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3-TYPES OF SIMPLICIAL GROUPS AND BRAIDED REGULAR CROSSED MODULES

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Abstract

In this work, we explain the relations among braided regular crossed modules, simplicial groups, 2-crossed modules, quadratic modules and crossed squares, and the role of hypercrossed complex pairings in these structures.

Introduction

As an algebraic model of connected (weak homotopy) 3-types, the notion of 2-crossed module was introduced by Conduché in [**16**], and these 2-crossed modules are equivalent to simplicial groups with Moore complex of length 2. Crossed squares and quadratic modules are other algebraic models of connected 3-types defined by Loday and Guin-Walery [**26**] and Baues [**6**] respectively. In [**5**], we explored the relations among 2-crossed modules, quadratic modules, crossed squares and simplicial groups, and the homotopy equivalences between these structures.

Brown and Gilbert in [**9**] introduced the notion of braided, regular crossed module as an alternative algebraic model of homotopy 3-types. They then showed that this structure is closely related to simplicial groups; they proved that the category of braided, regular crossed modules is equivalent to that of simplicial groups with Moore complex of length 2. They have also proved that braided, regular crossed modules are equivalent to Conduché's 2-crossed modules.

Related ideas of Conduché have been used by Carrasco and Cegarra [**14**] to study braided categorical groups (see also Garzon and Miranda [**20**]). Carrasco and Cegarra [**13**] defined the notion of n-hypercrossed complex as an algebraic model of connected $(n+1)$-types. The article [**4**] is one of a series in which the first author and Porter studied the higher dimensional Peiffer elements, called hypercrossed complex pairings $F_{\alpha,\beta}$, by using ideas of Conduché (cf. [**16**]) and techniques developed by Carrasco and Cegarra (cf. [**13**]), and they applied their results in various homological settings and then gave a reformulation of Conduché's result in terms of hypercrossed complex pairings for commutative algebras. Mutlu and Porter [**24**] have also adapted their method to simplicial groups. Castiglioni and Ladra [**15**] generalized the results proved in [**1**], [**4**] and [**24**].

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In the present article, we explain the relations among braided regular crossed modules and algebraic models of connected (weak homotopy) 3-types mentioned above.

Thus, the main points of this paper are:

(i) to give the relation between braided regular crossed modules and simplicial groups in terms of hypercrossed complex pairings $F_{\alpha,\beta}$ defined in [**24**],

(ii) to give the connection between crossed squares and braided regular crossed modules by using the relation between 2-crossed modules and crossed squares as described in [**5**, Section 4] in terms of bisimplicial nerve of the crossed square,

(iii) to give a construction of a quadratic module from a braided regular crossed module, by using the construction of a quadratic module from a 2-crossed module as given in [**5**, Section 5].

Therefore, the results of this paper can be summarized in the following diagram

$$\begin{tikzcd}
 & \mathbf{BRCM} \arrow[dl, leftrightarrow] \arrow[d, leftrightarrow, dashed, "F_{\alpha,\beta}"'] \arrow[dr, leftrightarrow, dotted] \arrow[drr, dotted] & & \\
\mathbf{X_2Mod} \arrow[r, leftrightarrow] \arrow[rrr, bend right] & \mathbf{SimpGrp}_{\leqslant 2} \arrow[r, leftrightarrow] & \mathbf{Crs^2} \arrow[r] & \mathbf{QM}
\end{tikzcd}$$

where the diagram is commutative, linking the broken arrows given below and the unbroken arrows given in [**5**].

1. Preliminaries

Simplicial groups

Denoting the usual category of finite ordinals by Δ, we obtain for each $k \geqslant 0$ a subcategory $\Delta_{\leqslant k}$ determined by the objects $[j]$ of Δ with $j \leqslant k$. A simplicial group is a (contravariant) functor from the opposite category Δ^{op} to the category of groups **Grp**. A *reduced* simplicial group is a simplicial group whose first component is trivial. A k-truncated simplicial group is a functor from $\Delta^{op}_{\leqslant k}$ to **Grp**. We will denote the category of simplicial groups by **SimpGrp** and the category of k-truncated simplicial groups by $\mathbf{Tr}_k\mathbf{SimpGrp}$. By a *$k$-truncation of a simplicial group,* we mean a k-truncated simplicial group $\mathbf{tr}_k\mathbf{G}$ obtained by forgetting dimensions of order $> k$ in a simplicial group $\mathbf{G}$. This gives a truncation functor $\mathbf{tr}_k\colon \mathbf{SimpGrp} \to \mathbf{Tr}_k\mathbf{SimpGrp}$ which admits a right adjoint $\mathbf{cosk}_k\colon \mathbf{Tr}_k\mathbf{SimpGrp} \to \mathbf{SimpGrp}$ called the *k-coskeleton functor,* and a left adjoint $\mathbf{sk}_k\colon \mathbf{Tr}_k\mathbf{SimpGrp} \to \mathbf{SimpGrp}$, called the *$k$-skeleton functor*. For explicit constructions of these, see [**18**].

Given a simplicial group $\mathbf{G}$, *the Moore complex* $(\mathbf{NG}, \partial)$ of $\mathbf{G}$ is the normal chain complex defined by

$$(NG)_n = \bigcap_{i=0}^{n-1} \ker d_i^n$$

with $\partial_n\colon NG_n \to NG_{n-1}$ induced from d_n^n by restriction.

The n^{th} *homotopy group* $\pi_n(\mathbf{G})$ of $\mathbf{G}$ is the n^{th} homology of the Moore complex of $\mathbf{G}$, i.e.

$$\pi_n(\mathbf{G}) \cong H_n(\mathbf{NG}, \partial) = \bigcap_{i=0}^{n} \ker d_i^n / d_{n+1}^{n+1}(\bigcap_{i=0}^{n} \ker d_i^{n+1}).$$

We say that the Moore complex $\mathbf{NG}$ of a simplicial group $\mathbf{G}$ is of *length* k if $NG_n = 1$ for all $n \geqslant k+1$, so that a Moore complex of length k is also of length l for $l \geqslant k$. We denote the category of simplicial groups with Moore complex of length k by $\mathbf{SimpGrp}_{\leqslant k}$.

The following lemma is due to Conduché [**16**].

Lemma 1.1. *Let $\mathbf{G}$ be a simplicial group. The Moore complex of its k-coskeleton $\mathbf{Cosk_k}(\mathbf{tr_k}(\mathbf{G}))$ is of length $k+1$, i.e.,*

$$N(\mathbf{Cosk_k}(\mathbf{tr_k}(\mathbf{G})))_i = 1 \text{ for } i > k+1,$$

and is identical to the Moore complex of $\mathbf{G}$ in dimension less than $k+1$. Moreover

$$N(\mathbf{Cosk_k}(\mathbf{tr_k}(\mathbf{G})))_{k+1} = \ker(\partial_k \colon NG_k \to NG_{k-1})$$

and the morphism $\partial_{k+1} \colon N(\mathbf{Cosk_k}(\mathbf{tr_k}(\mathbf{G})))_{k+1} \to N(\mathbf{Cosk_k}(\mathbf{tr_k}(\mathbf{G})))_k = NG_k$ is injective.

The poset of surjective maps

The following notation and terminology is derived from [**12**] and published version [**13**].

For the ordered set $[n] = \{0 < 1 < \cdots < n\}$, let $\alpha_i^n \colon [n+1] \to [n]$ be the increasing surjective map given by;

$$\alpha_i^n(j) = \begin{cases} j & \text{if } j \leqslant i, \\ j-1 & \text{if } j > i. \end{cases}$$

Let $S(n, n-r)$ be the set of all monotone increasing surjective maps from $[n]$ to $[n-r]$. This can be generated from the various α_i^n by composition. The composition of these generating maps is subject to the following rule: $\alpha_j\alpha_i = \alpha_{i-1}\alpha_j, j < i$. This implies that every element $\alpha \in S(n, n-r)$ has a unique expression as $\alpha = \alpha_{i_1} \circ \alpha_{i_2} \circ \cdots \circ \alpha_{i_r}$ with $0 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n-1$, where the indices i_k are the elements of $[n]$ such that $\{i_1, \ldots, i_r\} = \{i \colon \alpha(i) = \alpha(i+1)\}$. We thus can identify $S(n, n-r)$ with the set $\{(i_r, \ldots, i_1) \colon 0 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n-1\}$. In particular, the single element of $S(n, n)$, defined by the identity map on $[n]$, corresponds to the empty 0-tuple $()$ denoted by $\emptyset_n$. Similarly the only element of $S(n, 0)$ is $(n-1, n-2, \ldots, 0)$. For all $n \geqslant 0$, let

$$S(n) = \bigcup_{0 \leqslant r \leqslant n} S(n, n-r).$$

We say that $\alpha = (i_r, \ldots, i_1) < \beta = (j_s, \ldots, j_1)$ in $S(n)$ if $i_1 = j_1, \ldots, i_k = j_k$ but $i_{k+1} > j_{k+1}$, $(k \geqslant 0)$ or if $i_1 = j_1, \ldots, i_r = j_r$ and $r < s$. This makes $S(n)$ an ordered set.

