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HIGHER ORDER COHOMOLOGY OPERATIONS AND MINIMAL ATOMICITY

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(*communicated by Jesper Grodal*)

Abstract

We prove that $\Omega S^n_{(2)}$, $S^n\{2^r\}$, and $\Omega^2 S^n_{(2)}$ are minimal atomic spaces for appropriate values of n. We do this by using secondary and tertiary cohomology operations to prove that, above the Hurewicz dimension, no elements in the mod 2 homology of the cited spaces are in the image of the Hurewicz homomorphism. In the case of $\Omega^2 S^n$, we construct and exploit an appropriate filtration to facilitate the use of higher order cohomology operations. An appendix consisting of an examination of the coefficients in Adams' factorization is included.

1. Introduction

In this document, we study minimal atomic spaces, defined here in Section 2, at the prime 2. Introduced in [**HKM**] by Hu, Kriz, and May, minimal atomicity is a natural derivative of the atomicity concept which has been pervasive in the literature [**AK**], [**BM**], [**CMN**], [**HKM**], [**X**]. Baker and May studied minimal atomicity more extensively in [**BM**] with an appendix by the author. The authors restricted themselves to Hurewicz complexes, p-local CW spaces whose first non-trivial homotopy group is a cyclic module over $\mathbb{Z}_{(p)}$. The main result we use from that paper is its characterization of minimal atomic spaces as those Hurewicz complexes which have *no homotopy detected by mod 2 homology.* This criterion is verified by showing that the primitive elements of mod 2 homology fail to be in the image of the Hurewicz homomorphism.

Baker and May show that minimal atomic spaces are common; they provide a method for constructing a minimal atomic space from any atomic space. Yet, explicit examples of minimal atomic spaces are few. (Baker and May do provide explicit examples of minimal atomic spectra.) We show that the techniques of higher order cohomology operations can be applied to prove that a space is minimal atomic. This technique has unearthed a new minimal atomic space, $S^n\{2^r\}$, and reestablished minimal atomicity of $\Omega S^n_{(2)}$ and $\Omega^2 S^n_{(2)}$ for certain values of n and r.

Main Theorem. *Let n be a positive integer greater than 1. Higher order cohomology operations can be defined on the following spaces and used to show that they are minimal atomic:*

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(i) $\Omega S^n_{(2)}$*, for* $n \neq 2, 4, 8$*.*

(ii) $S^n\{2^r\}$*, for* $r > 1$ *and* $n \neq 2^s$ *for any* s*.*

(iii) $\Omega^2 S^n_{(2)}$*, for* $n \neq 2, 3, 4, 5, 8, 9$*.*

After isolating those primitive elements which could be spherical, we show that the dual indecomposable elements are in the target of a higher order operation. Thus, none of these primitive elements are spherical. In most cases, secondary cohomology operations will suffice, but occasionally we must appeal to tertiary cohomology operations to do the job.

The organization of this document is as follows. In Section 2, after reviewing some definitions, we give a proof using Hopf Invariant One of the minimal atomicity of ΩS^n for $n \neq 1, 2, 4, 8$ and outline the essence of the higher cohomology operation argument. Section 3 lays out the background of the higher order cohomology operations we use: the Brown-Peterson secondary cohomology operations, Adams' Hopf Invariant One secondary cohomology operations, and a tertiary cohomology operation which is defined using Adams' factorization of $Sq^{2^{r+1}}$. These operations are used to establish $\Omega S^n_{(2)}$ is minimal atomic when $n \neq 1, 2, 4, 8$ in Section 4. These higher order cohomology operation proofs pave the way for an analogous proof that $S^n\{2^r\}$ is minimal atomic in Section 5 as well as an examination of why these methods seem unable to show that $S^n\{2^r\}$ is minimal atomic when n is a power of 2. In Section 6 a filtration of $\Omega^2 S^n_{(2)}$ is developed based on the James construction filtration of $\Omega S^n_{(2)}$. This filtration, along with secondary and tertiary cohomology operation arguments, shows that $\Omega^2 S^n_{(2)}$ is minimal atomic in Section 7. The document ends with an appendix which discusses one computer program to obtain the aforementioned factorization of $Sq^{2^{r+1}}$.

Beginning with Section 2.2, spaces will be localized at the prime 2 unless otherwise specified, and all homology and cohomology will be taken with $\mathbb{F}_2$ coefficients; n will always denote a positive integer with $n \neq 1, 2, 4, 8$ unless otherwise specified. The notation $K(\pi, m)$ denotes an Eilenberg-MacLane space, and $K(m)$ denotes the Eilenberg-MacLane space $K(\mathbb{Z}/2\mathbb{Z}, m)$.

2. Strategies for proving minimal atomicity

In this section, we review the definitions related to the study of minimal atomicity. The section which follows covers the main ideas used in employing a higher order cohomology operations proof of minimal atomicity. The underpinning for all of these arguments is found in Theorem 2.6, which allows us to assess if a space is minimal atomic given information about which elements of its homology are spherical.

2.1. Definitions

We recall those definitions which were specified in [**BM**] that are relevant to this document. For a fixed prime p, minimal atomic spaces, X, must be p-local CW spaces in which the attaching maps are based maps whose domains are spheres localized at the prime p. All spaces X we consider must be simply-connected and localized at this prime p. The definitions below assume X satisfies these conditions.

Definition 2.1. Suppose X has the property that X is (n_0-1)-connected, but X is not n_0-connected. The **Hurewicz dimension** of X is n_0. If $\pi_{n_0}(X)$ is a cyclic module over $\mathbb{Z}_{(p)}$, X is a **Hurewicz complex**.

Definition 2.2. Suppose X and Y are Hurewicz complexes with Hurewicz dimension n_0. Let $f\colon Y\to X$ be such that $f_*\colon \pi_{n_0}(Y)\otimes\mathbb{F}_p\to\pi_{n_0}(X)\otimes\mathbb{F}_p$ is an isomorphism and all $f_*\colon\pi_n(Y)\to\pi_n(X)$ are monomorphisms. Then f is a **monomorphism** of Hurewicz complexes.

Definition 2.3. Suppose X is a Hurewicz complex with Hurewicz dimension n_0. Further, assume any self-map $f\colon X\to X$ which induces an isomorphism on $\pi_{n_0}(X)$ is an equivalence. Then X is **atomic**.

Definition 2.4. Suppose X is atomic. Then X is **minimal atomic** if any monomorphism $f\colon Y\to X$, with Y an atomic complex, is an equivalence.

Definition 2.5. Suppose X is a Hurewicz complex and the mod p Hurewicz homorphism $h\colon\pi_n(X)\to H_n(X;\mathbb{F}_p)$ is zero for all $n>n_0$. Then X **has no homotopy detected by mod p homology**.

The main result we use from [**BM**] is:

Theorem 2.6. *X is a minimal atomic space if and only if it is has no homotopy detected by mod p homology.*

2.2. James maps

Before we begin to use higher order cohomology operations, for completeness we recollect a proof that ΩS^n is minimal atomic which does not use higher order cohomology operations. The author is grateful to Fred Cohen for making her aware of this proof.

We have already alluded to the James construction on X denoted $J(X)$ which is equivalent to $\Omega\Sigma X$. We label the kth filtration of the James construction by $J_k(X)=X^k/\sim$ where

$$(x_1,\dots,x_{j-1},*,x_{j+1},\dots,x_k)\sim(x_1,\dots,x_{j-1},x_{j+1},\dots,x_k).$$

We utilize the James maps,

$$h_q\colon J(X)\to J(X^{(q)})$$

where

$$X^{(q)}=\underbrace{X\wedge\cdots\wedge X}_{q}.$$

Fred Cohen has pointed out that manipulating the James-Hopf map h_2 gives us one way to finish proving ΩS^n is minimal atomic. Let us denote the primitive elements of $H_*(\Omega S^n)$ by $(x_{n-1})^{2^q}$. Now, $h_2\colon\Omega S^n\to\Omega S^{2n-1}$ maps $(x_{n-1})^{2^q}$ to $(x_{2n-2})^{2^{q-1}}$ in homology. If $(x_{n-1})^{2^q}$ is spherical, then so is $(x_{2n-2})^{2^{q-1}}$. It follows that $x^2_{2^{q-1}(n-1)}$

is spherical. We shall show $(x_m)^2$ is not spherical for $m \neq 1, 3, 7$. Then, Theorem 2.6 allows us to conclude that ΩS^n is minimal atomic for $n \neq 2, 4, 8$.

For any m, let $S^m \xrightarrow{\eta} \Omega S^{m+1}$ be the unit of the (Σ, Ω) adjunction. Then $i_m \in H_m(S^m)$ maps to x_m under η_*. Suppose $(x_m)^2$ is spherical. Then, there exists a non-trivial map $S^{2m} \to \Omega S^{m+1}$ such that $(x_m)^2$ lies in the image of the map induced by homology. We may take the adjoint of this map to yield $S^{2m+1} \to S^{m+1}$, and we may loop this map to obtain $\Omega S^{2m+1} \to \Omega S^{m+1}$. By taking the cartesian product of this map with η we obtain

$$S^m \times \Omega S^{2m+1} \to \Omega S^{m+1} \times \Omega S^{m+1} \xrightarrow{\mu} \Omega S^{m+1} \tag{2.1}$$

where μ is the multiplication map. Then under the composition of maps in (2.1) we have,

$$i_m \otimes (x_{2m})^i \to x_m \times (x_m)^{2i} \to (x_m)^{2i+1}$$

and

$$1 \otimes (x_{2m})^i \to 1 \times (x_{2m})^{2i} \to (x_m)^{2i}.$$

We thus have an isomorphism in homology, and the composite of maps in (2.1) is an equivalence. In particular, S^m is a retract of ΩS^{m+1}. However, a retract X of an H-space Y is an H-space via the following commutative diagram:

$$\begin{tikzcd}
X \times X \arrow[r, "i\times i"] & Y \times Y \arrow[r, "\phi"] & Y \arrow[r] & X \\
X \vee X \arrow[u] \arrow[r, "i\vee i"] \arrow[dr, "\nabla"'] & Y \vee Y \arrow[u] \arrow[r, "\nabla"] & Y \arrow[u, equal] \arrow[r, "r"] & X \arrow[u, equal] \\
& X \arrow[ur, "i"] \arrow[urr, equal] & &
\end{tikzcd}$$

Thus S^m must be an H-space and so $m = 1, 3, 7$. Recall that if $(x_{n-1})^{2^q}$ is spherical, $(x_{2^{q-1}(n-1)})^2$ is spherical. Thus, from above, $2^{q-1}(n-1) = 1, 3, 7$. If $q \geqslant 2$ we obtain a contradiction, so it must be the case that $q = 1$ and $n - 1 = 1, 3, 7$. Thus the only possible spherical elements which exist above the Hurewicz dimension for ΩS^n are the classes $(x_1)^2 \in H_2(\Omega S^2)$, $(x_3)^2 \in H_6(\Omega S^4)$, and $(x_7)^2 \in H_{14}(\Omega S^8)$. Looping the Hopf maps $\eta\colon S^3 \to S^2$, $\nu\colon S^7 \to S^4$, and $\sigma\colon S^{15} \to S^7$ shows that these elements are spherical. Thus, ΩS^n is minimal atomic if and only if $n \neq 2, 4, 8$.

2.3. The higher order cohomology argument

Theorem 2.6 verifies that a space X is minimal atomic if no spherical elements of $H_*(X)$ exist above the Hurewicz dimension. Thus, the first step in the higher order cohomology argument is to calculate which primitive elements of $H_*(X)$ with dimension above the Hurewicz dimension are annihilated by the Steenrod algebra; all spherical elements we are interested in must satisfy these properties. We show that each of these candidates cannot be in the image of the Hurewicz homomorphism by using higher order cohomology operations and a naturality argument as follows.

Let $a \in H_i(X)$ be a spherical candidate with dual indecomposable element $\alpha \in H^i(X)$. To show a is not spherical, we will prove there exists $\beta \in H^j(X)$, with $j \neq i$, and a higher order cohomology operation $\Phi\colon H^j(X) \to H^i(X)/Q^i_\Phi(X)$ such that

$$\Phi(\beta) = \alpha + \gamma \neq 0.$$

Here $\gamma \in H^i(X)$ is a decomposable element which is possibly zero, and $Q^i_{\Phi(X)}$ is a submodule of $H^i(X)$ which is possibly zero, in which case Φ is defined "with zero indeterminacy". Some arguments will attest to the fact that Φ is defined on β and X.

For dimensional reasons, Φ will be defined on all spheres and will evaluate to zero with "zero indeterminacy"; in particular, Φ will be defined on S^i. Now, suppose that a is spherical. Then, there must exist a non-trivial map

$$f\colon S^i \to X$$

such that f^* maps α to the non-zero element of $H^i(S^i)$. Since our higher order cohomology operations are natural with respect to maps, we have the following commutative diagram:

$$\begin{tikzcd} H^j(X) \arrow[r, "f^*"] \arrow[d, "\Phi"] & H^j(S^i) \arrow[d, "\Phi"] \\ H^i(X) \arrow[r, "f^*"] & H^i(S^i) \end{tikzcd}$$

We see that $(f^* \circ \Phi)(\beta) = \alpha$ (modulo zero) while $(\Phi \circ f^*)(\beta) = 0$ (modulo zero). We have a contradiction, and we may conclude that a is not spherical. This basic argument will appear throughout the document.

3. Three kinds of higher order cohomology operations

We examine the secondary and tertiary cohomology operations which we will utilize. We recall the construction of the Brown-Peterson secondary cohomology operation which is based on a relation in the Steenrod algebra and prove such an operation has stable properties. We review Adams' secondary cohomology operations Φ_{ij}, pointing out similarities and differences with the Brown-Peterson operations. Finally, we construct a tertiary cohomology operation in the manner suggested by [**BP**] using Adams' factorization of $Sq^{2^{r+1}}$ into secondary cohomology operations.

3.1. Brown-Peterson secondary cohomology operations

Each secondary cohomology operation stems from a relation in the Steenrod algebra. We recall a particular secondary cohomology operation defined by Brown and Peterson in [**BP**].

Suppose for some fixed m we have a factorization of Steenrod operations

$$Sq^m = \sum_i Sq^{a_i} Sq^{b_i} \tag{3.1}$$

where each Sq^{a_i}, Sq^{b_i} has degree greater than 0. Let κ_m be the fundamental class of $K(m)$. For each Steenrod operation Sq^{b_i}, let $Sq^{b_i}\colon K(m) \to K(m+b_i)$ represent the element $Sq^{b_i}(\kappa_m)$. Then, define $f_1\colon K(m) \to \prod_i K(m+b_i)$ such that f_1^* takes the fundamental classes of $\prod_i K(m+b_i)$ to the elements $Sq^{b_i}(\kappa_m)$. Denote the homotopy fiber of f_1 by A_1, and let the fibration $A_1 \to K(m)$ be g_1. Let

$$h\colon \prod_i K(m+b_i-1) \to A_1$$

be the map of the fiber of g_1 into A_1. Note that the fundamental classes κ_{m+b_i-1} transgress to $Sq^{b_i}(\kappa_m)$ in the Serre spectral sequence of the fibration given by $h \circ g_1$. Then, $\sum Sq^{a_i}(\kappa_{m+b_i-1})$ transgresses to

$$\sum Sq^{a_i} Sq^{b_i}(\kappa_m) = Sq^m(\kappa_m) = \kappa_m^2 \neq 0.$$

If we loop our fibration, we obtain the new fibration

$$\prod_i K(m+b_i-2) \overset{\Omega h}{\to} \Omega A_1 \overset{\Omega g_1}{\to} K(m-1).$$

The fundamental class κ_{m+b_i-2} transgresses to $Sq^{b_i}(\kappa_{m-1})$ in the Serre spectral sequence for this fibration. By (3.1), $\sum_i Sq^{a_i}(\kappa_{b_i+m-2})$ transgresses to

$$\sum_i Sq^{a_i} Sq^{b_i}(\kappa_{m-1}) = Sq^m(\kappa_{m-1}) = 0.$$

Thus, the class $\sum_i Sq^{a_i}(\kappa_{m+b_i-2})$ survives in the spectral sequence and pulls back to an element of $H^{2m-2}(A_1)$ which we shall call ϕ_2.

Now, consider the diagram below:

$$\begin{tikzcd}
\Omega A_1 \arrow[r, "\phi_2"] \arrow[d, "\Omega g_1"] & K(2m-2) \\
K(m-1) \arrow[r, "\Omega f_1"] & \prod_i K(m+b_i-1)
\end{tikzcd}$$

This gives rise to a secondary cohomology operation Φ which is defined on those elements of $H^{m-1}(X)$ which are annihilated by the Sq^{b_i}. Given such an element τ, we abuse notation, thinking of τ as a map $\tau\colon X \to K(m-1)$ such that $\Omega f_1 \circ \tau$ is null-homotopic. We may then choose a lifting $\overline{\tau}\colon X \to \Omega A_1$. Then $\Phi(\tau)$ is defined to be the cohomology class represented by $\phi_2 \circ \overline{\tau}$, which is independent of the lifting when viewed as an element of $H^{2m-2}(X)/\oplus_i Sq^{a_i}(H^{2m-2-a_i}(X))$. Here

$\oplus_i Sq^{a_i}(H^{2m-2-a_i}(X))$ is the module of indeterminacy, a sum in a graded vector space; any two lifts

$$\overline{\tau_1}, \overline{\tau_2} \colon X \to \Omega A_1$$

will differ by a composite

$$X \to \textstyle\prod_i K(m+b_i-2) \to \Omega A_1 \xrightarrow{\phi_2} K_{2m-2}$$

which represents elements of the aforementioned module of indeterminacy.

Observe the following properties of Φ.

(i) Φ is natural with respect to maps of spaces.

(ii) For any sphere S^l, $\Phi(S^l)$ is zero modulo zero.

(iii) If Φ is defined on ΣX, there is a secondary cohomology operation defined on X which we denote by $\sigma\Phi$. The values of Φ and $\sigma\Phi$ are related by the evident commutative diagram via the cohomology suspension.

(iv) Φ satisfies the additivity formula, $\Phi(\tau+\gamma) = \Phi(\tau) + \Phi(\gamma) + \tau\gamma$.

Remarks 3.1. We shall prove (iii), but first we offer some remarks on the other observations.

