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SUPPORT VARIETIES: AN IDEAL APPROACH

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Abstract

We define support varieties in an axiomatic setting using the prime spectrum of a lattice of ideals. A key observation is the functoriality of the spectrum and that this functor admits an adjoint. We assign to each ideal its support and can classify ideals in terms of their support. Applications arise from studying abelian or triangulated tensor categories. Specific examples from algebraic geometry and modular representation theory are discussed, illustrating the power of this approach which is inspired by recent work of Balmer.

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1. Introduction

The spectrum of prime ideals and the support of objects like modules, sheaves, complexes, etc. belong to the fundamental concepts of algebraic geometry. In fact, the use of these concepts is not restricted to algebraic geometry; similar notions exist, for instance, in modular representation theory. In this paper we discuss a general approach which allows us to study prime ideal spectra and supports in various settings.

A prime ideal spectrum comes naturally equipped with a topology which is usually called the Zariski topology. However, there are various instances where it is more natural to consider another 'opposite' topology. It is one of the principal aims of this work to clarify the parallel use of two different topologies on a prime ideal spectrum. This is based on the notion of a spectral space, first introduced by Hochster [**12**].

We give a couple of motivating examples which illustrate the use of such different topologies. Let A be a commutative ring and denote by $\operatorname{Spec} A$ the set of prime ideals, together with the usual Zariski topology. We obtain another topology and write $\operatorname{Spec}^* A$ if we take the quasi-compact Zariski open sets as a basis of closed sets. Now suppose that A is noetherian. Let $\operatorname{mod} A$ denote the abelian category of all finitely generated A-modules. Then the assignment

$$\operatorname{mod} A \supseteq \mathcal{C} \mapsto \bigcup_{M \in \mathcal{C}} \operatorname{supp} M$$

induces a bijection between all Serre subcategories of $\operatorname{mod} A$ and all open subsets of $\operatorname{Spec}^* A$; see [**8**], and [**10**] for a recent generalization. Our second example arises from Ziegler's work on the model theory of modules [**23**]. The points of the Ziegler spectrum of A are the isomorphism classes of indecomposable pure-injective A-modules, and the closed subsets correspond to complete theories of modules. The indecomposable injective modules form a closed subset $\operatorname{Inj} A$, and the assignment

$$\operatorname{Spec}^* A \ni \mathfrak{p} \mapsto E(A/\mathfrak{p}) = \text{ injective envelope of } A/\mathfrak{p}$$

induces a bijection between all closed subsets of $\operatorname{Spec}^* A$ and all Ziegler closed subsets contained in $\operatorname{Inj} A$; see [**17**]. We do not comment any further on the second example, but the first example is explained in some detail when we discuss the abelian category of coherent sheaves and the triangulated category of perfect complexes on a scheme.

Now let us give a brief outline of the contents of this paper. At the beginning we introduce the notion of an ideal lattice and study its prime ideal spectrum. A key observation is the functoriality of the spectrum and that this functor admits an adjoint. We assign to each ideal its support and can classify ideals in terms of their support. Examples of ideal lattices arise from tensor categories which are abelian or triangulated. We provide a systematic treatment of such tensor categories and discuss a number of examples from algebraic geometry and modular representation theory. This is inspired by recent work of Balmer [**2**].

To be more specific, let $\mathcal{C}$ be a small abelian or triangulated tensor category with a tensor identity. We consider the ideal lattice of thick tensor ideals of $\mathcal{C}$ and its prime ideal spectrum $\operatorname{Spec} \mathcal{C}$. This space comes naturally equipped with a sheaf of rings

$\mathcal{O}_\mathcal{C}$ and we can describe the ringed space $(\operatorname{Spec}\mathcal{C}, \mathcal{O}_\mathcal{C})$ in some interesting examples. An important application says that every quasi-compact and quasi-separated scheme X can be reconstructed from the triangulated tensor category of perfect complexes on X. This is a slight generalization of a result of Balmer [**1, 2**] and based on the fundamental work of Thomason [**21, 22**]. On the other hand, it is the analogue—with almost identical proof—of the fact that a noetherian scheme X can be reconstructed from the abelian tensor category of coherent sheaves on X.

There are a number of interesting examples of triangulated tensor categories $\mathcal{C}$ where $(\operatorname{Spec}\mathcal{C}, \mathcal{O}_\mathcal{C})$ is actually a projective scheme. We provide a general criterion which explains those examples. For instance, this result establishes for a finite group G and a field k a conceptual link between the identical classifications of

- thick tensor ideals of the category of perfect complexes over the graded commutative cohomology ring $H^*(G, k)$, due to Hopkins and Neeman [**13, 16**], and
- thick tensor ideals of the stable category of finite dimensional k-linear representations of G, due to Benson, Carlson, and Rickard [**4**].

This generalizes to finite group schemes, by the recent work of Friedlander and Pevtsova [**7**].

Our personal motivation for this project stems from the work on support varieties in non-commutative settings, for instance for modular representations of finite dimensional algebras. It turns out that most parts of our theory do not require any commutativity assumptions. However, the product formula

$$\operatorname{supp}(ab) = \operatorname{supp}(a) \cap \operatorname{supp}(b) = \operatorname{supp}(ba)$$

for the support of two ideals a, b shows that commutativity is inherent to the subject, even though we allow $ab \neq ba$.

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2. The prime spectrum of an ideal lattice

2.1. Ideal lattices

In this section we introduce the notion of an ideal lattice. The collection of ideals of some fixed algebraic structure is usually equipped with two additional structures. We consider the partial ordering by inclusion and the internal multiplication. Recall that a *lattice* is a partially ordered set such that each non-empty finite set admits a supremum and an infimum.

Definition 2.1. An *ideal lattice* is by definition a partially ordered set $L = (L, \leqslant)$, together with an associative multiplication $L \times L \to L$, such that the following hold:

(L1) The poset L is a *complete lattice*, that is,

$$\bigvee_{a\in A} a = \sup A \quad \text{and} \quad \bigwedge_{a\in A} a = \inf A$$

exist in L for every subset $A \subseteq L$.

(L2) The lattice L is *compactly generated*, that is, every element in L is the supremum of compact elements. (An element $a \in L$ is *compact*, if for all $A \subseteq L$ with $a \leqslant \sup A$ there exists some finite $A' \subseteq A$ with $a \leqslant \sup A'$.)

(L3) We have for all $a, b, c \in L$

$$a(b \vee c) = ab \vee ac \quad \text{and} \quad (a \vee b)c = ac \vee bc.$$

(L4) The element $1 = \sup L$ is compact, and $1a = a = a1$ for all $a \in L$.

(L5) The product of two compact elements is again compact.

A *morphism* $\phi\colon L \to L'$ of ideal lattices is a map satisfying

$$\phi(\bigvee_{a\in A} a) = \bigvee_{a\in A} \phi(a) \quad \text{for} \quad A \subseteq L,$$

$$\phi(1) = 1 \quad \text{and} \quad \phi(ab) = \phi(a)\phi(b) \quad \text{for} \quad a, b \in L.$$

It is useful to think of a poset L as a category $\mathcal{L}$ where the objects of $\mathcal{L}$ are the elements of L and

$$\mathrm{Hom}_{\mathcal{L}}(a, b) \neq \emptyset \iff \mathrm{card}\,\mathrm{Hom}_{\mathcal{L}}(a, b) = 1 \iff a \leqslant b$$

for $a, b \in L$. Note that infimum and supremum in L correspond to product and coproduct, respectively, in $\mathcal{L}$. Thus a compactly generated complete lattice is precisely a locally finitely presentable category $\mathcal{L}$ (in the sense of [**9**]) satisfying

$$\mathrm{card}\,\mathrm{Hom}_{\mathcal{L}}(a, b) \leqslant 1 \quad \text{for all} \quad a, b \in \mathcal{L}.$$

Given an ideal lattice L, the multiplication $L \times L \to L$ corresponds to a tensor product $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$. A morphism $L \to L'$ of ideal lattices corresponds to a functor $\mathcal{L} \to \mathcal{L}'$ preserving all colimits and the tensor product.

Next observe that an ideal lattice L is essentially determined by its subset L^c of compact elements. To make this precise, let K be a poset and suppose that $\sup A$ exists for every finite subset $A \subseteq K$. A non-empty subset $I \subseteq K$ is an *ideal* of K if for all $a, b \in K$

(1) $a \leqslant b$ and $b \in I$ implies $a \in I$, and

(2) $a, b \in I$ implies $a \vee b \in I$.

Given $a \in K$, let $I(a) = \{x \in K \mid x \leqslant a\}$ denote the *principal ideal* generated by a. The set $\widehat{K}$ of all ideals of K is called the *completion* of K. This set is partially ordered by inclusion and in fact a compactly generated complete lattice. The map $K \to \widehat{K}$ sending $a \in K$ to $I(a)$ identifies K with $\widehat{K}^c$.

Lemma 2.2. *Let L be a compactly generated complete lattice. Then the map*

$$L \longrightarrow \widehat{L^c}, \quad a \mapsto I(a) \cap L^c = \{x \in L \mid x \leqslant a \text{ and } x \text{ compact}\},$$

is a lattice isomorphism.

Lattice theory	Ring theory
lattice (L)	ring (A)
element	ideal
$a \leqslant b$	$I \subseteq J$
$a \vee b$	$I + J$
$a \wedge b$	$I \cap J$
ab	IJ
$\sup L$	(1)
$\inf L$	(0)
prime element	prime ideal
compact element	finitely generated ideal
semi-prime element	semi-prime ideal

Table 1: A dictionary between lattice and ring theory

Proof. The inverse map sends an ideal $I \in \widehat{L^c}$ to $\sup I$ in L. □

We note some immediate consequences which we use frequently without further reference. Given elements a, b in a compactly generated complete lattice, we have

$$a = \bigvee_{\substack{a' \leqslant a \\ a' \text{ compact}}} a', \quad \text{and} \tag{2.1}$$

$$a \leqslant b \iff a' \leqslant b \text{ for all compact } a' \leqslant a.$$

Now let L be an ideal lattice. The multiplication $L \times L \to L$ restricts to a multiplication $L^c \times L^c \to L^c$. It turns out that all relevant structure is determined by the multiplication of compact elements. In our applications, we always have for $a, b \in L$ that $ab = \sup a'b'$ where $a' \leqslant a$ and $b' \leqslant b$ run through all compact elements.

2.2. The prime spectrum

Let L be an ideal lattice. We define the spectrum of prime elements in L and discuss some of its basic properties. An element $p \neq 1$ in L is called *prime* if $ab \leqslant p$ implies $a \leqslant p$ or $b \leqslant p$ for all $a, b \in L$. A subset $S \subseteq L$ is *multiplicative* if $ab \in S$ for all $a, b \in S$.

The basic example of an ideal lattice is the lattice of ideals of a commutative ring; see Proposition 7.4. A dictionary between lattice and ring theory is provided in Table 1.

Next we collect some elementary facts about prime elements.

Lemma 2.3. *An element $p \neq 1$ in L is prime if and only if $ab \leqslant p$ implies $a \leqslant p$ or $b \leqslant p$ for all compact $a, b \in L$.*

Proof. Use (2.1). □

Lemma 2.4. *Let $a \in L$ and $S \subseteq L$ be a non-empty multiplicative set of compact elements. Suppose that $s \nleqslant a$ for all $s \in S$. Then there exists a prime $p \in L$ such that $a \leqslant p$ and $s \nleqslant p$ for all $s \in S$.*

Proof. Consider the set A of all elements $x \in L$ such that $a \leqslant x$ and $s \nleqslant x$ for all $s \in S$. The set A is non-empty and for every chain $B \subseteq A$, we have $\sup B$ in A since the elements in S are compact. Thus A has a maximal element p, by Zorn's lemma. We claim that p is prime. Let $x, x' \in L$ with $xx' \leqslant p$. Suppose that $x \nleqslant p$ and $x' \nleqslant p$. Then we have $s, s' \in S$ such that $s \leqslant p \vee x$ and $s' \leqslant p \vee x'$, by the maximality of p. Therefore

$$ss' \leqslant (p \vee x)(p \vee x') = pp \vee px' \vee xp \vee xx' \leqslant p$$

which contradicts the fact that $p \in A$. Thus $x \leqslant p$ or $x' \leqslant p$, and therefore p is prime. □

An element $a \in L$ is *semi-prime* if $bb \leqslant a$ implies $b \leqslant a$ for all $b \in L$.