Hypercrossed Complex Pairings

In the following we recall from [**24**] hypercrossed complex pairings. The fundamental idea behind these can be found in Carrasco and Ceggarra (cf. [**12, 13**]). The construction depends on a variety of sources, mainly Conduché [**16**], Mutlu and Porter [**24**]. Define a set $P(n)$ consisting of pairs of elements (α, β) from $S(n)$ with $\alpha \cap \beta = \emptyset$ and $\beta < \alpha$, with respect to lexicographic ordering in $S(n)$ where $\alpha = (i_r, \dots, i_1), \beta = (j_s, \dots, j_1) \in S(n)$. The pairings that we will need,

$$\{F_{\alpha,\beta} \colon NG_{n-\sharp\alpha} \times NG_{n-\sharp\beta} \to NG_n \colon (\alpha, \beta) \in P(n), n \geqslant 0\},$$

are given as composites by the diagram

$$\begin{tikzcd} NG_{n-\sharp\alpha} \times NG_{n-\sharp\beta} \arrow[r, "F_{\alpha,\beta}"] \arrow[d, "s_\alpha \times s_\beta"'] & NG_n \\ G_n \times G_n \arrow[r, "\mu"'] & G_n \arrow[u, "p"'] \end{tikzcd}$$

where $s_\alpha = s_{i_r}, \dots, s_{i_1} \colon NG_{n-\sharp\alpha} \to G_n,\ s_\beta = s_{j_s}, \dots, s_{j_1} \colon NG_{n-\sharp\beta} \to G_n\ p \colon G_n \to NG_n$ is defined by composite projections $p(x) = p_{n-1} \dots p_0(x)$, where $p_j(z) = zs_j d_j(z)^{-1}$ with $j = 0, 1, \dots, n-1$. $\mu \colon G_n \times G_n \to G_n$ is given by commutator map and $\sharp\alpha$ is the number of the elements in the set of α, similarly for $\sharp\beta$. Thus

$$F_{\alpha,\beta}(x_\alpha, y_\beta) = p[s_\alpha(x_\alpha), s_\beta(x_\beta)].$$

Definition 1.2. ([**24**]) Let N_n, or more exactly N_n^G, be the normal subgroup of G_n generated by elements of the form $F_{\alpha,\beta}(x_\alpha, y_\beta)$ where $x_\alpha \in NG_{n-\sharp\alpha}$ and $y_\beta \in NG_{n-\sharp\beta}$.

This normal subgroup N_n^G depends functorially on G, but we will usually abbreviate N_n^G to N_n when no change of group is involved.

Mutlu and Porter in [**24**] illustrate this normal subgroup for $n = 2, 3, 4$, but we consider only $n = 3$.

Example 1.3. For all $x_1 \in NG_1, y_2 \in NG_2$, the corresponding generators of N_3 are:

$$F_{(1,0)(2)}(x_1, y_2) = [s_1 s_0 x_1, s_2 y_2][s_2 y_2, s_2 s_0 x_1],$$
$$F_{(2,0)(1)}(x_1, y_2) = [s_2 s_0 x_1, s_1 y_2][s_1 y_2, s_2 s_1 x_1][s_2 s_1 x_1, s_2 y_2][s_2 y_2, s_2 s_0 x_1]$$

and for all $x_2 \in NG_2, y_1 \in NG_1$,

$$F_{(0)(2,1)}(x_2, y_1) = [s_0 x_2, s_2 s_1 y_1][s_2 s_1 y_1, s_1 x_2][s_2 x_2, s_2 s_1 y_1]$$

whilst for all $x_2, y_2 \in NG_2$,

$$F_{(0)(1)}(x_2, y_2) = [s_0 x_2, s_1 y_2][s_1 y_2, s_1 x_2][s_2 x_2, s_2 y_2],$$
$$F_{(0)(2)}(x_2, y_2) = [s_0 x_2, s_2 y_2],$$
$$F_{(1)(2)}(x_2, y_2) = [s_1 x_2, s_2 y_2][s_2 y_2, s_2 x_2].$$

As a corollary, there is an equality

$$\partial_n(NG_n \cap D_n) = \partial_n(N_n^G \cap D_n).$$

The first author and Porter [**4**] have shown that if $\mathbf{E}$ is a simplicial commutative algebra with Moore complex $\mathbf{NE}$, and for $n > 0$ the ideal generated by the degenerate elements in dimension n is E_n, then

$$\partial_n(NE_n) \supseteq \sum_{I,J} K_I K_J.$$

This sum runs over those $\emptyset \neq I, J \subset [n-1] = \{0, \ldots, n-1\}$ with $I \cup J = [n-1]$, and $K_I = \bigcap_{i \in I} \ker d_i$. A similar result for simplicial Lie algebras was obtained by Akça and Arvasi in [**1**].

Mutlu and Porter [**24**] have adapted Arvasi's method to the case of simplicial groups. They gave the following result.

Proposition 1.4. ([**24**]) Let $\mathbf{G}$ be a simplicial group. Then for $n \geqslant 2$ and $I, J \subseteq [n-1]$ with $I \cap J = [n-1]$,

$$[\bigcap_{i \in I} \ker d_i, \bigcap_{j \in J} \ker d_j] \subseteq \partial_n(NG_n \cap D_n).$$

Castiglioni and Ladra [**15**] gave a general proof for the inclusions partially proved by Arvasi and Porter in [**4**], Arvasi and Akça in [**1**] and Mutlu and Porter in [**24**]. Their approach to the problem is different from that of the cited works. They have succeeded with a proof, for the case of algebras, over an operad by introducing a different description of the adjoint inverse of the normalization functor $\mathbf{N} \colon Ab^{\Delta^{op}} \to Ch_{\geqslant 0}$, and for the case of groups, they then adapted the construction for the adjoint inverse used for algebras to get a simplicial group $G \boxtimes \Lambda$ from the Moore complex of a simplicial group G.

The following is a result of Conduché [**16**].

Proposition 1.5. *Let $\mathbf{G}'$ be $(n-1)$-truncated simplicial group. Then there is a simplicial group $\mathbf{G}$ with $\mathbf{tr}_k\mathbf{G} \cong \mathbf{G}'$ if and only if $\mathbf{G}'$ satisfies the following property: For all nonempty sets of indices $(I \neq J), I, J \subset [n-1]$ with $I \cup J = [n-1]$,*

$$[\bigcap_{i \in I} \ker d_i, \bigcap_{j \in J} \ker d_j] = 1.$$

Proof. Since $\partial_n(NG'_n) = 1$, this follows from Proposition 1.4. $\square$

2-Crossed modules

Crossed modules were introduced by Whitehead in [**27**]. A *crossed module* is a group homomorphism $\partial \colon M \to P$ together with an action of P on M, written ${}^p m$ for $p \in P$ and $m \in M$, satisfying the conditions $\partial({}^p m) = p\partial(m)p^{-1}$ and ${}^{\partial m}m' = mm'm^{-1}$ for all $m, m' \in M, p \in P$. The last condition is called the 'Peiffer identity'.

The following definition of 2-crossed module is equivalent to that given by Conduché [**16**].

A *2-crossed module* of groups consists of a complex of groups

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$$

together with (a) actions of N on M and L so that ∂_2, ∂_1 are morphisms of N-groups, and (b) an N-equivariant function

$$\{\quad,\quad\}\colon M \times M \longrightarrow L$$

called a Peiffer lifting. This data must satisfy the following axioms:

$$\begin{array}{llrcl}
\textbf{2CM1:} & & \partial_2\{m,m'\} & = & \left({}^{\partial_1 m}m'\right) mm'^{-1}m^{-1} \\
\textbf{2CM2:} & & \{\partial_2 l, \partial_2 l'\} & = & [l', l] \\
\textbf{2CM3:} & (i) & \{mm', m''\} & = & {}^{\partial_1 m}\{m', m''\}\{m, m'm''m'^{-1}\} \\
 & (ii) & \{m, m'm''\} & = & \{m, m'\}^{mm'm^{-1}}\{m, m''\} \\
\textbf{2CM4:} & & \{m, \partial_2 l\}\{\partial_2 l, m\} & = & {}^{\partial_1 m}l l^{-1} \\
\textbf{2CM5:} & & {}^{n}\{m, m'\} & = & \{{}^{n}m, {}^{n}m'\}
\end{array}$$

for all $l, l' \in L$, $m, m', m'' \in M$ and $n \in N$.