(i) Given $f \colon Y \to X$ such that Φ is defined on Y, we see naturality satisfied in the following diagram, where $\overline{\tau} \circ f$ provides the desired lift.

$$\begin{tikzcd}
 & & \Omega A_1 \arrow[r, "\phi_2"] \arrow[d, "\Omega g_1"] & K(2m-2) \\
Y \arrow[r, "f"] & X \arrow[ur, "\overline{\tau}"] \arrow[r, "\tau"] & K(m-1) \arrow[r, "\Omega f_1"] & \prod_i K(m+b_i-1)
\end{tikzcd}$$

(ii) This follows because of dimensional reasons and the fact that the Steenrod algebra acts trivially on S^l.

(iv) This is proven in [**BP**] using the observation that ϕ_2 is not primitive.

Now, for any space K, let $\sigma_i \colon H^{i+1}(K) \to H^i(\Omega K)$ denote the cohomology suspension, a map which commutes with the Steenrod algebra action. We shall use σ_i to help us prove a notion of stability (iii) for these secondary cohomology operations. In the Serre spectral sequence for the fibration

$$\prod_i \Omega K(m+b_i-2) \xrightarrow{\Omega^2 h} \Omega^2 A_1 \xrightarrow{\Omega^2 g_1} \Omega K(m-1),$$

the element

$$\sum_i Sq^{a_i}(\sigma_{i+m-3}(\kappa_{b_i+m-2}))$$

transgresses to

$$\sum_i Sq^{a_i} Sq^{b_i}(\kappa_{m-2}) = Sq^m(\kappa_{m-2}) = 0.$$

Then, $\sum_i Sq^{a_i}(\sigma_{i+m-3}(\kappa_{b_i+m-2}))$ survives in the Serre spectral sequence for the looped fibration. This element pulls back to $\sigma_{2m-3}(\phi_2) \in H^*(\Omega^2 A_1)$. Observe that $\Omega\phi_2 \colon \Omega^2 A_1 \to K(2m-3)$ represents this element.

Then, looping the diagram above gives rise to another secondary cohomology operation, which we shall denote $\sigma\Phi$,

$$\begin{tikzcd}
\Omega^2 A_1 \arrow[r, "\Omega\phi_2"] \arrow[d, "\Omega g_1"] & K(2m-3) \\
K(m-2) \arrow[r, "\Omega^2 f_1"] & \prod_i K(m+b_i-2)
\end{tikzcd}$$

We observe that $\sigma\Phi$ is defined for all $\tau \in H^{m-2}(X)$ such that Sq^{a_i} annihilates τ; $\sigma\Phi$ takes values in $H^{2m-3}(X)/\oplus_i Sq^{a_i}(H^{2m-3-a_i}(X))$.

Now, suppose Φ is defined on an element $\tau \in H^{m-1}(\Sigma X)$. As above, we abuse notation and think of τ as a map $\tau \colon \Sigma X \to K(m-1)$. So, we have the following diagram where $\phi_2 \circ \overline{\tau}$ represents $\Phi(\tau)$,

$$\begin{tikzcd}
& \Omega A_1 \arrow[r, "\phi_2"] \arrow[d] & K(2m-2) \\
\Sigma X \arrow[ur, "\overline{\tau}"] \arrow[r, "\tau"] & K(m-1) \arrow[r] & \prod_i K(m+b_i-1)
\end{tikzcd}$$

Let $\eta \colon X \to \Omega\Sigma X$ be the canonical map. Then $\sigma\Phi$, defined on $\eta \circ \Omega\tau$, is $\Omega\phi_2 \circ \Omega\overline{\tau} \circ \eta$, as in the diagram below.

$$\begin{tikzcd}
& & \Omega^2 A_1 \arrow[r, "\Omega\phi_2"] \arrow[d] & \Omega K(2m-2) \\
X \arrow[r, "\eta"] & \Omega\Sigma X \arrow[ur, "\Omega\overline{\tau}"] \arrow[r, "\Omega\tau"] & \Omega K(m-1) \arrow[r] & \prod_i \Omega K(m+b_i-1)
\end{tikzcd}$$

Yet $\phi_2 \circ \overline{\tau}$ and $\Omega\phi_2 \circ \Omega\overline{\tau} \circ \eta$ are adjoints of each other. For any space X, let $s_i \colon H^{i+1}(\Sigma X) \to H^i(X)$ be the suspension homomorphism. Then, modulo indeterminacy, the following diagram commutes.

$$\begin{tikzcd}
H^{m-1}(\Sigma X) \arrow[r, "s_{m-2}"] \arrow[d, "\Phi"] & H^{m-2}(X) \arrow[d, "\Sigma\Phi"] \\
H^{2m-2}(\Sigma X) \arrow[r, "s_{m-3}"] & H^{2m-3}(X)
\end{tikzcd}$$

We shall use this notion of stability. Notice that if Φ is defined on $H^{m-1}(\Sigma X)$, then $\sigma\Phi$ is defined on $H^{m-2}(X)$.

3.2. Adams' secondary cohomology operations Φ_{ij}

In [**A**], Adams constructs secondary cohomology operations Φ_{ij} using minimal resolutions of $\mathbb{Z}/2\mathbb{Z}$ over the Steenrod algebra. We briefly look at these operations using the ideas of Brown and Peterson.

Instead of using an equation of the type of (3.1), Φ_{ij} is based on the relation, discussed in the appendix,

$$Sq^{2^i}Sq^{2^j} + \sum f_k Sq^{2^{g_k}} = 0 \tag{3.2}$$

where i, j are non-negative integers with $i \leqslant j$ and $i+1 \neq j$, f_k is a Steenrod operation, and $g_k < j$. As in Section 3.1, we may create a Postnikov system. Given any m, we let

$$f_1\colon K(m) \to \prod K(m+2^{g_k}) \times K(m+2^j)$$

be such that the fundamental classes on the right are mapped under f_1^* to the corresponding Steenrod operation

$$Sq^{2^{g_k}}(\kappa_m) \text{ or } Sq^{2^j}(\kappa_m).$$

We let A_1 be the homotopy fiber, denoting the fibration $A_1 \to K(m)$ by g_1. Let

$$h\colon \textstyle\prod_i K(m+2^{g_k}-1) \times K(m+2^j-1) \to A_1$$

be the map of the fiber of g_1 into A_1. We notice that the element

$$\sum f_k(\kappa_{m+2^{g_k}-1}) + Sq^{2^i}(\kappa_{m+2^j-1})$$

transgresses to 0, and thus must pull back to an element ϕ_{ij} of $H^*(A_1)$ in the Serre spectral sequence for g_1. Hence, we obtain the following diagram without looping:

$$\begin{tikzcd} A_1 \arrow[r, "\phi_{ij}"] \arrow[d, "g_1"] & K(m+2^i+2^j-1) \\ K(m) \arrow[r, "f_1"] & \prod K(m+2^{g_k}) \times K(m+2^j) \end{tikzcd}$$

As in Section 3.1, this diagram gives rise to a secondary cohomology operation which we denote Φ_{ij}. The fact that we need not loop the diagram gives rise to a stability property of Φ_{ij}. This also gives rise to the result that ϕ_{ij} is primitive, yielding a nice additivity formula for Φ_{ij}. We summarize some properties of this operation:

(i) Φ_{ij} is natural with respect to maps of spaces.

(ii) For any sphere S^l, $\Phi_{ij}(S^l)$ is zero modulo zero.

(iii) Φ_{ij} is a stable operation: Φ_{ij} is defined on X if and only if Φ_{ij} is defined on ΣX, and the results are related in the obvious way via the cohomology suspension.

(iv) Φ_{ij} satisfies the additivity formula, $\Phi_{ij}(\tau+\gamma) = \Phi_{ij}(\tau) + \Phi_{ij}(\gamma)$.

3.3. Constructing a tertiary cohomology operation

This tertiary operation is based on a relation between the stable secondary cohomology operations Φ_{ij} that Adams develops in [**A**]. There, he proves the relation,

$$Sq^{2^{r+1}}(\tau) = \sum_{i \leqslant j, i+1 \neq j} a_{ij}\Phi_{ij}(\tau) \tag{3.3}$$

for τ such that $Sq^{2^s}(\tau) = 0$ for $0 \leqslant s \leqslant r$ where a_{ij} are elements of the mod 2 Steenrod algebra. Let $Sq^i\colon K(2^{r+1}) \to K(2^{r+1}+i)$ represent the element, $Sq^i(\kappa_{2^{r+1}})$. To create Ψ, we look at the map

$$f_1\colon K(2^{r+1}) \to \prod_{0 \leqslant s \leqslant r} K(2^{r+1}+2^s)$$

such that

$$(f_1)^*(\kappa_{2^{r+1}+2^s}) = Sq^{2^s}(\kappa_{2^{r+1}}).$$

Denote the homotopy fiber of f_1 by A_1 and the map $A_1 \to K(2^{r+1})$ by g_1. Applying the Serre spectral sequence in cohomology to the fibration,

$$\prod_{0 \leqslant s \leqslant r} K(2^{r+1}+2^s-1) \to A_1 \to K(2^{r+1})$$

yields that the fundamental classes of the fiber, $\kappa_{2^{r+1}+2^s-1}$, transgress to $Sq^{2^s}(\kappa_{2^{r+1}})$ where $\kappa_{2^{r+1}}$ is the fundamental class of the base. Examining the spectral sequence shows that $\kappa_{2^{r+1}}$ survives to E_∞ and corresponds to the element $(g_1)^*(\kappa_{2^{r+1}})$. Furthermore, there are no elements of $H^*(A_1)$ with dimension between 2^{r+1} and 2^{r+2}. For $0 \leqslant s \leqslant r$, it must be the case that $Sq^{2^s}((g_1)^*(\kappa_{2^{r+1}})) = 0$ in $H^*(A_1)$. This condition allows us to define $\Phi_{ij}((g_1)^*(\kappa_{2^{r+1}})$ with zero indeterminacy. Representing these elements is a map $f_2\colon A_1 \to \prod_{i \leqslant j, i+1 \neq j} K(2^i+2^j-1+2^{r+1})$ as the degree of Φ_{ij} is 2^i+2^j-1.

We define the homotopy fiber of f_2 to be A_2 where $A_2 \to A_1$ is denoted g_2. Examining the cohomology Serre spectral sequence of

$$\prod_{i \leqslant j, i \neq j} K(2^i+2^j-1+2^{r+1}-1) \to A_2 \to A_1$$

shows that v_{ij} transgresses to $\Phi_{ij}((g_1)^*(\kappa_{2^{r+1}}))$, where v_{ij} is the fundamental class of $K(2^i+2^j-1+2^{r+1}-1)$. Inspired by (3.1) we compute $\sum_{i \leqslant j, i+1 \neq j} a_{ij}(v_{ij})$ which transgresses to $Sq^{2^{r+1}}((g_1)^*(\kappa_{2^{r+1}})) = ((g_1)^*(\kappa_{2^{r+1}}))^2 \neq 0$.

We loop the maps f_1, f_2, g_1, g_2 which provides us with a new set of fibrations, in particular,

$$\prod_{i \leqslant j, i \neq j} K(2^i+2^j-1+2^{r+1}-2) \to \Omega A_2 \to \Omega A_1.$$

We use $\overline{v_{ij}}$ to denote the fundamental classes in the fiber. Similar to above, we compute $\sum_{i \leqslant j, i+1 \neq j} a_{ij}(\overline{v_{ij}})$ which transgresses to $Sq^{2^{r+1}}((g_1)^*(\kappa_{2^{r+1}} - 1)) = 0$ for dimensional reasons. Thus, $\sum_{i \leqslant j, i+1 \neq j} a_{ij}(\overline{v_{ij}})$ must pull back to an element of $H^{2^{r+2}-2}(\Omega A_2)$ which we denote ψ_3. We consider the following diagram to better understand how to construct our tertiary operation:

$$\begin{tikzcd}
\Omega A_2 \arrow[r, "\phi_3"] \arrow[d, "\Omega g_2"] & K(2^{r+2}-2) \\
\Omega A_1 \arrow[r, "\Omega f_2"] \arrow[d, "\Omega g_1"] & \prod_{i \leqslant j, i+1 \neq j} K(2^{r+1} + 2^i + 2^j - 2) \\
K(2^{r+1}-1) \arrow[r, "\Omega f_1"] & \prod_{0 \leqslant s \leqslant r} K(2^s + 2^{r+1} - 1)
\end{tikzcd} \tag{3.4}$$

This diagram defines the tertiary operation Ψ, which is defined on elements of $H^{2^{r+1}-1}(X)$ with a trivial action under Φ_{ij}.

We list the properties of Ψ followed by an in-depth discussion of the last two properties.

(i) Ψ is natural with respect to maps of spaces.

(ii) For any sphere S^l, $\Psi(S^l)$ is zero modulo zero.

(iii) If Ψ is defined on ΣX, there is a tertiary cohomology operation defined on X which we denote by $\sigma\Psi$. The values of Ψ and $\sigma\Psi$ are related by the evident commutative diagram via the cohomology suspension.

(iv) Ψ satisfies the additivity formula, $\Psi(\tau + \gamma) = \Psi(\tau) + \Psi(\gamma) + \tau\gamma$.

3.3.1. Stability of Ψ

Similar to our previously developed $\sigma\Phi$, we can construct $\sigma\Psi$ with the property that for spaces ΣX in which $\sigma\Psi$ is defined, we have, modulo indeterminacy,

$$\begin{tikzcd}
H^{2^{r+1}-1}(\Sigma X) \arrow[r, "s_{2^{r+1}-2}"] \arrow[d, "\Phi"] & H^{2^{r+1}-2}(X) \arrow[d, "\Sigma\Phi"] \\
H^{2^{r+2}-2}(\Sigma X) \arrow[r, "s_{2^{r+2}-3}"] & H^{2^{r+2}-3}(X)
\end{tikzcd} \tag{3.5}$$

In (3.4), the bottom square corresponds to the construction for Φ_{ij}, which Adams has already proved is a stable operation. Thus, if we loop (3.4) the bottom square still corresponds to the construction of Φ_{ij}. In the Serre spectral sequence of the fibration

$$\prod_{i \leqslant j, i \neq j} \Omega K(2^i + 2^j - 1 + 2^{r+1} - 2) \to \Omega^2 A_2 \to \Omega^2 A_1,$$

the element $\sum_{i\leqslant j, i+1\neq j} a_{ij}(\sigma\overline{v_{ij}})$ transgresses to $Sq^{2^{r+1}}((g_1)^*(\sigma\kappa_{2^{r+1}}-1))=0$. Thus, $\sum_{i\leqslant j, i+1\neq j} a_{ij}(\sigma\overline{v_{ij}})$ must pull back to $\sigma\psi_3 \in H^{2^{r+2}-3}(\Omega A_2, \mathbb{F}_2)$, which is represented by $\Omega\psi_3\colon \Omega A_2 \to K(2^{r+2}-3)$. The resulting diagram results in a tertiary operation which we denote $\sigma\Psi$.

Suppose Ψ is defined on an element $\tau\colon \Sigma X \to K(2^{r+1}-1)$ so that lifts $\overline{\tau}\colon X \to \Omega A_1$ and $\overline{\overline{\tau}}\colon X \to \Omega A_2$ exist. Then $\sigma\Psi$ is defined on the element $\Omega\tau\circ\eta$, where $\eta\colon X \to \Omega\Sigma X$ is the canonical map. The relevant lifts $\Omega\overline{\tau}\circ\eta$ and $\Omega\overline{\overline{\tau}}\circ\eta$ exist, and in particular, $\Omega\overline{\overline{\tau}}\circ\eta$ is the adjoint of $\overline{\overline{\tau}}$. It follows that our desired diagram (3.5) exists.

3.3.2. An additivity formula for Ψ

Similar to the proof of the additivity formula of Φ in [**BP**], we must show that ϕ_3 is not primitive; we use the following result from [**W**, p. 383],

Lemma 3.2. *If ΩK is $(n-1)$-connected, then the module of primitives of $H^*(\Omega K)$ contained in $H^l(\Omega K)$ is equal to the image of σ_l as long as $l \leqslant 3n-1$.*

To apply Lemma 3.2, we compute the connectivity of ΩA_2. First, by looking at the long exact sequence of homotopy groups derived from

$$\prod_{0\leqslant k\leqslant r} K(2^k + 2^{r+1} - 1) \to A_1 \to K(2^{r+1}),$$

we see the connectivity of A_1 is $2^{r+1}-1$. Examining the long exact sequence of homotopy groups for $\prod_{i\leqslant j, i+1\neq j} K(2^{r+1}+2^i+2^j-1) \to A_2 \to A_1$ gives that the connectivity of A_2 is also $2^{r+1}-1$; thus, the connectivity of ΩA_2 is $2^{r+1}-2$. Since $r>3$,

$$2^{r+2}-2 < 3(2^{r+1})-1.$$

By Lemma 3.2 it follows that ϕ_3 is primitive if and only if $\phi_3 = \sigma_{2^{r+2}-2}(\psi)$ for some $\psi \in H^{2^{r+2}-1}(A_2)$. We assume ψ exists and show that

$$h^*(\psi) = \sum_{i\leqslant j, i+1\neq j} a_{ij}v_{ij}.$$

Then, $h^*(\psi)$ transgresses to zero in the spectral sequence induced by the fibration $\prod_{i\leqslant j, i+1\neq j} K(2^{r+1}+2^i+2^j-1) \xrightarrow{h} A_2 \to A_1$. However, we have seen $\sum_{i\leqslant j, i+1\neq j} a_{ij}v_{ij}$ transgresses to $Sq^{2^{r+1}}((g_1)^*(\kappa_{2^{r+1}})) = ((g_1)^*(\kappa_{2^{r+1}}))^2 \neq 0$. This gives us the desired contradiction, and hence, ϕ_3 is not primitive.

