X$ such that gf is the identity of X and fg is the identity of Y .

A dimap of PO-spaces which is a homeomorphism of underlying topological spaces need not be a dihomeomorphism, as in the following example.

Example 1.5. Let $I^2 = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ and $X = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1, y \leq x \leq y + 1\}$ with PO-space structures inherited from product partial order on \mathbb{R}^2 . The map $f : I^2 \rightarrow X$ defined by $f(x, y) = (x + y, y)$ is a homeomorphism with inverse $g : X \rightarrow I^2$ given by $g(x, y) = (x - y, y)$. While f is a dimap, g is not a dimap.

Definition 1.6. A map $f : X \rightarrow Y$ between PO-spaces is a *di-embedding* if f is a dihomeomorphism onto its image where $f(X)$ inherits its PO-space structure from Y .

Remark 1.7. A continuous map $q : [a, b] \rightarrow \mathbb{R}$, where $a < b$, is injective if and only if it is strictly monotone. Hence any orientation preserving embedding $q : [a, b] \rightarrow \mathbb{R}$ is a di-embedding.

Let X be a topological space. The *topological boundary* of $A \subset X$, denoted by $\text{bd}(A)$, is $\text{bd}(A) = \overline{A} \cap \overline{A^c}$ where A^c denotes the complement of A in X and for $C \subset X$, \overline{C} denotes the closure of C in X . For a subspace Y of X and $A \subset X$, let $\text{bd}_Y(Y \cap A)$ denote the topological boundary of $Y \cap A$ in the subspace Y .

We record the following elementary lemma for use in Lemma 2.3.

Lemma 1.8. *Let X be a topological space, Y a subspace of X and $A \subset X$. Then $\text{bd}_Y(Y \cap A) \subset \text{bd}_X(A)$.*

Proof. Observe that $y \in \text{bd}_Y(Y \cap A)$ if and only if for every neighborhood V_y of y in X we have $Y \cap V_y \cap A \neq \emptyset$ and $Y \cap V_y \cap A^c \neq \emptyset$ (where A^c is the complement of A in X). These conditions imply that $V_y \cap A \neq \emptyset$ and $V_y \cap A^c \neq \emptyset$ and thus $y \in \text{bd}_X(A)$. □

2. Di-embeddings of standard n -rectangles into \mathbb{R}^n

In this section we prove the main theorem of this paper (Theorem 2.5) and derive some consequences (Corollaries 2.7, 2.9, 2.10, 2.11).

Let $R \subset \mathbb{R}^n \times \mathbb{R}^n$ be the partial order relation for the product partial order on \mathbb{R}^n (see Example 1.2). Recall that $\pi_i : \mathbb{R}^n \rightarrow \mathbb{R}$ denotes projection to the i -th factor.

Lemma 2.1. *The topological boundary of R in $\mathbb{R}^n \times \mathbb{R}^n$ is given by*

$$\text{bd}(R) = \bigcup_{j=1}^n \{(x, y) \in R \mid \pi_j(x) = \pi_j(y)\}.$$

Proof. The complement of R in $\mathbb{R}^n \times \mathbb{R}^n$ is given by

$$R^c = \bigcup_{j=1}^n \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid \pi_j(x) > \pi_j(y)\}.$$

The closure of R^c is given by:

$$\overline{R^c} = \bigcup_{j=1}^n \overline{\{(x, y) \mid \pi_j(x) > \pi_j(y)\}} = \bigcup_{j=1}^n \{(x, y) \mid \pi_j(x) \geq \pi_j(y)\}.$$

Hence

$$\begin{aligned} \text{bd}(R) &= \overline{R} \cap \overline{R^c} = R \cap \overline{R^c} = \\ &= \left(\bigcap_{j=1}^n \{(x, y) \mid \pi_j(x) \leq \pi_j(y)\} \right) \cap \left(\bigcup_{j=1}^n \{(x, y) \mid \pi_j(x) \geq \pi_j(y)\} \right) \\ &= \bigcup_{j=1}^n \{(x, y) \in R \mid \pi_j(x) = \pi_j(y)\}. \quad \square \end{aligned}$$

For subset X of \mathbb{R}^n , write $R|_X = R \cap (X \times X)$. Note that $R|_X$ is the partial order relation for the restriction of the product partial order on \mathbb{R}^n to X .