Here we have used ${}^{m}l$ as a shorthand for $\{\partial_2 l, m\}l$ in condition **2CM3** (ii) where l is $\{m, m''\}$ and m is $mm'(m)^{-1}$. This gives a new action of M on L. Using this notation, we can split **2CM4** into two pieces, the first of which is tautologous:

$$\begin{array}{llrcl}
\textbf{2CM4:} & (a) & \{\partial_2 l, m\} & = & {}^{m}l(l)^{-1}, \\
 & (b) & \{m, \partial_2 l\} & = & ({}^{\partial_1 m}l)({}^{m}l^{-1}).
\end{array}$$

The old action of M on L, via ∂_1 and the N-action on L, is in general distinct from this second action, with $\{m, \partial_2 l\}$ measuring the difference (by **2CM4** (b)). An easy argument using **2CM2** and **2CM4** (b) shows that with this action, ${}^{m}l$, of M on L, (L, M, ∂_2) becomes a crossed module.

A morphism of 2-crossed modules can be defined in an obvious way. We thus define the category of 2-crossed modules, denoting it by $\mathbf{X_2Mod}$.

The following theorem, in some sense, is known. We do not give the proof since it exists in the literature, [**16**], [**23**] and [**24, 25**].

Theorem 1.6. *The category of 2-crossed modules is equivalent to the category of simplicial groups with Moore complex of length 2.* □

2. Braided, regular crossed modules and simplicial groups

Recall that a *groupoid* $\mathbf{C}$ is a small category in which every arrow is an isomorphism. We write a groupoid as (C_1, C_0), where C_0 is the set of objects and C_1 is the set of arrows, together with functions $s, t\colon C_1 \to C_0$ and $e\colon C_0 \to C_1$ such that $se = te = 1$. The functions s and t are sometimes called the *source* and *target* maps, respectively. If $a, b \in C_1$ and $t(a) = s(b)$, then a composite $a \circ b$ exists such that $s(a \circ b) = s(a)$ and $t(a \circ b) = t(b)$. Further, this composition is associative; the elements e_p, $p \in C_0$, act as identities; and each arrow a has an inverse a^{-1} with $s(a^{-1}) = t(a)$, $t(a^{-1}) = s(a)$, $a \circ a^{-1} = e_{s(a)}$ and $a^{-1} \circ a = e_{t(a)}$.

For any groupoid $\mathbf{C}$ and $p, q \in C_0$, the set of arrows a such that $s(a) = p$ and $t(a) = q$ is written $C_1(p, q)$ and termed a *hom-set*. If $C_1(p, q)$ is empty whenever p

and q are distinct (that is, if $s = t$) then $\mathbf{C}$ is called *totally disconnected*. We also write $C_1(p,p)$ as $C_1(p)$. For a survey of applications of groupoids and an introduction to their literature, see [**8**]. In any groupoid (C_1, C_0), for an element $p \in C_0$, the hom-set $C_1(p)$ is a group.

Now, we recall the notions of groupoid action and crossed module of groupoids from [**9**].

In the following we refer to C_1 and C_2 as the groupoids, when we of course mean (C_1, C_0) and (C_2, C_0).

Definition 2.1. Let C_1 and C_2 be groupoids over the same object set C_0 and let C_2 be totally disconnected. Then an *action* of C_1 on C_2 is given by a partially defined function

$$C_1 \times C_2 \quad \longrightarrow \quad C_2$$

written $(a, x) \mapsto x^a$, which satisfies:

1. x^a is defined if and only if $t(x) = s(a)$, and then $t(x^a) = t(a)$,
2. $(x \circ y)^a = x^a \circ y^a$ and ${e_p}^a = e_q$,
3. $x^{a \circ b} = (x^a)^b$ and $x^{e_p} = x$,

for all $x, y \in C_2(p)$ and $a \in C_1(p,q)$, $b \in C_1(q,r)$.

Definition 2.2. A *crossed module of groupoids* consists of a pair of groupoids C_1 and C_2 over the same object set C_0, with C_2 totally disconnected, and an action of C_1 on C_2, together with a functor $\delta\colon C_2 \to C_1$ which satisfies

CM1: $\delta(x^a) = a \circ (\delta x) \circ a^{-1}$

CM2: $x^{\delta y} = y \circ x \circ y^{-1}$

for all $x, y \in C_2(p)$ and $a \in C_1(p,q)$.

A crossed module of groupoids is often written diagrammatically as

$$C: \quad C_2 \xrightarrow{\delta} C_1 \overset{s}{\underset{t}{\rightrightarrows}} C_0$$

Note in particular that for each $p \in C_0$, $C_2(p) \to C_1(p)$ is a crossed module of groups.

Let U be a monoid. A *biaction* of U on the crossed module

$$C\colon C_2 \xrightarrow{\delta} C_1 \overset{s}{\underset{t}{\rightrightarrows}} C_0$$

consists of a pair of commuting left and right actions of U on the set C_0 and on the groupoids C_1 and C_2 compatible with all of the structure. Specifically, we have functions $U \times C_i \to C_i$ and $C_i \times U \to C_i$ for $i = 0, 1, 2$, denoted by $(u, c) \mapsto u \cdot c$ and $(c, u) \mapsto c \cdot u$ such that

BA1: each function $U \times C_i \to C_i$ determines a left action of U and each function $C_i \times U \to C_i$ determines a right action of U and these actions commute;

BA2: each action of U preserves the groupoid structure of C_1 over C_0 and in particular the source and target maps $s, t\colon C_1 \to C_0$ are U-equivariant relative to each action;

BA3: each action of U preserves the group operation in C_2 and if $x \in C_2(p)$ and $u \in U$ then $u \cdot x \in C_2(u \cdot p)$ and $x \cdot u \in C_2(p \cdot u)$;

BA4: each action of U is compatible with the action of C_1 on C_2 so that if $x \in C_2(p)$, $a \in C_1(p, q)$, and $u \in U$ then

$$u \cdot (x^a) = (u \cdot x)^{u \cdot a} \in C_2(u \cdot q),$$

$$(x^a) \cdot u = (x \cdot u)^{a \cdot u} \in C_2(q \cdot u);$$

BA5: the boundary homomorphism $\delta\colon C_2 \to C_1$ is U-equivariant relative to each action.

The crossed module

$$C\colon C_2 \xrightarrow{\delta} C_1 \underset{t}{\overset{s}{\rightrightarrows}} C_0$$

is *semiregular* if the object set C_0 is a monoid and there is a biaction of C_0 on C in which C_0 acts on itself in its left and right regular representations. A semiregular crossed module in which C_0 is a group is said to be *regular*. Note that every crossed module of groups is regular.

We now recall the definition of braided regular crossed modules from [**9**].

Definition 2.3. A braided regular crossed module of groupoids

$$C\colon C_2 \xrightarrow{\delta} C_1 \underset{t}{\overset{s}{\rightrightarrows}} C_0$$

is a regular crossed module of groupoids with the map $\{-,-\}\colon C_1 \times C_1 \to C_2$, called the braiding map, satisfying the following axioms:

B1: $\{a, b\} \in C_2((ta)(tb)), \{1_e, b\} = 1_{tb}, \{a, 1_e\} = 1_{ta}$ where $1_e \in C_1(e)$ is the identity arrow and e is the identity element of the group C_0;

B2: $\{a, b \circ b'\} = \{a, b\}^{ta \cdot b'} \circ \{a, b'\}$;

B3: $\{a \circ a', b\} = \{a', b\} \circ \{a, b\}^{a' \cdot tb}$;

B4: $\delta\{a, b\} = (ta \cdot b)^{-1} \circ (a^{-1} \cdot sb) \circ (sa \cdot b) \circ (a \cdot tb)$;

B5: $\{a, \delta y\} = (ta \cdot y)^{-1} \circ (sa \cdot y)^{a \cdot q}$ if $y \in C_2(q)$;

B6: $\{\delta x, b\} = ((x \cdot sb)^{p \cdot b})^{-1} \circ (x \cdot tb)$ if $x \in C_2(p)$;

B7: $p \cdot \{a, b\} = \{p \cdot a, b\}, \{a, b\} \cdot p = \{a, b \cdot p\}, \{a \cdot p, b\} = \{a, p \cdot b\}$ for all a, a', $b, b' \in C_1$, $x, y \in C_2$, and $p, q \in C_0$.

