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A CLASS OF LEFT IDEALS OF THE STEENROD ALGEBRA

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Abstract

We study the nested collection of left ideals of $\mathcal{A}$, the mod 2 Steenrod algebra, $L(k) := \mathcal{A}\{Sq^{2^0}, Sq^{2^1}, Sq^{2^2}, \dots, Sq^{2^k}\}$. We determine the smallest k such that $Sq^n \in L(k)$.

We discuss an application which improves upon the results of F. R. Cohen and the first author in their paper comparing the loop of the degree 2 map on a sphere and the H-space squaring map on the loop of a sphere.

1. Introduction and statement of result

The Steenrod algebra, first constructed by N. Steenrod [**10**], is an algebra of stable cohomology operations which acts on the $\mathbb{Z}/2$-cohomology groups of topological spaces. The Steenrod algebra is widely studied by mathematicians whose interests range from algebraic topology and homotopy theory to manifold theory, combinatorics, representation theory, and more. For more details and applications see [**5, 7**], and for a history of the study of the Steenrod algebra see [**11**], all three of which include extensive lists of additional references. In the study of self maps of loop spaces of spheres, F. Cohen and the first author encountered the algebraic question which is solved in Theorem 1.1. Its solution leads to more general and stronger results in [**2**] as stated in Theorem 1.2.

To formally define the Steenrod algebra, let M be the graded $\mathbb{Z}/2$-module such that $M_i = \mathbb{Z}/2$ is generated by the symbol Sq^i for $i \geqslant 0$. Let $T(M)$ be the tensor algebra of M. The mod 2 Steenrod algebra, $\mathcal{A}$, is defined as the quotient of the tensor algebra $T(M)$ by the two sided ideal generated by the Adem relations $R(a,b)$, and $Sq^0 + 1$, where for $0 < a < 2b$,

$$R(a,b) := Sq^a \otimes Sq^b + \sum_c \binom{b-c-1}{a-2c} Sq^{a+b-c} \otimes Sq^c$$

and the binomial coefficient is evaluated mod 2.

In this paper we look at a question regarding the following nested collection of

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left ideals of the Steenrod algebra,

$$L(k) := \mathcal{A}\{Sq^{2^0}, Sq^{2^1}, Sq^{2^2}, \dots, Sq^{2^k}\}.$$

Given an integer n, we are interested in finding the smallest k such that $Sq^n \in L(k)$. It is well known that Sq^{2^i}, $i \geqslant 0$, are indecomposable elements and $\{Sq^{2^i} \mid i \geqslant 0\}$ generates $\mathcal{A}$ as an algebra [**7**]. Hence the nested left ideals $L(k)$ limit to $\mathcal{A}$, so a smallest such k will always exist. The complexity of the Adem relations make the smallest k difficult to find. For example, Sq^{10} can be factored in the following two ways

$$Sq^{10} = Sq^4Sq^2Sq^4 + Sq^8Sq^2 + Sq^4Sq^5Sq^1, \quad \text{and} \quad Sq^{10} = Sq^2Sq^8 + Sq^9Sq^1.$$

The first factorization shows $Sq^{10} \in L(2)$, while this fact is not clear from the second factorization. One can easily check in this example that $k = 2$ is the smallest k such that $Sq^{10} \in L(k)$. The principal result of this paper is the following theorem, the proof of which appears in Section 2.

Theorem 1.1. *For $n \geqslant 1$, the smallest k such that $Sq^n \in L(k)$ is $k = f(n)$, where the function f is defined below.*

The following notation is used to define f. For a positive integer n, let $[n]$ denote the dyadic expansion of n viewed as a string of zeros and ones. This string is unique up to leading zeros. For example, $[13]$ can be viewed as 1101 or 0001101. Given a binary string α let $|\alpha|$ denote the integer with dyadic expansion α, and let $\operatorname{len}(\alpha)$ denote the length of the string α. For example, $|0001101| = |1101| = 13$, $\operatorname{len}(1101) = 4$, and $\operatorname{len}(0001101) = 7$. Of course expressions such as $\operatorname{len}([n])$ are not well-defined and will not be used. Given a string β we define $z(\beta)$ to be the number of non-trailing zeros in β; for example $z(000110010000) = 5$. Given strings α and β let $\alpha\beta$ denote their concatenation. With this notation f is defined as follows. Writing $[n] = \alpha\beta$ such that $|\alpha| < z(\beta)$ with $\operatorname{len}(\beta)$ minimal,

$$f(n) := \operatorname{len}(\beta) - 2.$$

As an example calculation consider $n = 13$. We write $[13]$ as 01101; with $\alpha = 0$ and $\beta = 1101$ the condition $|\alpha| < z(\beta)$ is satisfied and with $\alpha = 01$ and $\beta = 101$ this condition is not satisfied. Thus the β with minimal length such that the condition $|\alpha| < z(\beta)$ is satisfied is $\beta = 1101$. As $\operatorname{len}(1101) = 4$ we have $f(13) = 2$. This corresponds to the factorization

$$Sq^{13} = Sq^2Sq^5Sq^2Sq^4 + Sq^2Sq^5Sq^5Sq^1 + Sq^2Sq^9Sq^2 + Sq^{12}Sq^1 \in L(2).$$