We assume ψ exists. Then,

$$\begin{aligned}\sigma_{2^{r+2}-2}[(h)^*(\psi)] &= (\Omega h)^*(\sigma_{2^{r+2}-1}(\psi))\\ &= (\Omega h)^*(\phi_3)\\ &= \sum_{i\leqslant j, i+1\neq j} a_{ij}(\overline{v_{ij}})\\ &= \sigma_{2^{r+2}-2}[\sum_{i\leqslant j, i+1\neq j} a_{ij}(v_{ij})].\end{aligned}$$

We show $\sum_{i\leqslant j, i+1\neq j} a_{ij}(\overline{v_{ij}})$ is the only pre-image of $\sigma_{2^{r+2}-2}[\sum_{i\leqslant j, i+1\neq j} a_{ij}(v_{ij})]$ under

$$\sigma_{2^{r+2}-2}\colon H^{2^{r+2}-1}(\prod_{i\leqslant j, i+1\neq j} K(2^i+2^j+2^{r+1}-2)) \to H^{2^{r+2}-2}(\prod_{i\leqslant j, i+1\neq j} K(2^i+2^j+2^{r+1}-3)).$$

The Hurewicz dimension of the Eilenberg MacLane spaces in the product,

$$\prod_{i\leqslant j, i+1\neq j} K(2^i+2^j+2^{r+1}-2)$$

is at least 2^{r+1}, when $i=j=0$. Then, $H^{2^{r+2}-1}(\prod_{i\leqslant j, i+1\neq j} K(2^i+2^j+2^{r+1}-2))$ contains no decomposables, and so $\sigma_{2^{r+2}-2}$ is a monomorphism. It must be the case that $h^*(\psi) = \sum_{i\leqslant j, i+1\neq j} a_{ij}v_{ij}$, and we obtain the desired contradiction. We have established that ΩA_2 is $(2^{r+1}-2)$-connected. By the Künneth Theorem,

$$H^{2^{r+2}-2}(\Omega A_2\times\Omega A_2) = H^{2^{r+2}-2}(\Omega A_2)\oplus H^{2^{r+1}-1}(\Omega A_2)\otimes H^{2^{r+1}-1}(\Omega A_2)\oplus H^{2^{r+2}-2}(\Omega A_2).$$

Recall that we have a fibration, $\prod_{i\leqslant j, i\neq j} K(2^{r+1}+2^i+2^j-3) \xrightarrow{\Omega^2 f_2} \Omega A_2 \xrightarrow{\Omega g_2} \Omega A_1$. Examining the Serre spectral sequence applied to this fibration, we see that

$$(\Omega g_2)^*(\Omega g_1)^*(\kappa_{2^{r+1}-1})\otimes(\Omega g_2)^*(\Omega g_1)^*(\kappa_{2^{r+1}-1})$$

is the only non-trivial element of $H^{2^{r+1}-1}(\Omega A_2)\otimes H^{2^{r+1}-1}(\Omega A_2)$. Define $\upsilon\colon \Omega A_2\times\Omega A_2\to\Omega A_2$ to be the loop multiplication map. Thus, since ϕ_3 is not primitive,

$$\upsilon^*(\phi_3) = \phi_3\otimes 1 + (\Omega g_2)^*(\Omega g_1)^*(\kappa_{2^{r+1}-1})\otimes(\Omega g_2)^*(\Omega g_1)^*(\kappa_{2^{r+1}-1}) + 1\otimes\phi_3.$$

Let X be a space such that there exist $\tau,\gamma\in H^{2^{r+1}-1}(X,\mathbb{F}_2)$ such that Ψ is defined for τ and γ. Consider the maps that represent these elements

$$\tau\colon X\to K(2^{r+1}-1)$$

and

$$\gamma\colon X \to K(2^{r+1}-1).$$

By assumption, it must be the case that Sq^{2^k} acts trivially on τ and γ for $0 \leqslant k \leqslant r$, so we may find lifts $\overline{\tau}, \overline{\gamma}\colon X \to \Omega A_1$. Again, by assumption, Adams' Φ_{ij} are defined and zero on these elements, so we have lifts $\overline{\overline{\tau}}, \overline{\overline{\gamma}}\colon X \to \Omega A_2$. Let $\triangle\colon X \to X \times X$ be the diagonal map, and $\mu\colon \Omega K(2^{r+1}-1) \times \Omega K(2^{r+1}-1) \to \Omega K(2^{r+1}-1)$ be the multiplication map. Now,

$$\begin{aligned}\Omega g_1 \circ \Omega g_2 \circ \upsilon \circ (\overline{\overline{\tau}} \times \overline{\overline{\gamma}}) \circ \triangle &= \mu \circ (\tau \times \gamma) \circ \triangle \\ &= \tau + \gamma.\end{aligned}$$

So, $\upsilon \circ (\overline{\overline{\tau}} \times \overline{\overline{\gamma}}) \circ \triangle$ is a lift of $\tau + \gamma$ under the map $\Omega g_1 \circ \Omega g_2$ as well as a lift of $\upsilon \circ (\overline{\tau} + \overline{\gamma}) \circ \triangle$ through Ωg_2. The map $\upsilon \circ (\overline{\tau} + \overline{\gamma}) \circ \triangle$, itself, is a lift of $\tau + \gamma$ under Ωg_1. So modulo the indeterminacy of Ψ, we have

$$\begin{aligned}\Psi(\tau+\gamma) &= (\upsilon \circ (\overline{\overline{\tau}} \times \overline{\overline{\gamma}}) \circ \triangle)^* \phi_3 \\ &= (\overline{\overline{\tau}} \times \overline{\overline{\gamma}} \circ \triangle)^* (\phi_3 \otimes 1 + \xi \otimes \xi + 1 \otimes \phi_3) \\ &= \Psi(\tau) + \Psi(\gamma) + \tau\gamma,\end{aligned}$$

where $\xi = (\Omega g_2)^*(\Omega g_1)^*(\kappa_{2^{r+1}-1})$.

4. Showing ΩS^n is minimal atomic

We show for positive integers $n > 1$ that ΩS^n is minimal atomic for $n \neq 2, 4, 8$ by means of secondary and tertiary cohomology operations. These arguments will show in detail how certain cohomology classes are tied together. We first determine the spherical candidates of ΩS^n and show that above the Hurewicz dimension, these candidates are in the target of a higher order cohomology operation.

4.1. Spherical candidates of ΩS^n

To determine which elements of $H_*(\Omega S^n)$ are primitive and annihilated by the Steenrod algebra, we note that the Steenrod operations on ΩS^n are trivial. Discussed in [**S**, p. 85], the James construction applied to S^{n-1}, $J(S^{n-1})$, is homotopy equivalent to $\Omega\Sigma S^{n-1} = \Omega S^n$. Using the splitting property of the suspension of the James construction, we have $\Sigma\Omega\Sigma S^{n-1} = \bigvee_{k=1}^{\infty} \Sigma S^{k(n-1)}$. Since the Steenrod operations on $\Sigma S^{k(n-1)}$ are trivial, we deduce that the Steenrod operations are trivial on $\Sigma\Omega S^n$ and thus, on ΩS^n. So, all of the primitive elements above the Hurewicz homomorphism could conceivably lie in the image of the mod 2 Hurewicz homomorphism.

Computations with the Serre spectral sequence on the path fibration $\Omega S^n \to PS^n \to S^n$ allow us to conclude that $H^*(\Omega S^n) = \Gamma[\alpha_{n-1}]$ as a Hopf algebra. Alternatively, as an algebra, $H^*(\Omega S^n) = \bigotimes_{k \geqslant 0} P[\gamma_{2^k}(\alpha_{n-1})]/(\gamma_{2^k}(\alpha_{n-1}))^2$; the binomial coefficients appearing in the multiplication of the divided polynomial algebra reduce to give $\Gamma[\alpha_{n-1}] = \bigotimes_{k\geqslant 0} P[\gamma_{2^k}(\alpha_{n-1})]/(\gamma_{2^k}(\alpha_{n-1}))^2$. Also, observe that only one element exists in dimension $2^k(n-1)$, namely $\gamma_{2^k}(\alpha_{n-1})$. If there were another element of dimension $2^k(n-1)$, such an element would be a product of distinct generators $\gamma_{2^i}(\alpha_{n-1})$ with $i<k$. However, the largest degree achieved by elements of this form is $2^k(n-1)-1 = \left|\prod_{i=0}^{k-1}\gamma_{2^i}(\alpha_{n-1})\right|$. We will use this fact later. Let $a_k \in H_*(\Omega S^n)$ be the dual element to $\gamma_{2^k}(\alpha_{n-1})$.

4.2. $n \neq 2^{r+1}$

When $n \neq 2^{r+1}$, Sq^n has a factorization in the Steenrod algebra. This factorization, along with factorizations of $Sq^{2^k(n-1)}$ for $k \geqslant 1$, will be utilized to construct Brown-Peterson secondary cohomology operations necessary to show ΩS^n is minimal atomic. Now, to apply these operations to ΩS^n we must specify examples of our secondary cohomology operations Φ. Set $n = 2^r + a$ where 2^r is the largest power of 2 which appears in the binary representation of n. For now, we suppose that n is not a power of 2 so that $0 < a < 2^r$. By taking the binary representations of 2^r-1 and a, we see that $\binom{2^r-1}{a} = 1 \bmod 2$: Recall the calculation

$$\binom{i}{j} = \prod \binom{i_k}{j_k} \bmod 2$$

where i_k is the kth term in the binary representation of i and similarly for j_k. The binary representation of 2^r-1 consists of r uninterrupted 1's; the binary representation of a is at most r digits long. Applying the result above yields $\binom{2^r-1}{a} = 1$. Using the Adem relations, where the binomial coefficients are taken mod 2, gives

$$Sq^a Sq^{2^r} = Sq^n + \sum_{c>0} \binom{2^r-c-1}{a-2c} Sq^{n-c}Sq^c.$$

Then,

$$Sq^n = Sq^a Sq^{2^r} + \sum_{c\in S} \binom{2^r-c-1}{a-2c} Sq^{n-c}Sq^c, \tag{4.1}$$

where $S = \{c \mid 2c \leqslant a \text{ and } \binom{2^r-c-1}{a-2c} \neq 0 \bmod 2\ \}$.

As we have seen, this relation gives rise to a secondary cohomology operation Φ_0 which acts on elements of $H^{n-1}(\Omega S^n)$ that vanish under Sq^n and Sq^c for $c \in S$ above, and takes values in $H^{2n-2}(\Omega S^n)/Sq^a H^{2n-2-a}(\Omega S^n) \oplus \bigoplus_{c\in S} H^{2n-2-c}(\Omega S^n)$. Furthermore, we know that $\Phi_0(\tau+\gamma) = \Phi_0(\tau) + \Phi_0(\gamma) + \tau\gamma$ where Φ_0 is defined on τ and γ.

We have the loop multiplication map $\omega\colon \Omega S^n \times \Omega S^n \to \Omega S^n$. We note that since the Steenrod operations on $H^*(\Omega S^n)$ are trivial, $\Phi_0(\gamma_1(\alpha_{n-1}))$ is defined with zero indeterminacy. Then the Steenrod operations on $H^*(\Omega S^n \times \Omega S^n)$ are trivial, so Φ_0 is defined on $H^*(\Omega S^n \times \Omega S^n)$ with zero indeterminacy. Then,

$$\begin{aligned}\omega^*\Phi_0(\gamma_1(\alpha_{n-1})) &= \Phi_0(\omega^*(\gamma_1(\alpha_{n-1})))\\ &= \Phi_0(\gamma_1(\alpha_{n-1})\otimes 1 + 1\otimes\gamma_1(\alpha_{n-1}))\\ &= \Phi_0(\gamma_1(\alpha_{n-1}))\otimes 1 + 1\otimes\Phi_0(\gamma_1(\alpha_{n-1})) +\\ &\quad\ \gamma_1(\alpha_{n-1})\otimes\gamma_1(\alpha_{n-1}).\end{aligned}$$

Since $\gamma_1(\alpha_{n-1})\otimes\gamma_1(\alpha_{n-1}) \neq 0$, we have that $\omega^*(\Phi_0(\gamma_1(\alpha_{n-1}))$ is non-zero. Then, $\Phi_0(\gamma_1(\alpha_{n-1}))$ is non-zero on $H^{(n-1)}(\Omega S^n)$. For dimensional reasons, it must then be the case that

$$\Phi_0(\gamma_1(\alpha_{n-1})) = \gamma_2(\alpha_{n-1}).$$

Following Section 2.3, it must be the case that a_1 is not spherical.

We offer here a proof strictly using secondary cohomology operations to show that the remaining a_{k+1} (with $k > 0$) are also not spherical. In addition to showing that some elements of $H^*(\Omega S^n)$ are related by secondary cohomology operations the results of this argument will be useful when we look at $\Omega^2 S^n$.

Consider the Adem relation, where $k \geqslant 1$:

$$Sq^1 Sq^{2^k(n-1)} = \binom{2^k(n-1)-1}{1} Sq^{2^k(n-1)+1} = Sq^{2^k(n-1)+1}.$$

This yields,

$$Sq^{2^k(n-1)+1} = Sq^1 Sq^{2^k(n-1)}. \tag{4.2}$$

Let us call the secondary cohomology operations which stem from this relation Φ_k. So, Φ_k will act on elements of $H^{2^k(n-1)}(X)$ which vanish under $Sq^{2^k(n-1)}$, and will take values in $H^{2^{k+1}(n-1)}(X)/Sq^1(H^{2^{k+1}(n-1)}(X))$. Since the Steenrod operations act trivially on ΩS^n , Φ_k is defined on $H^{2^k(n-1)}(\Omega S^n)$ with no indeterminacy. To show that Φ_k is non-zero, we use the following result from [**Z**].

Lemma 4.1. *Let X and Y be CW complexes. Suppose $z = \sum_i \tau_i \otimes \gamma_i \in H^m(X \times Y)$ is in the domain of Φ_k with $|\tau_i|, |\gamma_i| > 0$. Let $\rho(\tau)$ and $\rho(\gamma)$ be the algebras over the Steenrod algebra generated by the τ_i's and γ_i's respectively. Then,*

$$\Phi_k(z) \cap [\rho(\tau)\otimes H^*(Y) + H^*(X)\otimes\rho(\gamma)] \neq \emptyset.$$

In order to apply this result, we study the action of Φ_k on $\Omega S^n \times \Omega S^n$. Again, the Steenrod operations act trivially on $\Omega S^n \times \Omega S^n$; so, Φ_k is defined with zero indeterminacy on $H^{2^k(n-1)}(\Omega S^n \times \Omega S^n)$. Recall that

$$\omega(\gamma_{2^k}(\alpha_{n-1})) = [1\otimes\gamma_{2^k}(\alpha_{n-1})] + [\gamma_{2^k}(\alpha_{n-1})\otimes 1] + [\textstyle\sum_i \tau_i\otimes\gamma_i]$$

where $2^k(n-1) > |\tau_i|, |\gamma_i| > 0$. Let $z, \rho(\tau), \rho(\gamma)$ be as in the statement of Lemma 4.1. We observe that $\gamma_{2^k}(\alpha_{n-1}) \notin \rho(\tau), \rho(\gamma)$ since $\gamma_{2^k}(\alpha_{n-1})$ is an indecomposable element of $H^*(\Omega S^n)$. Then,

$$\gamma_{2^k}(\alpha_{n-1}) \otimes \gamma_{2^k}(\alpha_{n-1}) \notin [\rho(\tau) \otimes H^*(\Omega S^n) + H^*(\Omega S^n) \otimes \rho(\gamma)].$$

Certainly z lies in the domain of Φ_k. Because there is no indeterminacy, $\Phi_k(z)$ is a singleton set, rather than a coset. So, applying Lemma 4.1, we have

$$\Phi_k(z) \in [\rho(\tau) \otimes H^*(\Omega S^n) + H^*(\Omega S^n) \otimes \rho(\gamma)].$$

So, $\gamma_{2^k}(\alpha_{n-1}) \otimes \gamma_{2^k}(\alpha_{n-1})$ is not a summand of $\Phi_k(z)$. Further,

$$\gamma_{2^k}(\alpha_{n-1}) \otimes \gamma_{2^k}(\alpha_{n-1})$$

is not a summand of

$$(\gamma_{2^k}(\alpha_{n-1}) \otimes 1 + 1 \otimes \gamma_{2^k}(\alpha_{n-1}))z$$

since $\gamma_{2^k}(\alpha_{n-1})$ is indecomposable and $z = \sum_i \tau_i \otimes \gamma_i$ with $|\tau_i|, |\gamma_i| > 0$. We evaluate

$$\begin{aligned}\omega^*\Phi_k(\gamma_{2^k}(\alpha_{n-1})) &= \Phi_k(\omega^*(\gamma_{2^k}(\alpha_{n-1})))\\ &= \Phi_k(\gamma_{2^k}(\alpha_{n-1}) \otimes 1 + 1 \otimes \gamma_{2^k}(\alpha_{n-1}) + z).\end{aligned}$$

By utilizing the the Additivity Formula, this expression evaluates to

$$\Phi_k(\gamma_{2^k}(\alpha_{n-1})) \otimes 1 + 1 \otimes \Phi_k(\gamma_{2^k}(\alpha_{n-1})) + \gamma_{2^k}(\alpha_{n-1}) \otimes \gamma_{2^k}(\alpha_{n-1}) + \Phi_k(z) + \\ (\gamma_{2^k}(\alpha_{n-1}) \otimes 1 + 1 \otimes \gamma_{2^k}(\alpha_{n-1}))z$$

From our observations above, we note that $\gamma_{2^k}(\alpha_{n-1}) \otimes \gamma_{2^k}(\alpha_{n-1})$ does not cancel out. Hence,

$$\omega^*\Phi_k(\gamma_{2^k}(\alpha_{n-1})) \neq 0.$$

Then, $\Phi_k(\gamma_{2^k}(\alpha_{n-1})) \neq 0$ and for dimensional reasons, it must be the case that

$$\Phi_k(\gamma_{2^k}(\alpha_{n-1})) = \gamma_{2^{k+1}}(\alpha_{n-1}). \tag{4.3}$$

As demonstrated in Section 2.3, we may conclude that a_{k+1} is not spherical. Thus, we have shown for all $k \geqslant 0$, a_{k+1} is not spherical. So, ΩS^n is minimal atomic for n not a power of 2.

