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THE ETA INVARIANT AND THE "TWISTED" CONNECTIVE K-THEORY OF THE CLASSIFYING SPACE FOR CYCLIC 2-GROUPS

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Abstract

Let $\ell = 2^\nu \geqslant 2$. We use the eta invariant to study the "twisted" connective real K-theory groups $ko_m(B\mathbb{Z}_\ell, \xi_1)$ of the classifying space $B\mathbb{Z}_\ell$ for the cyclic group $\mathbb{Z}_\ell$.

1. Introduction

Atiyah [**1**] expressed the complex K-theory of the classifying space of $\mathbb{Z}_\ell$ in terms of the complex representation ring of $\mathbb{Z}_\ell$. Looking for a "geometric" construction of Elliptic homology, Kreck and Stolz [**11**] gave a geometric characterization of connective real K-theory. Stolz used this characterization to study when a simply connected manifold admits a metric with positive scalar curvature, see [**12**] for details. Botvinnik and Gilkey [**6**] and Botvinnik, Gilkey, and Stolz [**7**] studied when a manifold with non-trivial fundamental group admits a metric with positive scalar curvature. They defined a "twisted" geometric version of connective real K-theory. In this paper we will express the "twisted" connective real K-theory of the classifying space $\mathbb{Z}_\ell$ in terms of the complex representation ring of $\mathbb{Z}_\ell$. Instead of using topological methods as Atiyah did, we shall use analytical methods; our fundamental tool is the eta invariant.

Let $\mathbb{Z}_\ell := \{\lambda \in \mathbb{C} : \lambda^\ell = 1\}$ be the cyclic group of order $\ell = 2^\nu \geqslant 2$. Let $\rho_s(\lambda) = \lambda^s$ define an irreducible linear representation of $\mathbb{Z}_\ell$. The ρ_s parametrize the irreducible representations of $\mathbb{Z}_\ell$. Let ξ_1 be the underlying real 2 plane bundle of the complex line bundle defined by the representation ρ_1. Let $D(\xi_1)$, $S(\xi_1)$ be the disk bundle and respectively sphere bundle with respect to some fiber metric on ξ_1. Let $T(\xi_1) = D(\xi_1)/S(\xi_1)$ be the Thom space associated with ξ_1. We use the Thom-Pontryagin construction to define the twisted equivariant spin bordism groups and twisted connective real K-theory groups by:

$$\begin{aligned} MSpin_m(B\mathbb{Z}_\ell, \xi_1) &:= \widetilde{MSpin}_{m+2}(T(\xi_1)) \\ ko_m(B\mathbb{Z}_\ell, \xi_1) &:= \widetilde{ko}_{m+2}(T(\xi_1)). \end{aligned}$$

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We refer to [**6, 7**] for further details. Notice that if we replace ξ_1 by the trivial line bundle ξ_0, then we get the standard untwisted equivariant bordism and real connective K-theory groups. In Section 2.2, we shall give a more geometric definition of these groups.

The calculations of [**6**] using the Adams Spectral Sequence show that the groups $ko_{8k+i}(B\mathbb{Z}_\ell, \xi_1)$ have finite order for $i = 1, 3, 5, 7$ and that $ko_m(B\mathbb{Z}_\ell, \xi_1) = 0$ otherwise. The additive structure of these groups has not been determined previously. We note that the additive structures of the groups $ko_*(B\mathbb{Z}_\ell)$ were studied in [**5**].

We can now state the main result of this paper. If $\tau : \mathbb{Z}_\ell \to U(2k)$ is a fixed point free representation, we denote the associated lens space by

$$L^{4k-1}(\ell;\tau) := S^{4k-1}/\tau(\mathbb{Z}_\ell).$$

Let $RU_0(\mathbb{Z}_\ell)$ be the augmentation ideal of the unitary group representation ring $RU(\mathbb{Z}_\ell)$. Let $\widetilde{KSp}$ and $\widetilde{KU}$ be the reduced symplectic and complex K-theory groups. Let

$$\mathcal{I} = \{\rho \in RU_0(\mathbb{Z}_\ell) : \rho(\bar{\lambda}) = -\rho(\lambda)\}.$$

Theorem 1.1. *Let $k \geqslant 1$.*

1. *We have that $ko_{4k+1}(B\mathbb{Z}_\ell, \xi_1) \approx \widetilde{KSp}(L^{4k+5}(\ell, \tau))$ for any suitable τ.*
2. *We have that $ko_{4k-1}(B\mathbb{Z}_\ell, \xi_1) \approx \mathcal{I}/(\mathcal{I} \cap RU_0(\mathbb{Z}_\ell)^{2k+2})$.*

Here is a brief guide to this paper. In Section 2.3, we review the properties concerning the eta invariant which we shall need. In Section 3, we prove the main Theorem.