This algebraic question arose from study of a topological question by F. Cohen and the first author in [**2**] regarding two natural self maps of Ω^kS^{2n+1}; the k-fold looping of the degree two map on an odd sphere

$$\Omega^k[2] \colon \Omega^kS^{2n+1} \to \Omega^kS^{2n+1},$$

and for $k \geqslant 1$, the H-space squaring map,

$$\Psi^k(2) \colon \Omega^kS^{2n+1} \to \Omega^kS^{2n+1}.$$

The degree two map induces multiplication by 2 on cohomology groups and the H-space squaring map induces multiplication by two on homotopy groups. The maps $\Omega[2]$ and $\Psi^1(2)$ are not homotopic for arbitrary $2n+1$, but these maps are stably homotopic. A natural question to ask is whether or not the maps $\Omega[2]$ and $\Psi^1(2)$ become homotopic after looping a sufficient number of times. Some of the history regarding this question along with partial answers can be found in [**1, 3, 4**].

In [**2**], for $2n+1 \neq 2^s - 1$ for any integer s, lower bounds are given on the number k of loops required for the maps $\Omega^k[2]$ and $\Psi^k(2)$ to be homotopic. Larger lower bounds are obtained if $Sq^{2n+2} \in L(m)$ for smaller m. Finding the smallest such m gives best possible lower bounds for $\Omega^k[2]$ and $\Psi^k(2)$ to be homotopic when using the techniques in [**2**].

Theorem 1.2. *For $k \geqslant 1$, let $\Omega^k[2]\colon \Omega^k S^{2n+1} \to \Omega^k S^{2n+1}$ denote the k-fold loop of the degree two map on S^{2n+1} and $\Psi^k(2)\colon \Omega^k S^{2n+1} \to \Omega^k S^{2n+1}$ denote the H-space squaring map.*

If $\Omega^k[2]$ and $\Psi^k(2)$ are homotopic, then $k \geqslant F(2n+2)$ where F is defined by

$$F(m) = m - 2^{f(m)} + 1$$

and the function f is defined as above.

We refer the reader to [**2**] for the proof of Theorem 1.2 as it follows directly from Theorem 1.1 and from the calculations in Section 2 of [**2**].

2. Proof of Theorem 1.1

We begin with the following definitions and two lemmas which reduce our problem about the aforementioned left ideals to a number theory problem.

Let χ denote the anti-automorphism of the Steenrod algebra defined recursively by

$$\chi(Sq^k) = \sum_{1 \leqslant i \leqslant k} Sq^i \chi(Sq^{k-i}), \text{ and } \chi(Sq^0) = 1.$$

Let χ^* denote the dual automorphism on the dual to the Steenrod algebra, $\mathcal{A}^*$, defined recursively by

$$\chi^*(\xi_k) = \sum_{i=0}^{k-1} \chi^*(\xi_i)^{2^{k-i}} \xi_{k-i}, \text{ and } \chi^*(\xi_0) = 1.$$

Additional information regarding the dual algebra, $\mathcal{A}^*$, which is a polynomial ring over $\mathbb{Z}/2$ generated by $\{\xi_i\}$, $i \geqslant 1$, and computing $\chi(Sq^k)$ and $\chi^*(\xi_k)$ via stripping can be found in [**6, 7, 11**]. Last, let $R(k) = \{Sq^{2^0}, Sq^{2^1}, \ldots, Sq^{2^k}\}\mathcal{A}$, denote the corresponding right ideal of the Steenrod algebra.