A *standard n -rectangle* is a subset P of \mathbb{R}^n of the form $P = \prod_{i=1}^n [a_i, b_i]$ where each $[a_i, b_i]$ is a closed interval with $a_i < b_i$. A straightforward adaptation of the proof of Lemma 2.2 yields:

Lemma 2.2. *Let P be a standard n -rectangle. The topological boundary of R in $P \times P$ is given by*

$$\text{bd}_{P \times P}(R|_P) = \bigcup_{j=1}^n \{(x, y) \in R|_P \mid \pi_j(x) = \pi_j(y)\}. \quad \square$$

For $1 \leq j \leq n$ and $c \in \mathbb{R}$, define $H_c^j = \pi_j^{-1}(\{c\})$, a hyperplane perpendicular to the j -th coordinate axis. For a subset $A \subset \mathbb{R}^n$, we write $H_c^j(A)$ for $H_c^j \cap A$. Note that $H_c^j \cap H_d^j \neq \emptyset$ if and only if $c = d$. Also observe that if $j \neq k$ then $H_c^j(P) \cap H_d^k(P) \neq \emptyset$ where $P = \prod_{i=1}^n [a_i, b_i]$ is a standard n -rectangle and $c \in [a_j, b_j]$ and $d \in [a_k, b_k]$.

Lemma 2.3. *Let $f: P \rightarrow \mathbb{R}^n$ be a di-embedding of a standard n -rectangle $P = \prod_{i=1}^n [a_i, b_i]$. Then for each index $1 \leq j \leq n$ and $c \in [a_j, b_j]$ there exists an index $1 \leq j(c) \leq n$ and a real number $q(c)$ such that $f(H_c^j(P)) \subset H_{q(c)}^{j(c)}$.*

Proof. Let $S = f(P)$. By hypothesis, $f: P \rightarrow S$ is a dihomeomorphism. In particular, $f: P \rightarrow S$ is a homeomorphism and so $f \times f: P \times P \rightarrow S \times S$ is also a homeomorphism. Let $R \subset \mathbb{R}^n \times \mathbb{R}^n$ be the partial order relation for the product partial order on \mathbb{R}^n (as in Lemma 2.1). Since $f: P \rightarrow S$ is a dihomeomorphism, for all $x, y \in P$, $f(x) \leq f(y)$ if and only if $x \leq y$ and hence $(f \times f)(R|_P) \subset R|_S$ and $(f \times f)^{-1}(R|_S) \subset R|_P$. It follows that $(f \times f)(R|_P) = R|_S$ because $f \times f$ is a

bijection. Since $f \times f: P \times P \rightarrow S \times S$ is homeomorphism,

$$(f \times f)(\text{bd}_{P \times P}(R|_P)) = \text{bd}_{S \times S}(R|_S). \tag{1}$$

Given $c \in [a_j, b_j]$, define points $a_c^j, b_c^j \in H_c^j(P)$ by $\pi_j(a_c^j) = \pi_j(b_c^j) = c$ and $\pi_i(a_c^j) = a_i, \pi_i(b_c^j) = b_i$ for $i \neq j$. Observe that $H_c^j(P) = \{x \in P \mid a_c^j \leq x \leq b_c^j\}$. By Lemma 2.2, $(a_c^j, b_c^j) \in \text{bd}_{P \times P}(R|_P)$ and so (1) implies that $(f(a_c^j), f(b_c^j)) \in \text{bd}_{S \times S}(R|_S)$. By Lemma 1.8, $\text{bd}_{S \times S}(R|_S) \subset \text{bd}(R)$ and so Lemma 2.1 implies there exists an index $j(c)$ such that $\pi_{j(c)}(f(a_c^j)) = \pi_{j(c)}(f(b_c^j))$; denote this real number by $q(c)$. Suppose $x \in H_c^j(P)$. Then $a_c^j \leq x \leq b_c^j$ and so $f(a_c^j) \leq f(x) \leq f(b_c^j)$ since f is a dimap. It follows that $\pi_{j(c)}(f(a_c^j)) \leq \pi_{j(c)}(f(x)) \leq \pi_{j(c)}(f(b_c^j))$ and thus $\pi_{j(c)}(f(x)) = q(c)$, i.e., $f(x) \in H_{q(c)}^{j(c)}$. \square

For a positive integer n , denote $\underline{n} = \{1, \dots, n\}$. Given a permutation σ of \underline{n} , i.e., a bijection $\sigma: \underline{n} \rightarrow \underline{n}$, the induced *coordinate permutation*, $\sigma_*: \mathbb{R}^n \rightarrow \mathbb{R}^n$, is the map given by $\pi_j(\sigma_*(x)) = \pi_{\sigma(j)}(x)$ for $x \in \mathbb{R}^n$. Note that σ_* is a dihomoemorphism with inverse $(\sigma^{-1})_*$.