Example 2.4. A braiding on a crossed module of groups

$$C_2 \xrightarrow{\delta} C_1$$

is a function $\{-,-\}\colon C_1 \times C_1 \to C_2$ satisfying the following axioms:

1. $\{a, bb'\} = \{a, b\}^{b'}\{a, b'\}$
2. $\{aa', b\} = \{a', b\}\{a, b\}^{a'}$
3. $\delta\{a, b\} = [b, a]$
4. $\{a, \delta x\} = x^{-1}x^a$

5. $\{\delta y, b\} = (y^{-1})^b y$, where $a, a', b, b' \in C_1$ and $x, y \in C_2$.

This example leads us to define a new category. This crossed module together with the braiding map is called the *braided crossed module*. In fact this is a special case of a braided regular crossed module and equivalent to Conduché's reduced 2-crossed module. Brown and Gilbert (cf. [**9**]) have stated as a corollary that the category of braided crossed modules of groups is equivalent to that of reduced simplicial groups with Moore complex of length 2. We have proved this result by using the $F_{\alpha,\beta}$ functions in [**3**].

In this section, we will extend this construction to regularity. That is 'a description of the passage from a simplicial group to a braided *regular* crossed module'. This is a reformulation of the Brown-Gilbert result [**9**]. Our aim is to show the role of the $F_{\alpha,\beta}$ functions in the structure. We will use the $F_{\alpha,\beta}$ functions in calculations of the axioms of braided *regular* crossed module.

Let $\mathbf{G}$ be a simplicial group with Moore complex $\mathbf{NG}$. We will construct a braided regular crossed module

$$C\colon C_2 \xrightarrow{\delta} C_1 \underset{t}{\overset{s}{\rightrightarrows}} C_0$$

from the simplicial group $\mathbf{G}$.

Let $C_0 = NG_0$ and $C_1 = NG_1 \rtimes s_0 NG_0$ together with source and target maps given by $s(g, s_0 p) = (d_1 g)p$ and $t(g, s_0 p) = p$ respectively. The groupoid composition in C_1 is given by

$$(g_1, s_0 p_1) \circ (g_2, s_0 p_2) = (g_1 g_2, s_0 p_2)$$

if $p_1 = (d_1 g_2) p_2$. Thus we have a groupoid (C_1, C_0). Furthermore C_0 acts on the left and on the right of the groupoid (C_1, C_0) by

$$\begin{aligned} p \cdot (g, s_0 q) &= (s_0 p g s_0 p^{-1}, s_0 p q), \\ (g, s_0 q) \cdot p &= (g, s_0 q p) \end{aligned}$$

for $p \in C_0$ and $(g, s_0 q) \in C_1$. This action gives a biaction of C_0 on the groupoid (C_1, C_0).

Now, we set $C_2(p) = NG_2/\partial_3(NG_3 \cap D_3) \rtimes s_1 s_0 NG_0 \subseteq G_2$ with the composition

$$(l_1, s_1 s_0 p) \circ (l_2, s_1 s_0 p) = (l_1 l_2, s_1 s_0 p)$$

and the source and target maps given by

$$s(l, s_1 s_0 p) = d_1^1 d_1^2 (l) p = p = d_0^1 d_0^2 (l) p = t(l, s_1 s_0 p).$$

Moreover, C_0 acts on the left and right of the groupoid (C_2, C_0) by

$$\begin{aligned} p \cdot (l, s_1 s_0 q) &= ((s_1 s_0 p) l (s_1 s_0 p^{-1}), s_1 s_0 p q), \\ (l, s_1 s_0 q) \cdot p &= (l, s_1 s_0 q p) \end{aligned}$$

for $p \in C_0$ and $(l, s_1 s_0 q) \in C_2(q)$. Thus, we see that $p \cdot (l, s_1 s_0 q) \in C_2(pq)$ and $(l, s_1 s_0 q) \cdot p \in C_2(qp)$. This action defines a biaction of C_0 on the groupoid (C_2, C_0).

The action of $(g, s_0q) \in C_1$ on $(l, s_1s_0p) \in C_2$ can be given by

$$(l, s_1s_0p)^{(g,s_0q)} = ((s_1g)l(s_1g^{-1}), s_1s_0q)$$

if $p = (d_1g)q$. Define the morphism $\delta\colon C_2 \to C_1$ by $\delta(l, s_1s_0p) = (d_2l, s_0p)$. Thus we can give the following proposition.

Proposition 2.5. *The diagram*

$$C_2 \xrightarrow{\delta} C_1 \underset{t}{\overset{s}{\rightrightarrows}} C_0$$

is a braided regular crossed module, where the braiding map is defined as follows:

$$\begin{aligned}\{\ \ ,\ \ \}\colon C_1 \times C_1 &\longrightarrow C_2\\ (a,b) &\longmapsto \{a,b\},\end{aligned}$$

for $a = (g_1, s_0p)$ and $b = (g_2, s_0q)$,

$$\begin{aligned}&\{(g_1, s_0p), (g_2, s_0q)\} =\\ &\qquad (\overline{s_1s_0ps_1g_2^{-1}s_1s_0p^{-1}s_1g_1^{-1}s_0g_1s_1s_0ps_1g_2s_1s_0p^{-1}s_0g_1^{-1}s_1g_1}, s_1s_0pq).\end{aligned}$$

Here, on the right hand side the overline denotes a coset in $NG_2/\partial_3(NG_3 \cap D_3)$ represented by an element in NG_2.

Proof. We display the elements omitting the overlines in our calculation to save complication. Firstly, we show that δ is a crossed module of groupoids by using the $F_{\alpha,\beta}$ functions.

CM1:

$$\begin{aligned}\delta((l, s_1s_0p)^{(g,s_0q)}) &= \delta((s_1g)l(s_1g^{-1}), s_1s_0q)\\ &= (d_2((s_1g)l(s_1g^{-1})), s_0q)\\ &= (g(d_2l)g^{-1}, s_0q),\end{aligned}$$

and

$$\begin{aligned}(g, s_0q) \circ \delta(l, s_1s_0p) \circ (g^{-1}, s_0q) &= (g, s_0q) \circ (d_2l, s_0p) \circ (g^{-1}, s_0q)\\ &= (g, s_0q) \circ ((d_2l)g^{-1}, s_0q)\\ &= (g(d_2l)g^{-1}, s_0q).\end{aligned}$$

Then we have

$$\delta((l, s_1s_0p)^{(g,s_0q)}) = (g, s_0q) \circ \delta(l, s_1s_0p) \circ (g^{-1}, s_0q).$$

CM2:

$$\begin{aligned}(l, s_1s_0p)^{\delta(l', s_1s_0p)} &= (l, s_1s_0p)^{(d_2l', s_0p)}\\ &= ((s_1d_2l')l(s_1d_2l'^{-1}), s_1s_0p)\\ &\equiv (l'l(l')^{-1}, s_1s_0p) \mod \partial_3(NG_3 \cap D_3)\\ &= ((l'), s_1s_0p) \circ (l, s_1s_0p) \circ (l'^{-1}, s_1s_0p).\end{aligned}$$

In the calculation of axiom **CM2**, we have used the $F_{\alpha,\beta}$ functions. Indeed, from [**24**], for $l, l' \in NG_2$, we have

$$\begin{aligned}F_{(1)(2)}(l',l) &= [s_1 l', s_2 l][s_2 l, s_2 l'] \\ &= s_1 l' s_2 l s_1 l'^{-1} s_2 l' s_2 l^{-1} s_2 l'^{-1} \in NG_3 \cap D_3,\end{aligned}$$

and from

$$\partial_3(F_{(1)(2)}(l',l)) = s_1 d_2(l') l s_1 d_2(l')^{-1}(l') l^{-1}(l')^{-1} \in \partial_3(NG_3 \cap D_3)$$

we have

$$s_1 d_2(l') l s_1 d_2(l')^{-1} \equiv l' l (l')^{-1} \bmod \partial_3(NG_3 \cap D_3).$$

Therefore the morphism δ is a crossed module of groupoids. Thus we have a regular crossed module of groupoids

$$C\colon C_2 \xrightarrow{\delta} C_1 \overset{s}{\underset{t}{\rightrightarrows}} C_0$$

together with the biaction of C_0 on C as given above.