Lemma 2.5. *An element $a \in L$ is semi-prime if and only if $a = \inf V$ for some set $V \subseteq L$ of prime elements.*

Proof. Suppose that a is semi-prime and let $V = \{p \in L \mid a \leqslant p \text{ and } p \text{ prime}\}$. For any compact $b \in L$ such that $b \nleqslant a$, consider the multiplicative set $\{b^n \mid n \geqslant 1\}$. It follows from Lemma 2.4 that there is a prime $p \in V$ such that $b \nleqslant p$. Thus $a = \inf V$. The other implication is clear. □

We denote by $\operatorname{Spec} L$ the set of prime elements in L and define for each $a \in L$

$$V(a) = \{p \in \operatorname{Spec} L \mid a \leqslant p\} \quad \text{and} \quad D(a) = \{p \in \operatorname{Spec} L \mid a \nleqslant p\}.$$

The subsets of $\operatorname{Spec} L$ of the form $V(a)$ are closed under forming arbitrary intersections and finite unions. More precisely,

$$V(\bigvee_{i \in \Omega} a_i) = \bigcap_{i \in \Omega} V(a_i) \quad \text{and} \quad V(ab) = V(a) \cup V(b).$$

Thus we obtain the *Zariski topology* on $\operatorname{Spec} L$ by declaring a subset of $\operatorname{Spec} L$ to be *closed* if it is of the form $V(a)$ for some $a \in L$. The set $\operatorname{Spec} L$ endowed with this topology is called the *prime spectrum* of L. Note that the sets of the form $D(a)$ with compact $a \in L$ form a basis of open sets. This is a consequence of the following lemma.

Lemma 2.6. *For $a \in L$, we have*

$$V(a) = \bigcap_{\substack{b \leqslant a \\ b \text{ compact}}} V(b) \quad \textit{and} \quad D(a) = \bigcup_{\substack{b \leqslant a \\ b \text{ compact}}} D(b).$$

Proof. Use (2.1). □

Proposition 2.7. *The assignments*

$$L \ni a \mapsto V(a) = \{p \in \operatorname{Spec} L \mid a \leqslant p\} \quad \textit{and} \quad \operatorname{Spec} L \supseteq Y \mapsto \inf Y$$

induce mutually inverse and order reversing bijections between

(1) *the set of all semi-prime elements in L, and*

(2) *the set of all closed subsets of* $\operatorname{Spec} L$.

Proof. Both maps are well-defined by Lemma 2.5. Given a semi-prime $a \in L$, the equality $\inf V(a) = a$ is clear since a is a join of prime elements, by Lemma 2.5. Now let $Y \subseteq \operatorname{Spec} L$. The inclusion $Y \subseteq V(\inf Y)$ is purely formal. Suppose that Y is of the form $Y = V(a)$ for some $a \in L$. If $p \in V(\inf Y)$, then $a \leqslant \inf Y \leqslant p$ and therefore $p \in Y$. Thus the proof is complete. □

Corollary 2.8. *The assignments*

$$L \ni a \mapsto D(a) = \bigcup_{\substack{b \leqslant a \\ b \text{ compact}}} D(b) \quad \text{and} \quad \operatorname{Spec} L \supseteq Y \mapsto \bigvee_{\substack{D(b) \subseteq Y \\ b \text{ compact}}} b$$

induce mutually inverse and order preserving bijections between

(1) *the set of all semi-prime elements in* L*, and*

(2) *the set of all open subsets of* $\operatorname{Spec} L$.

Proof. We apply Proposition 2.7 and need to check that for $V = \operatorname{Spec} L \setminus Y$ and $a \in L$, we have $a \leqslant \inf V$ if and only if $D(a) \subseteq Y$. This is clear since $a \leqslant p$ for all $p \in V$ is equivalent to $a \nleqslant q$ implies $q \in Y$. □

Remark 2.9. Let $a \in L$ and denote by $\sqrt{a} = \inf V(a)$ the smallest semi-prime in L containing a. Then we have

$$\sqrt{ab} = \inf(V(a) \cup V(b)) = \sqrt{ba} \quad \text{for} \quad a, b \in L,$$

even though we do not assume commutativity of the multiplication in L.

3. The prime spectrum is spectral

This section is devoted to recalling the definition of a spectral topological space and to showing that the space $\operatorname{Spec} L$ is spectral for an ideal lattice L.

In [**12**], a topological space is defined to be *spectral* if it is T_0 and quasi-compact, the quasi-compact open subsets are closed under finite intersections and form an open basis, and every non-empty irreducible closed subset has a generic point. Recall that a closed subset is *irreducible* if it cannot be written as the union of two proper closed subsets.

We have the following basic property of a spectral space.

Lemma 3.1. *Let* X *be a spectral space. Endow the underlying set with a new topology by taking as open sets those of the form* $Y = \bigcup_{i \in \Omega} Y_i$ *with quasi-compact open complement* $X \setminus Y_i$ *for all* $i \in \Omega$*, and denote the new space by* X^**. Then* X^* *is spectral and* $(X^*)^* = X$.

Proof. See [**12**, Prop. 8]. □

Let L be an ideal lattice. We now show that the space $\operatorname{Spec} L$ is spectral. We proceed in several steps.

Lemma 3.2. *An open subset of* $\operatorname{Spec} L$ *is quasi-compact if and only if it is of the form* $D(c)$ *for some compact* $c \in L$.

Proof. Fix an open subset $D(a)$ of $\operatorname{Spec} L$. Suppose first that $D(a)$ is quasi-compact. We have

$$D(a) = \bigcup_{\substack{b \leqslant a \\ b \text{ compact}}} D(b)$$

by Lemma 2.6, and therefore $D(a) = D(b)$ for some compact $b \in L$.

Now suppose that $a \in L$ is compact and $D(a) \subseteq \bigcup_{b \in B} D(b)$ for some subset $B \subseteq L$. We write $\bar{b} = \bigvee_{b \in B} b$ and have $D(a) \subseteq D(\bar{b})$. It follows from Lemma 2.4 that $a^n \leqslant \bar{b}$ for some $n \geqslant 1$. Thus $a^n \leqslant \bigvee_{b \in B'} b$ for some finite subset $B' \subseteq B$ since a^n is compact. This implies $D(a) \subseteq \bigcup_{b \in B'} D(b)$, and therefore $D(a)$ is quasi-compact. □

Lemma 3.3. *Let $p, q \in \operatorname{Spec} L$. Then $\overline{\{p\}} = V(p)$. In particular, $\overline{\{p\}} = \overline{\{q\}}$ implies $p = q$.*

Proof. Clear. □

Lemma 3.4. *Let $Y \subseteq \operatorname{Spec} L$ be a non-empty closed subset. If Y is irreducible, then $\inf Y$ is prime and $Y = \overline{\{\inf Y\}}$.*

Proof. First observe that we have $c \leqslant \inf Y$ for $c \in L$ if and only if $Y \subseteq V(c)$. To show that $\inf Y$ is prime, let $ab \leqslant \inf Y$. Then

$$Y \subseteq V(ab) = V(a) \cup V(b),$$

and we have $Y \subseteq V(a)$ or $Y \subseteq V(b)$ since Y is irreducible. Thus $a \leqslant \inf Y$ or $b \leqslant \inf Y$. Let $p = \inf Y$ and $Y = V(a)$. Then we have by construction $a \leqslant p$ and $Y \subseteq V(p)$. Thus $Y = V(p) = \overline{\{p\}}$, by Lemma 3.3. □

Proposition 3.5. *The prime spectrum* $\operatorname{Spec} L$ *of an ideal lattice L is spectral.*

Proof. The space $\operatorname{Spec} L$ is T_0 by Lemma 3.3. An open subset of $\operatorname{Spec} L$ is quasi-compact if and only if it is of the form $D(a)$ for some compact $a \in L$, by Lemma 3.2. Thus the definition of the topology implies that the quasi-compact open subsets form an open basis which is closed under finite intersections. Moreover, $\operatorname{Spec} L$ is quasi-compact since $1 = \sup L$ is compact. If Y is a non-empty irreducible closed subset, then $Y = \overline{\{p\}}$ for $p = \inf Y$ by Lemma 3.4. □

There is a close relation between spectral spaces and ideal lattices, and we make this more precise. Given a topological space X, we denote by $L_{\text{open}}(X)$ the lattice of open subsets of X and consider the multiplication map

$$L_{\text{open}}(X) \times L_{\text{open}}(X) \longrightarrow L_{\text{open}}(X), \quad (U, V) \mapsto UV = U \cap V.$$

Note that the lattice $L_{\text{open}}(X)$ is complete.

Lemma 3.6. *Let X be a space and $U \in L = L_{\text{open}}(X)$. Then*

(1) U is prime in L if and only if $X \setminus U$ is irreducible, and

(2) U is compact in L if and only if U is quasi-compact.

Proof. Clear. □

Proposition 3.7. *Let X be a spectral space. Then $L_{\text{open}}(X)$ is an ideal lattice and every ideal in $L_{\text{open}}(X)$ is semi-prime. Moreover, the map*

$$X \longrightarrow \operatorname{Spec} L_{\text{open}}(X), \quad x \mapsto X \setminus \overline{\{x\}},$$

is a homeomorphism.

Proof. Using Lemma 3.6, the properties of a spectral space can be translated into the defining properties of an ideal lattice. Clearly, every ideal is semi-prime, since $UU = U$ for all $U \in L_{\text{open}}(X)$. It is straightforward to check that the given map is a homeomorphism. □

Example 3.8. Let A be a commutative ring. Then the lattice $L_{\text{id}}(A)$ of ideals of A is an ideal lattice and therefore $\operatorname{Spec} A = \operatorname{Spec} L_{\text{id}}(A)$ is spectral. More generally, if X is a quasi-compact and quasi-separated scheme, then the underlying space of X is spectral.

4. An adjoint of the functor Spec

The prime spectrum of an ideal lattice satisfies a universal property which we discuss in this section. Then we view the assignment $L \mapsto \operatorname{Spec} L$ as a functor from ideal lattices to spectral topological spaces and study its adjoint.

Definition 4.1. A *spectrum* of an ideal lattice L is a pair (X, δ) where X is a topological space and δ is a map which assigns to each $a \in L$ an open subset $\delta(a) \subseteq X$, such that

$$\delta(\bigvee_{a \in A} a) = \bigcup_{a \in A} \delta(a) \quad \text{for} \quad A \subseteq L, \tag{4.1}$$

$$\delta(1) = X \quad \text{and} \quad \delta(ab) = \delta(a) \cap \delta(b) \quad \text{for} \quad a, b \in L.$$

A *morphism* $f\colon (X, \delta) \to (X', \delta')$ of spectra is a continuous map $f\colon X \to X'$ such that $\delta(a) = f^{-1}(\delta'(a))$ for all $a \in L$. Such a morphism is an isomorphism if and only if $f\colon X \to X'$ is a homeomorphism.