Lemma 2.4. *Let $P = \prod_{i=1}^k [a_i, b_i]$ be a standard k -rectangle, where $k \geq 1$, and let $u: P \rightarrow \mathbb{R}^k$ be a map of the form $u = \tau_* \left(\prod_{j=1}^k g_j \right)$, where $\tau_*: \mathbb{R}^k \rightarrow \mathbb{R}^k$ is a coordinate permutation and each $g_j: [a_j, b_j] \rightarrow \mathbb{R}$ is an embedding, $j = 1, \dots, k$. Let $v_i(P) \in P$, $i = 0, 1, \dots, k$, be the vertices $v_0(P) = (a_1, \dots, a_k)$ and $v_i(P) = (a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_k)$, $i = 1, \dots, k$. Then the vectors $u(v_i(P)) - u(v_0(P))$, $i = 1, \dots, k$, are non-zero and mutually orthogonal.*

Proof. Since the coordinate permutation $\tau_*: \mathbb{R}^k \rightarrow \mathbb{R}^k$ is an orthogonal linear transformation, it suffices to consider the case τ is the identity. Note that $u(v_i(P)) - u(v_0(P)) = (g_i(b_i) - g_i(a_i))e_i$, where e_i is the i -th standard unit basis vector for \mathbb{R}^k , is not zero because g_i is injective. The conclusion follows from the orthogonality of the standard basis. \square

Theorem 2.5. *Let $f: P \rightarrow \mathbb{R}^n$ be a di-embedding of a standard n -rectangle $P = \prod_{i=1}^n [a_i, b_i]$. Then there exists a coordinate permutation $\sigma_*: \mathbb{R}^n \rightarrow \mathbb{R}^n$ and orientation preserving embeddings $q_i: [a_i, b_i] \rightarrow \mathbb{R}$ such that $f = \sigma_* \left(\prod_{i=1}^n q_i \right)$.*

Proof. The proof proceeds by induction on n . The case $n = 1$ follows from Remark 1.7. Inductively, assume that the conclusion of the theorem is valid for di-embeddings of standard $(n - 1)$ -rectangles into \mathbb{R}^{n-1} . We will need the following strengthened version of Lemma 2.3:

Claim 2.6. *For each integer $1 \leq j \leq n$, there exists $1 \leq k \leq n$ such that for all $c \in [a_j, b_j]$ there exists a real number $q(c)$ such that $f(H_c^j(P)) \subset H_{q(c)}^k$.*

Proof of the Claim. By Lemma 2.3, given an integer $1 \leq j \leq n$ and $c \in [a_j, b_j]$ there exists an integer $1 \leq k(c) \leq n$ and a real number $q(c)$ such that $f(H_c^j(P)) \subset H_{q(c)}^{k(c)}$. Let $\phi_d^\ell: H_d^\ell \rightarrow \mathbb{R}^{n-1}$ denote the canonical dihomoemorphism given by

$$\phi_d^\ell(x_1, \dots, x_{\ell-1}, d, x_{\ell+1}, \dots, x_n) = (x_1, \dots, x_{\ell-1}, x_{\ell+1}, \dots, x_n).$$

Define $P_j = \prod_{i \neq j} [a_i, b_i]$ (a standard $(n - 1)$ -rectangle). Note that ϕ_c^j restricts to

a dihomeomorphism $H_c^j(P) \rightarrow P_j$ which we also denote by ϕ_c^j . By the inductive hypothesis applied to $\phi_{q(c)}^{k(c)} f(\phi_c^j)^{-1}$ and Lemma 2.4, the $n - 1$ vectors $w_i(c) = f((\phi_c^j)^{-1}(v_i(P_j))) - f((\phi_c^j)^{-1}(v_0(P_j)))$, $i = 1, \dots, n - 1$ (see Lemma 2.4 for the notation $v_i(P_j)$) are non-zero and orthogonal. Hence the cross product¹ $\text{CP}(w_1(c), \dots, w_{n-1}(c))$ is a vector in \mathbb{R}^n which is non-zero and orthogonal to the hyperplane $H_{q(c)}^{k(c)}$. The set of unit vectors which are orthogonal to some hyperplane of the form H_d^ℓ is the finite set $S = \{\pm e_j \mid j = 1, \dots, n\}$ where e_1, \dots, e_n is the standard basis for \mathbb{R}^n . The function $c \mapsto \text{CP}(w_1(c), \dots, w_{n-1}(c)) / \|\text{CP}(w_1(c), \dots, w_{n-1}(c))\|$, where $\|v\|$ denotes the length of v , is a continuous function with domain $[a_j, b_j]$ taking values in the discrete set S , hence it is a constant and so $k(c)$ is independent of c , proving the claim.