Now, we will show that all the axioms of a braided regular crossed module of groupoids are verified. We again use the $F_{\alpha,\beta}$ functions in the following calculations.

B1: For $a = (g_1, s_0 p)$, $b = (g_2, s_0 q)$ and $1_e = (1, s_0 e)$; $\{1_e, b\} = (s_1 g_2^{-1} s_1 g_2, s_1 s_0 q)$ $= (1, s_1 s_0\, q) = 1_{tb}$, $\{a, 1_e\} = (1, s_1 s_0\, p) = 1_{ta}$ and clearly $\{a, b\} \in C_2(pq) = C_2((ta)(tb))$.

B2: For $a = (g, s_0 p)$, $b = (h, s_0 q)$, $b' = (h', s_0 q')$ it must be that

$$\{a, b \circ b'\} = \{a, b\}^{ta \cdot b'} \circ \{a, b'\},$$

where $s(b') = d_1 h' q' = q = t(b)$. Then,

$$\begin{aligned}\{a, b \circ b'\} &= \{(g, s_0 p), (hh', s_0 q')\} \\ &= (\underbrace{s_1 s_0 p s_1 (h')^{-1} s_1 h^{-1} s_1 s_0 p^{-1} s_1 g^{-1} s_0 g s_1 s_0 p s_1 h}_{A} \\ &\qquad s_1 h' s_1 s_0 p^{-1} s_0 g^{-1} s_1 g, s_1 s_0 (pq')) \\ &= (A\, \underbrace{(s_1 s_0 p^{-1} s_0 g^{-1} s_1 g s_1 s_0 p s_1 h' s_1 s_0 p^{-1})}_{B} (s_1 s_0 p s_1 (h')^{-1} \\ &\qquad s_1 s_0 p^{-1} s_1 g^{-1} s_0 g s_1 s_0 p s_1 h' s_1 s_0 p^{-1} s_0 g^{-1} s_1 g, s_1 s_0 p q') \\ &= (AB, s_1 s_0 p q') \circ (s_1 s_0 p s_1 (h')^{-1} \\ &\qquad s_1 s_0 p^{-1} s_1 g^{-1} s_0 g s_1 s_0 p s_1 h' s_1 s_0 p^{-1} s_0 g^{-1} s_1 g, s_1 s_0 p q') \\ &= (AB, s_1 s_0 p q') \circ \{a, b'\},\end{aligned}$$

where

$$
\begin{aligned}
(AB, s_1s_0pq') &= (s_1s_0ps_1(h')^{-1}s_1h^{-1}s_1s_0p^{-1}s_1g^{-1}s_0gs_1s_0ps_1hs_1s_0p^{-1} \\
&\qquad s_0g^{-1}s_1gs_1s_0ps_1h's_1s_0p^{-1}, s_1s_0pq') \\
&= (s_1s_0ps_1(h')^{-1}s_1s_0p^{-1}s_1s_0ps_1h^{-1}s_1s_0p^{-1}s_1g^{-1}s_0g \\
&\qquad s_1s_0ps_1hs_1s_0p^{-1}s_0g^{-1}s_1gs_1s_0ps_1h's_1s_0p^{-1}, s_1s_0pq') \\
&= (s_1(s_0p(h')^{-1}s_0p^{-1})s_1s_0ps_1h^{-1}s_1s_0p^{-1}s_1g^{-1}s_0g \\
&\qquad s_1s_0ps_1hs_1s_0p^{-1}s_0g^{-1}s_1gs_1(s_0ph's_0p^{-1}), s_1s_0pq') \\
&= (s_1s_0ps_1h^{-1}s_1s_0p^{-1}s_1g^{-1}s_0gs_1s_0ps_1hs_1s_0p^{-1}s_0g^{-1}s_1g, \\
&\qquad s_1s_0pq)^{(s_0ph's_0p^{-1}, s_0pq')} \\
&= \{a,b\}^{ta\cdot b'}
\end{aligned}
$$

and then we have

$$\{a, b\circ b'\} = \{a,b\}^{ta\cdot b'} \circ \{a,b'\}.$$

B3: For $a=(g,s_0p), a'=(h,s_0q)$ and $b=(k,s_0r)$; (here $sa'=d_1hq=p=ta$)

$$
\begin{aligned}
\{a\circ a', b\} &= \{(gh, s_0q),(k,s_0r)\} \\
&= (s_1s_0qs_1k^{-1}s_1s_0q^{-1}s_1h^{-1}s_1g^{-1}s_0g \\
&\qquad s_0hs_1s_0qs_1ks_1s_0q^{-1}s_0h^{-1}s_0g^{-1}s_1gs_1h, s_1s_0qr) \\
&= (s_1s_0qs_1k^{-1}s_1s_0q^{-1}s_1h^{-1}s_1g^{-1}s_0gs_1s_0d_1h \\
&\qquad s_1s_0qs_1ks_1s_0q^{-1}s_1s_0d_1h^{-1}s_0g^{-1}s_1gs_1h, s_1s_0qr) \\
&= (\underbrace{s_1s_0qs_1k^{-1}s_1s_0q^{-1}s_1h^{-1}}_{A}s_1g^{-1}s_0g \\
&\qquad s_1s_0ps_1ks_1s_0p^{-1}s_0g^{-1}s_1gs_1h, s_1s_0qr) \\
&= (\underbrace{A(s_1s_0ps_1ks_1s_0p^{-1}s_1h)}_{B}s_1h^{-1}(s_1s_0ps_1k^{-1}s_1s_0p^{-1}s_1g^{-1}s_0g \\
&\qquad s_1s_0ps_1ks_1s_0p^{-1}s_0g^{-1}s_1g)s_1h, s_1s_0qr) \\
&= (B, s_1s_0qr)\circ \{(g,s_0p),(k,s_0r)\}^{(h,s_0q)\cdot r} \\
&= (B, s_1s_0qr)\circ \{a,b\}^{a'\cdot tb},
\end{aligned}
$$

where

$$
\begin{aligned}
(B, s_1s_0qr) &= (s_1s_0qs_1k^{-1}s_1s_0q^{-1}s_1h^{-1}s_1s_0ps_1ks_1s_0p^{-1}s_1h, s_1s_0qr) \\
&= s_1s_0qs_1k^{-1}s_1s_0q^{-1}s_1h^{-1}s_0hs_1s_0qs_1ks_1s_0q^{-1}s_0h^{-1}s_1h, s_1s_0qr) \\
&= \{(h,s_0q),(k,s_0r)\} \\
&= \{a',b\}
\end{aligned}
$$

since $s_1s_0p = s_0hs_1s_0q$. Thus we have

$$\{a\circ a', b\} = \{a',b\}\circ\{a,b\}^{a'\cdot tb}.$$

B4:

$$\begin{aligned}
\delta\{a,b\} &= \delta\{(g,s_0p),(h,s_0q)\}\\
&= (d_2(s_1s_0ps_1h^{-1}s_1s_0p^{-1}s_1g^{-1}s_0gs_1s_0ps_1hs_1s_0p^{-1}s_0g^{-1}s_1g),s_1s_0pq)\\
&= (s_0ph^{-1}s_0p^{-1}g^{-1}s_0d_1gs_0phs_0p^{-1}s_0d_1g^{-1}g,s_0pq)\\
&= (s_0ph^{-1}s_0p^{-1},s_0pq)\circ(g^{-1},s_0pq)\circ\\
&\qquad (s_0(d_1gp)hs_0(d_1gp)^{-1},s_0pq)\circ(g,s_0pq)\\
&= (ta\cdot b)^{-1}\circ(a^{-1}\cdot sb)\circ(sa\cdot b)\circ(a\cdot tb).
\end{aligned}$$

B5: For $a=(g,s_0p)\in C_1$ and $y=(l,s_1s_0q)\in C_2(q)$ it must be that

$$\{a,\delta y\}=(ta\cdot y)^{-1}\circ(sa\cdot y)^{a\cdot q}.$$

From [**24**], we have

$$\partial_3(F_{(2,0)(1)}(x,t))=[s_0x,s_1d_2t][s_1d_2t,s_1x][s_1x,t][t,s_0x]\in\partial_3(NG_3\cap D_3).$$