Theorem 4.2. *Let L be an ideal lattice. Then the pair* $(\operatorname{Spec} L, D)$ *is a spectrum of L. For every spectrum (X, δ) of L, there exists a unique continuous map $f\colon X \to$* $\operatorname{Spec} L$ *such that $\delta(a) = f^{-1}(D(a))$ for every $a \in L$. The map f is defined by*

$$f(x) = \bigvee_{\substack{x \notin \delta(c) \\ c \text{ compact}}} c \quad \textit{for} \quad x \in X.$$

Proof. Clearly, the pair $(\operatorname{Spec} L, D)$ is a spectrum. Now let (X, δ) be a spectrum of L. We show that for each $x \in X$ the element $f(x) = \bigvee_{x \notin \delta(c)} c$ is prime. First observe that $f(x) \neq 1$ since $\delta(1) = X$. Suppose that $ab \leqslant f(x)$, and we may assume that a, b are compact. Using (4.1), observe that the compact $c \in L$ with $x \notin \delta(c)$ form a directed set. Thus $ab \leqslant c$ for some compact $c \in L$ with $x \notin \delta(c)$. Using (4.1) again, we have

$$\delta(a) \cap \delta(b) = \delta(ab) \subseteq \delta(c)$$

and therefore $x \notin \delta(a)$ or $x \notin \delta(b)$. We conclude that $a \leqslant f(x)$ or $b \leqslant f(x)$. The definition of f implies $\delta(a) = f^{-1}(D(a))$ for every $a \in L$, since

$$x \in \delta(a) \iff a \nleqslant f(x) \iff f(x) \in D(a) \iff x \in f^{-1}(D(a)).$$

The continuity of f follows from the fact that the sets $D(a)$ with $a \in L$ are precisely the open subsets of $\operatorname{Spec} L$. Now let $f_1, f_2 \colon X \to \operatorname{Spec} L$ be two maps satisfying

$$f_1^{-1}(D(a)) = \delta(a) = f_2^{-1}(D(a))$$

for every $a \in L$. Fix $x \in X$. Then we have

$$\overline{\{f_1(x)\}} = \bigcap_{f_1(x) \notin D(a)} V(a) = \bigcap_{f_2(x) \notin D(a)} V(a) = \overline{\{f_2(x)\}}.$$

This implies $f_1(x) = f_2(x)$ by Lemma 3.3, and therefore the proof is complete. □

We observe that a spectrum (X, δ) is determined by its restriction to the subset of compact elements since

$$\delta(a) = \bigcup_{\substack{b \leqslant a \\ b \text{ compact}}} \delta(b) \quad \text{for} \quad a \in L.$$

This observation has the following consequence.

Corollary 4.3. *Let X be a topological space and δ_0 be a map which assigns to each compact $a \in L$ an open subset $\delta_0(a) \subseteq X$, such that*

$$\delta_0(a \vee b) = \delta_0(a) \cup \delta_0(b) \quad \textit{for} \quad a, b \in L,$$
$$\delta_0(1) = X \quad \textit{and} \quad \delta_0(ab) = \delta_0(a) \cap \delta_0(b) \quad \textit{for} \quad a, b \in L.$$

(1) There exists a unique continuous map $f \colon X \to \operatorname{Spec} L$ such that

$$\delta_0(a) = f^{-1}(D(a))$$

for every compact $a \in L$.

(2) There exists a unique spectrum (X, δ) of L such that $\delta(a) = \delta_0(a)$ for every compact $a \in L$.

Proof. Define the continuous map $f \colon X \to \operatorname{Spec} L$ as in Theorem 4.2 and define $\delta(a) = f^{-1}(D(a))$ for every $a \in L$. It is clear from the defining formula that (X, δ) is a spectrum of L. The uniqueness of δ follows from the formula

$$\delta(a) = \bigcup_{\substack{b \leqslant a \\ b \text{ compact}}} \delta_0(b) \quad \text{for} \quad a \in L.$$

□

Our next application says that Spec is actually a functor into the category of topological spaces.

Lemma 4.4. *A morphism of ideal lattices* $\phi\colon L \to L'$ *induces a unique continuous map* $\operatorname{Spec}\phi\colon \operatorname{Spec} L' \to \operatorname{Spec} L$ *such that*

$$D(\phi(a)) = (\operatorname{Spec}\phi)^{-1} D(a) \quad \textit{for} \quad a \in L.$$

Proof. The pair $(\operatorname{Spec} L', D \circ \phi)$ is a spectrum of L. Now apply Theorem 4.2. Note that we can compute more explicitly

$$(\operatorname{Spec}\phi)p = \sup\{a \in L \mid \phi(a) \leqslant p\} \quad \text{for} \quad p \in \operatorname{Spec} L'.$$

□

The universal property of the prime spectrum yields an adjoint functor for Spec.

Theorem 4.5. *We have an adjoint pair of contravariant functors*

$$\mathbf{Lat_{id}} \underset{L_{\mathrm{open}}}{\overset{\operatorname{Spec}}{\rightleftarrows}} \mathbf{Top_{sp}}$$

between the category of ideal lattices and the category of spectral spaces. More precisely, for an ideal lattice L *and a spectral space* X*, there are mutually inverse bijections*

$$\operatorname{Hom}_{\mathbf{Lat_{id}}}(L, L_{\mathrm{open}}(X)) \underset{\Lambda}{\overset{\Sigma}{\rightleftarrows}} \operatorname{Hom}_{\mathbf{Top_{sp}}}(X, \operatorname{Spec} L).$$

The functor L_{open} *is fully faithful, and an ideal lattice* L *is isomorphic to one of the form* $L_{\mathrm{open}}(X)$ *if and only if every ideal in* L *is semi-prime.*

Proof. Both functors are well-defined by Propositions 3.5 and 3.7. The maps Σ and Λ are defined by

$$(\Sigma\phi)(x) = \bigvee_{\substack{x \notin \phi(c) \\ c \text{ compact}}} c \quad \text{and} \quad (\Lambda f)(a) = f^{-1}(D(a)) \quad \text{for} \quad x \in X, a \in L.$$

It follows from Theorem 4.2 that both maps are mutually inverse bijections. Next observe that L_{open} is fully faithful. This follows from the fact that the adjunction morphism $X \to \operatorname{Spec} L_{\mathrm{open}}(X)$ is a homeomorphism; see also Proposition 3.7. It remains to describe the image of L_{open}. Clearly, every ideal in $L_{\mathrm{open}}(X)$ is semi-prime; see Proposition 3.7. Conversely, if every ideal in L is semi-prime, then the adjunction morphism $L \to L_{\mathrm{open}}(\operatorname{Spec} L)$ is an isomorphism, by Corollary 2.8. Thus the proof is complete. □

5. Support data

Let L be an ideal lattice. We have seen that the space $\operatorname{Spec} L$ is spectral and, in view of our applications, from this point on we consider the 'opposite' topology on $X = \operatorname{Spec} L$. To be precise, we let $\operatorname{Spec}^* L = X^*$ where the points of X^* and

X coincide and $Y \subseteq X^*$ is by definition *open* if $Y = \bigcup_{i\in\Omega} Y_i$ with quasi-compact Zariski-open complement $X \setminus Y_i$ for all $i \in \Omega$; see Lemma 3.1. For $a \in L$, we call

$$\operatorname{supp}(a) = \{p \in \operatorname{Spec}^* L \mid a \nleqslant p\}$$

the *support* of a and observe that $\operatorname{supp}(a)$ is closed if a is compact. Let us reformulate the classification of semi-prime ideals in terms of the topology $\operatorname{Spec}^* L$.

Proposition 5.1. *The assignments*

$$L \ni a \mapsto \operatorname{supp}(a) = \bigcup_{\substack{b \leqslant a \\ b \text{ compact}}} \operatorname{supp}(b) \quad \textit{and} \quad \operatorname{Spec}^* L \supseteq Y \mapsto \bigvee_{\substack{\operatorname{supp}(b) \subseteq Y \\ b \text{ compact}}} b$$

induce mutually inverse and order preserving bijections between

(1) the set of all semi-prime elements in L, and

(2) the set of all subsets $Y \subseteq \operatorname{Spec}^* L$ of the form $Y = \bigcup_{i\in\Omega} Y_i$ with quasi-compact open complement $\operatorname{Spec}^* L \setminus Y_i$ for all $i \in \Omega$.

Proof. Use Corollary 2.8 and observe that the subsets $Y = \bigcup_{i\in\Omega} Y_i$ with quasi-compact open complement $\operatorname{Spec}^* L \setminus Y_i$ are precisely the open subsets of $\operatorname{Spec} L = (\operatorname{Spec}^* L)^*$, by Lemma 3.1. □

Next we introduce for an ideal lattice the concept of a support datum. This is inspired by Balmer's definition of a support datum on a triangulated tensor category [**2**, Defn. 3.1]. The subsequent theorem is the analogue of [**2**, Thm. 3.2].

Definition 5.2. A *support datum* on an ideal lattice L is a pair (X, σ) where X is a topological space and σ is a map which assigns to each compact $a \in L$ a closed subset $\sigma(a) \subseteq X$, such that

$$\sigma(a \vee b) = \sigma(a) \cup \sigma(b) \quad \text{for} \quad a, b \in L, \tag{5.1}$$
$$\sigma(1) = X \quad \text{and} \quad \sigma(ab) = \sigma(a) \cap \sigma(b) \quad \text{for} \quad a, b \in L.$$

A *morphism* $f\colon (X, \sigma) \to (X', \sigma')$ of support data is a continuous map $f\colon X \to X'$ such that $\sigma(a) = f^{-1}(\sigma'(a))$ for all compact $a \in L$. Such a morphism is an isomorphism if and only if $f\colon X \to X'$ is a homeomorphism.

The following result complements the universal property of the pair $(\operatorname{Spec} L, D)$ of Theorem 4.2 and the proof is almost the same.

Theorem 5.3. *Let L be an ideal lattice. Then the pair* $(\operatorname{Spec}^* L, \operatorname{supp})$ *is a support datum on L. For every support datum (X, σ) on L, there exists a unique continuous map $f\colon X \to \operatorname{Spec}^* L$ such that $\sigma(a) = f^{-1}(\operatorname{supp}(a))$ for every compact $a \in L$. The map f is defined by*

$$f(x) = \bigvee_{\substack{x \notin \sigma(c) \\ c \text{ compact}}} c \quad \textit{for} \quad x \in X.$$

Proof. Clearly, the pair $(\operatorname{Spec}^* L, \operatorname{supp})$ is a support datum. Now let (X, σ) be a support datum on L. We show that for each $x \in X$ the element $f(x) = \bigvee_{x \notin \sigma(c)} c$ is prime. First observe that $f(x) \neq 1$ since $\sigma(1) = X$. Suppose that $ab \leqslant f(x)$, and we may assume that a, b are compact. Using (5.1), observe that the compact $c \in L$ with $x \notin \sigma(c)$ form a directed set. Thus $ab \leqslant c$ for some compact $c \in L$ with $x \notin \sigma(c)$. Using (5.1) again, we have

$$\sigma(a) \cap \sigma(b) = \sigma(ab) \subseteq \sigma(c)$$

and therefore $x \notin \sigma(a)$ or $x \notin \sigma(b)$. We conclude that $a \leqslant f(x)$ or $b \leqslant f(x)$. The definition of f implies $\sigma(a) = f^{-1}(\operatorname{supp}(a))$ for every compact $a \in L$, since

$$x \in \sigma(a) \iff a \nleqslant f(x) \iff f(x) \in \operatorname{supp}(a) \iff x \in f^{-1}(\operatorname{supp}(a)).$$

The continuity of f follows from the fact that the sets $\operatorname{supp}(a)$ with compact $a \in L$ form a basis of closed sets for the topology on $\operatorname{Spec}^* L$. Now let $f_1, f_2 \colon X \to \operatorname{Spec}^* L$ be two maps satisfying

$$f_1^{-1}(\operatorname{supp}(a)) = \sigma(a) = f_2^{-1}(\operatorname{supp}(a))$$

for every compact $a \in L$. Fix $x \in X$. Then we have

$$\overline{\{f_1(x)\}} = \bigcap_{\substack{f_1(x) \in \operatorname{supp}(a) \\ a \text{ compact}}} \operatorname{supp}(a) = \bigcap_{\substack{f_2(x) \in \operatorname{supp}(a) \\ a \text{ compact}}} \operatorname{supp}(a) = \overline{\{f_2(x)\}}.$$

This implies $f_1(x) = f_2(x)$ since the space $\operatorname{Spec}^* L$ is T_0, by Proposition 3.5 and Lemma 3.1. □

6. Classifying support data

Let L be an ideal lattice. A support datum (X, σ) on L is called *classifying* if the space X is spectral and the assignments

$$L \ni a \mapsto \bigcup_{\substack{b \leqslant a \\ b \text{ compact}}} \sigma(b) \quad \text{and} \quad X \supseteq Y \mapsto \bigvee_{\substack{\sigma(b) \subseteq Y \\ b \text{ compact}}} b$$

induce bijections between

(1) the set of all semi-prime elements in L, and
(2) the set of all subsets $Y \subseteq X$ of the form $Y = \bigcup_{i \in \Omega} Y_i$ with quasi-compact open complement $X \setminus Y_i$ for all $i \in \Omega$.