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2. The eta invariant and "twisted" connective real K-theory

2.1. Notational conventions

If ρ is a virtual unitary or symplectic representation, let V_ρ be the associated virtual flat bundle and let $[V_\rho]$ be the corresponding element in K-theory. Let $M = L^{4k-1}(\ell;\tau)$. The map $\rho \to [V_\rho]$ defines a surjective homomorphism from $RU_0(\mathbb{Z}_\ell)$ and $RSp_0(\mathbb{Z}_\ell)$ to $\widetilde{KU}(M)$ and $\widetilde{KSp}(M)$; see [**2, 10**].

2.2. A geometrical realization of connective K-theory and bordism

The equivariant spin bordism groups $MSpin_m(B\mathbb{Z}_\ell, \xi_1)$ can also be thought of as equivalence classes of triples (M, f, s) where M is a closed manifold of dimension m which need not be connected, f is a $\mathbb{Z}_\ell$ structure on M, and s is a *spin* structure on $T(M) \oplus f^*(\xi_1)$; such a manifold M admits a *spin*^{c} structure with determinant line bundle given by ρ_1, see [**4, 6**] for details. We define the relation $(M, f, s) \sim 0$ in $MSpin_m(B\mathbb{Z}_\ell, \xi_1)$ if there exists a compact manifold N with boundary M so that the structures s and f extend over N. Disjoint union defines the group structure; Cartesian product makes $MSpin_*(B\mathbb{Z}_\ell, \xi_1)$ into a $MSpin_*$ module. Let $T_m(B\mathbb{Z}_\ell, \xi_1)$ be the subgroup of $MSpin_m(B\mathbb{Z}_\ell, \xi_1)$ which is generated by the total spaces of geometrical fiber bundles $p : E \to B$ with fiber quaternionic projective space $\mathbb{HP}^2$; since $\mathbb{HP}^2$ is simply connected, we can identify $\mathbb{Z}_\ell$ structures on the base with those on the total space. We refer to Stolz [**13**] for the proof of the following lemma; it is fundamental to our study since it expresses the "twisted" connective real K-theory groups of $B\mathbb{Z}_\ell$ geometrically.

Theorem 2.1. $ko_m(B\mathbb{Z}_\ell, \xi_1) = MSpin_m(B\mathbb{Z}_\ell, \xi_1)/T_m(B\mathbb{Z}_\ell, \xi_1)$.

2.3. The eta invariant

Let $D = D(M, g, s)$ be the Dirac operator defined by the spin structure s and a Riemannian metric g, and let ρ be a complex representation of $\mathbb{Z}_\ell$. Let D_ρ be the Dirac operator with coefficients in the flat vector bundle V_ρ defined by ρ. Let $\{\lambda_n\}$ be the eigenvalues of D_ρ where each eigenvalue is repeated according to its multiplicity. The function $\eta(D_\rho, z) := \sum_{\lambda_n \neq 0} \operatorname{sign}(\lambda_n)|\lambda_n|^{-z}$ converges to a holomorphic function for $\Re e(z) \gg 0$. This function has a meromorphic extension to $\mathbb{C}$. The value $z = 0$ is a regular value and we define

$$\eta(M, f, s)(\rho) := \frac{1}{2}(\eta(D_\rho, z)|_{z=0} + \dim\ker(D_\rho))$$

as a measure of the spectral asymmetry of D_ρ. We refer to [**3, 9**] for details. The Atiyah-Patodi-Singer Index Theorem for manifolds with boundary [**3**] can be used to establish the following result. We refer to [**6**] (Thm 3.1), [**7**] for details:

Theorem 2.2. *If $m = 4k + 1$, let $\rho \in RU(\mathbb{Z}_\ell)$; if $m = 4k - 1$, let $\rho \in RU_0(\mathbb{Z}_\ell)$. The map $M \to \eta(M, f, s)(\rho)$ extends to a homomorphism η_ρ from $ko_m(B\mathbb{Z}_\ell, \xi_1)$ to $\mathbb{R}/\mathbb{Z}$.*