Thus, for $t=p\cdot l\in NG_2$ and $x=g^{-1}\in NG_1$, we have

$$\begin{aligned}
\{a,\delta y\} &= \{(g,s_0p),(d_2l,s_0q)\}\\
&= (s_1s_0ps_1d_2l^{-1}s_1s_0p^{-1}s_1g^{-1}s_0gs_1s_0ps_1d_2ls_1s_0p^{-1}s_0g^{-1}s_1g,s_1s_0pq)\\
&\equiv (s_1s_0pl^{-1}s_1s_0p^{-1}s_1g^{-1}s_0gs_1s_0pls_1s_0p^{-1}s_0g^{-1}s_1g,s_1s_0pq)\\
&\qquad \bmod\ \partial_3(NG_3\cap D_3)\\
&= ((s_1s_0p)l^{-1}(s_1s_0p)^{-1},s_1s_0pq)\circ(s_1s_0(d_1gp)ls_1s_0(d_1gp)^{-1},\\
&\qquad s_1s_0pq)^{(g,s_0pq)}\\
&= (ta\cdot y)^{-1}\circ(sa\cdot y)^{a\cdot q}.
\end{aligned}$$

B6: It must be that

$$\{\delta x,b\}=[(x\cdot sb)^{p\cdot b}]^{-1}\circ(x\cdot tb).$$

Similarly, from [**24**], we have

$$\partial_3(F_{(0)(2,1)}(k,y))=[s_0d_2k,s_1y][s_1k,s_1d_2x][x,s_1k]\in\partial_3(NG_3\cap D_3).$$

Thus, for $k=l^{-1}\in NG_2$ and $y=p\cdot g\in NG_1$ we have

$$\begin{aligned}
\{\delta x,b\} &= \{\delta(l,s_1s_0p),(g,s_0q)\}\\
&= \{(d_2l,s_0p),(g,s_0q)\}\\
&= (s_1s_0ps_1g^{-1}s_1s_0p^{-1}s_1d_2l^{-1}s_0d_2ls_1s_0ps_1gs_1s_0p^{-1}s_0d_2l^{-1}s_1d_2l,\\
&\qquad s_1s_0pq)\\
&\equiv (s_1s_0ps_1g^{-1}s_1s_0p^{-1}l^{-1}s_1s_0ps_1gs_1s_0p^{-1}l,s_1s_0pq)\\
&\qquad \bmod\ \partial_3(NG_3\cap D_3)\\
&= (s_1(s_0pg^{-1}s_0p^{-1})l^{-1}s_1(s_0pgs_0p^{-1})l,s_1s_0pq)\\
&= (l^{-1},s_1s_0pq)^{p\cdot(g^{-1},s_0q)}\circ(l,s_1s_0p)\cdot q\\
&= [(x\cdot sb)^{p\cdot b}]^{-1}\circ(x\cdot q).
\end{aligned}$$

B7: For $r \in C_0$, $a = (g, s_0p)$ and $b = (h, s_0q)$

$$\begin{aligned}\{a \cdot r, b\} &= \{(g, s_0pr), (h, s_0q)\} \\ &= (s_1s_0(pr)s_1h^{-1}s_1s_0(pr)^{-1}s_1g^{-1}s_0g \\ &\qquad s_1s_0(pr)s_1hs_1s_0(pr)^{-1}s_0g^{-1}s_1g, s_1s_0prq) \\ &= (s_1s_0ps_1(s_0rh^{-1}s_0r^{-1})s_1s_0p^{-1}s_1g^{-1}s_0g \\ &\qquad s_1s_0ps_1(s_0rhs_0r^{-1})s_1s_0p^{-1}s_0g^{-1}s_1g, s_1s_0prq) \\ &= \{(g, s_0p), (s_0rhs_0r^{-1}, s_0rq)\} \\ &= \{a, r \cdot b\}.\end{aligned}$$

Furthermore, $r \cdot \{a, b\} = \{r \cdot a, b\}$ and $\{a, b\} \cdot r = \{a, b \cdot r\}$ can be shown similarly. Therefore the braided regular crossed module axioms are verified. □

Remark 2.6. If the Moore complex of the simplicial group $\mathbf{G}$ is of length 2, we have $\partial_3(NG_3 \cap D_3) = 1$, and then we have a construction of a braided regular crossed module from a simplicial group with Moore complex of length 2.

Theorem 2.7. *The category of braided regular crossed modules is equivalent to that of simplicial groups with Moore complex of length 2.*

Proof. In the above proposition, a braided regular crossed module was already constructed by using the $F_{\alpha,\beta}$ functions from the Moore complex of a simplicial group. This defines a functor from simplicial groups to braided regular crossed modules

$$\Delta \colon \mathbf{SimpGrp} \to \mathbf{BRCM}.$$

Conversely, we suppose that

$$C \colon C_2 \xrightarrow{\delta} C_1 \overset{s}{\underset{t}{\rightrightarrows}} C_0$$

is a braided regular crossed module. We will construct a simplicial group $\mathbf{G}$ whose Moore complex has length 2 by using the $F_{\alpha,\beta}$ functions.

Let $G_0 = C_0$. The set

$$M = \{a \in C_1 \colon s(a) = e\}$$

is a group with the following operation

$$ab = a \circ t(a) \cdot b$$

for $a, b \in M$. The group C_0 acts on M as follows: for all $a \in M$ and $p \in C_0$, we set $a^p = p^{-1} \cdot a \cdot p$ by using the biaction of C_0 on C_1 and C_2. By using this action, we can create the semidirect product group

$$M \rtimes C_0 = G_1.$$

The group operation in G_1 is given by

$$(a, p)(a', p') = (a^{p'} \circ ((p')^{-1}t(a)p') \cdot a', pp')$$

for all $(a,p),(a',p') \in M \rtimes C_0$. Define the face and degeneracy maps by

$$d_0(a,p) = p, \quad d_1(a,p) = pt(a), \quad s_0(p) = (0_e, p).$$

These maps satisfy the simplicial identities. Indeed

$$d_0 s_0(p) = d_0(0_e, p) = p$$

$$d_1 s_0(p) = d_1(0_e, p) = pt(0_e) = pe = p.$$

Moreover, d_1 and d_0 are group homomorphisms, since

$$\begin{aligned} d_1((a,p)(a',p')) &= d_1((a^{p'}a', pp')) \\ &= pp't(a^{p'})t(a') \\ &= pt(a)p't(a') \\ &= d_1(a,p)d_1(a',p'), \end{aligned} \qquad \begin{aligned} d_0((a,p)(a',p')) &= pp' \\ &= d_0(a,p)d_0(a',p'). \end{aligned}$$

Furthermore, we know that $C_2(e)$ is a (vertex) group from [**9**]. An action of $a \in M$ on $y \in C_2(e)$ can be given by

$$y^a = t(a) \cdot y \circ \{a, \delta y\}$$

where $\{-,-\}$ is the braiding map. Then, by using this action we can create the semidirect product group $C_2(e) \rtimes M$.

Moreover, an action $(a,p) \in M \rtimes C_0$ on $(y,a') \in C_2(e) \rtimes M$ can be given by

$$\begin{aligned} (y,a')^{(0_e,p)} &= (y^p, (a')^p) \\ (y,a')^{(a,e)} &= (t(a) \cdot y \circ \{a, \delta y\}, a^{a'}). \end{aligned}$$

Using this action, we have the semidirect product

$$G_2 = (C_2(e) \rtimes M) \rtimes (M \rtimes C_0)$$

and homomorphisms

$$\begin{aligned} d_0(y,a,a',p) &= (a',p) \\ d_1(y,a,a',p) &= (a \circ t(a) \cdot a', p) = (aa', p) \\ d_2(y,a,a',p) &= (\delta(y) \circ t(\delta(y)) \cdot a, pt(a')) = (\delta(y)a, pt(a')) \\ s_0(a',p) &= (0_e, 0_e, a', p) \\ s_1(a,p) &= (0_e, a, 0_e, p). \end{aligned}$$

We thus have a 2-truncated simplicial group $\{G_2, G_1, G_0\}$ that looks like

$$\begin{tikzcd} (C_2(e) \rtimes M) \rtimes (M \rtimes C_0) \arrow[r, shift left=2, "d_0^2, d_1^2, d_2^2"] \arrow[r, shift left=1] \arrow[r] & (M \rtimes C_0) \arrow[l, shift left=1, "s_0^1, s_1^1"] \arrow[r, shift left, "d_0^1, d_1^1"] \arrow[r] & C_0 . \arrow[l, shift left, "s_0^0"] \end{tikzcd}$$