4.3. $n = 2^{r+1}$ for $r \geqslant 3$

Our method is similar to that above. To show a_{k+1} is not spherical for each $k \geqslant 0$, we exhibit a cohomology operation whose image hits $\gamma_{2^{k+1}}(\alpha_{n-1})$. Happily, our previous argument goes through in the case $k \geqslant 1$. The previously established Φ_k have the property that $\Phi_k(\gamma_{2^k}(\alpha_{n-1})) = \gamma_{2^{k+1}}(\alpha_{n-1})$. So, we still must show that a_1 is not spherical. To show that a_1 is not spherical, we show that the tertiary operation Ψ, constructed in Subsection 3.3 has the property that $\Psi(\gamma_1(\alpha_{n-1})) = \gamma_2(\alpha_{n-1})$.

Let us evaluate Ψ on the element, $\gamma_1(\alpha_{n-1}) \in H^{n-1}(\Omega S^n)$. To show that Ψ acts on $\gamma_1(\alpha_{n-1})$, we first recall that the Steenrod operations act trivially on ΩS^n. Thus, if $\gamma_1(\alpha_{n-1})$ is represented by a map $\tau\colon \Omega S^n \to K(2^{r+1}-1)$, we have a lift $\overline{\tau}\colon \Omega S^n \to \Omega A_1$. The composite $\Omega f_2 \circ \overline{\tau}$ represents the product of cohomology classes $\prod_{i\leqslant j, i+1\neq j} \Phi_{ij}(\tau)$. Recall that $\Sigma\Omega S^n$ is homotopy equivalent to $\bigvee_{i=1}^{\infty} S^{i(n-1)}$. We know that Φ_{ij} is trivial on $S^{i(n-1)}$ for $i \geqslant 1$, so Φ_{ij} is trivial on $\Sigma\Omega S^n$. Because Φ_{ij} is a stable cohomology operation, we have that Φ_{ij} is zero on ΩS^n. Thus, $\Phi_{ij}(\tau)$ is zero with zero indeterminacy, and we have the essential lift $\overline{\overline{\tau}}\colon \Omega S^n \to \Omega A_2$. That is, $\Psi(\tau)$ is defined.

Appealing to Section 4.2, with Ψ replaced by Φ_0, we see that $\Psi(\gamma_1(\alpha_{n-1}))$ is non-zero. For dimensional reasons, we have

$$\Psi(\gamma_1(\alpha_{n-1})) = \gamma_2(\alpha_{n-1}).$$

Thus, as outlined in Section 2.3, a_1, the dual element of $\gamma_2(\alpha_{n-1})$, cannot be spherical.

We have shown that no spherical elements of ΩS^n exist above the Hurewicz dimension. Thus, ΩS^n is minimal atomic for $n = 2^{r+1}$ where $r \geqslant 3$. In summary, we have shown that ΩS^n is minimal atomic for n where $n \neq 2^{r+1}$ for $r < 3$.

5. Showing $S^n\{2^r\}$ for $r > 1$ is minimal atomic

We show that $S^n\{2^r\}$ is minimal atomic when n is not a power of 2. When n is a power of 2 with $n \neq 1, 2, 4, 8$, we are not able to prove $S^n\{2^r\}$ is minimal atomic, but we carry out an examination using higher order cohomology operations and illustrate what is lacking to carry out a complete proof. Here, our spaces are automatically 2-local by inspection of homotopy groups.

Recall that $S^n\{2^r\}$ is defined to be the homotopy fiber of the degree 2^r map $f\colon S^n \to S^n$. A long exact sequence of homotopy groups arises:

$$\cdots \to \pi_{k+1}(S^n) \xrightarrow{2^r} \pi_{k+1}(S^n) \to \pi_k(S^n\{2^r\}) \to \pi_k(S^n) \xrightarrow{2^r} \pi_k(S^n) \to \cdots$$

By examining the cases when $k \leqslant n-1$, we see that $S^n\{2^r\}$ is $(n-2)$ connected with $\pi_{(n-1)}(S^n\{2^r\}) = \mathbb{Z}/2^r\mathbb{Z}$. This shows that $S^n\{2^r\}$ is a Hurewicz complex. We see that $\pi_n(S^n\{2^r\}) = \pi_{n+1}(S^n)/2^r\pi_{n+1}(S^n)$ for $k = n$. Since $\pi_{n+1}(S^n) = \mathbb{Z}/2\mathbb{Z}$, we have that $\pi_n(S^n\{2^r\}) = \mathbb{Z}/2\mathbb{Z}$. Now $\pi_{4n-1}(S^{2n})$ is a direct sum of $\mathbb{Z}$ and a finite group. Otherwise, $\pi_q(S^n)$ is finite if $q > n$. These facts allow us to deduce that $\pi_k(S^n\{2^r\})$ is composed strictly of 2-groups.

5.1. Spherical candidates of $S^n\{2^r\}$

Computations with the Serre spectral sequence on the induced fibration $\Omega S^n \to S^n\{2^r\} \to S^n$ allow us to conclude that

$$H^*(S^n\{2^r\}, \mathbb{F}_2) = P[\gamma_{2^k}(\alpha_{n-1})]/(\gamma_{2^k}(\alpha_{n-1}))^2 \otimes E[\beta_n]$$

with trivial Steenrod operation action. So, the primitive elements of $S^n\{2^r\}$ are the duals of the elements $\gamma_{2^k}(\alpha_{n-1})$ and β_n which we shall label a_k and b_n. Since the image of the Hurewicz homomorphism is contained in the primitive elements of $S^n\{2^r\}$, we must show for $k \geqslant 0$ that a_{k+1} and b_n are not spherical. First, we examine b_n.

If b_n is spherical, there exists a non-trivial map $j\colon S^n \to S^n\{2^r\}$ which induces an isomorphism in mod 2 homology. By naturality, we have the following diagram:

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "j_*"] \arrow[d, "\beta^r"] & H_n(S^n\{2^r\}) \arrow[d, "\beta^r"] \\
H_n(S^{n-1}) \arrow[r, "j_*"] & H_{n-1}(S^n\{2^r\})
\end{tikzcd}$$

where β^r is the rth Bockstein operator. Now, $\beta^r \circ j_*$ hits the element $\gamma_1(\alpha_{n-1})$ while $j_* \circ \beta^r$ is trivial. We have a contradiction, and so b_n cannot be spherical.

5.2. $n \neq 2^{r+1}$

Now, if n is not a power of 2, our proof of the minimal atomicity of ΩS^n applies exactly with ΩS^n replaced by $S^n\{2^r\}$. Since the Steenrod operations on $S^n\{2^r\}$ are trivial, we may use the same secondary cohomology operations, Φ_k for $k \geqslant 0$ to show that the a_{k+1} are not spherical.

5.3. $n = 2^{r+1}$ for $r \geqslant 3$

If $n = 2^r$ for some r with $n \neq 1, 2, 4, 8$ we will see that modelling the proof for ΩS^n has a stumbling block. As before, for $k \geqslant 1$, the secondary cohomology operations Φ_k show that a_{k+1} is not spherical. So, we still have some work to do to show that a_1 is not spherical. Fred Cohen has also pointed out that an analogous James-Hopf map $S^n\{2^r\} \to S^{2n+1}\{2^r\}$ exists which can be used inductively to show that the minimal atomicity of $S^n\{2^r\}$ reduces to the problem of whether a_1 is spherical.

If we try to apply our tertiary operation, Ψ, to $\gamma_1(\alpha_{n-1})$ we must check that the Φ_{ij} are zero on $\gamma_1(\alpha_{n-1})$. We observe that the Φ_{ij} are defined on $S^n\{2^r\}$ since the Steenrod operations act trivially on $S^n\{2^r\}$. Recall that $i, j \leqslant r-1$ and

$$|\Phi_{ij}(\gamma_1(\alpha_{n-1}))| = 2^i + 2^j + 2^r - 2 > 2^r - 1.$$

The largest value for $2^i + 2^j + 2^r - 2$ is $2^{r+1} - 2 = |\gamma_2(\alpha_{n-1})|$. Based solely on degree considerations, then, it is possible that $\Phi_{r-1,r-1}(\gamma_1(\alpha_{n-1}))$ is not zero since it might take on the value $\gamma_2(\alpha_{n-1})$. Other potential values of $\Phi_{ij}(\gamma_1(\alpha_{n-1}))$ are β_n and $\beta_n\gamma_1(\alpha_{n-1})$. By evaluating degrees, we see that it is possible that

$$\beta_n = \Phi_{0,0}(\gamma_1(\alpha_{n-1})).$$

On the other hand,

$$\beta_n\gamma_1(\alpha_{n-1}) \neq \Phi_{ij}(\gamma_1(\alpha_{n-1})).$$

So, to apply the tertiary cohomology argument, we must show

$$\Phi_{r-1,r-1}(\gamma_1(\alpha_{n-1})) \neq \gamma_2(\alpha_{n-1})$$

and

$$\Phi_{0,0}(\gamma_1(\alpha_{n-1})) \neq \beta_n.$$

However, if it is the case that $\Phi_{r-1,r-1}(\gamma_1(\alpha_{n-1})) = \gamma_2(\alpha_{n-1})$, then we can show $S^n\{2^r\}$ is minimal atomic without resorting to tertiary cohomology operations arguments. We simply apply the Section 2.3 argument using $\Phi_{r-1,r-1}$. Hence, if

$\Phi_{r-1,r-1}(\gamma_1(\alpha_{n-1})) = \gamma_2(\alpha_{n-1})$ we have shown that $S^n\{2^r\}$ is minimal atomic. Otherwise, if $\Phi_{r-1,r-1}(\gamma_1(\alpha_{n-1})) \neq \gamma_2(\alpha_{n-1})$, it must be the case that $\Phi_{r-1,r-1}(\gamma_1(\alpha_{n-1})) = 0$. In this scenario, we are able to show that $S^n\{2^r\}$ is minimal atomic only if $\Phi_{0,0}(\gamma_1(\alpha_{n-1})) \neq \beta_n$. Then $\Phi_{0,0}(\gamma_1(\alpha_{n-1})) = 0$, and we will be able to use our previous tertiary operation argument.

6. A filtration of $\Omega^2 S^n$

The James construction provides a filtration of ΩS^n. We shall use this to build a filtration of $\Omega^2 S^n$ with nice properties. Later, we will use this filtration to study the minimal atomicity of $\Omega^2 S^n$ when n is even.

6.1. The cohomology of F_k

Recall that the James construction, $J(S^{n-1})$, is the free monoid on S^{n-1} with basepoint the identity. Then, the kth-filtration, $J_k(S^{n-1})$ is the subspace of words of length at most k. Abbreviating $J_k(S^{n-1})$ by F_k, we have $\operatorname{colim} F_k \simeq J(S^{n-1})$. We recall since S^{n-1} is a connected CW-complex that $J(S^{n-1}) \simeq \Omega\Sigma S^{n-1}$. Furthermore, $F_k/F_{k-1} \simeq S^{k(n-1)}$. Now, the cofibration $F_k \to F_{k+1}$ induces a long exact sequence in homology,

$$\cdots \to \tilde{H}_*(F_{k-1}) \to \tilde{H}_*(F_k) \to \tilde{H}_*(S^{k(n-1)}) \to \cdots$$

Using the fact that $F_1 \simeq S^{n-1}$, we may proceed inductively to show

$$\tilde{H}_*(F_k) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{if } * = i(n-1) \text{ for } 1 \leqslant i \leqslant k; \\ 0, & \text{otherwise} \end{cases}$$

and

$$\tilde{H}_*(F_{k-1}) \to \tilde{H}_*(F_k) \text{ is a monomorphism.}$$

In this case, $\tilde{H}_*(F_{k-1}) \to \tilde{H}_*(\Omega S^n)$ is a monomorphism of coalgebras, and thus $H^*(\Omega S^n) \to H^*(F_k)$ is an epimorphism of algebras.

By taking the vector space dual of the computation of $\tilde{H}_*(F_k)$ above, we see there is one generator of $H^*(F_{2^k-1})$ in dimensions $i(n-1)$ for $0 \leqslant i \leqslant 2^k - 1$. These generators can be identified with $\gamma_i(\alpha_{n-1}) \in H^{i(n-1)}(\Omega S^n)$ for $0 \leqslant i \leqslant 2^k - 1$. Thus,

$$H^*(F_{2^k-1}) = E[\gamma_1(\alpha_{n-1}), \gamma_2(\alpha_{n-1}), \ldots, \gamma_{2^{k-1}}(\alpha_{n-1})].$$

We observe that $H^*(F_{2^k-1})$ inherits the Steenrod algebra structure of $H^*(\Omega S^n)$ via the map $H^*(\Omega S^n) \to H^*(F_{2^k-1})$. Thus, the Steenrod operations act trivially on $H^*(F_{2^k-1})$.

6.2. The cohomology of Y_k

Let Y_k be the homotopy fiber of $F_{2^k-1} \to F_{2^{k+1}-1}$.

Proposition 6.1. $H^*(Y_k) = \Gamma[\sigma\gamma_{2^k(\alpha_{n-1})}]$.

We prove Proposition 6.1 using the Eilenberg–Moore spectral sequence.

$$\begin{aligned} E_2 &= \mathrm{Tor}_{H^*(F_{2^{k+1}-1})}(H^*(F_{2^k-1}), \mathbb{Z}/2\mathbb{Z}) \\ &= \mathrm{Tor}_{E[\gamma_1(\alpha_{n-1}),\ldots,\gamma_{2^k}(\alpha_{n-1})]}(E[\gamma_1(\alpha_{n-1}),\ldots,\gamma_{2^{k-1}}(\alpha_{n-1})], \mathbb{Z}/2\mathbb{Z}) \end{aligned}$$

By the Change of Rings Theorem from [**Mc**], we have

$$\mathrm{Tor}_{R\otimes S}(R, \mathbb{Z}/2\mathbb{Z}) = \mathrm{Tor}_S(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}).$$

Applied here, we have

$$\mathrm{Tor}_{E[\gamma_1(\alpha_{n-1}),\ldots,\gamma_{2^k-1}(\alpha_{n-1})]\otimes E[\gamma_{2^k}(\alpha_{n-1})]}(E[\gamma_1(\alpha_{n-1}),\ldots,\gamma_{2^{k-1}}(\alpha_{n-1})], \mathbb{Z}/2\mathbb{Z})$$

evaluates to

$$\mathrm{Tor}_{E[\gamma_{2^k}(\alpha_{n-1})]}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \Gamma[\sigma\gamma_{2^k}(\alpha_{n-1})].$$

Inspection shows that there are no dimensional candidates for non-trivial differentials on these generators, so $E_2 = E\infty$.

We may conclude that modulo extensions, $H^*(Y_k) = \Gamma[\sigma\gamma_{2^k}(\alpha_{n-1})]$. Let us denote $\sigma\gamma_{2^k}(\alpha_{n-1})$ by ζ_k. To show that as an algebra $H^*(Y_k) = \Gamma[\zeta_k]$, we examine the Serre cohomology spectral sequence applied to $Y_k \to F_{2^k-1} \to F_{2^{k+1}-1}$. Dimensionally, for each $\gamma_i(\zeta_k)$ there must be a differential d_r such that

$$d_r(\gamma_i(\zeta_k)) = \gamma_{i-1}(\zeta_k) \otimes \gamma_{2^k}(\alpha_{n-1}).$$

An inductive proof on m shows that

$$\gamma_i(\zeta_k)\cdot\gamma_j(\zeta_k) = \tbinom{i+j}{i}\gamma_{i+j}(\zeta_k)$$

where $i + j = m$. We assume that for all $i + j = m - 1$,

$$\gamma_i(\zeta_k)\cdot\gamma_j(\zeta_k) = \tbinom{i+j}{i}\gamma_{i+j}(\zeta_k).$$

Given i, j such that $i + j = m$, using the Leibniz rule, we have

$$\begin{aligned} d_r(\gamma_i(\zeta_k)\cdot\gamma_j(\zeta_k)) &= \gamma_{i-1}(\zeta_k)\cdot\gamma_j(\zeta_k)\otimes\gamma_{2^k}(\alpha_{n-1}) + \\ &\quad \gamma_i(\zeta_k)\cdot\gamma_{j-1}(\zeta_k)\otimes\gamma_{2^k}(\alpha_{n-1}) \\ &= \binom{m-1}{j}\gamma_{m-1}(\zeta_k)\otimes\gamma_{2^k}(\alpha_{n-1}) + \\ &\quad \binom{m-1}{j-1}\gamma_{m-1}(\zeta_k)\otimes\gamma_{2^k}(\alpha_{n-1}) \\ &= \binom{i+j}{i}\gamma_{i+j-1}(\zeta_k)\otimes\gamma_{2^k}(\alpha_{n-1}). \end{aligned}$$

It must be the case that

$$\gamma_i(\zeta_k)\cdot\gamma_j(\zeta_k) = \tbinom{i+j}{i}\gamma_{i+j}(\zeta_k)$$

since d_r takes the same value on both. Then, $H^*(Y_k)$ is a divided polynomial algebra on ζ_k.

6.3. The cohomology of G_k

We denote $G_k = \Omega F_{2^k-1}$.

Proposition 6.2. $H^*(G_k) = \bigotimes_{i=0}^{k-1} \Gamma[\lambda_i]$ *where* $|\lambda_i| = 2^i(n-1) - 1$.

We show Proposition 6.2 by dualizing $H_*(G_k) = P[l_0, \ldots, l_{k-1}]$. This homology computation can be proved by induction by utilizing the following Proposition:

Proposition 6.3. *In a spectral sequence of coalgebras, given the smallest r such that $d^r \neq 0$, it must be the case that for the smallest degree element x with $d^r(x) \neq 0$, $d^r(x)$ must be primitive.*

Then, by dualizing, we have that as vector spaces $H^*(G_k) = \Gamma[(l_0)^*, (l_1)^*, \ldots, (l_{k-1})^*]$. Consider the cohomology Serre spectral sequence applied to the fibration $G_{k-1} \to G_k \to Y_{k-1}$. As vector spaces

$$\begin{aligned} E_2 &= H^*(G_{k-1}) \otimes H^*(Y_{k-1}) \\ &= \bigotimes_{i=0}^{k-2} \Gamma[\lambda_i] \otimes \Gamma[\zeta_{k-1}]. \end{aligned}$$

But, the spectral sequence converges to the cohomology of $H^*(G_k)$ and E_2 agrees with the cohomology of $H^*(G_k)$ as a vector space. Then, $E_2 = E_\infty$. We may identify ζ_{k-1} with λ_{k-1} under $H_*(G_k) \to H_*(Y_{k-1})$ and so, we have a splitting of algebras,

$$H^*(G_k) = H^*(G_{k-1}) \otimes H^*(Y_{k-1}), \tag{6.1}$$

that is,

$$H^*(G_k) = \bigotimes_{i=0}^{k-1} \Gamma[\lambda_i].$$

Now, λ_{k-1} is an indecomposable element. So, it is the dual of a primitive element of $H_*(G_k)$. The only possibility is that $(l_{k-1})^* = \lambda_{k-1}$, and so $H_*(G_k)$ is primitively generated by the permanent cycles l_i. We have the desired results. We see from above that $H_*(G_{k-1)} \hookrightarrow H_*(G_k)$. Then,

$$H_*(G_k) \to H_*(\Omega^2 S^n) \text{ is a monomorphism of Hopf algebras}$$

so that we may identify l_i with r_i. Then,

$$H^*(\Omega^2 S^n) \to H^*(G_k) \text{ is an epimorphism of Hopf algebras.}$$

We identify $\gamma_j(\rho_i)$ with $\gamma_j(\lambda_i)$. Thus, $H^*(G_k)$ inherits the Steenrod algebra structure of $H^*(\Omega^2 S^n)$ via the map $H^*(\Omega^2 S^n) \to H^*(G_k)$.