2.4. Lens spaces and lens space bundles

Let $\vec{a} = (a_1, ..., a_k)$ be a collection of odd indices and let $\tau = \tau(\vec{a}) := \rho_{a_1} \oplus ... \oplus \rho_{a_k}$ define a fixed point free representation from $\mathbb{Z}_\ell$ to $U(k)$. Let $L^{2k-1}(\ell, \tau) := S^{2k-1}/\tau(\mathbb{Z}_\ell)$ be the associated lens space. Let $H^{\otimes 2} \oplus (k-1)1$ be the Whitney sum of the tensor square of the complex Hopf line bundle with $(k-1)$ copies of the trivial complex line bundle over complex projective space $\mathbb{CP}^1$ which we identify with the sphere S^2. We let $\lambda \in S^1$ act by multiplication by λ^{a_ν} on the ν^{th} summand. This action restricts to a fixed point free action of $\mathbb{Z}_\ell$ on the associated sphere bundle. Let

$$X^{2k+1}(\ell; \tau) := S(H^{\otimes 2} \oplus (k-1)1)/\tau(\mathbb{Z}_\ell)$$

be the associated lens space bundle over S^2. We give $L^{2k-1}(\ell, \tau)$ and $X^{2k+1}(\ell; \tau)$ the natural $\mathbb{Z}_\ell$ structures. The lens space $L^{2k-1}(\ell; \tau)$ and the lens space bundles $X^{2k+1}(\ell; \tau)$ admit a natural *spin*^{c} structure; see [**4, 6, 7**] for details.

2.5. Combinatorial formulas for the eta invariant

Let $\tau = \tau(\vec{a})$. We define

1. If k is even, let $\mathcal{F}_L(\vec{a};\lambda) = \lambda^{-|\vec{a}|/2}\det(I - \tau(\vec{a})(\lambda))$.
2. If k is odd, let $\mathcal{F}_L(\vec{a};\lambda) = \lambda^{-(|\vec{a}|+1)/2}\det(I - \tau(\vec{a})(\lambda))$.
3. If $\lambda \neq 1$, let $\mathcal{G}_L(\vec{a};\lambda) = \mathcal{F}_L(\vec{a};\lambda)^{-1}$. If $\lambda = 1$, let $\mathcal{G}_L(\vec{a};\lambda) = 0$.
4. Let $\mathcal{G}_X(\vec{a};\lambda) = (1+\lambda^{a_1})(1-\lambda^{a_1})^{-1}\mathcal{G}_L(\vec{a};\lambda)$.

Let $\widetilde{\sum}_\lambda := \sum_{\lambda\in\mathbb{Z}_\ell,\lambda\neq 1}$. The following combinatorial formulas follow from work of Donnelly [**8**]:

Theorem 2.3. *1. We have* $\eta(L^{4k-1}(\ell;\tau))(\rho) = \ell^{-1}\widetilde{\sum}_\lambda \operatorname{Tr}(\rho(\lambda))\mathcal{G}_L(\tau)(\lambda)$.

2. We have $\eta(X^{4k+1}(\ell;\tau))(\rho) = \ell^{-1}\widetilde{\sum}_\lambda \operatorname{Tr}(\rho(\lambda))\mathcal{G}_X(\tau)(\lambda)$.

The eta invariant completely detects the odd dimensional "twisted" connective K-theory groups $ko_m(B\mathbb{Z}_\ell,\xi_1)$. We refer to [**6**] for the proof of the following result. Notice that these manifolds have positive scalar curvature, therefore $\hat{A} = 0$.

Theorem 2.4. *Let* $M := [(M,f,s)] \in ko_m(B\mathbb{Z}_\ell,\xi_1)$.

1. *If* $m \equiv 1 \bmod 8$, *then* $M = 0$ *if and only if* $\eta(M)(\rho) = 0$ *in* $\mathbb{R}/\mathbb{Z}$ *for all* $\rho \in RU(\mathbb{Z}_\ell)$.
2. *If* $m \equiv 3 \bmod 8$, *then* $M = 0$ *if and only if* $\eta(M)(\rho) = 0$ *in* $\mathbb{R}/\mathbb{Z}$ *for all* $\rho \in RU_0(\mathbb{Z}_\ell)$.
3. *If* $m \equiv 5 \bmod 8$, *then* $M = 0$ *if and only if* $\eta(M)(\rho) = 0$ *in* $\mathbb{R}/\mathbb{Z}$ *for all* $\rho \in RU(\mathbb{Z}_\ell)$.
4. *If* $m \equiv 7 \bmod 8$, *then* $M = 0$ *if and only if* $\eta(M)(\rho) = 0$ *in* $\mathbb{R}/\mathbb{Z}$ *for all* $\rho \in RU_0(\mathbb{Z}_\ell)$.

Let $c_{Sp} : RSp(\mathbb{Z}_\ell) \to RU(\mathbb{Z}_\ell)$ be the natural injective homomorphism obtained by forgetting the symplectic structure to get a complex structure.

We refer to [**2, 9, 10**] for the proof of the following result.

Theorem 2.5. *Let* $M := L^{2k-1}(\ell;\tau)$.