There is a $\mathbf{Cosk}_2$ functor from the category of 2-truncated simplicial groups to that of simplicial groups. We set $\mathbf{G}' = \mathbf{Cosk}_2\{G_2, G_1, G_0\}$ and claim $NG'_3 = 1$. We now give the sketch of the argument. From the result of Conduché as given in

Lemma 1.1 of this paper, for the 2-truncated simplicial group $\{G_2, G_1, G_0\}$, we have $N(\mathbf{Cosk}_2\{G_2, G_1, G_0\})_3 = 1$. From [**24**], by using the image of $F_{\alpha,\beta}$ functions, we have that $\partial_3(NG_3 \cap D_3)$ is the product of $[\ker d_2, \ker d_0 \cap \ker d_1]$, $[\ker d_1, \ker d_0 \cap \ker d_2]$, $[\ker d_0, \ker d_1 \cap \ker d_2]$, $[\ker d_0 \cap \ker d_2, \ker d_0 \cap \ker d_1]$, $[\ker d_1 \cap \ker d_2, \ker d_0 \cap \ker d_1]$ and $[\ker d_1 \cap \ker d_2, \ker d_0 \cap \ker d_2]$. A direct calculation using the descriptions of the face maps and the actions above shows that these are all trivial, so $\partial_3(NG_3 \cap D_3) = 1$, but again ∂_3 is a monomorphism so $NG'_3 = 1$ as required. □

3. Crossed Squares and Braided, Regular Crossed Modules

Loday and Guin-Walery [**26**] introduced the notion of crossed square as an algebraic model of connected 3-types.

A *crossed square* of groups is a commutative square of groups;

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd}$$

together with left actions of P on L, M, N and a function $h\colon M \times N \to L$. Let M and N act on M, N and L via P. The structure must satisfy the following axioms for all $l \in L$, $m, m' \in M$, $n, n' \in N$, $p \in P$;

- **(i)** the homomorphisms $\mu, \nu, \lambda, \lambda'$ and $\mu\lambda$ are crossed modules and both λ, λ' are P-equivariant,
- **(ii)** $h(mm', n) = h({}^m m', {}^m n)h(m, n)$,
- **(iii)** $h(m, nn') = h(m, n)h({}^n m, {}^n n')$,
- **(iv)** $\lambda h(m, n) = m{}^n m^{-1}$
- **(v)** $\lambda' h(m, n) = {}^m n n^{-1}$,
- **(vi)** $h(\lambda l, n) = l{}^n l^{-1}$,
- **(vii)** $h(m, \lambda' l) = {}^m l l^{-1}$,
- **(viii)** $h({}^p m, {}^p n) = {}^p h(m, n)$.

Conduché constructed (private communication to Brown in 1984; see also published version [**17**]) a 2-crossed module from a crossed square

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd}$$

as

$$L \xrightarrow{(\lambda^{-1}, \lambda')} M \rtimes N \xrightarrow{\mu\nu} P.$$

In [**5**], by taking the bisimplicial nerve of the crossed square and using the Artin-Mazur codiagonal functor (cf. [**2**]), we obtained a 2-truncated simplicial group $\mathbf{G}^{(2)}$

that looks like

$$\mathbf{G}^{(2)}\colon (L \rtimes (N \rtimes M)) \rtimes (N \rtimes (M \rtimes P)) \underset{s_0^1, s_1^1}{\overset{d_0^2, d_1^2, d_2^2}{\rightrightarrows}} N \rtimes (M \rtimes P) \underset{s_0^0}{\overset{d_0^1, d_1^1}{\rightrightarrows}} P$$

with the faces and degeneracies as given in [**5**]. We also showed that the Moore complex of this 2-truncated simplicial group

$$NG_2 \xrightarrow{\partial_2} NG_1 \xrightarrow{\partial_1} NG_0$$

is isomorphic to the mapping cone complex

$$L \xrightarrow{(\lambda^{-1}, \lambda')} M \rtimes N \xrightarrow{\mu\nu} P \tag{1}$$

of the crossed square, and that this mapping cone has a 2-crossed module structure.

In this section, we will construct a braided regular crossed module as

$$C\colon C_2 \xrightarrow{\delta} C_1 \underset{t}{\overset{s}{\rightrightarrows}} C_0$$

by applying the Brown-Gilbert functor from 2-crossed modules to braided regular crossed modules to this mapping cone complex (1).

Clearly we have $C_0 = P \cong G_0^{(2)}$, and by using the action of P on $M \rtimes N$, we have $C_1 = (M \rtimes N) \rtimes P \cong NG_1^{(2)} \rtimes s_0(G_0^{(2)})$. The source and target maps are given by $s(m, n, p) = \mu(m)\nu(n)p$ and $t(m, n, p) = p$ for $m \in M, n \in N, p \in P$. The groupoid composition in C_1 is given by

$$(m, n, p) \circ (m', n', p') = (m^{\nu(n)} m', nn', p')$$

if $\mu(m')\nu(n')p' = p$. We have

$$t((m, n, p) \circ (m', n', p')) = p' = t(m', n', p')$$

and

$$\begin{aligned} s((m, n, p) \circ (m', n', p')) &= s((m^{\nu(n)} m', nn', p')) \\ &= \mu(m^{\nu(n)} m')\nu(n)\nu(n')p' \\ &= \mu(m)\nu(n)\mu(m')\nu(n')p' \\ &= \mu(m)\nu(n)p \\ &= s(m, n, p), \end{aligned}$$

and thus we have a groupoid structure (C_1, C_0). The biaction of the group C_0 on the groupoid (C_1, C_0) can be given by

$$\begin{aligned} p \cdot (m, n, q) &= ({}^p m, {}^p n, pq) \\ (m, n, q) \cdot p &= (m, n, qp) \end{aligned}$$

for $(m, n, q) \in C_1$ and $p \in C_0$.

We also have $C_2 = L \rtimes P \cong NG_2^{(2)} \rtimes s_0s_0(NG_0^{(2)})$, and the groupoid composition can be given by

$$(l,p)\circ(l',p) = (ll',p),$$

and the biaction of C_0 on the groupoid (C_2, C_0) can be given by

$$p\cdot(l,q) = ({}^pl, pq), \quad (l,q)\cdot p = (l, qp).$$

Thus we have a regular crossed module

$$C\colon L \rtimes P \xrightarrow{\delta'} (M \rtimes N) \rtimes P \overset{s}{\underset{t}{\rightrightarrows}} P$$

from the mapping cone complex (1), where the morphism δ' is given by $(l,p)\mapsto (\lambda^{-1}(l), \lambda'(l), p)$. The braiding map on this structure is given by

$$\{(m,n,p),(m',n',p')\} = \left(h({}^{\nu(n^{-1})}m^{-1}, n^{-1}({}^pm')n), pp'\right)$$

for $m, m' \in M$, $n, n' \in N$ and $p, p' \in P$, where h is the h-map of the crossed square.

Thus, if given a crossed square

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd}$$

its associated braided regular crossed module is

$$L \rtimes P \xrightarrow{\delta'} (M \rtimes N) \rtimes P \overset{s}{\underset{t}{\rightrightarrows}} P$$

as described above.

Now, let

$$C\colon C_2 \xrightarrow{\delta} C_1 \overset{s}{\underset{t}{\rightrightarrows}} C_0$$

be any braided regular crossed module. Consider its associated 2-truncated simplicial group

$$\mathbf{G}'\colon (C_2(e) \rtimes M) \rtimes (M \rtimes C_0) \overset{d_0^2, d_1^2, d_2^2}{\underset{s_0^1, s_1^1}{\rightrightarrows}} (M \rtimes C_0) \overset{d_0^1, d_1^1}{\underset{s_0^0}{\rightrightarrows}} C_0$$

together with the face and degeneracy maps as given in Theorem 2.7.

We investigate the Moore complex of this 2-truncated simplicial group. Clearly $NG_0' = C_0$. By the definition of d_0^1, we have $\ker d_0^1 = NG_1' = M$, and by the definition of d_1^1, we have

$$\ker d_1^1 = \overline{M} = \{(m,p)\colon p = t(m)^{-1}, m \in M, p \in P\}.$$

Similarly, by the definition of the face maps d_0^2 and d_1^2, we have $\ker d_0^2 \cap \ker d_1^2 = NG_2' = C_2(e)$.