6.4. The equivalence of Y_k with $\Omega S^{2^k(n-1)}$

Note that $H^*(Y_k) \simeq H^*(\Omega S^{2^k(n-1)})$. In fact, we will show $Y_k \simeq \Omega S^{2^k(n-1)}$. Our first aim will be to show that a map between $\Omega S^{2^k(n-1)}$ and Y_k exists. To accomplish this consider the diagram below where the top horizontal map is the defining fibration for Y_k, the bottom horizontal map is a standard path fibration, and the middle vertical arrow is the constant map,

$$\begin{tikzcd} Y_k \arrow[r] \arrow[d, dashed] & F_{2^k-1} \arrow[r] \arrow[d] & F_{2^{k+1}-1} \arrow[d, dashed] \\ \Omega S^{2^k(n-1)} \arrow[r] & * \arrow[r] & S^{2^k(n-1)} \end{tikzcd} \quad . \tag{6.2}$$

We show that the vertical right arrow exists, thereby giving the existence of a left vertical arrow which makes the whole diagram commute up to homotopy.

Now, it is known that F_k is a CW complex which is a subcomplex of F_{k+1}. We recall an explicit description of the cofibration $F_k \to F_{k+1}$. Since $S^{n-1} \simeq D^{n-1}/S^{n-2}$, we have the collapsing map, $D^{n-1} \to S^{n-1}$. By taking Cartesian products, we have $(D^{n-1})^{k+1} \to (S^{n-1})^{k+1}$. Given $x \in (D^{n-1})^{k+1}$, x may be thought of as a word in D^{n-1} with $k+1$ letters, while the boundary of $(D^{n-1})^{k+1}$ consists of those words in which at least one letter lies on the boundary of D^{n-1}. So, we have a map, $\partial(D^{n-1})^{k+1} \to F_k$. But $(D^{n-1})^{k+1} \simeq D^{(n-1)(k+1)}$, so we have $S^{(n-1)(k+1)-1} \to F_k$. Then, F_{k+1} is the pushout of the following diagram,

$$\begin{tikzcd} S^{(n-1)(k+1)-1} \arrow[r] \arrow[d, "I_k"] & D^{(n-1)(k+1)} \arrow[d] \\ F_k \arrow[r] & F_{k+1} \end{tikzcd}$$

Suppose we have a map $f_k \colon F_k \to X$. A sufficient condition for f_k to extend to $f_{k+1} \colon F_{k+1} \to X$ is that $f_k \circ l_k \colon S^{(n-1)(k+1)-1} \to X$ is null-homotopic. If so, there exists a homotopy $H \colon S^{(n-1)(k+1)-1} \times I \to F_k$ from $f_k \circ l_k$ to the constant basepoint map. Then H induces a map, $D^{(n-1)(k+1)} \to X$ and by the pushout property, a map f_{k+1} exists with the desired properties. We apply this principle multiple times to obtain the vertical map $F_{2^{k+1}-1} \to S^{2^k(n-1)}$ in (6.2).

Observe that there exists a map $F_{2^k} \to S^{2^k(n-1)}$ which maps the word $x_1 x_2 \cdots x_{2^k}$ to $x_1 \wedge x_2 \wedge \cdots \wedge x_{2^k}$, where $x_i \in S^{n-1}$, and we use $(S^{n-1})^{(m)} \simeq S^{m(n-1)}$. Let f_{2^k} denote this map, $F_{2^k} \to S^{2^k(n-1)}$; the composite $F_{2^k-1} \to F_{2^k} \xrightarrow{f_{2^k}} S^{2^k(n-1)}$ is trivial (as F_{2^k-1} is mapped to the basepoint). We show that extensions $f_{2^k+1}, \ldots, f_{2^{k+1}-1}$ exist as in the diagram below,

$$\begin{tikzcd} F_{2^k-1} \arrow[r] \arrow[d, "0"] & F_{2^k} \arrow[r] \arrow[dl, "f_{2^k}"] & F_{2^k+1} \arrow[r] \arrow[dll, dashed, "f_{2^k+1}"] & \cdots \arrow[r] & F_{2^{k+1}-1} \arrow[dllll, dashed, "f_{2^{k+1}-1}"] \\ S^{2^k(n-1)} \end{tikzcd} \quad . \tag{6.3}$$

Based on our study of extensions above, for $2^k+1 \leqslant i \leqslant 2^{k+1}-1$, to show the existence of f_i, we must examine

$$f_{i-1} \circ l_{i-1} \in \pi_{(n-1)i-1}(S^{2^k(n-1)}).$$

By the Freudenthal suspension theorem[**M**, p. 83], we know that when

$$(n-1)i - 1 < 2(2^k(n-1)) - 1,$$

we have an isomorphism

$$\Sigma\colon \pi_{(n-1)i-1}(S^{2^k(n-1)}) \to \pi_{(n-1)i}(S^{2^k(n-1)+1}).$$

Since $n > 1$,

$$\begin{aligned}(n-1)i - 1 &\leqslant (n-1)\cdot(2^{k+1}-1) - 1\\ &= 2(2^k(n-1)) - (n-1) - 1\\ &< 2(2^k(n-1)) - 1.\end{aligned}$$

So we are in the range where $\Sigma\colon \pi_{(n-1)i-1}(S^{2^k(n-1)}) \to \pi_{(n-1)i}(S^{2^k(n-1)+1})$ is an isomorphism. Because we are interested in $f_{k+i} \circ l_{k+i} \in \pi_{(n-1)i-1}(S^{2^k(n-1)})$, we may investigate the corresponding element in $\pi_{(n-1)i}(S^{2^k(n-1)+1})$. To do this, we look at what happens to our extension problem when we apply the suspension to our spaces and maps. Consider the diagram,

$$\begin{tikzcd}
\Sigma F_{2^k-1} \arrow[r] \arrow[d, "0"] & \Sigma F_{2^k} \arrow[r] \arrow[dl, "\Sigma f_{2^k}"'] & \Sigma F_{2^k+1} \arrow[r] \arrow[dll, dashed, "\Sigma f_{2^k+1}"'] & \cdots \arrow[r] & \Sigma F_{2^{k+1}-1} \arrow[dllll, dashed, "\Sigma f_{2^{k+1}-1}"] \\
\Sigma S^{2^k(n-1)} & & & &
\end{tikzcd}$$

However, for all positive j,

$$\Sigma F_j \simeq \bigvee_{i=1}^{j} \sum (S^{(n-1)})^{(j)}.$$

In particular $\Sigma f_{2^k-1}\colon \Sigma F_{2^k-1} \to S^{2^k(n-1)+1}$ is the identity on $S^{2^k(n-1)+1}$ and maps all other points in the domain to the basepoint. It is clear that the extensions Σf_{2^k+i} exist. Similar to Σf_{2^k}, Σf_{2^k+i} is the identity on $S^{2^k(n-1)+1}$ and maps all other points in the domain to the basepoint. So, we have

$$\begin{tikzcd}
\bigvee_{i=1}^{2^k-1} S^{i(n-1)+1} \arrow[r] \arrow[d, "0"] & \bigvee_{i=1}^{2^k} S^{i(n-1)+1} \arrow[r] \arrow[dl, "\Sigma f_{2^k}"'] & \cdots \arrow[r] & \bigvee_{i=1}^{2^{k+1}-1} S^{i(n-1)+1} \arrow[dlll, "\Sigma f_{2^{k+1}-1}"] \\
S^{2^k(n-1)+1} & & &
\end{tikzcd} \qquad (6.4)$$

Recall that $l_k\colon S^{(n-1)(k+1)-1} \to F_k$ is the attaching map for F_{k+1} in its CW decomposition. Then, Σl_k is the attaching map for ΣF_{k+1}. We consider the diagram below, where the square is a pushout diagram,

$$\begin{tikzcd}
S^{(n-1)(k+1)} \arrow[r] \arrow[d, "\Sigma l_k"] & D^{(n-1)(k+1)+1} \arrow[d] \\
\Sigma F_k \arrow[r] \arrow[d, "\Sigma f_k"] & \Sigma F_{k+1} \arrow[dl, "\Sigma f_{k+1}"] \\
X &
\end{tikzcd}$$

Since the extension Σf_{k+1} exists, we can show that the composite $\Sigma f_k \circ \Sigma l_k$ must be null-homotopic. We have the composite $\alpha\colon D^{(n-1)(k+1)} \to \Sigma F_{k+1} \xrightarrow{\Sigma f_{k+1}} X$. But, $D^{(n-1)(k+1)-1}$ is just $CS^{(n-1)(k+1)}$. This gives a homotopy from the constant map to the restriction of α to the boundary. This restriction is the map $\Sigma f_k \circ \Sigma l_k$. Thus, $\Sigma f_k \circ \Sigma l_k \in \pi_{(n-1)(k+1)}(X)$ corresponds to the zero element.

Applying these ideas to the extensions that exist in (6.4) yields that for $2^k + 1 \leqslant i \leqslant 2^{k+1} - 1$,

$$\Sigma l_{i-1} \circ \Sigma f_{i-1} = 0 \in \pi_{(n-1)i}(S^{2^k(n-1)+1}).$$

Yet because we are in the range where the Freudenthal suspension is an isomorphism, we have that

$$l_{i-1} \circ f_{i-1} = 0 \in \pi_{(n-1)i-1}(S^{2^k(n-1)}).$$

This shows that for $2^k + 1 \leqslant i \leqslant 2^{k+1} - 1$, f_i exists, and hence we have a map

$$f_{2^{k+1}-1}\colon F_{2^{k+1}-1} \to S^{2^k(n-1)}.$$

Furthermore, by examination of (6.3), we see that the composite

$$F_{2^k-1} \to F_{2^k} \to \cdots \to F_{2^{k+1}-1} \to S^{2^k(n-1)}$$

is null homotopic. So we have shown the existence of a square which commutes up to homotopy, as in the rightmost square of (6.2). Thus, there must exist a map

$$y_k\colon Y_k \to \Omega S^{2^k(n-1)}$$

such that (6.2) commutes up to homotopy.

Let E denote the cohomology Serre spectral sequence of the fibration

$$Y_k \to F_{2^k} \to F_{2^{k+1}-1}.$$

Let ${}'E$ denote the cohomology Serre spectral sequence of the fibration

$$\Omega S^{2^k(n-1)} \to * \to S^{2^k(n-1)}.$$

We have the following commutative diagram.

$$\begin{tikzcd}
{}'E^{0,p} \arrow[r, "{}'d_r"] \arrow[d, "(y_k)^*"] & {}'E^{r,p+r-1} \arrow[d, "(y_k)^* \otimes (f_{2^{k+1}-1})^*"] \\
E^{0,p} \arrow[r, "d_r"] & E^{r,p+r-1}
\end{tikzcd}$$

Studying E, one can see that there are differentials d_r such that

$$d_r(\gamma_i(\zeta_k)) = \gamma_{i-1}(\zeta_k) \otimes \gamma_{2^k}(\alpha_{n-1}).$$

As algebras, $H^*(\Omega S^{2^k(n-1)}) = \Gamma(\alpha_{2^k(n-1)-1})$ and $H^*(S^{2^k}(n-1)) = E[\iota_{2^k(n-1)}]$. Examining ${}^{\backprime}E$ we see there are differentials ${}^{\backprime}d_r$ such that

$${}^{\backprime}d_r(\gamma_i(\alpha_{2^k(n-1)-1})) = \gamma_{i-1}(\alpha_{2^k(n-1)-1}) \otimes \iota_{2^k(n-1)}.$$

Using inductive reasoning, knowing that $(f_{2^{k+1}-1})^*(\gamma_{2^k}(\alpha_{n-1})) = \iota_{2^k(n-1)}$, yields the following diagram of elements.

$$\begin{tikzcd}
\gamma_i(\alpha_{2^k(n-1)-1}) \arrow[r, "{}^{\backprime}d_r"] \arrow[d, dashed, "(y_k)^*"] & \gamma_{i-1}(\alpha_{2^k(n-1)-1}) \otimes \iota_{2^k(n-1)} \arrow[d, "(y_k)^* \otimes (f_{2^{k+1}-1})^*"] \\
\gamma_i(\zeta_k) \arrow[r, "d_r"] & \gamma_{i-1}(\zeta_k) \otimes \gamma_{2^k}(\alpha_{n-1})
\end{tikzcd}$$

As depicted, it must be the case that

$$(y_k)^*(\gamma_i(\alpha_{2^k(n-1)-1})) = \gamma_i(\zeta_k).$$

It follows that $(y_k)^*$ is an isomorphism in cohomology. Thus, since our spaces are localized at the prime 2, we have that y_k is an equivalence and

$$Y_k \simeq \Omega S^{2^k(n-1)}.$$

In particular, the splitting of algebras in (6.1) (with k replaced by $k+1$) becomes

$$H^*(G_{k+1}) = H^*(G_k) \otimes H^*(\Omega S^{2^k(n-1)}) \tag{6.5}$$

with $H^*(\Omega S^{2^k(n-1)}) = \Gamma[\zeta_k]$.

7. Showing $\Omega^2 S^n$ is minimal atomic

We prove the minimal atomicity of $\Omega^2 S^n$ by breaking the argument into the cases when n is odd and when n is even. The case where n is odd resembles the proof that ΩS^n is minimal atomic. Yet, for n even, there are too many elements to contend with to use previous arguments alone, and we instead combine the methods of higher order cohomology operations with the filtration of Section 6 to obtain the desired conclusion. Finally, we look at the cases when $\Omega^2 S^n$ is not minimal atomic and isolate the elements which are obstructions to minimal atomicity.

7.1. Spherical candidates of $\Omega^2 S^n$

From [**KA**], we see $H_*(\Omega^2(S^n))$ is a polynomial ring having generators r_0 and $Q^J(r_0)$ where $|r_0| = n-2$ and Q^J are the Dyer–Lashof operations, where J is an admissible sequence such that $e(J) > n-2$ and $l(J) < n$. By an induction argument, we can show that any admissible J's satisfying these constraints must be of the form

$$((n-1)2^k,(n-1)2^{k-1},(n-1)2^{k-2},\dots,(n-1)).$$

We refer to the generator $Q^J(r_0)$ with

$$J=((n-1)2^k,(n-1)2^{k-1},(n-1)2^{k-2},\dots,(n-1))$$

by r_{k+1}. Using the Nishida relations, we can compute the action of the dual of the Steenrod algebra on $H_*(\Omega^2 S^n)$ to obtain the result as in [**C**, p. 29]:

$$Sq_*^{2^s}((r_k)^{2^t})=\begin{cases}0 & \text{if } s\neq t \text{ or } k=0 \text{ or if } n \text{ even with } s=t \text{ and } k=1\\ r_{k-1}^{2^{s+1}} & \text{if } s=t \text{ and } k>1 \text{ for } n \text{ even or if } s=t \text{ and } k\geqslant 1\\ & \text{for } n \text{ odd}\end{cases}$$

Now, $H^*(\Omega^2(S^n))=\bigotimes_{i\geqslant 0,j\geqslant 0} P[\gamma_{2^i}(\rho_j)]/(\gamma_{2^i}(\rho_j))^2$ where the dual of $(r_j)^{2^i}$ is $\gamma_{2^i}(\rho_j)$. Using this information, we can compute the action of the Steenrod operations on $H^*(\Omega^2 S^n)$:

$$Sq^{2^s}(\gamma_{2^t}(\rho_k))=\begin{cases}0 & \text{if } s+1\neq t, \text{ or if } n \text{ is even with } s+1=t \text{ and } k=0\\ \gamma_{2^s}(\rho_{k+1}) & \text{if } s+1=t \text{ with } n \text{ odd, or } s+1=t \text{ with } n \text{ even and}\\ & k\geqslant 1\end{cases}$$

7.2. n odd

Now, in the case where n is odd, each $(r_k)^{2^t}$ is not an A-annihilated primitive when $k\geqslant 1$. Thus, these elements cannot be spherical. The only candidates for spherical elements above the Hurewicz dimension are $(r_0)^{2^t}$ for $t\geqslant 1$. We shall refer to these elements as p_t. We apply the techniques of higher order cohomology operations to prove that p_t is not spherical. For each $t>0$, we exhibit a higher order cohomology operation whose image includes $\gamma_{2^t}(\rho_0)$.

Let us establish how the Steenrod operations act on $\gamma_{2^t}(\rho_0)$ for $t\geqslant 0$. First, for $t=0$, we see from above that no Steenrod operation of the form Sq^{2^i} acts non-trivially on $\gamma_1(\rho_0)$. Since operations of the type Sq^{2^i} generate the Steenrod algebra, we conclude that all Steenrod operations annihilate $\gamma_1(\rho_0)$.

For $t>0$, the only generator of the Steenrod algebra which acts non-trivially on $\gamma_{2^t}(\rho_0)$ is $Sq^{2^{t-1}}$; the result of this action is $\gamma_{2^{t-1}}(\rho_1)$. If $t-1>0$, again, we have exactly one generator of the Steenrod algebra which acts non-trivially on $\gamma_{2^{t-1}}(\rho_1)$, $Sq^{2^{t-2}}$. Proceeding in this fashion, we see that the Steenrod operations which act non-trivially on $\gamma_{2^t}(\rho_0)$ are of the form Sq^I where $I=(2^m,2^{m+1},\dots,2^{t-1})$ where $0\leqslant m\leqslant t-1$. In particular, the highest dimensional Steenrod operation which can act on $\gamma_{2^t}(\rho_0)$ is of degree $1+2+\ldots+2^{t-1}=2^t-1$.