1. *Let* $\rho \in RU_0(\mathbb{Z}_\ell)$. *The following conditions are equivalent:*
   (a) $\rho \in RU_0(\mathbb{Z}_\ell)^k$.
   (b) $\eta(M)(\rho\tilde{\rho}) \in \mathbb{Z}\ \forall\ \tilde{\rho} \in RU_0(\mathbb{Z}_\ell)$.
   (c) $[V_\rho] = 0$ *in* $\widetilde{KU}(M)$.
2. *Let* $\gamma = c_{Sp}(\rho)$ *for* $\rho \in RSp_0(\mathbb{Z}_\ell)$. *If* $2k-1 \equiv 7 \bmod 8$, *the following conditions are equivalent:*
   (a) $\gamma \in \psi^k c_{Sp} RSp_0(\mathbb{Z}_\ell)$ *where* $\psi = (\rho_0 - \rho_1)^2\rho_{-1}$.
   (b) $\eta(M)(\gamma\tilde{\rho}) \in \mathbb{Z}\ \forall\ \tilde{\rho} \in RU_0(\mathbb{Z}_\ell)$ *and* $(\ell/2)\eta(M)(\gamma) \in \mathbb{Z}$.
   (c) $[V_\rho] = 0$ *in* $\widetilde{KSp}(M)$.

3. *Let* $\gamma = c_{Sp}(\rho)$ *for* $\rho \in RSp_0(\mathbb{Z}_\ell)$. *If* $2k-1 \equiv 3 \bmod 8$, *the following conditions are equivalent:*
   (a) $\gamma \in RU_0(\mathbb{Z}_\ell)^k$.
   (b) $\eta(M)(\gamma\tilde{\rho}) \in \mathbb{Z} \ \forall \ \tilde{\rho} \in RU_0(\mathbb{Z}_\ell)$.
   (c) $[V_\rho] = 0$ *in* $\widetilde{KSp}(M)$.

3. The proof of the main theorem

We consider the free Abelian group generated by the lens spaces $L^{4k+1}(\ell;\vec{a})$ and by the lens space bundles $X^{4k-1}(\ell;\vec{a})$; we give these manifolds the natural structures and omit these structures from the notation in the interests of notational simplicity. Define:

1. $\mathcal{B}L^{4k+1}(\ell;\cdot,3) := 3L^{4k+1}(\ell;\cdot,3) - L^{4k+1}(\ell;\cdot,1)$.
2. $\mathcal{B}X^{4k-1}(\ell;\cdot,3) := 3X^{4k-1}(\ell;\cdot,3) - X^{4k-1}(\ell;\cdot,1)$.

This defines $\mathcal{B}$ on any lens space and lens space bundle. We extend $\mathcal{B}$ to the free Abelian group generated by these manifolds. We define $M^L_{m,j}$ for $2j-1 \leqslant m$. When considering the lens space bundles, we assume the index "3" in question is not the first index. Thus we define $M^X_{m,j}$ for $2j-1 \leqslant m-4$. The eta invariant is additive with respect to direct sums and extends to this setting.

1. $M^L_{4k+1,j} := \mathcal{B}^j L^{4k+1}(\ell;3,...,3)$.
2. $M^X_{4k-1,j} := \mathcal{B}^j X^{4k-1}(\ell;1,3,...,3)$.

Theorem 3.1. *We have*

$$ko_{4k+1}(B\mathbb{Z}_\ell,\xi_1) = \operatorname{span}\{M^L_{4k+1,j} : 0 \leqslant j \leqslant 2k+1\}, \text{ and}$$
$$ko_{4k-1}(B\mathbb{Z}_\ell;\xi_1) = \operatorname{span}\{M^X_{4k-1,j} : 0 \leqslant j \leqslant 2k-2\}$$

Proof. We prove this theorem with the following lemmas.

We refer to [**6**] (Lemma 4.2) for the proof of the following result.

Lemma 3.2. *Let* $\widetilde{\sum}_\lambda := \sum_{\lambda\in\mathbb{Z}_\ell,\lambda\neq 1}$.

1. *If* k *is even, then* $L^{2k-1}(\ell;\vec{a})$ *and* $X^{2k+1}(\ell;\vec{a})$ *admit spin structures.*
2. *If* k *is odd, then* $L^{2k-1}(\ell;\vec{a})$ *and* $X^{2k+1}(\ell;\vec{a})$ *have spin*^{c} *structures with determinant line bundle given by* ρ_1.
3. *We have* $\eta(L^{2k-1}(\ell;\vec{a}))(\rho) = \ell^{-1}\widetilde{\sum}_\lambda \operatorname{Tr}(\rho)\mathcal{G}_L(\vec{a};\lambda) \in \mathbb{Q}$.
4. *We have* $\eta(X^{2k+1}(\ell;\vec{a}))(\rho) = \ell^{-1}\widetilde{\sum}_\lambda \operatorname{Tr}(\rho)\mathcal{G}_X(\vec{a};\lambda) \in \mathbb{Q}$.