Lemma 2.1. *For all $k \geqslant 1$, $\chi^*(\xi_k) = \xi_1^{2^k-1} +$ other terms.*

Proof. We proceed by induction on k. $\chi^*(\xi_1) = \chi^*(\xi_0)^2\xi_1 = \xi_1$. Assume $\chi^*(\xi_k) = \xi_1^{2^k-1}+$ o.t. (other terms), then

$$\begin{aligned}
\chi^*(\xi_{k+1}) &= \sum_{i=0}^{k} \chi^*(\xi_i)^{2^{k+1-i}}\xi_{k+1-i} \\
&= \chi^*(\xi_k)^{2^1}\xi_1 + \left(\sum_{i=0}^{k-1} \chi^*(\xi_i)^{2^{k+1-i}}\xi_{k+1-i}\right) \\
&= (\xi_1^{2^k-1} + \text{ o.t. })^2\xi_1 + \left(\sum_{i=0}^{k-1} \chi^*(\xi_i)^{2^{k+1-i}}\xi_{k+1-i}\right) \\
&= \xi_1^{2^{k+1}-1} + \text{ o.t. } + \left(\text{o.t. not equal to } \xi_1^{2^{k+1}-1}\right), \\
&= \xi_1^{2^{k+1}-1} + \text{ other terms.}
\end{aligned}$$

□

Lemma 2.2. *$Sq^n \in L(k)$ if and only if n is not a non-negative linear combination of the numbers $\{2^{k+1}, 2^{k+1}+2^k, 2^{k+1}+2^k+2^{k-1}, \ldots, 2^{k+2}-1\}$.*

Proof. A theorem of Negishi [**8**] states that the annihilator of $R(k)$, denoted by $R(k)^t$, is given by

$$R(k)^t = \mathbb{Z}/2[\xi_1^{2^{k+1}}, \xi_2^{2^k}, \xi_3^{2^{k-1}}, \ldots, \xi_{k+1}^2, \xi_{k+2}, \xi_{k+3}, \ldots].$$

Now $Sq^n \in L(k)$ if and only if $\chi(Sq^n) \in R(k)$, if and only if $\langle \chi(Sq^n), R(k)^t\rangle = 0$, if and only if $\langle Sq^n, \chi^*(R(k)^t)\rangle = 0$. By Lemma 2.1, $\chi^*(\xi_i) = \xi_1^{2^i-1}+$ o.t. (other terms). Thus

$$\chi^*(R(k)^t) = \mathbb{Z}/2[\xi_1^{(2-1)2^{k+1}}, \xi_1^{(2^2-1)2^k} + \text{ o.t.}, \xi_1^{(2^3-1)2^{k-1}} + \text{ o.t.},$$

$$\ldots, \xi_1^{2^{k+2}-2} + \text{ o.t.}, \xi_1^{2^{k+2}-1} + \text{ o.t.}, \xi_1^{2^{k+3}-1} + \text{ o.t.}, \ldots]$$

Consulting the incidence matrix for $\mathcal{A}\otimes\mathcal{A}^* \to \mathbb{Z}/2$, we see that $\langle Sq^n, \alpha\rangle \neq 0$ if and only if $\alpha = \xi_1^n +$ other terms [**6**]. Hence $\langle Sq^n, \chi^*(R(k)^t)\rangle = 0$ if and only if

$$\langle Sq^n, \mathbb{Z}/2[\xi_1^{(2-1)2^{k+1}}, \xi_1^{(2^2-1)2^k}, \xi_1^{(2^3-1)2^{k-1}}, \ldots \xi_1^{2^{k+2}-2}, \xi_1^{2^{k+2}-1}, \xi_1^{2^{k+3}-1} \ldots]\rangle = 0.$$

The result then follows for dimensional reasons and the fact that $2^{k+3}-1, 2^{k+4}-1, \ldots$ can all be expressed as sums of elements of the set $\{2^{k+1}, 2^{k+1}+2^k, 2^{k+1}+2^k+2^{k-1}, \ldots, 2^{k+2}-1\}$. □

Given a positive integer n, the previous lemma reduces the question of understanding Adem relations to a number theory question of finding the smallest k such that n cannot be written as a linear combination of the numbers $\{2^{k+1}, 2^{k+1}+2^k, \ldots, 2^{k+2}-2, 2^{k+2}-1\}$ with non-negative integer coefficients.

We begin with the following notation. For each integer $k \geqslant -1$ let

$$G(k) := \{2^{k+1}, 2^{k+1} + 2^k, 2^{k+1} + 2^k + 2^{k-1}, \dots, 2^{k+2} - 1\},$$

and let

$$S(k) := \{n \in \mathbb{N} \mid n \text{ is not a sum of elements of } G(k)\}.$$

Note that $G(-1) = \{1\}$ and $S(-1) = \emptyset$. For $n \in \mathbb{N}$ we define the function $f(n) := \min\{k \mid n \in S(k)\}$.

The following lemma essentially gives a recursive description of the set $S(k)$ in terms of $S(k-1)$.

Lemma 2.3. *Let $n \in \mathbb{N}$.*

1. *If n is even, then $n \in S(k)$ if and only if $\frac{n}{2} \in S(k-1)$.*
2. *If n is odd, then $n \in S(k)$ if and only if $n < 2^{k+2} - 1$ or $n - (2^{k+2} - 1) \in S(k)$.*

The proof of Lemma 2.3 is straightforward.