7.2.1. Showing p_{t+1} is not spherical when $t>0$

Consider the relation $Sq^{2^t(n-2)+1}=Sq^1Sq^{2^t(n-2)}$ which is similar to (4.2) with $n-1$ replaced by $n-2$ when $t>0$. This relation gives rise to a secondary cohomology operation which we shall call Θ_t. Then, for appropriate θ and γ,

$$\Theta_t(\tau+\gamma)=\Theta_t(\tau)+\Theta_t(\gamma)+\tau\gamma.$$

The defining cohomology class $\phi_2 \in H^{2^{t+1}(n-2)}(\Omega E_2)$ is primitive. Θ_t is defined on elements $\tau \in H^{2^t(n-2)}(X)$ for which $Sq^{2^t(n-2)}$ has a trivial action, and the image

$$\Theta_t(\tau) \in H^{2^{t+1}(n-2)}(X)/Sq^1(H^{2^{t+1}(n-2)-1}(X).$$

In particular, we see that $Sq^{2^t(n-2)}$ acts trivially on $\gamma_{2^t}(\rho_0)$ since

$$|\gamma_{2^t}(\rho_0)| = 2^t(n-2)$$

and

$$(\gamma_{2^t}(\rho_0))^2 = 0.$$

So, Θ_t is defined on $\gamma_{2^t}(\rho_0)$ for $t > 0$. Then,

$$\gamma_{2^{t+1}}(\rho_0) \notin Sq^1(H^{2^{t+1}(n-2)-1}(\Omega^2 S^n)).$$

For $t > 0$, $\gamma_{2^{t+1}}(\rho_0)$ clearly is not in the image of Sq^1 and does not belong to the indeterminacy of Θ_t.

Let $f\colon \Omega S^{n-1} \to \Omega^2 S^n$ be the result of looping the canonical map η. Then, f maps the indecomposable elements $\gamma_{2^t}(\rho_0)$ of $\Omega^2 S^n$ to the indecomposable elements $\gamma_{2^t}(\alpha_{n-2})$ of ΩS^{n-1}. The following diagram commutes:

$$\begin{tikzcd}
H^{(n-2)\cdot 2^t}(\Omega^2 S^n) \arrow[r, "f^*"] \arrow[d, "\Theta_t"] & H^{(n-2)\cdot 2^t}(\Omega S^{n-1}) \arrow[d, "\Theta_t"] \\
H^{(n-2)\cdot 2^{t+1}}(\Omega^2 S^n) \arrow[r, "f^*"] & H^{(n-2)\cdot 2^{t+1}}(\Omega S^{n-1}))
\end{tikzcd}$$

Now, $\Theta_t(f^*(\gamma_{2^t}(\rho_0))_{=}\gamma_{2^{t+1}}(\alpha_{n-2})$. By chasing the diagram, we see that modulo $Sq^1(H^{2^{t+1}(n-2)-1}(\Omega^2 S^n))$, we have $\Theta_t(\gamma_{2^t}(\rho_0)) = \gamma_{2^{t+1}}(\rho_0) + \tau$ where τ is in the kernel of $H^{(n-2)2^{t+1}}(\Omega^2 S^n) \to H^{(n-2)2^{t+1}}(\Omega S^{n-1})$; τ must be a decomposable element since the only elements τ of degree $(n-2)2^{t+1}$ besides $\gamma_{2^{t+1}}(\rho_0)$ are decomposable elements of $H^{(n-2)2^{t+1}}(\Omega^2 S^n)$. So $\tau = \sum a_i b_i$ where $|a_i|, |b_i| > 0$. As we have noted the module of indeterminacy $Sq^1(H^{(n-1)2^{t+1}-1}(\Omega^2 S^n))$ does not contain $\gamma_{2^{t+1}}(\rho_0)$, so it must contain only decomposables. Thus,

$$\Theta_t(\gamma_{2^t}(\rho_0)) = \gamma_{2^{t+1}}(\rho_0) + \tau$$

where $\tau \in H^{(n-2)2^{t+1}}(\Omega^2 S^n)$ is a sum of decomposable elements (possibly zero).

Appealing to Section 2.3, we have for $t > 0$, p_{t+1} is not spherical.

7.2.2. Showing p_1 is not spherical when $n-1 \neq 2^r$

We show that p_1 is not spherical. To accomplish this, we exhibit a higher order cohomology operation which we evaluate on $\gamma_1(\rho_0)$ to yield $\gamma_2(\rho_0)$. First, we take the case that $n-1$ is not a power of 2. Then, Sq^{n-1} has a factorization as in (3.1), with

$$Sq^{n-1} = Sq^a Sq^{2^r} + \sum_{c>0} \binom{2^r - c - 1}{a - 2c} Sq^{(n-1)-c} Sq^c$$

where 2^r is the largest power of 2 which appears in the binomial expansion of $n-1$ and $(n-1) = a + 2^r$. Given this primary relation, we may consider the Brown and Peterson secondary cohomology operation which stems from this, which we shall call Θ_0. Θ_0 is defined on elements of $H^{n-2}(X)$ which are annihilated by the Sq^c above, and its image lies in $H^{2(n-2)}(X)$ with indeterminacy $\sum Sq^{(n-1)-c} H^{n-3+c}(X)$ where the sum runs over those c such that $0 \leqslant c \leqslant \left[\dfrac{n-1}{2}\right]$ with $\binom{2^r-c-1}{a-2c} \neq 0$.

If we set $X = \Omega^2 S^n$, we see Θ_0 is defined on $\gamma_1(\rho_0)$ because $\gamma_1(\rho_0)$ is annihilated by all Steenrod operations. By considering the map, as above, $\Omega S^{n-1} \to \Omega^2 S^n$, we may show that $\Theta_0(\gamma_1(\rho_0)) = \gamma_2(\rho_0) + \sum a_i b_i$ for some $a_i, b_i \in H^*(\Omega^2 S^n)$ such that $a_i b_i \in H^{2(n-2)}(\Omega^2 S^n)$, and $|a_i|, |b_i| > 0$. Thus, if we suppose p_1 is spherical, we must get a contradiction as in Section 2.3.

7.2.3. Showing p_1 is not spherical when $n - 1 = 2^{r+1}$ for $r \geqslant 3$

Let us take the case when $n-1$ is a power of two greater than 8. Then, $n - 1 = 2^{r+1}$ for $r \geqslant 3$. As before, we provide a higher cohomology operation which acts on $\gamma_1(\rho_0)$ to yield $\gamma_2(\rho_0)$. We may apply our tertiary operation Ψ to this situation. First, we check that Ψ is defined on $\gamma_1(\rho_0)$ by showing that

1. Sq^{2^i} annihilates $\gamma_1(\rho_0)$ for $i \leqslant r$
2. Φ_{ij} annihilates $\gamma_1(\rho_0)$ for $i, j \leqslant r$

We have already seen that (1) is true. This result enables us to define $\Phi_{ij}(\gamma_1(\rho_0))$; now we try to compute it. We review the construction of Adams' Φ_{ij}. Recall Φ_{ij} is a secondary cohomology operation based on a primary relation of the form $\sum_{0 \leqslant m \leqslant j} b_m Sq^{2^m} = 0$ where b_m is an element of the Steenrod algebra. To obtain this relation, we apply the Adem relation to $Sq^{2^i} Sq^{2^j}$ and express the right-hand Steenrod operation of each summand in terms of the Steenrod operations, Sq^{2^m}. We note that each term of this summand has degree $2^i + 2^j$ and $b_j = Sq^{2^i}$.

For any l, let $Sq^{2^m} \colon K(l) \to K(l + 2^m)$ represent the element $Sq^{2^m}(\kappa_l)$ where κ_l is the fundamental class of $K(l)$. Then, consider $f_1 \colon K(l) \to \prod_{0 \leqslant m \leqslant j} K(l + 2^m)$ which is given by $f_1^*(\kappa_{l+2^m}) = Sq^{2^m}(\kappa_l)$. Denote the homotopy fiber of f_1 by A_1. We have the map $g_1 \colon A_1 \to K(l)$ with fiber $\prod_{0 \leqslant m \leqslant j} K(l + 2^m - 1)$. The elements κ_{l+2^m-1} transgress to $Sq^{2^m}(\kappa_l)$. Then,

$$\sum_{0 \leqslant m \leqslant j} b_m \kappa_{l+2^m-1}$$

transgresses to

$$\sum_{0 \leqslant m \leqslant j} b_m Sq^{2^m} \kappa_l = 0.$$

So, $\displaystyle\sum_{0 \leqslant m \leqslant j} b_m \kappa_{l+2^m-1}$ survives in the Serre spectral sequence of the fibration

$$\prod_{0\leqslant m\leqslant j} K(l+2^m-1) \to A_1 \xrightarrow{g_1} K(l),$$

and there exists $\phi_{ij} \in H^{2^i+2^j+l-1}(A_1)$. We may represent this cohomology class by a map $\phi_{ij}\colon A_1 \to K_{2^i+2^j+l-1}$. If we replace l by $n-1$, effectively, the diagram below captures the action of Φ_{ij} on cohomology elements in degree $n-1$.

$$\begin{tikzcd} A_1 \arrow[r, "\phi_{ij}"] \arrow[d, "g_1"] & K_{n-1+2^i+2^j-1} \\ K(n-1) \arrow[r, "f_1"] & \prod_{0\leqslant m\leqslant j} K(n-1+2^m) \end{tikzcd}$$

Observe that if we loop this diagram, we will obtain a diagram which represents the action of Φ_{ij} on cohomology elements in degree $n-2$ because Φ_{ij} is stable. We have discussed that Φ_{ij} is defined on $\gamma_1(\alpha_{n-1}) \in H^{n-1}(\Omega S^n)$. By replacing l above by $n-1$ and by representing $\gamma_1(\alpha_{n-1})$ by the map $f: \Omega S^n \to K(n-1)$, we have that $\Phi_{ij}(\gamma_1\rho_1)$ corresponds to $\phi_{ij} \circ \widetilde{f}$:

$$\begin{tikzcd} & A_1 \arrow[r, "\phi_{ij}"] \arrow[d, "g_1"] & K_{n-1+2^i+2^j-1} \\ \Omega S^n \arrow[ur, dashed, "\widetilde{f}"] \arrow[r, "f"] & K(n-1) \arrow[r, "f_1"] & \prod_{0\leqslant m\leqslant j} K(n-1+2^m) \end{tikzcd}$$

We exhibited earlier that $\Phi_{ij}(\gamma_1(\alpha_{n-1})) = 0$ with zero indeterminacy. Thus, $\phi_{ij} \circ \widetilde{f}$ is null homotopic.

Now, looping this diagram enables us to study

$$\Phi_{ij}(\gamma_1(\rho_0)) = \Omega\phi_{ij} \circ \Omega\widetilde{f}.$$

So

$$\begin{tikzcd} & \Omega A_1 \arrow[r, "\Omega\phi_{ij}"] \arrow[d, "\Omega g_1"] & \Omega K_{n-1+2^i+2^j-1} \\ \Omega^2 S^n \arrow[ur, dashed, "\Omega\widetilde{f}"] \arrow[r, "\Omega f"] & \Omega K(n-1) \arrow[r, "\Omega f_1"] & \prod_{0\leqslant m\leqslant j} \Omega K(n-1+2^m) \end{tikzcd}$$

Since $\phi_{ij} \circ \widetilde{f}$ is null homotopic, $\Omega\phi_{ij} \circ \Omega\widetilde{f}$ is null homotopic. Thus, $\Phi_{ij}(\gamma_1(\rho_0))$ is zero modulo indeterminacy. Via dimensional arguments, we show the indeterminacy is zero. Now

$$\Phi_{ij}(\gamma_1(\rho_0)) \in H^{2^i+2^j-1+n-2}(\Omega^2 S^n).$$

By definition, the indeterminacy of the secondary cohomology operation Φ_{ij} is

$$\sum_{0\leqslant m\leqslant r} b_m H^{2^i+2^j-1+n-2-|b_m|}(\Omega^2 S^n).$$

Yet, $|b_m| = 2^i + 2^j - 2^m$, so the indeterminacy lies in

$$\sum_{0\leqslant m\leqslant r} b_m H^{n-3+2^m}(\Omega^2 S^n) = \sum_{0\leqslant m\leqslant r} b_m H^{2^{r+1}+2^m-2}(\Omega^2 S^n). \tag{7.1}$$

To understand indeterminacy, we consider $H^{2^{r+1}+2^m-2}(\Omega^2 S^n)$ for $0 \leqslant m \leqslant r$. Now, $\gamma_1(\rho_0)$ belongs to $H^{2^{r+1}-1}(\Omega^2 S^n)$, but all other $\gamma_a(\rho_b)$, and thus products of the generators of $H^*(\Omega^2 S^n)$, have degree too large to be in $H^{2^{r+1}+2^m-2}(\Omega^2 S^n)$. Since $\gamma_1(\rho_0)$ is annihilated by all Steenrod operations b_0, the indeterminacy must be zero. Thus, $\Phi_{ij}(\gamma_1(\rho_0))$ is zero with zero indeterminacy for each i, j with $0 \leqslant i \leqslant j \leqslant r$ and $i \neq j+1$. Since Sq^{2^i} annihilates $\gamma_1(\rho_0)$ for $i \leqslant r$, we may proceed in applying the tertiary operation Ψ to $\gamma_1(\rho_0)$.

Again, we use the map $f\colon \Omega S^{n-1} \to \Omega^2 S^n$ noting that $f^*(\gamma_{2^t}(\rho_0)) = \gamma_{2^t}(\alpha_{n-2})$. The following diagram commutes:

$$\begin{array}{ccc} H^{n-2}(\Omega^2 S^n) & \xrightarrow{f^*} & H^{n-2}(\Omega S^{n-1}) \\ \downarrow \Psi & & \downarrow \Psi \\ H^{2\cdot(n-2)}(\Omega^2 S^n) & \xrightarrow{f^*} & H^{2\cdot(n-2)}(\Omega S^{n-1})) \end{array}$$

Now, $\Psi(f^*(\gamma_1(\rho_0))) = \gamma_2(\alpha_{n-2})$ modulo zero indeterminacy. By chasing the diagram, we see that modulo the indeterminacy of Ψ, $\Psi(\gamma_1(\rho_0)) = \gamma_2(\rho_0) + \tau$, where $\tau \in$ kernel of $H^{(n-2)\cdot 2}(\Omega^2 S^n) \to H^{(n-2)\cdot 2}(\Omega S^{n-1})$. However, there are no elements of degree $(n-2)\cdot 2$ in $H^*(\Omega^2 S^n)$ besides $\gamma_2(\rho_0)$. So τ must be zero. Also, $\gamma_2(\rho_0)$ is not in the image of the Steenrod algebra action on $H^*(\Omega^2 S^n)$ so $\gamma_2(\rho_0)$ is not in the indeterminacy of Ψ. Hence for dimension reasons the indeterminacy module of Ψ must be zero, modulo zero,

$$\Psi(\gamma_1(\rho_0)) = \gamma_2(\rho_0).$$

Thus, following the ideas of Section 2.3, p_1, the dual element of $\gamma_2(\rho_0)$, is not spherical. Thus, we have shown that in the case when n is odd and $n-1 \neq 2^r$ where $r > 3$, $\Omega^2 S^n$ is minimal atomic.

7.3. n even

To determine which elements of $H_*(\Omega^2 S^n)$ are spherical when n is even, we determine which primitive elements are annihilated by the dual of the Steenrod algebra. From our work above, we see that the only elements fitting this description are $(r_0)^{2^t}$ and $(r_1)^{2^t}$ for $t \geqslant 0$. Thus, we establish how the Steenrod operations act on $\gamma_{2^t}(\rho_0)$ and $\gamma_{2^t}(\rho_1)$ so we may use our higher order cohomology techniques.

We see from our previous work that $Sq^{2^s}(\gamma_{2^t}(\rho_0)) = 0$ and so the entire Steenrod algebra annihilates $\gamma_{2^t}(\rho_0)$. Also, $\gamma_1(\rho_1)$ has a trivial action under the Steenrod algebra.

For $t > 0$, the only generator of the Steenrod algebra which acts non-trivially on $\gamma_{2^t}(\rho_1)$ is $Sq^{2^{t-1}}$ yielding $\gamma_{2^{t-1}}(\rho_2)$. If $t-1 > 0$, again, we have exactly one generator of the Steenrod algebra which acts non-trivially on $\gamma_{2^{t-1}}(\rho_1)$, $Sq^{2^{t-2}}$. Proceeding in this fashion, we see that the Steenrod algebra elements that act non-trivially on $\gamma_{2^t}(\rho_1)$ are of the form Sq^I where $I = (2^m, 2^{m+1}, \ldots, 2^{t-1})$ where $0 \leqslant m \leqslant t-1$. In particular, the highest dimensional Steenrod operation which can act on $\gamma_{2^t}(\rho_1)$ is of degree $1 + 2 + \ldots + 2^{t-1} = 2^t - 1$.

As in Section 7.2, we use the same secondary cohomology operation Θ_t defined on elements, $\gamma \in H^{2^t(n-2)}(X)$ such that $Sq^{2^t(n-2)}(\gamma) = 0$ with image contained in $H^{2^{t+1}(n-2)}(X)/Sq^1(H^{2^{t+1}(n-2)-1}(X))$. Then, as before, for $t > 0$, Θ_t is defined on $\gamma_{2^t}(\rho_0)$ and $\gamma_{2^{t+1}}(\rho_0) \notin Sq^1(H^{2^{t+1}(n-2)-1}(\Omega^2 S^n))$. That is, for $t > 0$, $\gamma_{2^{t+1}}(\rho_0)$ does not belong to the indeterminacy of Θ_t.