We have the following integrality theorem; we refer to [**6**] (Lemma 4.2) for the proof.

Lemma 3.3. *Let* $\rho \in RU_0(\mathbb{Z}_\ell)^j$. *Let* $m < 2j+1$. *Then*

$$\eta(L^m(\ell;\cdot))(\rho) \in \mathbb{Z} \text{ and } \eta(X^m(\ell;\cdot))(\rho) \in \mathbb{Z}.$$

Lemma 3.4. *Let $\rho \in RU(\mathbb{Z}_\ell)$ and let $\psi := (\rho_0 - \rho_1)^2\rho_{-1}$.*

1. *$\eta(\mathcal{B}M)(\rho) = \eta(M)(\psi\rho)$ for M a lens space or suitable lens space bundle.*
2. *$\eta(M^L_{m,j})(\rho) = \eta(M^L_{m,0})(\psi^j\rho)$ and $\eta(M^X_{m,j})(\rho) = \eta(M^X_{m,0})(\psi^j\rho)$*

Proof. We see that

$$\mathcal{G}_L(\vec{a}, 1; \lambda) - 3\mathcal{G}_L(\vec{a}, 3; \lambda) = \psi(\lambda)\mathcal{G}_L(\vec{a}, 3; \lambda).$$

Consequently,

$$\eta(\mathcal{B}L^{2k+1}(\ell; \vec{a}, 3))(\rho) = \eta(L^{2k+1}(\ell; \vec{a}, 3))(\psi\rho)$$

and assertions concerning lens spaces follow. Similarly,

$$\eta(\mathcal{B}X^{2k+3}(\ell; \vec{a}, 3))(\rho) = \eta(X^{2k+3}(\ell; \vec{a}, 3))(\psi\rho)$$

provided that the index "3" is not the first index; the first index plays a distinguished role in the definition of $\mathcal{G}_X$. □

Lemma 3.5. *Let $\alpha := \rho_{-3}(\rho_0 - \rho_3)^2 \in RU_0(\mathbb{Z}_\ell)^2$.*

1. *$\eta(M^L_{m,j})(\alpha\rho) = \eta(M^L_{m-4,j})(\rho)$.*
2. *$\eta(M^X_{m,j})(\alpha\rho) = \eta(M^X_{m-4,j})(\rho)$.*
3. *$\eta(M^L_{5,0})(\alpha\rho_{-2}) = (\ell - 1)/2\ell$.*
4. *$\eta(M^X_{5,0})(\alpha(\rho_0 - \rho_3)\rho_{-2}) = (\ell - 2)/\ell$.*
5. *If $\rho \in RU(\mathbb{Z}_\ell)$, then $\eta(M^L_{4k+1,k})(\alpha\rho) \in \mathbb{Z}$. Furthermore there exists γ^L_{4k+1} so that $\eta(M^L_{4k+1,k})(\alpha\rho)(\gamma^L_{4k+1}) = (\ell - 1)/2\ell$.*
6. *If $\rho \in RU(\mathbb{Z}_\ell)$, then $\eta(M^X_{4k+1,k})(\alpha\rho) \in \mathbb{Z}$. Furthermore there exists γ^X_{4k+1} so that $\eta(M^L_{4k+1,k})(\alpha\rho)(\gamma^X_{4k+1}) = (\ell - 2)/\ell$.*

Proof. Since $\mathcal{F}(\vec{a}, 3, 3; \lambda) = \alpha(\lambda)\mathcal{F}(\vec{a}; \lambda)$,

$$\eta(L^{m+4}(\ell; \vec{a}, 3, 3))(\alpha\rho) = \eta(L^m(\ell; \vec{a}))(\rho)$$
$$\eta(X^{m+4}(\ell; \vec{a}, 3, 3))(\alpha\rho) = \eta(X^m(\ell; \vec{a}))(\rho).$$

The first two assertions now follow. We prove the second two assertions by computing:

$$\begin{aligned}\eta(M^L_{5,0})(\alpha\rho_{-2}) &= \ell^{-1}\widetilde{\sum}_\lambda(1 - \lambda^3)^{-1}\\ &= (2\ell)^{-1}\widetilde{\sum}_\lambda((1 - \lambda)^{-1} + (1 - \bar{\lambda})^{-1})\\ &= (2\ell)^{-1}\widetilde{\sum}_\lambda 1 = (\ell - 1)/(2\ell)\end{aligned}$$

and

$$\eta(M^X_{5,0})(\alpha(\rho_0 - \rho_3)\rho_{-2}) = \ell^{-1}\widetilde{\sum}_\lambda(1 + \lambda^3) = \ell^{-1}\widetilde{\sum}_\lambda(1 + \lambda) = (\ell - 2)/\ell.$$