Note that $(\operatorname{Spec}^* L, \operatorname{supp})$ is a classifying support datum by Proposition 5.1.

Proposition 6.1. *Let $f \colon (X, \sigma) \to (X', \sigma')$ be a morphism of support data. If both support data are classifying, then the map $f \colon X \to X'$ is a homeomorphism.*

Proof. Let $Y \subseteq X$ and $Y' \subseteq X'$ be subsets which are unions of subsets with quasi-compact open complement, and suppose

$$\bigvee_{\substack{\sigma(b) \subseteq Y \\ b \text{ compact}}} b = a = \bigvee_{\substack{\sigma'(b) \subseteq Y' \\ b \text{ compact}}} b.$$

Then we have

$$Y = \bigcup_{\substack{b \leqslant a \\ b \text{ compact}}} \sigma(b) \quad \text{and} \quad Y' = \bigcup_{\substack{b \leqslant a \\ b \text{ compact}}} \sigma'(b).$$

This implies

$$f^{-1}(Y') = \bigcup_{\substack{b\leqslant a \\ b \text{ compact}}} f^{-1}(\sigma'(b)) = \bigcup_{\substack{b\leqslant a \\ b \text{ compact}}} \sigma(b) = Y.$$

It follows that the map $Y \mapsto f^{-1}(Y)$ induces an inclusion preserving bijection between the open subsets of X^* and $(X')^*$. In fact, we use that X, X' are spectral and apply Lemma 3.1. Thus f is a homeomorphism $X^* \to (X')^*$ and therefore also a homeomorphism $X \to X'$. □

The following consequence is the analogue of [**2**, Thm. 5.2]. Note that we do not assume that the support space is noetherian.

Corollary 6.2. *A support datum (X, σ) on L is classifying if and only if the canonical morphism $(X, \sigma) \to (\operatorname{Spec}^* L, \operatorname{supp})$ is an isomorphism.*

7. Thick tensor ideals

In this section we consider an additive category with a tensor product and study its collection of ideals. If there is an additional abelian or triangulated structure, then we consider those tensor ideals which are also thick subcategories.

In this paper, all categories are assumed to be *small*, that is, the isomorphism classes of objects form a set (in some fixed universe).

7.1. Sublattices of an ideal lattice

Let L be an ideal lattice. We fix a subset $L' \subseteq L$ satisfying the following conditions.

(L∧) If $A \subseteq L'$, then $\inf A \in L'$.

(L∨) If $A \subseteq L'$ is directed, then $\sup A \in L'$.

We consider on L' the partial order induced from the partial order on L and define the map

$$\pi\colon L \longrightarrow L', \quad a \mapsto \bigwedge_{a\leqslant a' \in L'} a'.$$

Note that we have

$$\pi(a) \leqslant a' \iff a \leqslant a' \quad \text{for} \quad a \in L,\, a' \in L'. \tag{7.1}$$

Thus π is a left adjoint of the inclusion $L' \to L$ if we think of posets as categories. Moreover, we have $1 = \inf \emptyset \in L'$.

Lemma 7.1. *The poset L' is a complete and compactly generated lattice. Every compact element in L' is of the form $\pi(a)$ for some compact $a \in L$.*

Proof. Let $A \subseteq L'$. Then we use (L∧) to compute the infimum $\inf A$ in L' and have

$$\sup A = \inf\{a' \in L' \mid a \leqslant a' \text{ for all } a \in A\}.$$

It follows from (7.1) and (L∨) that π preserves compactness and that each element in L' is the supremum of compact elements. Thus L' is compactly generated. If $a' \in L'$

is compact, write $a' = \bigvee_i a_i$ as directed union of all compact elements $a_i \leqslant a'$ in L and use that $a' = \pi(a') = \bigvee_i \pi(a_i)$ equals $\pi(a_i)$ for some index i. □

Given $a, b \in L'$, we define their product in L' as

$$a \cdot b = \pi(ab)$$

and use a dot to distinguish it from the product in L. We make a further assumption.

(Lπ) Given $a, b \in L$, we have $\pi(a\pi(b)) = \pi(ab) = \pi(\pi(a)b)$.

Lemma 7.2. *Let $a, b, c \in L'$. Then we have*

(1) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,

(2) $a \cdot 1 = a = 1 \cdot a$,

(3) $a \cdot (b \vee c) = (a \cdot b) \vee (a \cdot c)$ *and* $(a \vee b) \cdot c = (a \cdot c) \vee (b \cdot c)$.

Proof. Clear. □

Proposition 7.3. *Let L be an ideal lattice and L' be a subset satisfying the conditions* (L$\wedge$), (L$\vee$), *and* (Lπ). *Then L' inherits from L the structure of an ideal lattice.*

Proof. We apply Lemmas 7.1 and 7.2. Thus L' satisfies (L1)–(L4). To check (L5), let $a', b' \in L'$ be compact and choose compact elements $a, b \in L$ with $\pi(a) = a'$ and $\pi(b) = b'$. Now we obtain

$$a' \cdot b' = \pi(a'b') = \pi(\pi(a)\pi(b)) = \pi(ab).$$

The element ab is compact in L, and π preserves compactness by (L$\vee$). Thus $a' \cdot b'$ is compact in L'. □

7.2. The ideal lattice of a semi-ring

Let $A = (A, +, \cdot)$ be a *semi-ring*, that is, A is a set together with two associative binary operations with identities (denoted by 0 and 1) such that the addition is commutative and distributivity holds. A subset $I \subseteq A$ containing 0 is by definition an *ideal* if for all $x, y \in A$

(1) $x \in I$ and $y \in I$ implies $x + y \in I$, and

(2) $x \in I$ or $y \in I$ implies $xy \in I$.

The ideals of A are partially ordered by inclusion and form a lattice which we denote by $L_{\mathrm{id}}(A)$. Given $I, J \in L_{\mathrm{id}}(A)$, we define

$$IJ = \{\sum_i x_i y_i \mid x_i \in I, y_i \in J\} \quad \text{and} \quad I + J = \{x + y \mid x \in I, y \in J\}.$$

Note that $I + J = I \vee J$.

Proposition 7.4. *Let A be semi-ring. Then the lattice $L_{\mathrm{id}}(A)$ of ideals satisfies the conditions* (L1)–(L4). *An ideal in $L_{\mathrm{id}}(A)$ is compact if and only if it is finitely generated. If A is commutative, then condition* (L5) *is satisfied.*

Proof. The proof is straightforward. To identify the compact elements, one uses that $\bigvee_i I_i = \bigcup_i I_i$ for any directed set of ideals I_i. To show (L5), let $I = \langle I_0 \rangle$ and $J = \langle J_0 \rangle$ be ideals generated by subsets I_0 and J_0, respectively. If A is commutative, then $IJ = \langle xy \mid x \in I_0, y \in J_0 \rangle$. Therefore (L5) holds. □

Example 7.5. Let A be a (not necessarily commutative) ring and suppose A satisfies the ascending chain condition on ideals. Then the lattice $L_{\mathrm{id}}(A)$ of ideals is an ideal lattice. Note that we have the following weak commutativity: $\sqrt{IJ} = \sqrt{JI}$ for any pair I, J of ideals.

7.3. Thick tensor ideals

Let $\mathcal{C} = (\mathcal{C}, \otimes, e)$ be an additive category with a *tensor product*. To be precise, we have an additive bifunctor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a natural isomorphism $(x \otimes y) \otimes z \xrightarrow{\sim} x \otimes (y \otimes z)$. In addition, we require the existence of a *tensor identity* e, that is, we have natural isomorphisms $x \otimes e \xrightarrow{\sim} x$ and $e \otimes y \xrightarrow{\sim} y$ satisfying the Pentagon Axiom and the Triangle Axiom.

Denote by C the set of isomorphism classes of objects in $\mathcal{C}$, and let $x + y = x \amalg y$ and $xy = x \otimes y$ for $x, y \in C$. Then C is a semi-ring, and we shall identify $\mathcal{C}$ and C whenever it is convenient. A *tensor ideal* of $\mathcal{C}$ is a full additive subcategory $\mathcal{D}$ such that for all $x, y \in \mathcal{C}$, we have $x \otimes y \in \mathcal{D}$ if $x \in \mathcal{D}$ or $y \in \mathcal{D}$. Note that the tensor ideals in $\mathcal{C}$ are precisely the ideals of the semi-ring C. We denote by $L_{\mathrm{id}}(\mathcal{C})$ the lattice of tensor ideals of $\mathcal{C}$ and define the multiplication of tensor ideals as in $L_{\mathrm{id}}(C)$.

Now suppose that there exists some additional exact or triangulated structure on $\mathcal{C}$. A full subcategory of $\mathcal{C}$ is called *thick* if it is 'compatible' with this additional structure; see below. We view the thick tensor ideals as a subset of $L_{\mathrm{id}}(\mathcal{C})$ and denote it by $L_{\mathrm{thick}}(\mathcal{C})$. We say that an ideal $\mathcal{D} \in L_{\mathrm{thick}}(\mathcal{C})$ is *generated* by a class $\mathcal{D}_0$ of objects and we write

$$\mathcal{D} = \langle \mathcal{D}_0 \rangle$$

if $\mathcal{D}$ is the smallest thick tensor ideal containing $\mathcal{D}_0$. The product of $\mathcal{D}_1, \mathcal{D}_2$ in $L_{\mathrm{thick}}(\mathcal{C})$ is by definition $\langle \mathcal{D}_1 \mathcal{D}_2 \rangle$ where $\mathcal{D}_1 \mathcal{D}_2$ is computed in $L_{\mathrm{id}}(\mathcal{C})$.

7.4. Abelian and triangulated tensor categories

Let $\mathcal{C}$ be an abelian category, or more generally, an exact category in the sense of Quillen. A full subcategory $\mathcal{D}$ is called *thick* if for every exact sequence $0 \to x' \to x \to x'' \to 0$ in $\mathcal{C}$, we have $x \in \mathcal{D}$ if and only if $x', x'' \in \mathcal{D}$. Now suppose that there is a tensor product $\otimes$ defined on $\mathcal{C}$.

Proposition 7.6. *Let $\mathcal{C}$ be an abelian category, or more generally an exact category, with a tensor product which is exact in each variable. Suppose that either the tensor product is commutative, or that there exists an object $c \in \mathcal{C}$ such that there is no proper thick subcategory of $\mathcal{C}$ containing c. Then the thick tensor ideals in $\mathcal{C}$ form an ideal lattice. Moreover, an ideal is compact if and only if it is generated by a single object.*

Proof. It is straightforward to check the conditions (L∧) and (L∨) for the subset of thick tensor ideals $L_{\mathrm{thick}}(\mathcal{C})$ in $L_{\mathrm{id}}(\mathcal{C})$. To verify (Lπ), observe that for a tensor ideal $\mathcal{D} \in L_{\text{id}}(\mathcal{C})$, the thick subcategory generated by $\mathcal{D}$ equals $\pi(\mathcal{D})$. Here we use the exactness of the tensor product. We deduce from Propositions 7.3 and 7.4 that $L_{\text{thick}}(\mathcal{C})$ is an ideal lattice. Note that a finitely generated ideal $\langle x_1, \ldots, x_n \rangle$ is generated by $x_1 \amalg \ldots \amalg x_n$. Thus compact and cyclic ideals coincide. Finally, if $\otimes$ is not commutative, then (L5) follows from the identity $\langle x \rangle \langle y \rangle = \langle x \otimes c \otimes y \rangle$, assuming that c generates $\mathcal{C}$. □

Now let $\mathcal{C}$ be a triangulated category. A full subcategory $\mathcal{D}$ is called *thick* if $\mathcal{D}$ is a triangulated subcategory and for each $x \in \mathcal{D}$ a decomposition $x = x_1 \amalg x_2$ implies $x_1, x_2 \in \mathcal{D}$. Now suppose that there is a tensor product $\otimes$ defined on $\mathcal{C}$.