7.3.1. Showing $(r_0)^{2^t}$ is not spherical for $t > 0$

Again, we use $f\colon \Omega S^{n-1} \to \Omega^2 S^n$ noting that $f^*(\gamma_{2^t}(\rho_0)) = \gamma_{2^t}(\alpha_{n-2})$. The following diagram commutes:

$$\begin{array}{ccc} H^{(n-2)\cdot 2^t}(\Omega^2 S^n) & \xrightarrow{f^*} & H^{(n-2)\cdot 2^t}(\Omega S^{n-1}) \\ \downarrow{\scriptstyle \Theta_t} & & \downarrow{\scriptstyle \Theta_t} \\ H^{(n-2)\cdot 2^{t+1}}(\Omega^2 S^n) & \xrightarrow{f^*} & H^{(n-2)\cdot 2^{t+1}}(\Omega S^{n-1}) \end{array}$$

As in the case when n is odd, chasing the diagram leads to the conclusion that $\Theta_t(\gamma_{2^t}(\rho_0)) = \gamma_{2^{t+1}}(\rho_0) + \tau$ where $\tau \in H^{(n-2)2^{t+1}}(\Omega^2 S^n)$ is a sum of decomposable elements (possibly zero). Thus, using Section 2.3 for $t > 0$, $(r_0)^{2^{t+1}}$ is not spherical.

To show $(r_0)^2$ is not spherical, we observe that in the case when n is even, we can factor Sq^{n-1}. Further, $\gamma_2(\rho_0)$ is annihilated by the Steenrod algebra and is not in the image of the Steenrod algebra. Thus, repeating the argument in Section 7.2.2 shows that $\gamma_2(\rho_0) + \sum a_i b_i$ is in the image of Θ_0. Thus, $(r_0)^2$ is not spherical and for all $t > 0$, $(r_0)^{2^t}$ is not spherical.

7.3.2. Showing $(r_1)^{2^t}$ is not spherical when $t > 0$

Consider the Steenrod algebra relation,

$$Sq^{2^t(2(n-1)-1)+1} = Sq^1 Sq^{2^t(2(n-1)-1)}.$$

Analogous to the results of Section 4.2, for each t this relation gives rise to a secondary cohomology operation, which we shall denote Φ_t. In particular, Φ_t is defined on $H^*(\Omega S^{2(n-1)}) = H^*(Y_1)$ with zero indeterminacy and similar to (4.3) we obtain

$$\Phi_t(\gamma_{2^t}(\rho_1)) = \gamma_{2^{t+1}}(\rho_1). \tag{7.2}$$

Considering $\gamma_{2^t}(\rho_1)$ as an element of $H^*(\Omega^2 S^n)$, we deduced in Section 7.3 that the highest dimensional Steenrod operation which acts non-trivially on $\gamma_{2^t}(\rho_1)$ has degree $2^t - 1$. Then $Sq^{2^t(2(n-1)-1)}$ must act trivially on $\gamma_{2^t}(\rho_1)$. Since $H^*(G_2)$

inherits its Steenrod algebra action from $H^*(\Omega^2 S^n)$, $Sq^{2^t(2(n-1)-1)}$ acts trivially on $\gamma_{2^t}(\rho_1)$ considered as an element of $H^*(G_2)$. Thus $\Phi_t(\gamma_{2^t}(\rho_1))$ is defined on $H^*(\Omega^2 S^n)$ and $H^*(G_2)$.

Let $g_1\colon G_2 \to Y_1$ be the map which occurs in the fibration $G_1 \to G_2 \to Y_1$. In our Serre spectral sequence analysis, we saw that $(g_1)^*\colon H^*(Y_1) \hookrightarrow H^*(G_2)$ and so $(g_1)^*(\gamma_{2^i}(\rho_1)) = \gamma_{2^i}(\rho_1)$. Then, by naturality, we have a diagram

$$\begin{tikzcd} H^*(Y_1) \arrow[r, "(g_1)^*"] \arrow[d, "\Phi_t"] & H^*(G_2) \arrow[d, "\Phi_t"] \\ H^*(Y_1) \arrow[r, "(g_1)^*"] & H^*(G_2) \end{tikzcd}$$

Then, $\Phi_t((g_1)^*(\gamma_{2^t}(\rho_1))) = (g_1)_*(\Phi_t(\gamma_{2^t}(\rho_1)))$. So, modulo indeterminacy in $H^*(G_2)$,

$$\Phi(\gamma_{2^t}(\rho_1)) = \gamma_{2^{t+1}}(\rho_1).$$

Since $H^*(G_2)$ inherits its Steenrod algebra structure from $H^*(\Omega^2 S^n)$ we can see that $\gamma_{2^{t+1}}(\rho_1)$ is not in the image of Sq^1. Furthermore, there are no indecomposable elements of $H^*(\Omega^2 S^n)$ with degree $2^{t+1}(2(n-1)-1) = |\gamma_{2^{t+1}}(\rho_1)|$. So, only decomposables are in the indeterminacy of Φ_t, $Sq^1(H^{2^{t+1}(2(n-1)-1)-1}(G_2))$.

Using the map $g_2\colon G_2 \to \Omega^2 S^n$, we have following commutative diagram.

$$\begin{tikzcd} H^*(\Omega^2 S^n) \arrow[r, "(g_2)^*"] \arrow[d, "\Phi_t"] & H^*(G_2) \arrow[d, "\Phi_t"] \\ H^*(\Omega^2 S^n) \arrow[r, "(g_2)^*"] & H^*(G_2) \end{tikzcd}$$

Then, modulo the indeterminacy of $Sq^1(H^{2^{t+1}(2(n-1)-1)-1}(\Omega^2 S^n))$,

$$\begin{aligned} ((g_2)^*)(\Phi_t(\gamma_{2^t}(\rho_1))) &= \Phi_t((g_2)^*(\gamma_{2^t}(\rho_1))) \\ &= \Phi_t(\gamma_{2^t}(\rho_1)) = \gamma_{2^{t+1}}(\rho_1). \end{aligned}$$

As before, we observe that $\gamma_{2^{t+1}}(\rho_1)$ does not belong to the indeterminacy for Φ_t defined on $H^*(\Omega^2 S^n)$. Also, the only elements that belong to the indeterminacy must be decomposables. The kernel of

$$(g_2)^*\colon H^{2^{t+1}(2(n-1)-1)}(\Omega^2 S^n) \to H^{2^{t+1}(2(n-1)-1)}(G_2)$$

consists of decomposable elements. Thus, it must be the case that for some decomposable element $\tau \in H^*(\Omega^2 S^n)$

$$\Phi_t(\gamma_{2^t}(\rho_1)) = \gamma_{2^{t+1}}(\rho_1) + \tau.$$

Using Section 2.3, we see that $(r_1)^{2^{t+1}}$ cannot be spherical.

To show that $(r_1)^2$ is not spherical, we refer to Section 4.2 replacing n with $|r_1| = 2(n-1)$. We observe that since n is even, $2(n-1)$ is not a power of 2, and a factorization of $Sq^{2(n-1)}$ exists. Thinking of $\gamma_1(\rho_1), \gamma_2(\rho_2)$ as elements of $H^*(Y_1)$, the argument of Section 4.2 shows that

$$\Phi_0(\gamma_1(\rho_1)) = \gamma_2(\rho_1) + \sum a_i b_i. \tag{7.3}$$

We would like a version of (7.3) to hold in $H^*(G_2)$ and $H^*(\Omega^2 S^n)$. Observe that $\gamma_1(\rho_1) \in H^*(G_2)$ is annihilated by all the Steenrod operations, so Φ_0 can legitimately be defined on $\gamma_1(\rho_1)$. Since $\gamma_2(\rho_1) \in H^*(G_2)$ is not in the image of any Steenrod operations, $\gamma_2(\rho_1)$ will not be cancelled out when taking into account indeterminacy of Φ_0 defined on $H^*(G_2)$. Then using naturality of Φ_0 applied to g_1, we have that modulo the indeterminacy of $H^*(G_2)$, (7.3) holds. Applying naturality of Φ_0 with respect to g_2, we obtain (7.3) thinking of $\gamma_1(\rho_1), \gamma_2(\rho_1)$ as elements of $H^*(\Omega^2 S^n)$. Thus, it must be the case that $(r_1)^2$ is not spherical, and thus for all $t > 0$, $(r_1)^{2^t}$ is not spherical.

7.3.3. Showing r_1 is not spherical when n is not a power of 2

We show that $\gamma_1(\rho_0)$ hits $\gamma_1(\rho_1)$ via a higher order cohomology operation. We suppose that n is not a power of 2. For $k \geqslant 2$, let us take the adjoint of the identity map on ΩF_{2^k-1}, $f\colon \Sigma G_k \to F_{2^k-1}$. Then,

$$f^*(\gamma_{2^i}(\alpha_{n-1})) = \Sigma\gamma_1(\sigma\gamma_{2^i}(\alpha_{n-1})) = \Sigma\gamma_1(\rho_i).$$

We have shown that there exist secondary cohomology operations Φ_i such that $\Phi_0(\gamma_1(\alpha_{n-1})) = \gamma_2(\alpha_{n-1})$ on $H^*(\Omega S^n)$. Applying naturality of Φ_0 to the map $F_{2^k-1} \to \Omega S^n$ gives us the same result for $H^*(F_{2^k-1})$. Also, the Steenrod operations annihilate $\gamma_1(\rho_i)$, so $\Sigma\gamma_1(\rho_i)$ is annilated by the Steenrod operations. In particular, Φ_0 is defined on $\gamma_1(\rho_i)$. We have a commutative diagram.

$$\begin{tikzcd} H^{n-1}(F_{2^k-1}) \arrow[r, "f^*"] \arrow[d, "\Phi_0"] & H^{n-1}(\Sigma G_k) \arrow[d, "\Phi_0"] \\ H^{2(n-1)}(F_{2^k-1}) \arrow[r, "f^*"] & H^{2(n-1)}(\Sigma G_k) \end{tikzcd}$$

Thus, $f^*(\Phi_0(\gamma_1(\alpha_{n-1}))) = \Phi_0(f^*(\gamma_1(\alpha_{n-1})))$, and so modulo the indeterminacy of Φ_0 on $H^*(\Sigma G_k)$, we have

$$\Phi_0(\Sigma\gamma_1(\rho_0)) = \Sigma\gamma_1(\rho_1).$$

As $\gamma_1(\rho_1)$ is not in the image of the Steenrod algebra action on $H^*(G_k)$, $\Sigma\gamma_1(\rho_1)$ is not in the image of the Steenrod algebra action on $H^*(\Sigma G_k)$. Recall the suspension homomorphism s_l has the property that $s_{2^i(n-1)-1}(\Sigma\gamma_1(\rho_i)) = \gamma_1(\rho_i)$. Following the discussion of $\sigma\Phi$ in Section 3.1, we have

$$\begin{tikzcd} H^{n-1}(\Sigma G_k) \arrow[r, "s_{n-2}"] \arrow[d, "\Phi_0"] & H^{n-2}(G_k) \arrow[d, "\sigma\Phi_0"] \\ H^{2n-2}(\Sigma G_k) \arrow[r, "s_{2n-3}"] & H^{2n-3}(G_k) \end{tikzcd}$$

Then,

$$s_{2n-3}(\sigma\Phi_0(\Sigma\gamma_1(\rho_0))) = \Phi_0(s_{n-2}(\Sigma\gamma_1(\rho_0)).$$

So, modulo the indeterminacy of $\sigma\Phi_0$ on $H^*(G_k)$,

$$\sigma\Phi_0(\gamma_1(\rho_0)) = \gamma_1(\rho_1).$$

Since $\gamma_1(\rho_1)$ is not in the image of the Steenrod algebra action on $H^*(G_k)$, $\gamma_1(\rho_1)$ is not in the indeterminacy module of $\sigma\Phi_0$ on $H^*(G_k)$.

We have the following naturality diagram for $k = 2$, observing that the Steenrod algebra action on $\gamma_1(\rho_0) \in H^*(\Omega^2 S^n)$ enables $\sigma\Phi_0(\gamma_1(\rho_0))$ to be defined:

$$\begin{tikzcd} H^*(\Omega^2 S^n) \arrow[r, "(g_2)^*"] \arrow[d, "\sigma\Phi_0"] & H^*(G_2) \arrow[d, "\sigma\Phi_0"] \\ H^*(\Omega^2 S^n) \arrow[r, "(g_2)^*"] & H^*(G_2) \end{tikzcd}$$

Then, modulo the indeterminacy of $\sigma\Phi_0$ on $H^*(\Omega^2 S^n)$,

$$\begin{aligned} ((g_2)_*)(\sigma\Phi_0(\gamma_1(\rho_0))) &= \sigma\Phi_0((g_2)_*(\gamma_1(\rho_0))) \\ &= \sigma\Phi_0(\gamma_1(\rho_0)) \\ &= \gamma_1(\rho_1). \end{aligned}$$

Using the naturality argument from Section 2.3 gives the result that r_1 is not spherical.

7.3.4. Showing r_1 is not spherical when n is a power of 2

We take the case where $n = 2^{r+1}$ where $r \geqslant 3$. Then, in Section 4.3, we have shown that a tertiary cohomology operation Ψ exists such that

$$\Psi(\gamma_1(\alpha_{n-1})) = \gamma_2(\alpha_{n-1})$$

with zero indeterminacy on $H^*(\Omega S^n)$. By naturality, we have

$$\Psi(\gamma_1(\alpha_{n-1})) = \gamma_2(\alpha_{n-1})$$

on $H^*(F_{2^k-1})$ for $k \geqslant 2$ with zero indeterminacy.

We must check that Adams secondary cohomology operations Φ_{ij} with $i, j \leqslant r$ evaluate to zero on $\Sigma\gamma_1(\rho_0) \in H^*(\Sigma G_2)$. We have the commutative diagram,

$$\begin{tikzcd} H^{n-1}(F_{2^2-1}) \arrow[r, "f^*"] \arrow[d, "\Phi_{ij}"] & H^{n-1}(\Sigma G_2) \arrow[d, "\Phi_{ij}"] \\ H^{n+2^i+2^j-2}(F_{2^2-1}) \arrow[r, "f^*"] & H^{n+2^i+2^j-2}(\Sigma G_2) \end{tikzcd} .$$

Then, modulo the indeterminacy of Φ_{ij} on $H^*(\Sigma G_2)$, we have

$$\Phi_{ij}(f^*(\gamma_1(\alpha_{n-1}))) = f^*(\Phi_{ij}(\gamma_1(\alpha_{n-1}))) = 0.$$

For each i, j, the indeterminacy of Φ_{ij} takes the form

$$\Sigma_m b_m H^{2^i+2^j-1+n-1-|b_m|}(\Sigma G_2)$$

where b_m are Steenrod operations of degree larger than 0. A lower bound for

$$2^i + 2^j - 1 + n - 1 - |b_m|$$

is $n-1$, and an upper bound is $2n-3$. By dimensional arguments, we see only two elements of $H^*(\Sigma G_2)$ have dimensions falling within these bounds, namely $\Sigma\gamma_1(\rho_0)$ and $\Sigma\gamma_2(\rho_0)$ with dimensions $n-1$ and $2n-3$ respectively. However all Steenrod operations annihilate these elements when n is even. Thus, it must be the case that the module of indeterminacy is zero. So, with zero indeterminacy,

$$\Phi_{ij}(\Sigma\gamma_1(\rho_0)) = 0.$$

Then, Ψ is defined on $\Sigma\gamma_1(\rho_0)$. We have the commutative diagram

$$\begin{tikzcd}
H^{n-1}(F_{2^2-1}) \arrow[r, "f^*"] \arrow[d, "\Psi"] & H^{n-1}(\Sigma G_2) \arrow[d, "\Psi"] \\
H^{2(n-1)}(F_{2^2-1}) \arrow[r, "f^*"] & H^{2(n-1)}(\Sigma G_2)
\end{tikzcd}$$

Thus, $\Psi(f^*(\gamma_1(\alpha_{n-1}))) = f^*(\Psi(\gamma_1(\alpha_{n-1}))) = \Sigma\gamma_1(\rho_1)$. That is, modulo indeterminacy,

$$\Psi(\Sigma\gamma_1(\rho_0)) = \Sigma\gamma_1(\rho_1).$$

We observe that $\Sigma\gamma_1(\rho_1)$, with dimension $2n-2$, is not in the image of Steenrod operations, so $\Sigma\gamma_1(\rho_1)$ is not in the indeterminacy for Ψ. Using dimensional criteria, the only other candidates in the indeterminacy are Steenrod operations applied to $\Sigma\gamma_1(\rho_0)$ and $\Sigma\gamma_2(\rho_0)$. Since Steenrod operations annihilate these elements, the indeterminacy is zero.

Since Φ_{ij} is stable, $\Phi_{ij}(\gamma_1(\rho_0)) = 0$ on $H^*(G_2)$. Thus Ψ is defined on $\gamma_1(\rho_0)$. Then, we have the commutative diagram

$$\begin{tikzcd}
H^{n-1}(\Sigma G_2) \arrow[r, "s_{n-2}"] \arrow[d, "\Psi"] & H^{n-2}(G_2) \arrow[d, "\sigma\Psi"] \\
H^{2n-2}(\Sigma G_2) \arrow[r, "s_{2n-3}"] & H^{2n-3}(G_2)
\end{tikzcd}$$

So,

$$\sigma\Psi(\gamma_1(\rho_0)) = \sigma\Psi(s_{n-2}(\Sigma\gamma_1(\rho_0))) = s_{2n-3}(\Psi(\Sigma\gamma_1(\rho_0))) = \gamma_1(\rho_1). \tag{7.4}$$

Now, $\gamma_1(\rho_1))$ is not in the image of the Steenrod action on $H^*(\Omega^2 S^n)$. Thus, using naturality of $\sigma\Psi$ with respect to g_2 yields the relation (7.4), considering $\gamma_1(\rho_0)$ and $\gamma_1(\rho_1)$ as elements of $H^*(\Omega^2 S^n)$. Section 2.3 shows that r_1 is not spherical. Thus, we have shown that $\Omega^2 S^n$ is minimal atomic for $n \geqslant 6$ unless $n = 8, 9$.

7.4. The cases where $\Omega^2 S^n$ is not minimal atomic

We show that $\Omega^2 S^n$ is not minimal atomic for $n = 2, 3, 4, 5, 8, 9$. Recall that $(x_1)^2 \in H_2(\Omega S^2)$, $(x_3)^2 \in H_6(\Omega S^4)$, and $(x_7)^2 \in H_{14}(\Omega S^8)$.