In Claim 2.6, the integer k is uniquely determined by j . In order to verify this assertion, suppose that for $c \in [a_j, b_j]$, $f(H_c^j(P)) \subset H_{q(c)}^k$ and $f(H_c^\ell(P)) \subset H_{r(c)}^\ell$. It follows that $f(H_{a_j}^j(P)) \subset H_{q(a_j)}^k \cap H_{r(a_j)}^\ell$. If $k \neq \ell$ then $H_{q(a_j)}^k \cap H_{r(a_j)}^\ell \cong \mathbb{R}^{n-2}$ (where “ \cong ” denotes homeomorphism). $H_{a_j}^j(P)$ is homeomorphic to a standard $(n - 1)$ -rectangle and the restriction of f to $H_{a_j}^j(P)$ thus yields an embedding of a standard $(n - 1)$ -rectangle into \mathbb{R}^{n-2} contradicting Invariance of Domain². Note that the function $c \mapsto q(c)$ with the property $f(H_c^j(P)) \subset H_{q(c)}^k$ for every $c \in [a_j, b_j]$ is uniquely determined by j and f . Suppose $c \mapsto p(c)$ is another function with the property $f(H_c^j(P)) \subset H_{p(c)}^k$. Then $H_{q(c)}^k \cap H_{p(c)}^k$ is not empty since it contains $f(H_c^j(P))$ and so $q(c) = p(c)$. We will use the notation $q_j(c)$ for $q(c)$ to indicate its dependence on j .

Claim 2.6 yields a function $\sigma: \underline{n} \rightarrow \underline{n}$, where $\sigma(j)$ is the unique integer for which $f(H_c^j(P)) \subset H_{q_j(c)}^{\sigma(j)}$ for all $c \in [a_j, b_j]$. We show that σ is injective. Suppose $\sigma(j) = \sigma(k)$. Then $f(H_c^j(P)) \subset H_{q_j(c)}^{\sigma(j)}$ for all $c \in [a_j, b_j]$ and $f(H_d^k(P)) \subset H_{q_k(d)}^{\sigma(k)}$ for all $d \in [a_k, b_k]$. If $j \neq k$ then $H_c^j(P) \cap H_d^k(P) \neq \emptyset$ and thus $H_{q_j(c)}^{\sigma(j)} \cap H_{q_k(d)}^{\sigma(k)} \neq \emptyset$. Since $\sigma(j) = \sigma(k)$, it follows that $q_j(c) = q_k(d)$ for all $c \in [a_j, b_j]$ and $d \in [a_k, b_k]$. In particular, $q_j(c)$ is independent of c and so $f(H_c^j(P)) \subset H_{q_j(a_j)}^{\sigma(j)}$ for all $c \in [a_j, b_j]$. Since $P = \cup_c H_c^j(P)$, it follows that $f(P) \subset H_{q_j(a_j)}^{\sigma(j)}$ yielding an embedding of the standard n -rectangle P into $H_{q_j(a_j)}^{\sigma(j)} \cong \mathbb{R}^{n-1}$, contradicting Invariance of Domain. Thus σ is injective and hence bijective, i.e. a permutation, since \underline{n} is finite. Define $h: P \rightarrow \mathbb{R}^n$ by $h = (\sigma^{-1})_* f$. Then $h(H_c^j(P)) \subset H_{q_j(c)}^j$ for all $c \in [a_j, b_j]$ and $1 \leq j \leq n$. This condition is equivalent to $\pi_j h = q_j(\pi_j|_P)$ for $1 \leq j \leq n$ where $\pi_j|_P: P \rightarrow [a_j, b_j]$ is projection. Hence each $q_j: [a_j, b_j] \rightarrow \mathbb{R}$ is continuous

¹For $k \geq 2$, the cross product of a $(k - 1)$ -tuple of vectors w_1, \dots, w_{k-1} in \mathbb{R}^k , denoted $\text{CP}(w_1, \dots, w_{k-1})$, is the vector $\sum_{j=1}^k (-1)^j \det(A_j) e_j$, where e_1, \dots, e_k is the standard basis for \mathbb{R}^k and $\det(A_j)$ is the determinant of the $(k - 1) \times (k - 1)$ matrix A_j obtained from the $(k - 1) \times k$ matrix A whose i -th row is the vector w_i by deleting the j -th column. If w_1, \dots, w_{k-1} in \mathbb{R}^k are linearly independent, then $\text{CP}(w_1, \dots, w_{k-1})$ is non-zero and orthogonal to each w_i .