Mutlu and Porter in [**24**] defined a functor from simplicial groups to crossed squares, denoting it by

$$\mathbf{M}(-,2)\colon \mathbf{SimpGrp} \to \mathbf{Crs^2}.$$

We now briefly explain this functor.

Let $\mathbf{G}$ be a simplicial group. Then the following diagram

$$\begin{tikzcd} NG_2/\partial_3 NG_3 \arrow[r, "\partial_2"] \arrow[d, "\partial_2'"'] & NG_1 \arrow[d, "\mu"] \\ \overline{NG}_1 \arrow[r, "\mu'"'] & G_1 \end{tikzcd}$$

is the underlying square of a crossed square. The extra structure is given as follows; $NG_1 = \ker d_0^1$ and $\overline{NG}_1 = \ker d_1^1$. Since G_1 acts on $NG_2/\partial_3 NG_3, \overline{NG}_1$ and NG_1, there are actions of $\overline{NG}_1$ on $NG_2/\partial_3 NG_3$ and NG_1 via μ', and NG_1 acts on $NG_2/\partial_3 NG_3$ and $\overline{NG}_1$ via μ. Both μ and μ' are inclusions, and all actions are given by conjugation. The h-map is

$$\begin{aligned} h\colon NG_1 \times \overline{NG}_1 &\longrightarrow NG_2/\partial_3 NG_3 \\ (x, \overline{y}) &\longmapsto h(x,y) = [s_1x, s_1ys_0y^{-1}]\partial_3 NG_3. \end{aligned}$$

Here x and y are in NG_1 as there is a bijection between NG_1 and $\overline{NG}_1$.

Now, we apply this functor to the 2-truncated simplicial group $\mathbf{G'}$ given above. From Section 2, we have $\partial_3(NG_3') = 1$. In the above calculations, we have $NG_0' = C_0$, $NG_1' = M$, $\overline{NG_1'} = \overline{M}$ and $NG_2' = C_2(e)$. Thus we have a crossed square $\mathbf{M}(\mathbf{G'}, 2)$ that looks like

$$\begin{tikzcd} C_2(e) \arrow[r, "\delta"] \arrow[d, "\delta'"'] & M \arrow[d, "\mu"] \\ \overline{M} \arrow[r, "\mu'"'] & M \rtimes C_0. \end{tikzcd}$$

The braiding map $\{-,-\}\colon C_1 \times C_1 \to C_2$ induces the h-map of the crossed square.

Thus, if given a braided regular crossed module

$$C\colon C_2 \xrightarrow{\delta} C_1 \overset{s}{\underset{t}{\rightrightarrows}} C_0$$

its associated crossed square is

$$\begin{tikzcd} C_2(e) \arrow[r, "\delta"] \arrow[d, "\delta'"'] & M \arrow[d, "\mu"] \\ \overline{M} \arrow[r, "\mu'"'] & M \rtimes C_0 \end{tikzcd}$$

as described above.

4. Quadratic modules from braided regular crossed modules

Quadratic modules of groups were initially defined by Baues in [**6**] as models for connected 3-types. In [**5**, Section 5], we constructed a functor from 2-crossed modules to quadratic modules. Furthermore, in [**5**], by using the $F_{\alpha,\beta}$ functions, we gave a construction of a quadratic module from a simplicial group.

Recall that a *nil(2)-module* is a pre-crossed module $\partial\colon M \to N$ with an additional "nilpotency" condition. This condition is $P_3(\partial) = 1$, where $P_3(\partial)$ is the subgroup of M generated by Peiffer commutator $\langle x_1, x_2, x_3\rangle$ of length 3.

The Peiffer commutator in a pre-crossed module $\partial\colon M \to N$ is defined by

$$\langle x, y\rangle = ({}^{\partial x}y)xy^{-1}x^{-1}$$

for $x, y \in M$.

For a group G, the group

$$G^{ab} = G/[G, G]$$

is the abelianisation of G and

$$\partial^{cr}\colon M^{cr} = M/P_2(\partial) \to N$$

is the crossed module associated to the pre-crossed module $\partial\colon M \to N$. Here $P_2(\partial) = \langle M, M\rangle$ is the Peiffer subgroup of M.

The following definition is due to Baues (cf. [**6**]).

Definition 4.1. A quadratic module $(\omega, \delta, \partial)$ is a diagram

$$\begin{tikzcd} & C \otimes C \arrow[dl, "\omega"'] \arrow[d, "w"] & \\ L \arrow[r, "\delta"'] & M \arrow[r, "\partial"'] & N \end{tikzcd}$$

of homomorphisms between groups such that the following axioms are satisfied.

QM1: The homomorphism $\partial\colon M \to N$ is a nil(2)-module with Peiffer commutator map w defined above. The quotient map $M \twoheadrightarrow C = (M^{cr})^{ab}$ is given by $x \mapsto \overline{x}$, where $\overline{x} \in C$ denotes the class represented by $x \in M$ and $C = (M^{cr})^{ab}$ is the abelianisation of the associated crossed module $M^{cr} \to N$.

QM2: The boundary homomorphisms ∂ and δ satisfy $\partial\delta = 1$ and the quadratic map ω is a lift of the Peiffer commutator map w, that is $\delta\omega = w$ or equivalently

$$\delta\omega(\overline{x} \otimes \overline{y}) = ({}^{\partial x}y)xy^{-1}x^{-1} = \langle x, y\rangle$$

for $x, y \in M$.

QM3: L is an N-group and all homomorphisms of the diagram are equivariant with respect to the action of N. Moreover, the action of N on L satisfies the formula $(a \in L, x \in M)$

$${}^{\partial x}a = \omega\big(\big(\overline{x} \otimes \overline{\delta a}\big)\big(\overline{\delta a} \otimes \overline{x}\big)\big)a.$$

QM4: Commutators in L satisfy the formula $(a, b \in L)$

$$\omega(\overline{\delta a} \otimes \overline{\delta b}) = [b, a].$$

Now, consider the braided regular crossed module

$$C\colon C_2 \xrightarrow{\delta} C_1 \underset{t}{\overset{s}{\rightrightarrows}} C_0$$

and its associated 2-crossed module

$$C_2(e) \xrightarrow{\delta} M \xrightarrow{t} C_0 \tag{2}$$

as given in [**9**]. Applying the functor from 2-crossed modules to quadratic modules as described in [**5**, Section 5] to this associated 2-crossed module (2) gives a quadratic module

$$\begin{tikzcd} & C \otimes C \arrow[dl, "\omega"'] \arrow[d, "w"] & \\ C_2(e)/P_3' \arrow[r, "\delta'"'] & M/P_3 \arrow[r, "\partial'"'] & C_0 \end{tikzcd}$$

where P_3 is the normal subgroup of M generated by elements of the form

$$\langle x, \langle y, z\rangle\rangle \text{ and } \langle\langle x, y\rangle, z\rangle$$

for $x, y, z \in M$. The Peiffer elements in M are given by $\langle x, y\rangle = ({}^{t(x)}y)xy^{-1}x^{-1}$ for $x, y \in M$. Also, P_3' is the normal subgroup of $C_2(e)$ generated by elements of the form

$$\{x, \langle y, z\rangle\} \text{ and } \{\langle x, y\rangle, z\}$$

where $\{-, -\}$ is the Peiffer map of the associated 2-crossed module (2).

Since $t(\langle x, \langle y, z\rangle\rangle) = 1$ and $t(\langle\langle x, y\rangle, z\rangle) = 1$, the map $\partial'\colon M/P_3 \to C_0$ given by $\partial'(mP_3) = t(m)$ is a well defined group homomorphism. In addition, since $\delta\{x, \langle y, z\rangle\} = \langle x, \langle y, z\rangle\rangle$ and $\delta\{\langle x, y\rangle, z\} = \langle\langle x, y\rangle, z\rangle$, the map $\delta'\colon C_2(e)/P_3' \to M/P_3$ given by $\delta'(xP_3') = \delta(x)P_3$ is a well defined group homomorphism. The braiding map induces the quadratic map ω.

Thus, if given a braided regular crossed module

$$C\colon C_2 \xrightarrow{\delta} C_1 \underset{t}{\overset{s}{\rightrightarrows}} C_0 \ ,$$

its associated quadratic module is

$$\begin{tikzcd} & C \otimes C \arrow[dl, "\omega"'] \arrow[d, "w"] & \\ C_2(e)/P_3' \arrow[r, "\delta'"'] & M/P_3 \arrow[r, "\partial'"'] & C_0 \end{tikzcd}$$

as described above.

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