Proposition 7.7. *Let $\mathcal{C}$ be a triangulated category with a tensor product which is exact in each variable. Suppose that either the tensor product is commutative, or that there exists an object $c \in \mathcal{C}$ such that there is no proper thick subcategory of $\mathcal{C}$ containing c. Then the thick tensor ideals in $\mathcal{C}$ form an ideal lattice. Moreover, an ideal is compact if and only if it is generated by a single object.*

Proof. The proof is the same as that of Proposition 7.6. □

Remark 7.8. Let $(\mathcal{C}, \otimes, e)$ be an abelian or triangulated tensor category with exact tensor product and tensor identity e. Let $\mathcal{E}$ be the thick subcategory generated by e. Then we have $x \otimes y \in \mathcal{E}$ for all $x, y \in \mathcal{E}$ and therefore $(\mathcal{E}, \otimes, e)$ is a category satisfying the assumptions from Propositions 7.6 or 7.7.

7.5. Support data

Let $\mathcal{C}$ be an abelian or triangulated tensor category. We assume from now on that the lattice of thick tensor ideals of $\mathcal{C}$ is an ideal lattice, for instance by imposing the assumptions from Propositions 7.6 or 7.7. We write

$$\operatorname{Spec} \mathcal{C} = \operatorname{Spec}^* L_{\text{thick}}(\mathcal{C})$$

for the spectrum of prime ideals. Note that we keep the 'opposite' of the Zariski topology in view of our applications. This practice is in accordance with Balmer's notion of a spectrum in [**2**]. The compact ideals in $L_{\text{thick}}(\mathcal{C})$ are precisely the ideals $\langle x \rangle$ generated by a single object $x \in \mathcal{C}$. We write

$$\operatorname{supp}(x) = \operatorname{supp}(\langle x \rangle) = \{\mathcal{P} \in \operatorname{Spec} \mathcal{C} \mid x \notin \mathcal{P}\} \quad \text{for} \quad x \in \mathcal{C}$$

and call this subset of $\operatorname{Spec} \mathcal{C}$ the *support* of x. It is convenient to work with support data defined on objects of $\mathcal{C}$ instead of support data defined on ideals of $\mathcal{C}$. This motivates the following definition from [**2**].

Definition 7.9. A *support datum* on $\mathcal{C}$ is a pair (X, τ) where X is a topological space and τ is a map which assigns to each object $x \in \mathcal{C}$ a closed subset $\tau(x) \subseteq X$, such that for all $x, y \in \mathcal{C}$

$$\tau(x) = \bigcup_{x' \in \langle x \rangle} \tau(x'), \quad \tau(x \amalg y) = \tau(x) \cup \tau(y),$$
$$\tau(e) = X \quad \text{and} \quad \tau(x \otimes y) = \tau(x) \cap \tau(y).$$

A *morphism* $f \colon (X, \tau) \to (X', \tau')$ of support data is a continuous map $f \colon X \to X'$ such that $\tau(x) = f^{-1}(\tau'(x))$ for all $x \in \mathcal{C}$.

Lemma 7.10. *Let $\mathcal{C}$ be an abelian or triangulated tensor category satisfying the assumptions from* Propositions 7.6 or 7.7.

(1) *If (X, τ) is a support datum on $\mathcal{C}$, then $\sigma(\langle x\rangle) = \tau(x)$ defines a support datum on the lattice of thick tensor ideals of $\mathcal{C}$.*

(2) *If (X, σ) is a support datum on the lattice of thick tensor ideals of $\mathcal{C}$, then $\tau(x) = \sigma(\langle x\rangle)$ defines a support datum on $\mathcal{C}$.*

Proof. We start with a support datum (X, τ) on $\mathcal{C}$. The map σ on compact ideals of $\mathcal{C}$ is well-defined because of the condition $\tau(x) = \bigcup_{x'\in\langle x\rangle} \tau(x')$. Now compute for compact ideals $\langle x\rangle$ and $\langle y\rangle$

$$\sigma(\langle x\rangle) \cup \sigma(\langle y\rangle) = \tau(x) \cup \tau(y) = \tau(x \amalg y) = \sigma(\langle x \amalg y\rangle) = \sigma(\langle x\rangle \vee \langle y\rangle),$$

and if the tensor product is commutative

$$\sigma(\langle x\rangle) \cap \sigma(\langle y\rangle) = \tau(x) \cap \tau(y) = \tau(x \otimes y) = \sigma(\langle x \otimes y\rangle) = \sigma(\langle x\rangle\langle y\rangle),$$

using that $\langle x\rangle\langle y\rangle = \langle x \otimes y\rangle$. In the non-commutative case, we have $\langle x\rangle\langle y\rangle = \langle x \otimes c \otimes y\rangle$ for some $c \in \mathcal{C}$ with $\langle c\rangle = \mathcal{C}$. Thus

$$\begin{aligned}\sigma(\langle x\rangle) \cap \sigma(\langle y\rangle) = \tau(x) \cap \tau(y) &= \tau(x) \cap \tau(c) \cap \tau(y)\\ &= \tau(x \otimes c \otimes y) = \sigma(\langle x \otimes c \otimes y\rangle) = \sigma(\langle x\rangle\langle y\rangle).\end{aligned}$$

Finally, we have

$$\sigma(1) = \sigma(\langle e\rangle) = \tau(e) = X.$$

We conclude that (X, σ) is a support datum on the lattice of thick tensor ideals of $\mathcal{C}$. The proof of the converse is analogous. □

From now on we do not distinguish between support data on the lattice of thick tensor ideals $L_{\text{thick}}(\mathcal{C})$ and support data on $\mathcal{C}$. We leave it to the interested reader to reformulate our general results about ideal lattices for the lattice $L_{\text{thick}}(\mathcal{C})$ and its spectrum $\operatorname{Spec}\mathcal{C}$.

8. The structure sheaf of a tensor category

Let $\mathcal{C}$ be an abelian or triangulated tensor category with tensor identity e. Following [**2**], we define a structure sheaf on $\operatorname{Spec}\mathcal{C}$ as follows. For an open subset $U \subseteq \operatorname{Spec}\mathcal{C}$, let

$$\mathcal{C}_U = \{x \in \mathcal{C} \mid \operatorname{supp}(x) \cap U = \emptyset\}$$

and observe that $\mathcal{C}_U$ is a thick tensor ideal. We denote by $\mathcal{C}/\mathcal{C}_U$ the corresponding quotient category and observe that e is the tensor identity of this category. Thus one obtains a presheaf of rings on $\operatorname{Spec}\mathcal{C}$ by

$$U \mapsto \operatorname{End}_{\mathcal{C}/\mathcal{C}_U}(e).$$

If $V \subseteq U$ are open subsets, then the restriction map $\operatorname{End}_{\mathcal{C}/\mathcal{C}_U}(e) \to \operatorname{End}_{\mathcal{C}/\mathcal{C}_V}(e)$ is induced by the quotient functor $\mathcal{C}/\mathcal{C}_U \to \mathcal{C}/\mathcal{C}_V$. The sheafification is called the *structure sheaf* of $\mathcal{C}$ and is denoted by $\mathcal{O}_{\mathcal{C}}$. Note that the endomorphism ring of a tensor identity is commutative, if the tensor product is commutative, or if $\mathcal{C}$ is a suspended tensor category; see for instance [**20**, Thm. 1.7]. Next observe that

$$\mathcal{O}_{\mathcal{C},\mathcal{P}} \cong \operatorname{End}_{\mathcal{C}/\mathcal{P}}(e) \quad \text{for each} \quad \mathcal{P} \in \operatorname{Spec}\mathcal{C}.$$

This is an immediate consequence of the following lemma.

Lemma 8.1. *Let $\mathcal{C}$ be an abelian or triangulated tensor category and $\mathcal{P} \in \operatorname{Spec}\mathcal{C}$. Then*

$$\varinjlim_{\mathcal{P}\in U} \operatorname{Hom}_{\mathcal{C}/\mathcal{C}_U}(x,y) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}/\mathcal{P}}(x,y) \quad \textit{for all} \quad x,y \in \mathcal{C},$$

where U runs through all (quasi-compact) open subsets containing $\mathcal{P}$.

Proof. Use that $\mathcal{P} = \bigcup_{\mathcal{P}\in U} \mathcal{C}_U$. □

Now we discuss briefly the functoriality of the spectrum.

Lemma 8.2. *Let $F\colon \mathcal{C} \to \mathcal{C}'$ be an exact tensor functor.*

(1) F induces a unique continuous map $f\colon \operatorname{Spec}\mathcal{C}' \to \operatorname{Spec}\mathcal{C}$ such that

$$\operatorname{supp}(Fx) = f^{-1}(\operatorname{supp}(x)) \quad \textit{for} \quad x \in \mathcal{C}.$$

The map sends $\mathcal{P} \in \operatorname{Spec}\mathcal{C}'$ to $F^{-1}(\mathcal{P})$.

(2) F induces a morphism of ringed spaces

$$(f, f^\sharp)\colon (\operatorname{Spec}\mathcal{C}', \mathcal{O}_{\mathcal{C}'}) \to (\operatorname{Spec}\mathcal{C}, \mathcal{O}_{\mathcal{C}}).$$

Proof. (1) The map sending $x \in \mathcal{C}$ to $\operatorname{supp}(Fx)$ is a support datum on $\mathcal{C}$. Now apply Theorem 5.3 to obtain a continuous map $\operatorname{Spec}\mathcal{C}' \to \operatorname{Spec}\mathcal{C}$.

(2) Let $U \subseteq \operatorname{Spec}\mathcal{C}$ be open. Then F maps $\mathcal{C}_U$ to $\mathcal{C}_{f^{-1}U}$ and induces a functor $\mathcal{C}/\mathcal{C}_U \to \mathcal{C}'/\mathcal{C}'_{f^{-1}U}$. This functor induces a homomorphism

$$\operatorname{End}_{\mathcal{C}/\mathcal{C}_U}(e) \to \operatorname{End}_{\mathcal{C}'/\mathcal{C}'_{f^{-1}U}}(e').$$

Thus we obtain a morphism $f^\sharp\colon \mathcal{O}_{\mathcal{C}} \to f_*\mathcal{O}_{\mathcal{C}'}$. □

9. Applications to schemes

9.1. Coherent sheaves on a scheme

We consider a noetherian scheme X and reconstruct it from the abelian tensor category $\operatorname{coh} X$ of coherent $\mathcal{O}_X$-modules. This is based on the following well-known classification of all thick subcategories of $\operatorname{coh} X$. Given $x \in \operatorname{coh} X$, we write

$$\operatorname{supp}_X(x) = \{P \in X \mid x_P \neq 0\}.$$

Proposition 9.1. *Let X be a noetherian scheme. The assignments*

$$\operatorname{coh} X \supseteq \mathcal{D} \mapsto \bigcup_{x\in\mathcal{D}} \operatorname{supp}_X(x) \quad \textit{and} \quad X \supseteq Y \mapsto \{x \in \operatorname{coh} X \mid \operatorname{supp}_X(x) \subseteq Y\}$$

induce bijections between

(1) the set of all thick subcategories of $\operatorname{coh} X$*, and*

(2) the set of all subsets $Y \subseteq X$ *of the form* $Y = \bigcup_{i\in\Omega} Y_i$ *with quasi-compact open complement* $X \setminus Y_i$ *for all* $i \in \Omega$*.*

Proof. See [**8**, Prop. VI.4]. □

Note that every open subset of a noetherian space is quasi-compact. Nonetheless, the above formulation is appropriate because it generalizes to schemes which are not necessarily noetherian; see for instance [**11**].

The abelian category $\operatorname{coh} X$ carries a commutative tensor product $\otimes_{\mathcal{O}_X}$, and we deduce from the classification of thick subcategories the following properties.