We complete the proof by establishing the final two assertions. We use Lemma 3.4 to compute

$$\eta(M^L_{4k+1,k})(\alpha\rho) = \eta(M^L_{4k+1,0})(\alpha\psi^k\rho), \text{ and}$$
$$\eta(M^X_{4k+1,k})(\alpha\rho) = \eta(M^X_{4k+1,0})(\alpha\psi^k\rho).$$

Then $\rho\alpha\psi^k \in RU_0(\mathbb{Z}\ell)^{2k+2}$. Since $\dim(M^L_{4k+1,0}) = \dim(M^X_{4k+1,0}) = 2(2k+1)-1$, these eta invariants take values in $\mathbb{Z}$ by Lemma 3.3. Similarly, we compute:

$$\eta(M^L_{4k+1,k})(\gamma_{k,L}) = \eta(M^L_{m,0})(\gamma_{k,L}\psi^k), \text{ and}$$
$$\eta(M^X_{4k+1,k})(\gamma_{k,X}) = \eta(M^X_{m,0})(\gamma_{k,X}\psi^k).$$

We have $\psi^k R(\mathbb{Z}_\ell) = \alpha^k R(\mathbb{Z}_\ell)$. Thus we may choose $\gamma_{k,L}$ so that $\gamma_{k,L}\psi^k = \alpha^k\rho_{-2}$; let $\gamma_{k,X} = \gamma_{k,L}(\rho_0 - \rho_3)$. Then

$$\begin{aligned}\eta(M^L_{m,0})(\gamma_{k,L}\psi^k) = \eta(M^L_{m,0})(\alpha^k\rho_{-2}) &= \eta(M^L_{5,0})(\alpha\rho_{-2})\\ &= (\ell-1)/2\ell\\ \eta(M^X_{m,0})(\gamma_{k,X}\psi^k) &= \eta(M^L_{m,0})(\alpha^k(\rho_0-\rho_3)\rho_{-2})\\ &= \eta(M^L_{5,0})(\alpha(\rho_0-\rho_3)\rho_{-2}) = (\ell-2)/\ell.\end{aligned}$$

□

Let $k \geqslant 0$. We define

1. $\mathcal{M}^L_{4k+1}(\ell) := \text{span}_{0\leqslant j\leqslant 2k+1}\{M^L_{4k+1,j}\} \subset ko_{4k+1}(B\mathbb{Z}_\ell, \xi_1)$.
2. $\mathcal{M}^X_{4k-1}(\ell) := \text{span}_{0\leqslant j\leqslant 2k-2}\{M^X_{4k-1,j}\} \subset ko_{4k-1}(B\mathbb{Z}_\ell, \xi_1)$.

The Pontryagin dual A^* of an Abelian group A is the group of homomorphisms to $\mathbb{R}/\mathbb{Z}$. Thus, for example, $\mathbb{Z}/\ell\mathbb{Z}$ is the Pontryagin dual of $\mathbb{Z}_\ell$. Let $\eta^*(M)$ be the homomorphism which sends ρ to $\eta(M)(\rho)$. By Theorem 2.5, the eta invariant extends to connective K-theory:

$$\eta^* : ko_{4k+1}(B\mathbb{Z}_\ell, \xi_1) \to RU(\mathbb{Z}_\ell)^*, \text{ and } \eta^* : ko_{4k-1}(B\mathbb{Z}_\ell, \xi_1) \to RU_0(\mathbb{Z}_\ell)^*.$$

The homomorphism which sends ρ to $\alpha\rho$ defines a dual map α^* from $RU(\mathbb{Z}_\ell)^*$ to $RU(\mathbb{Z}_\ell)^*$.

Lemma 3.6. *Let $k \geqslant 0$ and assume $\ell \geqslant 4$.*

1. $|\eta^*\mathcal{M}^L_{4k+5}(\ell)| \geqslant (2\ell)^{k+2}$.
2. $ko_{4k+5}(B\mathbb{Z}_\ell, \xi_1) = \mathcal{M}^L_{4k+5}(\ell)$.

Proof. It is immediate that

$$|\eta^*\mathcal{M}^L_m(\ell)| \geqslant |\alpha^*\eta^*\mathcal{M}^L_m(\ell)| \cdot |\ker\alpha^* \cap \eta^*\mathcal{M}^L_m(\ell)|$$

We use Lemma 3.5 to see that

$$\begin{aligned}|\alpha^*\eta^*\mathcal{M}^L_m(\ell)| &\geqslant |\eta^*\mathcal{M}^L_{m-4}(\ell)|,\\ |\alpha^*\eta^*\mathcal{M}^L_5(\ell)| &\geqslant 2\ell\\ |\ker\alpha^* \cap \eta^*\mathcal{M}^L_m(\ell)| &\geqslant 2\ell.\end{aligned}$$

This proves the first assertion and gives a lower bound for $ko_m(B\mathbb{Z}_\ell, \xi_1)$ if $m \equiv 1 \mod 4$.