To determine whether or not a given integer n is in the set $S(k)$ we use the following theorem regarding binary strings α and β. For nonempty binary strings α and β, we define the star notation,

$$\alpha * \beta \Leftrightarrow |\alpha| < z(\beta),$$

where $|\alpha|$ is the integer with binary representation α and $z(\beta)$ is the number of non-trailing zeros in β. As an example of how the star notation is used, consider the binary string 1000110100. Then $10 * 00110100$ is true, as $2 < 3$; however, $100 * 0110100$ is false since $4 \not< 2$.

Theorem 2.4. *For nonempty strings α and β with $|\alpha\beta| \neq 0$,*

$$\alpha * \beta \Leftrightarrow |\alpha\beta| \in S(\operatorname{len}(\beta) - 2).$$

Proof. We proceed by induction on $n := |\alpha\beta|$. Set $k := \operatorname{len}(\beta) - 2$, so the theorem asserts $\alpha * \beta \Leftrightarrow n \in S(k)$.

If $n = 1$, then $|\alpha| = 0$ and $z(\beta) = k + 1$. So $\alpha * \beta \Leftrightarrow k + 1 > 0 \Leftrightarrow k \geqslant 0 \Leftrightarrow 1 \in S(k)$, the last equivalence being easy to check.

Now assume $n > 1$ and the theorem holds for all smaller values.

(i) Suppose n is even.
The case $\beta = 0$ is handled easily, so assume $\operatorname{len}(\beta) > 1$, write $\beta = \beta'0$, and note $\operatorname{len}(\beta') = k + 1$, $z(\beta) = z(\beta')$. Then

$$\alpha * \beta \Leftrightarrow \alpha * \beta' \Leftrightarrow n/2 = |\alpha\beta'| \in S(k-1) \Leftrightarrow n \in S(k),$$

where the second equivalence follows by induction and the last follows from Lemma 2.3.

(ii) Suppose n is odd. There are three cases:

(a) Suppose $z(\beta) = 0$. Then $|\beta| = 2^{k+2} - 1$ and $n = |\alpha| \cdot 2^{k+2} + 2^{k+2} - 1 \notin S(k)$ as n is visibly a sum of elements of $G(k)$. So in this case, $\alpha * \beta$ and $n \in S(k)$ are both false.

(b) Suppose $z(\beta) > 0$ and $|\alpha| = 0$. Then $n = |\beta| < 2^{k+2} - 1$ and n is odd, so by Lemma 2.3, $n \in S(k)$. So in this case, $\alpha * \beta$ and $n \in S(k)$ are both true.

(c) Suppose $z(\beta) > 0$ and $|\alpha| > 0$. Write $\beta = \beta' 0 \gamma$ where β' is possibly empty and γ is a string of ones with $\mathrm{len}(\gamma) = m \leqslant k+1$. We can express $n - (2^{k+2} - 1)$ in binary notation as $[|\alpha| - 1]\, \beta' 1 \epsilon$, where ϵ is a string of m zeros; note here that $z(\beta' 1 \epsilon) = z(\beta) - 1$ and $\mathrm{len}(\beta' 1 \epsilon) = \mathrm{len}(\beta) = k+2$. So

$$\alpha * \beta \Leftrightarrow [|\alpha| - 1] * \beta' 1 \epsilon \Leftrightarrow n - (2^{k+2} - 1) \in S(k) \Leftrightarrow n \in S(k)$$

where the second equivalence follows from the inductive assumption and the last follows from Lemma 2.3. $\square$

Theorem 1.1 follows immediately from Lemma 2.2 and Theorem 2.4.

Here are a few example calculations. Notice that to apply Theorem 1.1 it may be necessary to write a binary string with as many as two leading zeros.

For $n = 24 = |11000| = |0011000|$: We have $0 * 011000$ but not $00 * 11000$, so $f(24) = 4$. Hence $Sq^{24} \in L(4)$ and $Sq^{24} \notin L(3)$. A factorization showing $Sq^{24} \in L(4)$ is $Sq^{24} = Sq^8 Sq^{16} + Sq^{23} Sq^1 + Sq^{22} Sq^2 + Sq^{20} Sq^4$.

For $n = 50 = |110010|$: We have $1 * 10010$ but not $11 * 0010$, so $f(50) = 3$. Hence $Sq^{50} \in L(3)$ and $Sq^{50} \notin L(2)$.

Returning to the example $n = 10$ from the introduction, $10 = |1010| = |01010|$: We have $0 * 1010$ but not $01 * 010$, thus $f(10) = 2$, which implies $Sq^{10} \in L(2)$ and $Sq^{10} \notin L(1)$.

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