Proposition 9.2. *Let* X *be a noetherian scheme and* $\mathcal{C} = \operatorname{coh} X$*. Then every thick subcategory of* $\mathcal{C}$ *is a tensor ideal and the thick tensor ideals of* $\mathcal{C}$ *form an ideal lattice.*

Proof. We apply Proposition 9.1. The formula

$$\operatorname{supp}_X(x \otimes_{\mathcal{O}_X} y) = \operatorname{supp}_X(x) \cap \operatorname{supp}_X(y)$$

shows that every thick subcategory is a tensor ideal. The space X is spectral because the scheme is noetherian. Thus X^* is spectral and $L_{\text{open}}(X^*)$ is an ideal lattice, by Proposition 3.7. We have an isomorphism $L_{\text{open}}(X^*) \cong L_{\text{thick}}(\mathcal{C})$, and therefore $L_{\text{thick}}(\mathcal{C})$ is an ideal lattice. □

It would be interesting to have a direct proof (not involving a classification) that the thick tensor ideals of $\operatorname{coh} X$ form an ideal lattice. Note that Proposition 7.6 does not apply because the tensor product $\otimes_{\mathcal{O}_X}$ is exact only in trivial cases.

Theorem 9.3. *Let* X *be a noetherian scheme and consider the abelian tensor category* $\operatorname{coh} X$ *of coherent* $\mathcal{O}_X$*-modules. The pair* $(X, \operatorname{supp}_X)$ *is a classifying support datum on* $\operatorname{coh} X$ *and there is an induced isomorphism*

$$(X, \mathcal{O}_X) \xrightarrow{\sim} (\operatorname{Spec} \operatorname{coh} X, \mathcal{O}_{\operatorname{coh} X})$$

of ringed spaces.

Proof. Let $\mathcal{C} = \operatorname{coh} X$. It follows from well-known properties of the support $\operatorname{supp}_X(x)$ that $(X, \operatorname{supp}_X)$ is a support datum on $\mathcal{C}$. Thus we obtain a continuous map $f \colon X \to \operatorname{Spec} \mathcal{C}$ satisfying $\operatorname{supp}_X(x) = f^{-1}(\operatorname{supp}(x))$ for each $x \in \mathcal{C}$, by Theorem 5.3. The classification of thick subcategories of $\mathcal{C}$ from Proposition 9.1 shows that the support datum $(X, \operatorname{supp}_X)$ is classifying. Here we use in addition that the underlying space of X is spectral. It follows from Corollary 6.2 that f is a homeomorphism.

It remains to construct an isomorphism $f^\sharp \colon \mathcal{O}_{\mathcal{C}} \to f_*\mathcal{O}_X$. Observe that for each open $U \subseteq \operatorname{Spec} \mathcal{C}$, the restriction $\operatorname{coh} X \to \operatorname{coh} f^{-1}U$ induces an equivalence $\mathcal{C}/\mathcal{C}_U \xrightarrow{\sim} \operatorname{coh} f^{-1}U$; see [**8**, Prop. VI.2]. Thus we obtain for $e = \mathcal{O}_X$ an isomorphism

$$\mathcal{O}_{\mathcal{C}}(U) = \operatorname{End}_{\mathcal{C}/\mathcal{C}_U}(e) \xrightarrow{\sim} \mathcal{O}_X(f^{-1}U)$$

which yields the isomorphism $f^\sharp \colon \mathcal{O}_{\mathcal{C}} \xrightarrow{\sim} f_*\mathcal{O}_X$. □

9.2. Perfect complexes on a scheme

We consider a quasi-compact and quasi-separated scheme X and its triangulated tensor category $\mathbf{D}^{\mathrm{per}}(X)$ of perfect complexes with tensor product $\otimes^{\mathbf{L}}_{\mathcal{O}_X}$; see [**21**, Sec. 2] for a concise discussion of these concepts. For instance, every noetherian scheme is quasi-compact and quasi-separated. Let us recall Thomason's classification of thick tensor ideals. Given $x \in \mathbf{D}^{\mathrm{per}}(X)$, we write

$$\operatorname{supp}_X(x) = \{P \in X \mid x_P \neq 0\}.$$

Proposition 9.4. *Let X be a quasi-compact and quasi-separated scheme. The assignments*

$$\mathbf{D}^{\mathrm{per}}(X) \supseteq \mathcal{D} \mapsto \bigcup_{x \in \mathcal{D}} \operatorname{supp}_X(x) \quad \textit{and} \quad X \supseteq Y \mapsto \{x \in \mathbf{D}^{\mathrm{per}}(X) \mid \operatorname{supp}_X(x) \subseteq Y\}$$

induce bijections between

(1) *the set of all thick tensor ideals of $\mathbf{D}^{\mathrm{per}}(X)$, and*

(2) *the set of all subsets $Y \subseteq X$ of the form $Y = \bigcup_{i \in \Omega} Y_i$ with quasi-compact open complement $X \setminus Y_i$ for all $i \in \Omega$.*

Proof. See [**22**, Thm. 4.1]. □

We observe that the thick tensor ideals of $\mathbf{D}^{\mathrm{per}}(X)$ form an ideal lattice by Proposition 7.7. The following result shows that a quasi-compact and quasi-separated scheme can be reconstructed from the triangulated tensor category of perfect complexes; it is a slight generalization of [**2**, Thm. 6.3] which assumes the scheme to be topologically noetherian.

Theorem 9.5. *Let X be a quasi-compact and quasi-separated scheme and consider the triangulated tensor category $\mathbf{D}^{\mathrm{per}}(X)$ of perfect complexes on X. The pair $(X, \operatorname{supp}_X)$ is a classifying support datum on $\mathbf{D}^{\mathrm{per}}(X)$ and there is an induced isomorphism*

$$(X, \mathcal{O}_X) \xrightarrow{\sim} (\operatorname{Spec} \mathbf{D}^{\mathrm{per}}(X), \mathcal{O}_{\mathbf{D}^{\mathrm{per}}(X)})$$

of ringed spaces.

Proof. The proof is essentially the same as that of Theorem 9.3, with $\mathcal{C} = \operatorname{coh} X$ replaced by $\mathcal{C} = \mathbf{D}^{\mathrm{per}}(X)$. Note that the assumption on X implies that the underlying space is spectral. We use the classification of thick tensor ideals from Proposition 9.4. For the equivalence $\mathcal{C}/\mathcal{C}_U \xrightarrow{\sim} \mathbf{D}^{\mathrm{per}}(f^{-1}U)$, up to direct factors, when $U \subseteq \operatorname{Spec} \mathcal{C}$ is quasi-compact open, we refer to [**21**, Sec. 5]. □

10. A projective scheme

There are a number of interesting examples of triangulated tensor categories $\mathcal{C}$ where $(\operatorname{Spec} \mathcal{C}, \mathcal{O}_{\mathcal{C}})$ is actually a projective scheme. Here, we present a general criterion which explains those examples. We fix a triangulated tensor category $\mathcal{C}$ with tensor identity e. As before, we assume that the tensor product is exact in each variable. This implies in particular that the tensor category is suspended in the sense of [**20**], that is, the tensor product and the suspension are compatible. Let us start with some preparation. For $x, y \in \mathcal{C}$, we write

$$\operatorname{Hom}^*_{\mathcal{C}}(x, y) = \coprod_{n \in \mathbb{Z}} \operatorname{Hom}_{\mathcal{C}}(x, \Sigma^n y)$$

where Σ denotes the suspension of $\mathcal{C}$. The graded endomorphism ring $\operatorname{End}^*_{\mathcal{C}}(x)$ acts on $\operatorname{Hom}^*_{\mathcal{C}}(x, y)$ from the right and $\operatorname{End}^*_{\mathcal{C}}(y)$ acts from the left. We use the graded ring homomorphism

$$\phi_x \colon \operatorname{End}^*_{\mathcal{C}}(e) \longrightarrow \operatorname{End}^*_{\mathcal{C}}(x), \quad \alpha \mapsto \alpha \otimes x.$$

Note that $\operatorname{End}^*_{\mathcal{C}}(e)$ acts on $\operatorname{Hom}^*_{\mathcal{C}}(x, y)$ from the right via ϕ_x and from the left via ϕ_y, with

$$\alpha \cdot \beta = (-1)^{|\alpha||\beta|} \beta \cdot \alpha$$

for homogeneous elements $\alpha \in \operatorname{End}^*_{\mathcal{C}}(e)$ and $\beta \in \operatorname{Hom}^*_{\mathcal{C}}(x, y)$ with $|\gamma|$ denoting the degree of a homogeneous element γ. This follows from arguments similar to those in [**20**]. In particular, $\operatorname{End}^*_{\mathcal{C}}(e)$ is graded commutative.

10.1. Cohomological localization

We need a basic result about the localization of triangulated categories. Under appropriate assumptions, we show that first taking cohomology and then localizing is the same as first localizing and then taking cohomology. For a homogeneous element $\sigma \colon \Sigma^n x \to x$ in $\operatorname{End}^*_{\mathcal{C}}(x)$, we denote by x/σ its cofiber in $\mathcal{C}$.

Proposition 10.1. *Let $\mathcal{C}$ be a triangulated category and $\mathcal{D} \subseteq \mathcal{C}$ a full triangulated subcategory. Let $c \in \mathcal{C}$ be an object and $\phi \colon H \to \operatorname{End}^*_{\mathcal{C}}(c)$ be a graded ring homomorphism such that H is graded commutative. Fix a subset $S \subseteq H$ of homogeneous elements and consider for each $x \in \mathcal{C}$ the following commutative diagram of canonical homomorphisms in the category of graded H-modules.*

$$\begin{array}{ccc} \operatorname{Hom}^*_{\mathcal{C}}(c, x) & \xrightarrow{\ \mu\ } & S^{-1} \operatorname{Hom}^*_{\mathcal{C}}(c, x) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle S^{-1}\pi} \\ \operatorname{Hom}^*_{\mathcal{C}/\mathcal{D}}(c, x) & \xrightarrow{\ \nu\ } & S^{-1} \operatorname{Hom}^*_{\mathcal{C}/\mathcal{D}}(c, x) \end{array}$$

(1) If $\{c/\phi(\sigma) \mid \sigma \in S\} \subseteq \mathcal{D}$, then ν is an isomorphism.

(2) If $\mathcal{D} \subseteq \{x \in \mathcal{C} \mid S^{-1} \operatorname{Hom}^*_{\mathcal{C}}(c, x) = 0\}$, then $S^{-1}\pi$ is an isomorphism.

Proof. We assume that H is graded commutative because we need in (2) that localization of graded H-modules with respect to S is an exact functor.

(1) Assume $\{c/\phi(\sigma) \mid \sigma \in S\} \subseteq \mathcal{D}$. Let $Q \colon \mathcal{C} \to \mathcal{C}/\mathcal{D}$ denote the quotient functor. Then H acts on $\operatorname{Hom}^*_{\mathcal{C}/\mathcal{D}}(c, x)$ via Q, and each $\sigma \in S$ acts invertibly since $Q\phi(\sigma)$ is invertible. Thus the canonical map

$$\operatorname{Hom}^*_{\mathcal{C}/\mathcal{D}}(c, x) \to S^{-1} \operatorname{Hom}^*_{\mathcal{C}/\mathcal{D}}(c, x)$$

is invertible.