²Invariance of Domain is the assertion that a subset of \mathbb{R}^n which is homeomorphic to an open subset of \mathbb{R}^n is itself an open subset of \mathbb{R}^n ; see Theorem 16 in Section 4.7 of [8]

and $h = \prod_{j=1}^n q_j$. Since h is an embedding and a dimap, so is each q_j ; furthermore, by Remark 1.7 each q_j is an orientation-preserving embedding. We conclude $f = \sigma_*\left(\prod_{j=1}^n q_j\right)$. \square

The special case $n = 2$ of Theorem 2.5 was proved in [4].

Corollary 2.7. *Let $f: P \rightarrow \mathbb{R}^n$ be a di-embedding of a standard n -rectangle $P = \prod_{i=1}^n [a_i, b_i]$. Then $f(P)$ is a standard n -rectangle.*

Proof. The conclusion of the corollary is clear for a map of the form $f = \sigma_*\left(\prod_{j=1}^n q_j\right)$ as in the conclusion of Theorem 2.5. \square

Remark 2.8. The proof of Theorem 2.5 shows that the di-embedding $f: P \rightarrow \mathbb{R}^n$ uniquely determines the permutation σ and the orientation-preserving embeddings $q_j: [a_j, b_j] \rightarrow \mathbb{R}$, $j = 1, \dots, n$, such that $f = \sigma_*\left(\prod_{j=1}^n q_j\right)$.

A *standard open n -rectangle* is a subset P of \mathbb{R}^n of the form $P = \prod_{i=1}^n (a_i, b_i)$ where each (a_i, b_i) is an open interval with $a_i < b_i$.

Corollary 2.9. *Let $f: P \rightarrow \mathbb{R}^n$ be a di-embedding of a standard open n -rectangle $P = \prod_{i=1}^n (a_i, b_i)$. Then there exists a coordinate permutation $\sigma_*: \mathbb{R}^n \rightarrow \mathbb{R}^n$ and orientation-preserving embeddings $q_i: (a_i, b_i) \rightarrow \mathbb{R}$ such that $f = \sigma_*\left(\prod_{i=1}^n q_i\right)$.*

Proof. Let $\delta = \frac{1}{3} \min\{|b_i - a_i| \mid i = 1, \dots, n\}$ and let $P_m = \prod_{i=1}^n [a_i + \delta/m, b_i - \delta/m]$ for $m \geq 1$. Theorem 2.5 applied to the restriction of f to P_m , $m \geq 1$, and Remark 2.8 yields the conclusion. \square

Let $I^n = \prod_{i=1}^n [0, 1] \subset \mathbb{R}^n$ denote the unit n -cube. The group of dihomeomorphisms of I^n , denoted by $\text{DiHomeo}(I^n)$, is a subgroup of the group of homeomorphisms of I^n . Let $\text{Homeo}^+(I^1)$ denote the group of orientation preserving homeomorphisms of I^1 .

Corollary 2.10. *There is an isomorphism of topological groups:*

$$\text{DiHomeo}(I^n) \cong \left(\text{Homeo}^+(I^1)\right)^n \rtimes \Sigma_n \quad (\text{semidirect product})$$

where the symmetric group Σ_n of permutations of \underline{n} acts on $(\text{Homeo}^+(I^1))^n$ by permuting factors.

Proof. By Theorem 2.5,

$$\text{DiHomeo}(I^n) = \left\{ \sigma_*\left(\prod_{i=1}^n q_i\right) \mid \sigma \in \Sigma_n, q_i \in \text{Homeo}^+(I^1), i = 1, \dots, n \right\}.$$

The map $\text{DiHomeo}(I^n) \rightarrow \Sigma_n$ given by $\sigma_*\left(\prod_{i=1}^n q_i\right) \mapsto \sigma$ is a group homomorphism, which is split by $\sigma \mapsto \sigma_*$, and its kernel is $(\text{Homeo}^+(I^1))^n$. \square

Let BG denote the classifying space of the topological group G . Since $\text{Homeo}^+(I^1)$ is contractible, Corollary 2.10 yields the following result.

Corollary 2.11. *$B\text{DiHomeo}(I^n)$ is homotopy equivalent to $B\Sigma_n$.* \square

Remark 2.12. By Corollary 2.9, the corresponding versions of Corollaries 2.10 and 2.11 are valid for open cubes in place of closed cubes.

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