The following estimates were established in Botvinnik and Gilkey [**6**].

1. $|ko_{8k+1}(\mathbb{Z}_\ell, \xi_1)| = (2\ell)^{2k+1}$.
2. $|ko_{8k+3}(\mathbb{Z}_\ell, \xi_1)| = (\ell/2)^{2k+1}$.
3. $|ko_{8k+5}(\mathbb{Z}_\ell, \xi_1)| = (2\ell)^{2k+2}$.
4. $|ko_{8k+7}(\mathbb{Z}_\ell, \xi_1)| = (\ell/2)^{2k+2}$.

The final assertion follows from these estimates. □

Again, we begin our discussion with a technical Lemma.

Lemma 3.7. *1.* $\eta(M^L_{3,0})((\rho_0 - \rho_3)\rho_{-2}) = (\ell - 1)/2\ell$.

2. $\eta(M^X_{3,0})((\rho_0 - \rho_3)^2\rho_{-2}) = (\ell - 2)/\ell$.

Proof. We prove the first assertion by computing:

$$\begin{aligned}\eta(M^L_{3,0})((\rho_0 - \rho_3)\rho_{-2}) &= \ell^{-1}\widetilde{\sum}_\lambda(1 - \lambda^3)^{-1}\\ &= (2\ell)^{-1}\widetilde{\sum}_\lambda((1 - \lambda)^{-1} + (1 - \bar\lambda)^{-1})\\ &= (2\ell)^{-1}\widetilde{\sum}_\lambda 1 = (\ell - 1)/(2\ell).\end{aligned}$$

and

$$\eta(M^X_{3,0})((\rho_0 - \rho_3)^2\rho_{-2}) = \ell^{-1}\widetilde{\sum}_\lambda(1 + \lambda^3) = \ell^{-1}\widetilde{\sum}_\lambda(1 + \lambda) = (\ell - 2)/\ell.$$

□

Lemma 3.8. *If* $k \geqslant 0$ *and if* $\ell \geqslant 4$, *then*

1. $|\eta^*(\mathcal{M}^X_{4k+3}(\ell)| \geqslant (\ell/2)^{k+1}$.

2. $ko_{4k+3}(B\mathbb{Z}_\ell, \xi_1) = \mathcal{M}^X_{4k+3}(\ell)$.

Proof. The first assertion follows from Lemma 3.7 if $k = 0$, so we assume $k \geqslant 1$ henceforth. Let $\delta = (\rho_0 - \rho_3)\rho_{-2}$ and let $m = 4k + 3$. Then,

$$\eta(M^X_{m,j}(\ell))(\delta\rho) = \eta(M^X_{m-2,j}(\ell))(\rho).$$

Thus,

$$\eta^*\mathcal{M}^X_{4k+1} \subset \delta^*\eta^*\mathcal{M}^X_{4k+3}.$$

We use this equation and Lemma 3.7 to complete the proof of the first two assertions for $k \geqslant 1$ by computing:

$$(\ell/2)^{k+1} \leqslant |\eta^*(\mathcal{M}^X_{4k+1}(\ell))| \leqslant |\eta^*(\mathcal{M}^X_{4k+3}(\ell))|$$

□

We can now complete the proof of the main Theorem. Let

$$\psi = -(\rho_0 - \rho_{-1})(\rho_0 - \rho_1) = (\rho_0 - \rho_1)^2\rho_{-1} \in RU_0(\mathbb{Z}_\ell)^2.$$

We define $W^{4k+5} = L^{4k+5}(\ell; 3, 3, ..., 3, 1, -1)$. Then

$$\eta(M^L_{4k+1,j})(\rho) = \eta(M^L_{4k+1,0})(\psi^j\rho) = \eta(W^{4k+5})(\psi^{j+1}\rho).$$

Let $\mathcal{A}_{4k+1}$ be the linear span of the manifolds $M^L_{4k+1,j}$ for $0 \leqslant j \leqslant 2k+1$. Let $\sigma(M^L_{4k+1,j}) := \psi^{j+1}$. We extend σ linearly to $\mathcal{A}_{4k+1}$. We then have

$$\eta(M)(\rho) = \eta(W^{4k+5})(\sigma(M)\rho)\ \forall\ M \in \mathcal{A}_{4k+1}.$$

If $[M] = 0$ in $ko_{4k+1}(B\mathbb{Z}_\ell, \xi_1)$, then $\eta(M)(\rho) \in \mathbb{Z}$ for all $\rho \in RU_0(\mathbb{Z}_\ell)$ by 2.5 so

$$\eta(W^{4k+5})(\sigma(M)\rho) \in \mathbb{Z} \qquad \forall\rho \in RU_0(\mathbb{Z}_\ell).$$