(2) Assume $\mathcal{D} \subseteq \{x \in \mathcal{C} \mid S^{-1} \operatorname{Hom}^*_{\mathcal{C}}(c, x) = 0\}$. We embed $\mathcal{C}$ into the category $\operatorname{Mod} \mathcal{C}$ of additive functors $\mathcal{C}^{\text{op}} \to \text{Ab}$ via the Yoneda functor

$$\mathcal{C} \longrightarrow \operatorname{Mod} \mathcal{C}, \quad x \mapsto \operatorname{Hom}_{\mathcal{C}}(-, x).$$

Note that every cohomological functor $F : \mathcal{C} \to \mathcal{A}$ into an abelian Grothendieck category $\mathcal{A}$ extends uniquely to an exact and coproduct preserving functor $\bar{F} \colon \operatorname{Mod} \mathcal{C} \to \mathcal{A}$; see [**15**, Lem. 2.2]. Now take the composition

$$\mathcal{C} \xrightarrow{\operatorname{Hom}^*_{\mathcal{C}}(c,-)} \operatorname{Mod} H \xrightarrow{S^{-1}} \operatorname{Mod} H$$

which annihilates $\mathcal{D}$ by our assumption. We obtain the following commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F = \operatorname{Hom}^*_{\mathcal{C}}(c,-)} & \operatorname{Mod} H \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle S^{-1}} \\ \mathcal{C}/\mathcal{D} & \xrightarrow{G} & \operatorname{Mod} H \end{array}$$

which can be extended to the following commutative diagram of exact and coproduct preserving functors.

$$\begin{array}{ccc} \operatorname{Mod} \mathcal{C} & \xrightarrow{\bar{F}} & \operatorname{Mod} H \\ {\scriptstyle Q^*}\downarrow & & \downarrow{\scriptstyle S^{-1}} \\ \operatorname{Mod} \mathcal{C}/\mathcal{D} & \xrightarrow{\bar{G}} & \operatorname{Mod} H \end{array}$$

Note that

$$\bar{F}(M) \cong \coprod_n M(\Sigma^{-n} c) \cong \coprod_n \operatorname{Hom}_{\mathcal{C}}(\operatorname{Hom}_{\mathcal{C}}(-, \Sigma^{-n} c), M)$$

for M in $\operatorname{Mod} \mathcal{C}$. The first isomorphism is clear for each representable functor $M = \operatorname{Hom}_{\mathcal{C}}(-, x)$. Then observe that every $M \in \operatorname{Mod} \mathcal{C}$ is a colimit of representable functors and both functors preserve colimits. The second isomorphism follows from Yoneda's lemma. The functor Q^* has a right adjoint

$$Q_* \colon \operatorname{Mod} \mathcal{C}/\mathcal{D} \longrightarrow \operatorname{Mod} \mathcal{C}, \quad M \mapsto M \circ Q,$$

and the adjunction morphism $Q^* Q_* M \to M$ is an isomorphism for all $M \in \operatorname{Mod} \mathcal{C}/\mathcal{D}$, since Q is a quotient functor. Now consider for $x \in \mathcal{C}$ the adjunction morphism

$$\eta_x \colon \operatorname{Hom}_{\mathcal{C}}(-, x) \to Q_* Q^* \operatorname{Hom}_{\mathcal{C}}(-, x).$$

First observe that $Q^* \eta_x$ is an isomorphism. On the other hand, $\bar{F} \eta_x$ equals π up to an isomorphism, since

$$\begin{aligned}\bar{F}(Q_*Q^*\operatorname{Hom}_{\mathcal{C}}(-,x)) &\cong \coprod_n \operatorname{Hom}_{\mathcal{C}}(\operatorname{Hom}_{\mathcal{C}}(-,\Sigma^{-n}c), Q_*Q^*\operatorname{Hom}_{\mathcal{C}}(-,x))\\ &\cong \coprod_n \operatorname{Hom}_{\mathcal{C}/\mathcal{D}}(Q^*\operatorname{Hom}_{\mathcal{C}}(-,\Sigma^{-n}c), Q^*\operatorname{Hom}_{\mathcal{C}}(-,x))\\ &\cong \coprod_n \operatorname{Hom}_{\mathcal{C}/\mathcal{D}}(\operatorname{Hom}_{\mathcal{C}/\mathcal{D}}(-,\Sigma^{-n}c), \operatorname{Hom}_{\mathcal{C}/\mathcal{D}}(-,x))\\ &\cong \coprod_n \operatorname{Hom}_{\mathcal{C}/\mathcal{D}}(\Sigma^{-n}c, x)\\ &= \operatorname{Hom}^*_{\mathcal{C}/\mathcal{D}}(c,x).\end{aligned}$$

Thus $S^{-1}\pi \cong S^{-1}(\bar{F}\eta_x) = \bar{G}(Q^*\eta_x)$ is an isomorphism, and this finishes the proof. □

We formulate an immediate consequence.

Corollary 10.2. *Let $\mathcal{C}$ be a triangulated category. Let $c \in \mathcal{C}$ be an object such that its graded endomorphism ring $\operatorname{End}^*_{\mathcal{C}}(c)$ is graded commutative and fix a homogeneous prime ideal $\mathfrak{p}$. Let*

$$\mathcal{C}_{\mathfrak{p}} = \{x \in \mathcal{C} \mid \operatorname{Hom}^*_{\mathcal{C}}(c,x)_{\mathfrak{p}} = 0\}.$$

Then we have a natural isomorphism $\operatorname{Hom}^*_{\mathcal{C}/\mathcal{C}_{\mathfrak{p}}}(c,x) \xrightarrow{\sim} \operatorname{Hom}^*_{\mathcal{C}}(c,x)_{\mathfrak{p}}$ for all $x \in \mathcal{C}$.

10.2. Cohomological support

We keep fixed a triangulated tensor category $\mathcal{C}$ with tensor identity e and suppose that $H = \operatorname{End}^*_{\mathcal{C}}(e)$ is concentrated in non-negative degrees. Let $\operatorname{Proj} H$ denote the set of homogeneous prime ideals of H which do not contain $H^+ = \coprod_{n>0} H^n$. This set is endowed with the Zariski topology and $\mathcal{O}_H$ denotes the corresponding structure sheaf. We define the *cohomological support* of an object $x \in \mathcal{C}$ as

$$\operatorname{supp}_H(x) = \{\mathfrak{p} \in \operatorname{Proj} H \mid \operatorname{End}^*_{\mathcal{C}}(x)_{\mathfrak{p}} \neq 0\},$$

where H acts on $\operatorname{End}^*_{\mathcal{C}}(x)$ via the canonical ring homomorphism $H \to \operatorname{End}^*_{\mathcal{C}}(x)$ taking an element α to $\alpha \otimes x$. It is useful to observe that for each $\mathfrak{p} \in \operatorname{Proj} H$

$$\operatorname{End}^*_{\mathcal{C}}(x)_{\mathfrak{p}} = 0 \quad\iff\quad \operatorname{Hom}^*_{\mathcal{C}}(c,x)_{\mathfrak{p}} = 0 \text{ for all } c \in \mathcal{C}. \qquad (10.1)$$

10.3. A projective scheme

We provide a criterion for $(\operatorname{Spec}\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ to be a projective scheme. We assume that the tensor product on $\mathcal{C}$ is commutative or that $\mathcal{C}$ is generated by a single object as a triangulated category. Thus the thick tensor ideals of $\mathcal{C}$ form an ideal lattice, by Proposition 7.7. The following elementary observation will be useful.

Lemma 10.3. *Let $\mathcal{C}$ be an abelian or triangulated tensor category and (X,σ) be a support datum on $\mathcal{C}$. Then $\mathcal{C}_0 = \{x \in \mathcal{C} \mid \sigma(x) = \emptyset\}$ is a thick tensor ideal and (X,σ) induces a support datum (X,σ') on the quotient $\mathcal{C}/\mathcal{C}_0$ such that $\sigma'(x) = \sigma(x)$ for all $x \in \mathcal{C}$.*

The following is the main result.

Theorem 10.4. *Let $\mathcal{C}$ be a triangulated tensor category with tensor identity e and suppose that $H = \operatorname{End}^*_{\mathcal{C}}(e)$ is concentrated in non-negative degrees. Define for each $x \in \mathcal{C}$*

$$\operatorname{supp}_H(x) = \{\mathfrak{p} \in \operatorname{Proj} H \mid \operatorname{End}^*_{\mathcal{C}}(x)_{\mathfrak{p}} \neq 0\}$$

and suppose that

$$\operatorname{supp}_H(x \otimes y) = \operatorname{supp}_H(x) \cap \operatorname{supp}_H(y) \quad \text{for all} \quad x, y \in \mathcal{C}.$$

Then $\mathcal{C}_0 = \{x \in \mathcal{C} \mid \operatorname{supp}_H(x) = \emptyset\}$ is a thick tensor ideal of $\mathcal{C}$ and $(\operatorname{Proj} H, \operatorname{supp}_H)$ induces a support datum on the quotient $\bar{\mathcal{C}} = \mathcal{C}/\mathcal{C}_0$. We obtain an induced morphism

$$f\colon (\operatorname{Proj} H, \mathcal{O}_H) \to (\operatorname{Spec} \bar{\mathcal{C}}, \mathcal{O}_{\bar{\mathcal{C}}})$$

of ringed spaces which induces a ring isomorphism

$$\mathcal{O}_{\bar{\mathcal{C}}, f(\mathfrak{p})} \xrightarrow{\sim} \mathcal{O}_{H,\mathfrak{p}}$$

for all $\mathfrak{p} \in \operatorname{Proj} H$. In particular, f is an isomorphism if and only if the support datum $(\operatorname{Proj} H, \operatorname{supp}_H)$ on $\bar{\mathcal{C}}$ is classifying.

Proof. The condition

$$\operatorname{supp}_H(x \otimes y) = \operatorname{supp}_H(x) \cap \operatorname{supp}_H(y)$$

implies that $(\operatorname{Proj} H, \operatorname{supp}_H)$ is a support datum on $\mathcal{C}$, and therefore also on $\bar{\mathcal{C}}$ by Lemma 10.3. Thus we obtain a continuous map $f\colon \operatorname{Proj} H \to \operatorname{Spec} \bar{\mathcal{C}}$ satisfying

$$\operatorname{supp}_H(x) = f^{-1}(\operatorname{supp}(x))$$

for all $x \in \bar{\mathcal{C}}$, by Theorem 5.3.

We need to construct a morphism of sheaves $f^\sharp\colon \mathcal{O}_{\bar{\mathcal{C}}} \to f_*\mathcal{O}_H$. First observe that for each $x \in \bar{\mathcal{C}}$, the ring H acts on $\operatorname{Hom}^*_{\bar{\mathcal{C}}}(e,x)$ via the quotient functor $\mathcal{C} \to \bar{\mathcal{C}}$. In particular,

$$\operatorname{Hom}^*_{\mathcal{C}}(e,x)_{\mathfrak{p}} \xrightarrow{\sim} \operatorname{Hom}^*_{\bar{\mathcal{C}}}(e,x)_{\mathfrak{p}} \quad \text{for} \quad \mathfrak{p} \in \operatorname{Proj} H$$

by Proposition 10.1. Now fix an open subset $U \subseteq \operatorname{Spec} \bar{\mathcal{C}}$ and consider the composition of the functors

$$F\colon \bar{\mathcal{C}} \xrightarrow{\operatorname{Hom}^*_{\bar{\mathcal{C}}}(e,-)} \operatorname{Mod} H \xrightarrow{(\tilde{-})} \operatorname{Qcoh} \operatorname{Proj} H \xrightarrow{(-)|_{f^{-1}(U)}} \operatorname{Qcoh} f^{-1}(U).$$

Here, we denote for any H-module M by $\tilde{M}$ its associated sheaf. Note that the stalk of $\tilde{M}$ at a homogeneous prime $\mathfrak{p}$ equals the degree zero part $M_{(\mathfrak{p})}$ of the localized module $M_{\mathfrak{p}}$. We claim that F annihilates $\bar{\mathcal{C}}_U$. In fact, $x \in \bar{\mathcal{C}}_U$ implies $f^{-1}(\operatorname{supp}(x)) \cap f^{-1}(U) = \emptyset$ and therefore $\operatorname{supp}_H(x) \cap f^{-1}(U) = \emptyset$. Thus $\operatorname{Hom}^*_{\bar{\mathcal{C}}}(e,x)_{(\mathfrak{p})} = 0$ for all $\mathfrak{p} \in f^{-1}(U)$ and therefore $Fx = 0$. It follows that F factors through $\bar{\mathcal{C}}/\bar{\mathcal{C}}_U$ and induces a map $\operatorname{End}_{\bar{\mathcal{C}}/\bar{\mathcal{C}}_U}(e) \to \mathcal{O}_H(f^{-1}(U))$ which extends to a map

$$\mathcal{O}_{\bar{\mathcal{C}}}(U) \to \mathcal{O}_H(f^{-1}(U)).$$

This yields the morphism of sheaves $f^\sharp\colon \mathcal{O}_{\bar{\mathcal{C}}} \to f_*\mathcal{O}_H$.