7.4.1. $n = 3, 5, 9$

We take the case where $n = 3, 5, 9$. Here we will show there exists a spherical class with degree above the Hurewicz dimension for $\Omega^2 S^3, \Omega^2 S^5$, and $\Omega^2 S^9$. Consider the canonical map, $S^{n-1} \xrightarrow{\eta} \Omega S^n$. Here, $i_{n-1} \in H_{n-1}(S^{n-1})$ maps to a_{n-1} where $H_*(\Omega S^n) = \mathbb{F}_2[a_{n-1}]$. For each n we have shown there exists $f\colon S^{2n-1} \to \Omega S^{n-1}$ such that $i_{2(n-1)}$ maps to a_{n-2}^2. We hope to understand the map $\Omega S^{n-1} \xrightarrow{\Omega\eta} \Omega^2 S^n$. Recall that $H_*(\Omega^2 S^n) = \mathbb{F}_2[r_0, r_1, r_2, \ldots]$ where $|r_i| = 2^i(n-1) - 1$.

Let $`E$ be the homology Serre spectral sequence for the fibration $\Omega S^{n-1} \to * \to S^{n-1}$, and let $E$ be the homology Serre spectral sequence for the fibration $\Omega^2 S^n \to * \to \Omega S^n$. The canonical map $\eta$ induces a map $`E \to E$ of spectral sequences.

Then, $\Omega\eta$ has the property that

$$(\Omega\eta)_*(a_{n-2}) = r_0$$

and

$$(\Omega\eta)_*(a_{n-2})^2 = (r_0)^2.$$

Thus, in the composition

$$S^{2(n-2)} \xrightarrow{f} \Omega S^{n-1} \xrightarrow{\Omega\eta} \Omega^2 S^n$$

we have

$$i_{2(n-2)} \xrightarrow{f_*} (a_{n-2})^2 \xrightarrow{(\Omega\eta)_*} (r_0)^2.$$

It follows that $(r_0)^2$ is spherical and $\Omega^2 S^3, \Omega^2 S^5$ and ΩS^9 are not minimal atomic. Observe that in the cases where $\Omega^2 S^{2^q-1}$ is minimal atomic, our proof in Section 7.2.3 focused on showing that the element $(r_0)^2$ is not spherical.

7.4.2. $n = 2, 4, 8$

We show that for the cases $n = 2, 4, 8$, $\Omega^2 S^n$ is not minimal atomic. For each n we have shown there exists $f\colon S^{2n-1} \to \Omega S^n$ with $i_{2(n-1)}$ mapping to $(a_{n-1})^2$. Looping f, we obtain

$$\Omega S^{2(n-1)} \xrightarrow{\Omega f} \Omega^2 S^n.$$

Now, $H_*(\Omega S^{2(n-1)}) = \mathbb{F}_2[a_{2(n-1)-1}]$. Let $`E$ be the homology Serre spectral sequence for the fibration $\Omega S^{2(n-1)} \to * \to S^{2(n-1)}$, and let $E$ be the homology Serre spectral sequence for the fibration $\Omega^2 S^n \to * \to \Omega S^n$. The map $f$ induces a map $`E \to E$ of spectral sequences, which allows us to see that $(\Omega f)_*(a_{2(n-1)-1}) = r_1$. Let

$$g\colon S^{2(n-1)-1} \to \Omega S^{2(n-1)}$$

be the non-trivial map such that $i_{2(n-1)-1} \in H_*(S^{2(n-1)-1})$ maps to $a_{2(n-1)-1}$ under g_*. Then the composite,

$$S^{2(n-1)-1} \xrightarrow{g} \Omega S^{2(n-1)-1} \xrightarrow{\Omega f} \Omega^2 S^n$$

maps the fundamental class to r_1. Thus, r_1 is spherical and $\Omega^2 S^2, \Omega^2 S^4$ and $\Omega^2 S^8$ are not minimal atomic. Looking back on our proof in Section 7.3.4, observe that we had to introduce a special argument to show that r_1 is not spherical in the cases where n is a power of 2 greater than 8.

Appendix A. Factoring $Sq^{2^{r+1}}$

Let $r \geqslant 3$. For any space X, let $\tau \in H^*(X)$ be such that $Sq^{2^s}(\tau) = 0$ for $0 \leqslant s \leqslant r$. Recall (3.3), which states that there exist Steenrod operations a_{ij} such that

$$Sq^{2^{r+1}}(\tau) = \sum_{i \leqslant j, i+1 \neq j} a_{ij} \Phi_{ij}(\tau).$$

To obtain a specific factorization of Sq^{2^s} we need to calculate the coefficients a_{ij} that satisfy (3.3). In [**LW**], one factorization of Sq^{16} is given, but it is worth noting that there are several factorizations of Sq^{16} and one factorization might be preferable to another depending on context. A computer program has been implemented in Maple which builds upon [**MN**]. This new program calculates all a_{ij} which satisfy (3.3). In this appendix, we examine the mathematics of these coefficients discussed in [**A**]. This provides a constructive approach to finding a_{ij} which forms the basis of the aforementioned computer program. It should be noted that the program does have flaws—it yields a_{ij} for factorizations of Sq^{16}, but for values of r larger than 3, the program stalls ostensibly due to memory constraints. Perhaps this can be improved in the future.

Let C_0 be the free module over the Steenrod algebra with basis element c of degree 1, and let $\epsilon\colon C_0 \longrightarrow \mathbb{Z}/2\mathbb{Z}$ be the non-trivial map. Let C_1 be the free module over the Steenrod algebra with basis elements c_i of degree 2^i. Define $d_1\colon C_1 \to C_0$ by setting

$$d_1(c_i) = Sq^{2^i}(c).$$

We may construct a minimal resolution of $\mathbb{Z}/2\mathbb{Z}$ over the Steenrod algebra with these starting terms,

$$\cdots \longrightarrow C_s \xrightarrow{d_s} C_{s-1} \xrightarrow{d_{s-1}} C_{s-2} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{d_1} C_0 \xrightarrow{\epsilon} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

Let $I(A) = \sum_{q>0} A_q$ where A is the Steenrod algebra and A_q consists of those elements of A with degree q. Then,

$$\mathrm{Tor}^A_{s,t}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z} \otimes_A C)_{s,t}$$

and

$$\mathrm{Ext}_A^{s,t}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong Hom^t_A(C_s, \mathbb{Z}/2\mathbb{Z}).$$

Now for each C_s, let $J(C_s) = I(A) \cdot C_s$ and $Z(s) = Ker(d_s) \cap J(C_s)$. We observe that for any partial minimal resolution,

$$C_s \xrightarrow{d_s} C_{s-1} \xrightarrow{d_{s-1}} C_{s-2} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{d_1} C_0 \xrightarrow{\epsilon} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

we may define a homomorphism

$$\theta\colon Z(s) \to \mathrm{Tor}^A_{s+1}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}).$$

by the following: Extend the partial minimal resolution to another partial minimal resolution by adjoining an appropriate (C_{s+1}, d_{s+1}). For any $z \in Z(s)$, given $w \in C_{s+1}$ such that $d_{s+1}(w) = z$,

$$\theta(z) = \{1 \otimes_A w\}.$$

Adams shows that we can choose a cycle $z_{ij} \in C_1$ with $i \leqslant j$ and $i + 1 \neq j$ such that

$$h_i h_j(\theta z_{ij}) = 1 \tag{A.1}$$

where h_i, h_j are the basis elements of $\mathrm{Ext}^{1,t}_A(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ of degree 2^i and 2^j respectively. Now, define C_2 to be the free module over the Steenrod algebra on generators $c_{i,j}$ such that $d_2(c_{ij}) = z_{ij}$. Adams shows that

$$C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{\epsilon} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

is a partial minimal resolution over $\mathbb{Z}/2\mathbb{Z}$ over A. Adams continues to show that there must exist $z \in C_2$ such that

$$h_0 h_r^2(\theta z) = 1. \tag{A.2}$$

We write z in the C_2 basis, so that

$$z = \sum a_{ij} c_{ij}.$$

The coefficients here are the desired a_{ij}.

To obtain explicit coefficients a_{ij}, we require explicit representations of z_{ij} in the C_1 basis. Adams describes how to obtain such a representation in [**A**]. We review the procedure, and check that the result satisfies (A.1) using Lemma A.1 below. This lemma will also be helpful in trying to find an element z which satisfies (A.2).

Let t' be a positive integer, and suppose we are given a function $\alpha\colon A \to \mathbb{Z}/2\mathbb{Z}$ of degree $-t'$ such that $\alpha(ab) = 0$ for $a, b \in I(A)$. Then the composite

$$C_1 \xrightarrow{d_1} A \xrightarrow{\alpha} \mathbb{Z}/2\mathbb{Z}$$

defines an element $h_\alpha \in \mathrm{Ext}^{1,t'}_A(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$. We have the following lemma from [**A**].

Lemma A.1. *Given any $h \in \mathrm{Ext}^{s,t}_A(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ and any $x \in Z(s) \cap C_{s,t+t'}$ with $x = \sum a_i c'_i$*

$$(h_\alpha h)(\theta x) = \textstyle\sum(\alpha a_i)(h\{1 \otimes_A c'_i\})$$

where $\{c'_i\}$ is the basis of C_s.

We will show that for each i, j with $i \leqslant j$ and $i+1 \neq j$, there exists an equation of the type,

$$Sq^{2^i} Sq^{2^j} + \sum f_k Sq^{2^{g_k}} = 0. \tag{A.3}$$

Then, it must be the case that

$$d_1(Sq^{2^i} c_j + \textstyle\sum f_k c_{g_k}) = 0.$$

We set $z_{ij} \in C_1$ to be $Sq^{2^i} c_j + \sum f_k c_{g_k}$ and show using Lemma A.1 that (A.1) is satisfied.

To obtain (A.3), recall the following standard result.

Lemma A.2. *For $m \geqslant 1$ with m not a power of 2, there exist finitely many Steenrod algebra elements b_k and non-negative integers c_k such that*

$$Sq^m = \textstyle\sum b_k Sq^{2^{c_k}}$$

It is clear that such a decomposition exists since $\{Sq^{2^k}\}$ comprises a set of generators for A, but the proof that follows reviews a method for obtaining the desired decomposition, which can be easily translated into computer code.

Proof. Let 2^d be the largest power of 2 which occurs in the binary representation of m. Then $2^d > m - 2^d$. In Section 4.2, we saw that an application of the Adem relations to $Sq^{m-2^d} Sq^{2^d}$ yields

$$Sq^m = Sq^{m-2^d} Sq^{2^d} + \sum Sq^{l_k} Sq^{m_k}$$

where $m_k \leqslant m - 2^d < 2^d$. If we apply a similar factorization to those Sq^{m_k} for which m_k is not a power of 2, and iterate this process, we will obtain the desired result of Lemma A.2. $\square$

To obtain z_{ij} for any i, j with $i \leqslant j$ and $i+1 \neq j$, observe that the Adem relations apply to give a factorization

$$Sq^{2^i} Sq^{2^j} = \sum Sq^{m_k} Sq^{n_k}. \tag{A.4}$$

If $i = j$, then the binomial coefficient of $Sq^{2^i+2^j}$ is $\binom{2^i-1}{2^i} = 0$, so in the factorization above all $m_k, n_k \neq 0$ and $n_k < 2^i, 2^j$.

If $i \neq j$, then the binomial coefficient of $Sq^{2^i+2^j}$ is $\binom{2^j-1}{2^i} = 1$ by (4.1). If we apply the Adem relations to factor $Sq^{2^{i+1}} Sq^{2^j-2^i}$, then $Sq^{2^i+2^j}$ appears in the factorization because $\binom{2^j-2^i-1}{2^{i+1}} = 1$. The other summands in this factorization are of the form $Sq^{u_k} Sq^{v_k}$ and $u_k, v_k \neq 0$ with $v_k < 2^{i+1} < 2^j$. Thus, if we add these two factorizations together, the $Sq^{2^i+2^j}$ terms cancel each other out. Thus, we may obtain (after relabelling the m_k and n_k) (A.4) above such that $m_k, n_k \neq 0$ and $n_k < 2^j$. Thus, in each case we have the same kind of factorization of $Sq^{2^i} Sq^{2^j}$. We apply Lemma A.2 to each Sq^{n_k} to rewrite $Sq^{2^i} Sq^{2^j}$ as $\sum f_k Sq^{2^{g_k}}$ where $f_k \in A$ and $g_k < j$ is a non-negative integer. Thus, we have verified (A.3). Further, we can replicate this process using a computer program because the processes are no more difficult than applying the Adem relations multiple times.

To show z_{ij} satisfies (A.1), consider the Milnor basis element $\xi_1^{2^i}$ of the dual of A. We apply Lemma A.1 with α taken to be $\xi_1^{2^i}$ observing that in the notation of that lemma, $h_{\xi_1^{2^i}} = h_i$. Thus,

$$h_i h_j(\theta z_{ij}) = \xi_1^{2^i}(Sq^{2^i}) h_j(\{1 \otimes c_j\}) + \sum \xi_1^{2^i}(f_k) h_j(\{1 \otimes c_{g_k}\})$$

Now, for dimensional reasons $h_j(\{1 \otimes c_{g_k}\}) = 0$. Also for dimensional reasons, $\{1 \otimes c_j\}$ is the single non-trivial element of $\mathrm{Tor}^A_{1,2^j}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$, and since h_j is the single non-trivial element of $\mathrm{Ext}_A^{1,2^j}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$, it must be the case that $h_j(\{1 \otimes c_j\}) = 1$. Thus, $h_i h_j(\theta z_{ij}) = 1$ and (A.1) is satisfied.

Having shown that a computer program would be able to find explicit examples of z_{ij}, we turn to the more complicated issue of how to find $z \in Z(2)$ which satisfies (A.2).

We observe that any $z \in Z(2)$ satisfies (A.2) if and only if $\xi_1(a_{rr}) = 1$. Again, in the notation of Lemma A.1, we see that h_{ξ_1} is the basis element h_0. Then,

$$(h_0 h_r^2)(\theta z) = \textstyle\sum_i (\xi_1 a_{ij})(h_r h_r \{1 \otimes_A c_{ij}\}).$$

For dimensional reasons,

$$h_r h_r \{1 \otimes_A c_{rr}\} = 1$$

while for $(i, j) \neq (r, r)$,

$$h_r h_r \{1 \otimes_A c_{ij}\} = 0.$$

Thus, $h_0 h_r^2(\theta z) = 1$ if and only if $\xi_1(a_{r,r}) = 1$. That is,

$$a_{rr} = Sq^1. \tag{A.5}$$

Hence, any $z = \sum a_{ij} c_{ij}$ with $z \in Z(2)$ such that $a_{rr} = Sq^1$, has the property that its coefficents a_{ij} give the desired factorization of $Sq^{2^{r+1}}$.

Compared with the information we have regarding z_{ij}, the data we have related to z is quite meager. Adams does add to this small repository by observing that our desired z must also satisfy

$$\xi_1^{2^r}(a_{0r}) = 1.$$

Following Lemma A.1, we see that $h_{\xi_1^{2^r}}$ is the basis element h_r. Since the group $\mathrm{Ext}_A(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ is commutative [**MT**, p. 193] we have

$$1 = (h_0 h_r^2)(\theta z) = (h_r(h_0 h_r))(\theta z) = \textstyle\sum_i (\xi_1^{2^r} a_{ij})(h_0 h_r \{1 \otimes_A c_{ij}\}).$$

As above, $h_0 h_r \{1 \otimes_A c_{ij}\}$ is non-zero (and equal to 1) only when $i = 0$ and $j = r$. Then we have that $\xi_1^{2^r}(a_{0r}) = 1$ and so,

$$a_{0r} = Sq^{2^r} + a_{2^r} \tag{A.6}$$

where $a_{2^r} \in A_{2^r}$ and does not have Sq^{2^r} as a summand.

Now, the computer program identifies valid z's by trial and error. It picks different choices of coefficients a_{ij} and tests whether or not the resulting z satisfies the conditions that $z \in Z(2)$ and $h_0 h_r^2(\theta z) = 1$. The mathematics that we have illustrated makes this job a bit easier. Since the degree of z is $2^{r+1}+1$, and $|c_{ij}| = 2^i + 2^j$, it must be that $|a_{ij}| = 2^{r+1} + 1 - 2^i - 2^j$; we restrict the guesses for a_{ij} to be Steenrod operations of the appropriate degree. Further, (A.5) and (A.6) put further constraints on the coefficients a_{rr} and a_{0r} which allow us to proceed by an educated version of trial and error.

We may write z_{ij} in the C_1 basis as before with $z_{ij} = Sq^{2^i} c_j + \sum f_k c_{g_k}$ where $g_k < j \leqslant r$. In particular, each z_{ij} is represented in the C_1 basis using only the basis elements $c_0, c_1, \ldots, c_r$ (and not necessarily all of these basis elements.) Each of the $r+1$ basis elements gives rise to an equation which must be zero: The fact that $\sum a_{ij} z_{ij} = d(z) = 0$ is equivalent to stating that the coefficient of each basis element in $\sum a_{ij} z_{ij}$ must sum to zero. This observation plays a role in the computer code. Instead of looking for a whole system of coefficients a_{ij} which simultaneously satisfies our conditions for z, the program looks for coefficients one basis element at a time. First, it searches for which a_{ij} actually even appear as coefficients of c_r in $\sum a_{ij} z_{ij}$. Then, it uses the previously mentioned trial and error to look for a combination of those a_{ij} which force the sum of coefficients of c_r to be zero. Once the program has found coefficients that work it fixes those values of a_{ij} and examines which remaining a_{ij} appear as coefficients of c_{r-1}. Using trial and error, the program picks choices for these new coefficients. If the program finds a set of a_{ij} that works for both c_r and c_{r-1} it proceeds in the same fashion to look for coefficients of c_{r-2}. If it is not able to find coefficients for c_{r-1} which incorporate the previous coefficient settings of c_r, the program returns to examining c_r. It looks for a new set of a_{ij} which ensure that the sum of coefficients of c_r is zero and continues again. The process eventually outputs a full set of a_{ij} when it finds a complete set of coefficients which has been tested against each basis element c_r. Then, it repeats the process with untested settings of a_{ij} in order to produce all combinations of a_{ij} which satisfy (3.3).

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