By Theorem 2.5(1), $\sigma(M) \in RU_0(\mathbb{Z}_\ell)^{2k+3}$. Thus by Lemma 3.6, σ induces a map

$$\sigma : ko_{4k+1}(B\mathbb{Z}_\ell, \xi_1) \to RU_0(\mathbb{Z}_\ell)/RU_0(\mathbb{Z}_\ell)^{2k+3}.$$

Suppose first that $m = 4k + 1$. If $\sigma([M]) = \sigma(M) = 0$ then

$$\eta(M)(\rho) \in \mathbb{Z} \qquad \forall\rho \in RU_0(\mathbb{Z}_\ell).$$

Since we have that $\mathcal{G}_L(\ell; \vec{a})(\bar{\lambda}) = -\mathcal{G}_L(\ell; \vec{a})(\lambda)$, $\eta(M)(\rho_0) = 0$. Thus we have $\eta(M)(\rho) = 0$ for all $\rho \in RU(\mathbb{Z}_\ell)$ so by Theorem 2.4, $[M] = 0$ in $ko_{4k+1}(\mathbb{Z}_\ell; \xi_1)$. Thus σ is injective. We prove the first assertion of the main Theorem by noticing that:

$$c_{Sp}(RSp_0(\mathbb{Z}_\ell)) = \mathrm{span}\{\psi^{j+1} : j \geqslant 0\}.$$

Suppose next that $m = 4k - 1$. Let $\mathcal{A}_{4k-1}$ be the free group generated by $M^X_{4k-1,j}$ for $0 \leqslant j \leqslant 2k - 2$. Let $Y^{4k+3} := L^{4k+3}(\ell; 1, 3, 3, ..., 3, 1, -1)$. We then have

$$\eta(M^X_{4k-1,j})(\rho) = \eta(M^X_{4k-1,0})(\psi^j\rho) = \eta(Y^{4k+3})(\psi^j\theta\rho).$$

Where $\theta = (\rho_0 - \rho_{-1})(\rho_0 + \rho_1)$. Define $\sigma(M^X_{4k-1,j}) = \psi^j\theta$ and extend σ linearly to $\mathcal{A}_{4k-1}$. We then have

$$\eta(M)(\rho) = \eta(Y^{4k+3})(\sigma(M)\rho) \qquad \text{for all } M \in \mathcal{A}_{4k-1}.$$

If $[M] = 0$ in $ko_{4k-1}(B\mathbb{Z}_\ell, \xi_1)$, then $\eta(M)(\rho) = 0$ in $\mathbb{R}/\mathbb{Z}$ for all $\rho \in RU_0(\mathbb{Z}_\ell)$. Thus $\eta(Y^{4k+3})(\sigma(M)\rho) = 0$ in $\mathbb{R}/\mathbb{Z}$ for all $\rho \in RU_0(\mathbb{Z}_\ell)$, so $\sigma(M) \in RU_0(\mathbb{Z}_\ell)^{2k+2}$. Thus we may regard σ a well defined map

$$\sigma : ko_{4k-1}(B\mathbb{Z}_\ell, \xi_1) \to RU_0(\mathbb{Z}_\ell)/RU_0(\mathbb{Z}_\ell)^{2k+2}.$$

If $\sigma([M]) = 0$, then $\eta(M)(\rho) = 0$ for all $\rho \in RU_0(\mathbb{Z}_\ell)$ and by Theorem 2.4, $[M] = 0$ in $ko_{4k-1}(B\mathbb{Z}_\ell, \xi_1)$. Thus σ is injective. The set $\mathcal{I}$ is generated by $\rho_s - \rho_{-s}$ and by ψ^j for $j > 0$. Since

$$\theta = \rho_1 - \rho_{-1} = (\rho_0 + \rho_1)(\rho_0 - \rho_{-1}),$$

$\mathcal{I}$ is generated by $\psi^j\theta$ for $j \geqslant 0$. As we can work modulo $RU_0(\mathbb{Z}_\ell)^{2k+2}$, we can restrict $0 \leqslant j \leqslant k+1 < 2k+2$. Notice that

$$\sigma(ko_{4k-1}(B\mathbb{Z}_\ell,\xi_1)) + RU_0(\mathbb{Z}_\ell)^{2k+2} = \mathcal{I} + RU_0(\mathbb{Z}_\ell)^{2k+2}.$$

Thus $ko_{4k-1}(B\mathbb{Z}_\ell,\xi_1) \approx \mathcal{I}/(\mathcal{I} \cap RU_0(\mathbb{Z}_\ell)^{2k+2})$. This completes the proof of the Theorem. □

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