Now fix a point $\mathfrak{p} \in \operatorname{Proj} H$. Then $f^\sharp$ induces a map $f^\sharp_\mathfrak{p} \colon \mathcal{O}_{\bar{\mathcal{C}}, f(\mathfrak{p})} \to \mathcal{O}_{H,\mathfrak{p}}$. We have an isomorphism $\mathcal{O}_{\bar{\mathcal{C}}, f(\mathfrak{p})} \cong \operatorname{End}_{\bar{\mathcal{C}}/f(\mathfrak{p})}(e)$ by Lemma 8.1. Next observe that

$$f(\mathfrak{p}) = \{x \in \bar{\mathcal{C}} \mid \operatorname{End}^*_{\mathcal{C}}(x)_\mathfrak{p} = 0\}.$$

We have

$$\{e/\sigma \in \bar{\mathcal{C}} \mid \sigma \in H \setminus \mathfrak{p}\} \subseteq f(\mathfrak{p}) \subseteq \{x \in \bar{\mathcal{C}} \mid \operatorname{Hom}^*_{\bar{\mathcal{C}}}(e, x)_\mathfrak{p} = 0\}.$$

This follows from (10.1), and we obtain a second isomorphism

$$\operatorname{End}_{\bar{\mathcal{C}}/f(\mathfrak{p})}(e) \cong \operatorname{End}^*_{\bar{\mathcal{C}}}(e)_{(\mathfrak{p})} \cong \operatorname{End}^*_{\mathcal{C}}(e)_{(\mathfrak{p})} = \mathcal{O}_{H,\mathfrak{p}}$$

from Proposition 10.1. We conclude that $f^\sharp_\mathfrak{p}$ is an isomorphism. It follows that f is an isomorphism of ringed spaces if and only if the map $\operatorname{Proj} H \to \operatorname{Spec} \bar{\mathcal{C}}$ is a homeomorphism. This last condition is satisfied if and only if the support datum $(\operatorname{Proj} H, \operatorname{supp}_H)$ is classifying, by Corollary 6.2. □

Note that our Theorem 10.4 gives a partial answer to Balmer's question when $(\operatorname{Spec} \mathcal{C}, \mathcal{O}_\mathcal{C})$ is a scheme [**2**, Rem. 6.4]. The result is best illustrated by the following example from representation theory; see [**7**, Thm. 7.3] for an alternative discussion.

Example 10.5. Let k be a field and let $A = kG$ be the group algebra of a finite group G or more generally a finite group scheme. We consider the category $\operatorname{mod} A$ of finite dimensional A-modules and its bounded derived category $\mathbf{D}^b(\operatorname{mod} A)$. The tensor product $\otimes_k$ on $\operatorname{mod} A$ induces a tensor product on $\mathbf{D}^b(\operatorname{mod} A)$ which is exact in each variable. The trivial representation k is the tensor identity and its graded endomorphism ring equals the group cohomology ring

$$H = H^*(G, k) = \operatorname{Ext}^*_A(k, k).$$

Note that for $x \in \mathbf{D}^b(\operatorname{mod} A)$, we have $\operatorname{supp}_H(x) = \emptyset$ if and only if x belongs to the thick tensor ideal $\mathbf{D}^{\mathrm{per}}(A)$ of perfect complexes. The composite

$$\operatorname{mod} A \xrightarrow{\text{inc}} \mathbf{D}^b(\operatorname{mod} A) \xrightarrow{\text{can}} \mathbf{D}^b(\operatorname{mod} A)/\mathbf{D}^{\mathrm{per}}(A)$$

induces an equivalence $\underline{\operatorname{mod}}\, A \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod} A)/\mathbf{D}^{\mathrm{per}}(A)$, where $\underline{\operatorname{mod}}\, A$ denotes the stable module category of A; see for instance [**18**, Thm 2.1]. The thick tensor ideals of $\underline{\operatorname{mod}}\, A$ have been classified for the case where G is a finite group and k is algebraically closed, by Benson, Carlson, and Rickard in [**4**, Thm. 3.4], and for a finite group scheme over an arbitrary field, by Friedlander and Pevtsova in [**7**, Thm. 6.3]. The classification implies that $(\operatorname{Proj} H, \operatorname{supp}_H)$ is a classifying support datum on $\underline{\operatorname{mod}}\, A$, and therefore we have an isomorphism

$$(\operatorname{Proj} H^*(G, k), \mathcal{O}_{H^*(G,k)}) \xrightarrow{\sim} (\operatorname{Spec} \underline{\operatorname{mod}}\, kG, \mathcal{O}_{\underline{\operatorname{mod}}\, kG})$$

of ringed spaces by Theorem 10.4.

The following example shows a triangulated tensor category which arises in modular representation theory. The tensor product is not necessarily commutative. This category can be used to define support varieties of representations of finite dimensional algebras, generalizing the classical case of a group algebra; see [**19**].

Example 10.6. Let A be a finite dimensional algebra over a field k and let $A^e = A \otimes_k A^{\mathrm{op}}$ be its enveloping algebra. We consider the category $\operatorname{mod} A^e$ of finite dimensional A^e-modules and the full subcategory $\mathcal{B}$ of A^e-modules which are projective when restricted to A or A^{op}. Note that $\mathcal{B}$ carries an exact structure which is induced from the natural exact structure of $\operatorname{mod} A^e$. The inclusion $\mathcal{B} \to \operatorname{mod} A^e$ induces a fully faithful exact functor $\mathbf{D}^b(\mathcal{B}) \to \mathbf{D}^b(\operatorname{mod} A^e)$. The tensor product $\otimes_A$ on $\operatorname{mod} A^e$ is exact in each variable when restricted to $\mathcal{B}$ and therefore induces an exact tensor product on $\mathbf{D}^b(\mathcal{B})$. The A^e-module A, viewed as a complex concentrated in degree zero, is a tensor identity of $\mathbf{D}^b(\mathcal{B})$. The tensor product restricts to a tensor product on the thick subcategory $\mathcal{C} \subseteq \mathbf{D}^b(\mathcal{B})$ which is generated by A. We therefore have a triangulated tensor category $(\mathcal{C}, \otimes_A, A)$ and the lattice of thick tensor ideals is an ideal lattice, by Proposition 7.7. Note that the tensor product of $\mathcal{C}$ is not necessarily commutative. The graded endomorphism ring of the tensor identity

$$\operatorname{End}^*_{\mathcal{C}}(A) = \operatorname{Ext}^*_{A^e}(A, A) = \operatorname{HH}^*(A)$$

equals the Hochschild cohomology ring of A.

11. Decompositions of ideals

In this final section we sketch how the decomposition of objects and ideals of an additive tensor category are reflected by the decomposition of their supports. The prototypical result in this direction is Carlson's theorem from modular representation theory which says that the variety of an indecomposable module is connected [**5**].

11.1. Decompositions of ideals

Let L be an ideal lattice and write $0 = \inf L$. A non-zero element $a \in L$ is called *indecomposable* if $a = a_1 \vee a_2$ and $a_1 \wedge a_2 = 0$ implies $a_1 = 0$ or $a_2 = 0$.

Proposition 11.1. *Let L be an ideal lattice and suppose that the space* $\operatorname{Spec}^* L$ *is noetherian. Given a semi-prime $a \in L$, there exists a unique decomposition $a = \bigvee_{i\in\Omega} a_i$ such that*

(1) a_i is indecomposable and semi-prime for all $i \in \Omega$, and

(2) $a_i \wedge a_j = 0$ for all $i \neq j$ in Ω.

Proof. First observe that every open subset of $\operatorname{Spec}^* L$ is quasi-compact since $\operatorname{Spec}^* L$ is noetherian. Thus Proposition 5.1 provides a bijection $b \mapsto \operatorname{supp}(b)$ between all semi-primes in L and all subsets of $\operatorname{Spec}^* L$ which are unions of closed subsets. Under this bijection, a decomposition $a = \bigvee_i a_i$ satisfying (1) and (2) corresponds to a disjoint union

$$\operatorname{supp}(a) = \bigcup_i \operatorname{supp}(a_i).$$

Now observe that the unions of closed subsets are closed under arbitrary intersections. Thus there exists a partition $\operatorname{supp}(a) = \bigcup_i Y_i$ into unions of closed subsets which admits no proper refinement; see Lemma 11.2 below. We obtain the decomposition $a = \bigvee_i a_i$ by taking for a_i the semi-prime satisfying $\operatorname{supp}(a_i) = Y_i$. □

Lemma 11.2. *Let X be a set and $\mathcal{Y}$ be a family of subsets which is closed under forming intersections. Then there exists for each $Y \in \mathcal{Y}$ a unique partition $Y = \bigcup_{i\in\Omega} Y_i$ into non-empty subsets from $\mathcal{Y}$ which admits no proper refinement. More precisely, for all i, a disjoint union $Y_i = Y_{i1} \cup Y_{i2}$ with $Y_{i1}, Y_{i2} \in \mathcal{Y}$ implies $Y_{i1} = \emptyset$ or $Y_{i2} = \emptyset$.*

Proof. Let $(\bigcup_{i\in\Omega_s} Y_{si})_{s\in\Sigma}$ be the family of all partitions of Y with $Y_{si} \in \mathcal{Y}$ for all s, i. For each $x \in Y$, let

$$Y_x = \bigcap_{\substack{s\in\Sigma \\ x\in Y_{si}}} Y_{si}.$$

Then $Y = \bigcup_{x\in Y} Y_x$ is a partition which admits no proper refinement. □

Remark 11.3. There are refinements of Proposition 11.1 which do not require the space $\operatorname{Spec}^* L$ to be noetherian. For instance, if $a \in L$ is compact and the space $\operatorname{supp}(a)$ is noetherian (with the induced topology), then we have a unique decomposition $a = \bigvee_{i=1}^n a_i$ into indecomposables such that $a_i \wedge a_j = 0$ for all $i \neq j$.

11.2. Decompositions in tensor categories

Let $\mathcal{C}$ be an abelian or triangulated tensor category. We consider the lattice $L_{\text{thick}}(\mathcal{C})$ of thick tensor ideals of $\mathcal{C}$ and recall the following definition from [**14**]. Given a thick tensor ideal $\mathcal{D}$, a family $(\mathcal{D}_i)_{i\in\Omega}$ of thick tensor ideals is a *decomposition* of $\mathcal{D}$ if

(1) the objects in $\mathcal{D}$ are the finite coproducts of objects from the $\mathcal{D}_i$, and

(2) $\mathcal{D}_i \cap \mathcal{D}_j = 0$ for all $i \neq j$.

Such a decomposition is denoted by $\mathcal{D} = \coprod_{i\in\Omega} \mathcal{D}_i$, and we call $\mathcal{D}$ *indecomposable* if $\mathcal{D} \neq 0$ and any decomposition $\mathcal{D} = \mathcal{D}_1 \amalg \mathcal{D}_2$ implies $\mathcal{D}_1 = 0$ or $\mathcal{D}_2 = 0$.

The decomposition $\mathcal{D} = \bigvee_i \mathcal{D}_i$ of a thick tensor ideal (as discussed in Proposition 11.1) amounts to a decomposition $\mathcal{D} = \coprod_i \mathcal{D}_i$, provided that

$$\mathcal{D}_1 \wedge \mathcal{D}_2 = 0 \quad \Longrightarrow \quad \mathcal{D}_1 \vee \mathcal{D}_2 = \mathcal{D}_1 \amalg \mathcal{D}_2$$

for every pair of thick tensor ideals $\mathcal{D}_1, \mathcal{D}_2$. This property holds if $\mathcal{C}$ admits an appropriate internal Hom-functor, because then $\mathcal{D}_1 \wedge \mathcal{D}_2$ implies $\operatorname{Hom}_{\mathcal{C}}(\mathcal{D}_1, \mathcal{D}_2) = 0$. We do not go into details, but refer to the literature. A treatment of decompositions in the stable module category $\underline{\operatorname{mod}}\, kG$ can be found in [**14**], where kG denotes the group algebra of a finite group G. For further discussions, see the recent work of Balmer [**3**] and Chebolu [**6**].

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